Constructions of low-degree and error-correcting $\epsilon$-biased sets

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Abstract

In this work we give two new constructions of $\epsilon$-biased generators. Our first construction answers an open question of Dodis and Smith [DS05], and our second construction significantly extends a result of Mossel et al. [MST03]. In particular we obtain the following results:

1. We construct a family of asymptotically good binary codes such that the codes in our family are also $\epsilon$-biased sets for an exponentially small $\epsilon$. Our encoding and decoding algorithms run in polynomial time in the block length of the code. This answers an open question of Dodis and Smith [DS05].

2. We construct a degree $k$ $\epsilon$-biased generator, $G : \{0,1\}^m \rightarrow \{0,1\}^n$, for every $k = o(\log n)$. For $k$ constant we get that $n = \Omega(m/\log(1/\epsilon))^k$, which is nearly optimal. Our result also separates degree $k$ generators from generators in $NC^0_k$, showing that the stretch of the former can be much larger than the stretch of the latter. This problem of constructing degree $k$ generators was introduced by Mossel et al. [MST03] who gave a construction only for the case of degree 2 generators.

1 Introduction

A subset $S \subseteq \{0,1\}^n$ is called an $\epsilon$-biased set if its bias with respect to any linear test is at most $\epsilon$. Namely, for every non-zero vector $w \in \{0,1\}^n$ we have that $|\Pr_{s \in S}[\langle w, s \rangle = 1] - 1/2| \leq \epsilon$. In other words, for every hyperplane $H \subseteq \{0,1\}^n$ it holds that $|S \cap H| - \frac{|S|}{2} \leq \epsilon |S|$. An $\epsilon$-biased generator is a mapping $G : \{0,1\}^m \rightarrow \{0,1\}^n$ whose image is an $\epsilon$-biased set. A subset $C \subseteq \{0,1\}^n$ is called a good error correcting code\footnote{To be completely accurate we have to speak about family of codes, and we do it in a later section.} if it has an exponential size and the Hamming distance between any two of its elements is linear.

In this paper we give two constructions of $\epsilon$-biased sets. The first is a construction of an $\epsilon$-biased generator such that each of its output bits is a low degree polynomial. Our second

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1To be completely accurate we have to speak about family of codes, and we do it in a later section.
result is a construction of a family of (efficiently encodable and decodable) error correcting codes that are also \(\epsilon\)-biased sets for an exponentially small \(\epsilon\).

**Background.** The notion of \(\epsilon\)-biased sets (or more accurately of an \(\epsilon\)-biased distribution) was first defined by Naor and Naor [NN93] who also gave the first constructions of such distributions and demonstrated the power of \(\epsilon\)-biased sets for several applications. The construction of Naor and Naor was later improved in a series of papers [AGHP92, AIK+90, RSW93, EGL+92, AM95]. Since their first appearance \(\epsilon\)-biased sets have found many applications in different areas of theoretical computer science including: derandomization of algorithms such as fast verification of matrix multiplication [NN93]; construction of almost \(k\)-wise independent distributions [NN93, MNN94]; inapproximability results for quadratic equations over GF(2) [HPS93]; learning theory [AM95]; explicit constructions of Ramsey graphs [Nao92]; explicit constructions of Cayley expanders [AR94, MW02]; construction of efficient low degree tests and short PCPs [BSSVW03]; and construction of two-source extractors [Raz05].

In several recent works \(\epsilon\)-biased sets were studied from a different perspective. In [CM01] Cryan and Miltersen ask whether there exist an \(NC^0\) construction of an \(\epsilon\)-biased generator for which \(n\) is super-linear in \(m\). This question was answered affirmatively by Mossel et al. [MST03] who gave a construction of a generator in \(NC^0\) with \(n = m^{O(\sqrt{k})}\). Mossel et al. also raised the question of constructing degree \(k\) \(\epsilon\)-biased generators - i.e. generators that each of their output bits is a degree \(k\) polynomial in the input bits. They were also able to give a construction of a degree 2 generator with a near optimal stretch (i.e. \(n = \Omega(m^2)\) and \(\epsilon = \exp(-O(n))\)).

In [DS05] Dodis and Smith ask ”Does there exist an explicitly-constructible ensemble of good codes with small bias and polytime encoding and decoding algorithms (ideally, codes with linear rate and minimum distance, and negligible bias)?”. Namely, Dodis and Smith raise the question of constructing a family of good codes that are also \(\epsilon\)-biased sets for an exponentially (in the block length of the code) small \(\epsilon\). Such a family of codes was needed for the construction of a cryptographic scheme that will enable two parties to securely correct errors in a shared secret string. Dodis and Smith managed to give a protocol for correcting errors without leaking information without constructing a family of \(\epsilon\)-biased good codes, however such a construction can simplify their scheme.

**Our results.** Our first result is a construction of degree \(k\) \(\epsilon\)-biased generators of maximal stretch (up to a constant). Namely we give a construction of a degree \(k\) generator from \(m\) bits to \(n = \Omega((m/\log(1/\epsilon))^k)\) bits, for any fixed \(k\). Thus, for every fixed \(\epsilon\) the output length is \(\Omega(m^k)\), and clearly the output length cannot exceed \(m^k\) (as there are only \(O(m^k)\) linearly independent polynomials of degree \(k\) in \(m\) variables).

**Theorem 1** For every integer \(0 < k \) and every large enough\(^2\) integer \(m\) and every \(\epsilon > 0\).

\(^2\)More precisely, there exists \(m_0\), independent of \(k\), such that for every \(m \geq m_0\)...
\[
\exp(-O\left(\frac{m}{k^2 \log(1/\epsilon)}\right)), \text{ there is a mapping } G : \{0,1\}^m \rightarrow \{0,1\}^n, \text{ where } n = \Omega\left((\frac{m}{k^2 \log(1/\epsilon)})^k\right), \text{ such that } G \text{ is a degree } k \text{ generator with bias at most } \epsilon.
\]

We note that as \(k\) is constant the output is of length \(\Omega((m/\log(1/\epsilon))^k)\). In particular this gives a separation between \(\epsilon\)-biased generators in \(NC_0^k\) and degree \(k\) generators: Theorem 6 of [MST03] shows that the stretch (i.e. \(n - m\)) of any \(\epsilon\)-biased generator in \(NC_0^k\), for \(\epsilon < 2^{-2k}\), is at most \(O\left(2^k m^{k/2}\right)\). In contrast we can get a stretch of \(\Omega\left((\frac{m}{k^2 \log(1/\epsilon)})^k\right)\) that for a fixed \(k\) and a not too small \(\epsilon\) (say \(\epsilon = 2^{-m^2}\)) is at least \(m^{k-o(1)}\).

Our second result is a construction of a family of \(\epsilon\)-biassed good codes. Namely, we give a construction of a family of good codes (constant relative rate, constant relative distance, efficient encoding and decoding algorithms) such that the codes in the family have an exponential small bias. Thus our construction answers affirmatively the open question of [DS05].

**Theorem 2** Let \(a = 0.0595/2\). Then for every large enough integers \(0 \leq t, n\) where \(n = 189 - 8^t\) there is a polynomial time constructible generator \(G : \{0,1\}^m \rightarrow \{0,1\}^n\), where \(m \geq an/48\), such that its image \(C = G(\{0,1\}^m)\) is a code with the following properties. \(C\) has relative rate \(\geq a/48\); relative distance \(\geq a^2/24\); a polynomial time decoding algorithm that can fix \(a^2/48\) fraction of errors; and the bias of \(C\) (and hence of \(G\)) is \(\epsilon = \exp(-O(n))\).

**Motivation.** One prosaic motivation for studying these questions is that they were studied before and remained open. However, we feel that there is a stronger argument in favor of studying these problems: both problems are very natural and combine notions that were found to have many applications in theoretical computer science.

It is very desirable to give explicit constructions of low complexity for combinatorial objects. For example, in [Häs87, Gol00a, CM01, KL01, MST03, AIK04] questions regarding the existence of objects such as \(\epsilon\)-biased generators, one-way functions and pseudo-random generators in \(NC_0\) were studied. As low degree polynomial are a natural "low complexity" class we feel that constructing low degree \(\epsilon\)-biased generators is a natural question.

Error correcting codes (ECCs) have many applications in theoretical computer science (cf. [Fei95, Gur01, Tre04]). Finding explicit constructions of ECCs is an extensively studied question (cf. [MS77, Tre04]). In recent years the focus in the theoretical computer science community is on giving explicit constructions of ECCs that have additional properties, for example: codes that have efficient list-decoding algorithms (cf. survey of Trevisan [Tre04]), quantum codes (cf. [NC00, KSV02]), codes that are locally testable and codes that are locally decodable (cf. survey of Goldreich [Gol00b]). We believe that as error correcting codes and \(\epsilon\)-biased sets are such important objects it is natural to combine them and to construct an \(\epsilon\)-biased error correcting codes. We expect that our constructions will find new applications.

**Methods.** Our constructions are similar in spirit to the constructions of \(\epsilon\)-biased generators of Mossel et al. [MST03]. In the proofs of both Theorem 1 and Theorem 2 we first construct
a generator $G^{(h)}$ that is (almost) unbiased w.r.t. heavy tests\(^3\), i.e. the bias of $G^{(h)}$ w.r.t. any test $w$ of large weight is small. Then we construct a generator $G^{(l)}$ that is (almost) unbiased w.r.t. light tests and then we take their XOR, on two independent inputs, to get the final generator $G(x, y) = G^{(h)}(x) \oplus G^{(l)}(y)$. A significant difference from [MST03] is in the way that we construct $G^{(l)}$. Previously $G^{(l)}$ was constructed "from scratch", i.e. there was no connection between the construction of $G^{(l)}$ and of $G^{(h)}$. We show a novel use of error correcting codes that enables us to transform any generator that is (almost) unbiased with respect to heavy tests to a generator that is (almost) unbiased w.r.t. light tests. In the heart of this transformation lies the observation that from the generating matrix of a linear error correcting code of relative rate 1/2 one can construct a linear transformation, from $\{0, 1\}^n$ to itself, that sends "light" vectors to "heavy" vectors (where "light" and "heavy" depend on the properties of the code).

**Organization.** In Section 2 we give the basic notations and definitions. In particular in subsection 2.1 we give the basic definitions of error correcting codes and recall a construction of self-dual error correcting codes, and in subsection 2.2 we give the basic definitions regarding $\epsilon$-biased sets and $\epsilon$-biased generators and give the proofs of some well known facts. In section 3 we prove Theorem 1, and in section 4 we prove Theorem 2.

## 2 Preliminaries

We shall denote with $\log(x)$ the natural logarithm of $x$, i.e. $\log(x) = \log_e(x)$. We denote $\exp(x) = e^x$. We use the notation of $|I|$ to denote the size of the set $I$. For a random variable $X$ that is distributed according to a distribution $D$ we denote with $E_D[X]$ the expectation of $X$. For a matrix $A$ we denote with $A^t$ the transpose of $A$.

For a vector $v \in \{0, 1\}^n$ we denote with $v_i$ the $i$’th coordinate of $v$. Namely, $v = (v_1, \ldots, v_n)$. For two vectors $u, v \in \{0, 1\}^n$ we denote with $\text{dist}(u, v)$ the hamming distance between $u$ and $v$, i.e. $\text{dist}(u, v) = |\{i : u_i \neq v_i\}|$. We also denote with $\text{wt}(v)$ (the weight of $v$) the number of non-zero coordinates of $v$. In other words, $\text{wt}(v) = \text{dist}(v, \vec{0})$, where $\vec{0}$ is the zero vector. For $v, u \in \{0, 1\}^n$ we denote with $\langle v, u \rangle$ their inner product modulo 2, i.e. $\langle v, u \rangle = v_1 \cdot u_1 + \ldots + v_n \cdot u_n$.

For two vectors $v, u \in \{0, 1\}^n$ we denote $v \oplus u = (v_1 \oplus u_1, \ldots, v_n \oplus u_n)$, i.e. it is the coordinate-wise XOR of $v$ and $u$. For two multisets $S_1, S_2 \subseteq \{0, 1\}^n$ we denote $S_1 \oplus S_2 = \{v \oplus u | v \in S_1 \text{ and } u \in S_2\}$. For two functions $G_1 : \{0, 1\}^{m_1} \rightarrow \{0, 1\}^n$ and $G_2 : \{0, 1\}^{m_2} \rightarrow \{0, 1\}^n$ we denote with $G = G_1 \oplus G_2$ the function $G(x, y) : \{0, 1\}^{m_1} \times \{0, 1\}^{m_2} \rightarrow \{0, 1\}^n$ satisfying $G(x, y) = G_1(x) \oplus G_2(y)$. Usually we write $\{0, 1\}^{m_1+m_2}$ instead of $\{0, 1\}^{m_1} \times \{0, 1\}^{m_2}$.

\(^3\)As mentioned earlier, every test can be identified with a binary vector of length $n$. The weight of the test is the number of non zero coordinates of this vector.
2.1 Error correcting codes

Let \( E : \{0,1\}^k \to \{0,1\}^n \). Denote with \( C = E(\{0,1\}^n) \) the image of \( E \). Then \( C \) is called an \([n,k,d]\)-code if for any two codewords \( E(v), E(u) \in C \), where \( u \neq v \), we have that \( \text{dist}(E(u), E(v)) \geq d \). We denote with \( R = k/n \) the relative rate of \( C \) and with \( \delta = d/n \) the relative minimal distance of \( C \), and say that \( C \) is an \([R, \delta]\)-code. When \( E \) is a linear mapping we say that \( C \) is a linear code. A map \( D : \{0,1\}^n \to \{0,1\}^k \) can correct \( t \) errors if for any \( v \in \{0,1\}^k \) and any \( w \in \{0,1\}^n \) such that \( \text{dist}(E(v), w) \leq t \) we have that \( D(w) = v \). Such a \( D \) is called a decoding algorithm for \( C \). A family of codes \( \{C_i\} \), where \( C_i \) is an \([R_i, \delta_i]\)-code of block length \( n_i \), has constant rate if there exists a constant \( 0 < R \) such that for all codes in the family it holds that \( R_i \geq R \). The family has a linear distance if there exists a constant \( 0 < \delta \) such that for all codes in the family we have \( \delta_i \geq \delta \). In such a case we say that the family is a family of \([R, \delta]\) codes. If a family of codes as above has \( \lim_{n \to \infty} n_i = \infty \), a constant rate and a linear minimal distance then we say that the family is a family of good codes and that the codes in the family are good. Similarly, we say that the family of codes has a decoding algorithm for a fraction \( \tau \) of errors if for each \( C_i \) there is a decoding algorithm \( D_i \) that can decode from \( \tau \cdot n_i \) errors.

When \( C \) is a linear code we define the dual of \( C \) in the following way:

\[
C^\perp \triangleq \{ y \in \{0,1\}^n : \forall x \in C \quad \langle y, x \rangle = 0 \}.
\]

A family of codes is said to have good dual codes if the family of the dual codes is good.

In our constructions we will need to use a family of good error correcting codes with good dual codes. The following result of Matsumoto[Mat02] (after a result of Ashikhmin et al. [ALT01]) gives an explicit construction of such a family. As this result is implicit in [Mat02] we give a sketch of the proof.

**Theorem 3 (Matsumoto)** For an integer \( t \) let \( n_t = \frac{189}{4} 8^t \). Then, for a large enough \( t \), there is a polynomial time constructible self-dual \([\frac{1}{2}, 0.0595]\)-code of block length \( n_t \). Moreover, the code has a polynomial time (in the block length) decoding algorithm that can correct from \( > 0.0595/2 \) fraction of errors.

**Proof** In the proof we use the notations of [Mat02]. Consider proposition 2 of [Mat02]. Let \( m = 3 \) then \( n_t = \frac{63}{8} 8^t \). Now let \( j = 0 \) and \( t \) large enough so that \( \frac{n_t}{n_i} \leq \frac{1}{2} + 0.0001 \). Then the code \( C_t = C(G_0 + 0 \cdot P_\infty) \) is an \([n_t, n_t/2, n_t/2 - n_t/7 - 0.0001 n_t] \) self-dual code over the field \( \text{GF}(2^6) \). By expanding this code w.r.t. a self dual basis of \( \text{GF}(2^6) \) over \( \text{GF}(2) \) (see section III of [ALT01]), we get a \([6n_t, 3n_t, n_t/2 - n_t/7 - 0.0001 n_t] \) binary code \( D_t \). The block length is thus \( \frac{189}{4} 8^t \), the relative rate of this code is \( 1/2 \) and its relative minimum distance is larger than 0.0595. As \( C_t \) was self-dual then so is \( D_t \). The fact regarding the decoding algorithm follows from the work of Feng and Rao [FR93]. \( \square \)
2.2 $\epsilon$-biased sets

**Definition 4** Let $S \subseteq \{0, 1\}^n$ be a set (or a multi-set). For a non-zero vector $w \in \{0, 1\}^n$ we denote with $\text{bias}_w(S)$ the bias of $S$ w.r.t. $w$. That is,

$$\text{bias}_w(S) \triangleq \left| \frac{1}{2} - \frac{1}{|S|} \sum_{s \in S} \langle w, s \rangle \right| = \left| \frac{1}{2} - \frac{1}{|S|} \sum_{s \in S} \rho_w(s) \right|.$$

The bias of $S$ is equal to the maximal bias w.r.t. any non-zero test:

$$\text{bias}(S) = \max_{\emptyset \neq w \in \{0, 1\}^n} \text{bias}_w(S).$$

In particular, $S$ is $\epsilon$-biased if for every $\emptyset \neq w \in \{0, 1\}^n$ it holds that $\text{bias}_w(S) \leq \epsilon$.

In order to define $\epsilon$-biased generator it will be convenient to speak about $\epsilon$-biased distributions. In the following we identify vectors $w \in \{0, 1\}^n$ with a subset $W \subseteq \mathbb{N}$ in the usual manner ($i \in W$ if and only if $w_i = 1$).

**Definition 5** Let $G = (G_1, \ldots, G_n)$ be a random variable ranging over $\{0, 1\}^n$. The bias of $G$ w.r.t. a non-zero vector $w \in \{0, 1\}^n$ is defined to be

$$\text{bias}_w(G) \triangleq \left| \frac{1}{2} - \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} [\bigoplus_{i \in W} G_i(x) = 1] \right| = \left| \frac{1}{2} - \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} \langle G(x), w \rangle \right|.$$

The bias of $G$ is equal to the maximal bias w.r.t. any non-zero test:

$$\text{bias}(G) = \max_{\emptyset \neq w \in \{0, 1\}^n} \text{bias}_w(G).$$

In particular, $G$ is $\epsilon$-biased if for every non-zero $w$ it holds that $\text{bias}_w(G) \leq \epsilon$.

When $G : \{0, 1\}^m \to \{0, 1\}^n$ is a map we define the bias of $G$ to be the bias of the random variable $G = (G_1, \ldots, G_n)$, where $G_i$ is the $i$'th output bit of $G$. Notice that the bias of $G$ is equal to the bias of the multi-set $G(\{0, 1\}^n)$.

The following well known lemma gives information about the bias of the XOR of two independent random variables (or the XOR of two different sets).

**Lemma 6** Let $G_1, G_2$ be two independent random variables taking values in $\{0, 1\}^n$. Then $\text{bias}(G_1 \oplus G_2) \leq 2 \text{bias}(G_1) \text{bias}(G_2) \leq \min_i \text{bias}(G_i)$. Similarly, if $S_1, S_2 \subseteq \{0, 1\}^n$ we have that $\text{bias}(S_1 \oplus S_2) \leq 2 \text{bias}(S_1) \text{bias}(S_2) \leq \min_i \text{bias}(S_i)$.

**Proof** We give the proof only for the case of two random variables as the proof for two sets is basically the same. Fix $\emptyset \neq w \in \{0, 1\}^n$. As $G_1$ and $G_2$ are independent it follows that

$$\text{Pr}[(\bigoplus_{i \in W} (G_1 \oplus G_2)_i) = 1] = \text{Pr}[(\bigoplus_{i \in W} G_1)_i = 1] + \text{Pr}[(\bigoplus_{i \in W} G_2)_i = 1] - 2 \text{Pr}[(\bigoplus_{i \in W} (G_1)_i) = 1] \cdot \text{Pr}[(\bigoplus_{i \in W} (G_2)_i) = 1].$$
In particular we get that
\[
\bias_w(G_1 \oplus G_2) = \left| \frac{1}{2} - \Pr[\oplus_{i \in S_w}(G_1 \oplus G_2)_i = 1] \right|
= 2 \left( \frac{1}{2} - \Pr[\oplus_{i \in S_w}(G_1)_i = 1] \right) \left( \frac{1}{2} - \Pr[\oplus_{i \in S_w}(G_2)_i = 1] \right)
= 2 \bias_w(G_1) \bias_w(G_2).
\]

As it is always the case that \( \bias_w(G_i) \leq 1/2 \), it follows that for every \( \vec{0} \neq w \in \{0,1\}^n \) we have that \( \bias_w(G_1 \oplus G_2) \leq \min(\bias_w(G_1), \bias_w(G_2)) \), and the claim easily follows. \( \square \)

The following lemma is an immediate corollary of Lemma 6.

**Lemma 7** Let \((G_i)_{i \in I}\) be independent random variables taking values in \(\{0,1\}^n\). Then
\[
\bias(\oplus_{i \in I} G_i) \leq 2^{|I| - 1} \prod_{i \in I} \bias(G_i).
\]

Similarly, if \((S_i)_{i \in I}\) is a family of subsets of \(\{0,1\}^n\) we have that
\[
\bias(\oplus_{i \in I} S_i) \leq 2^{|I| - 1} \prod_{i \in I} \bias(S_i).
\]

As a special case of this lemma we get the well known estimate for the bias of a sum of independent random coins.

**Lemma 8** Let \(X_1, \ldots, X_t\) be independent 0/1 random variables. Assume that for some \(0 < \delta < 1/2\) and for every \(i\) we have that \(\delta \leq \Pr[X_i = 1] \leq 1 - \delta\), then
\[
\bias\left(\oplus_{i=1}^t X_i\right) \leq \frac{1}{2} \left(1 - 2\delta\right)^t.
\]

Another basic fact that we shall need is an estimate on the bias of a degree \(k\) polynomial, which follows immediately from the famous Schwartz-Zippel theorem [Sch80, Zip79].

**Lemma 9 (Schwartz-Zippel)** Let \(f : \{0,1\}^m \to \{0,1\}\) be a non-constant degree \(k\) polynomial, then
\[
\frac{1}{2^k} \leq \Pr[f(x) = 1] \leq 1 - \frac{1}{2^k}.
\]

Equivalently,
\[
\bias(f(x)) \leq \frac{1}{2} - \frac{1}{2^k}.
\]
3 Low degree $\epsilon$-biased generator

In this section we construct an $\epsilon$-bias generator where each of its output bits is a low degree polynomial. Similarly to the construction in the paper of Mossel et al. [MST03] our generator is the combination of two other generators. One generator has a low bias w.r.t. "heavy" tests (tests that involve a large number of output bits) and the other has a low bias against "light" tests (i.e. tests that involve a small number of output bits). The final generator is obtained by XOR-ing the output of the two generators on independent seeds.

We first construct a generator that is unbiased against heavy tests, and then show a general method for transforming generators that are unbiased with respect to heavy tests into generators that are unbiased with respect to light tests.

3.1 Construction for heavy tests

The following theorem gives a construction of a generator that is unbiased w.r.t. heavy tests. We try to give the most general statement so it is a bit cumbersome. The reader should have the following "relaxed" statement in mind: For every $k, m$ and every $0 < \epsilon$ (such that $\epsilon$ is not too small as a function of $m$, say $\epsilon > 2^{-\sqrt{m}}$), we can construct a degree $k$ $\epsilon$-biased generator (that is unbiased w.r.t. heavy tests) from $m$ bits to $n$ bits where $n = \Omega((m/k \log(1/\epsilon))^k)$. The difference of the relaxed statement from the formal one, is that we need to make all the parameters involved in the construction integers, and we aim to make the bound on $\epsilon$ as small as possible. We now give the formal statement of the theorem.

**Theorem 10** For every two integers $k, m$ and $0 < a, \epsilon$ such that $\epsilon \geq \exp(-O(m^{1-k^2}))$, we define $\gamma = \gamma_{k, \epsilon, a} = \frac{2 \epsilon}{2\sqrt{\log(1/2\epsilon)} + 4a}, s_0 = \lfloor \gamma m \rfloor, i_0 = \lfloor m/s_0 \rfloor$. Assume that $n$ is such that $\frac{1}{8}i_0(\binom{s_0}{k}) \leq n \leq i_0(\binom{s_0}{k})$. Then there is an explicit map $G^{(h)} : \{0, 1\}^m \rightarrow \{0, 1\}^n$ with the following properties:

- For every $w \in \{0, 1\}^n$ such that $wt(w) \geq \gamma n$ we have that $\text{bias}_w(G^{(h)}) \leq \epsilon$.
- Each output bit of $G^{(h)}$ is a degree $k$ polynomial in the input bits.

**Proof** We first give a sketch of the proof. Partition the $m$ input bits to $k$ sets of roughly equal sizes. For each subset of inputs $B$ we define a set of output bits that depend only on the input bits in $B$. Thus, output bits that correspond to different sets are independent (as random variables when the input is chosen uniformly at random from $\{0, 1\}^m$). The output bits that depend on a given set of inputs are in $1-1$ correspondence with multilinear monomials of degree $k$. Namely, every output bit is the evaluation of a degree $k$ monomial on the input bits of the relevant set. Now, given a "heavy" linear combination of the output bits, we can present it as a linear combination of "many" degree $k$ polynomials, where each polynomial is defined on a different subset of inputs. As these polynomials are defined on distinct sets of
Indeed let the same way as the coordinates of the supports of at least get that the bias of each the set of indices for which the output bits corresponds to a different monomial from the variables of size. Each of the output bits corresponds to a different monomial from \( M \) evaluated on the variables of \( B_i \). We denote with \( G_i \) the set of output bits corresponding to \( B_i \) and with \( g_{i,M} \) the output bit corresponding to the monomial \( M \). With these notations we have that

\[
g_{i,M}(x_1, \ldots, x_m) = M(B_i) = M(x_{s_0(i-1)+1}, \ldots, x_{s_0i}),
\]

\[
G_i = (g_{i,M})_{M \in M_k},
\]

\[
G^{(h)} = (G_1, \ldots, G_{s_0}) = (g_{i,M})_{i=1, \ldots, s_0, M \in M_k}.
\]

The length of the output of \( G^{(h)} \) is clearly \( s_0 = \binom{n}{k} \). Note that we ignore the input bits that fell in the leftover set.

For example, let \( m = 7, \gamma = 1/2 \) and \( k = 2 \). Then \( s_0 = 3, i_0 = 2, M_3 = \{y_1y_2, y_1y_3, y_2y_3\}, B_1 = \{x_1, x_2, x_3\} \) and \( B_2 = \{x_4, x_5, x_6\} \). We get that \( G_1 = (x_1x_2, x_1x_3, x_2x_3), G_2 = (x_4x_5, x_4x_6, x_5x_6) \), and \( G^{(h)} = (x_1x_2, x_1x_3, x_2x_3, x_4x_5, x_4x_6, x_5x_6) \).

We now show that for every \( w \in \{0, 1\}^n \) such that \( \text{wt}(w) \geq an \) we have that \( \text{bias}_w(G^{(h)}) \leq \epsilon \). Indeed let \( w \) be such that \( \text{wt}(w) \geq an \). For convenience we enumerate the coordinates of \( w \) in the same way as the coordinates of \( G^{(h)} \), that is \( w = (w_{i,M})_{i=1, \ldots, s_0, M \in M_k} \). We also partition \( w \) to \( i_0 \) disjoint sets \( w = (w_1, \ldots, w_{i_0}) \), where \( w_i = (w_{i,M})_{M \in M_k} \). We note that as \( \text{wt}(w) \geq 8n \) then the supports of at least \( \left\lfloor a \cdot \frac{n}{i_0} \right\rfloor = \left\lfloor a \cdot i_0 \right\rfloor \) of the \( w_i \)'s are not empty. We now have that,

\[
\langle w, G^{(h)} \rangle = \sum_{i=1}^{i_0} \langle w_i, G_i \rangle = \sum_{i=1}^{i_0} \left( \sum_{M \in M_k} w_i M g_{i,M} \right) = \sum_{i=1}^{i_0} \sum_{B_i} p_i(B_i),
\]

where each \( p_i(B_i) \) is a degree \( k \) polynomial, over \( GF(2) \), in the variables of \( B_i \). Denote with \( I \) the set of indices for which \( p_i \neq 0 \). As each \( p_i \) is a sum of different degree \( k \) monomials we have that the size of \( I \) is equal to the number of non empty \( w_i \)'s. Thus,

\[
|I| \geq \left\lfloor a \cdot i_0 \right\rfloor \geq a \cdot i_0 = a \left\lfloor \frac{m}{\gamma m} \right\rfloor \geq a \left( \frac{m}{\gamma m} - 1 \right) \geq a \left( \frac{1}{\gamma} - 1 \right) > 2^{k-1} \log(1/2\epsilon).
\]

As the sets \( B_i \) are disjoint the polynomials \( p_i(B_i) \) for \( i \in I \), viewed as random variables in the input bits, are independent random variables. By the Schwartz-Zippel lemma (Lemma 9), we get that the bias of each \( p_i \) for \( i \in I \) is at most \( \frac{1}{2} - \frac{1}{2^{k-1}} \), and so by Lemma 7 we get that

\[
bias_w(G^{(h)}) < \frac{1}{2} \left( 1 - \frac{2}{2\epsilon} \right)^{|I|} < \frac{1}{2} \left( 1 - \frac{1}{2^{k-1}} \right)^{2^{k-1} \log(1/2\epsilon)} \leq \epsilon.
\]
When \( n < i_0 \binom{s_0}{k} \) we make a small modification to the construction. Let \( t_0 = \left\lceil \frac{n}{i_0} \right\rceil \). Let \( i_1 = n - i_0 \cdot t_0 \). Clearly \( i_1 < i_0 \). For every set in the partition we shall define \( t_0 \) output bits, and for the first \( i_1 \) sets we shall define an additional output bit. The total number of output bits is \( i_0 \cdot t_0 + i_1 = n \) as required. Instead of computing the value of every monomial of \( M_k \) on every set of the partition, we only output the value of the first \( t_0 + 1 \) monomials (according to, say, the lexicographical ordering of their exponent vector) on the first \( i_1 \) sets of the partition, and the value of the first \( t_0 \) monomials on the rest of the sets of the partition (i.e. on the remaining \( i_0 - i_1 \) sets). In this way the output length is \( i_1 \cdot (t_0 + 1) + (i_0 - i_1) \cdot t_0 = i_0 \cdot t_0 + i_1 = n \).

The analysis of the bias also requires a small modification. As before we consider the partition of \( w \) to disjoint sets, where the first \( i_1 \) sets are of size \( t_0 + 1 \) and the last \( i_0 - i_1 \) sets are of size \( t_0 \). We define \( p_i \) and \( I \) as before. We have that

\[
|I| \geq \left\lceil \frac{a \cdot n}{t_0 + 1} \right\rceil \geq \frac{a \cdot n}{i_0} + 1 \geq a(i_0 - 1) \geq a\left(\frac{1}{\gamma} - 2\right) = 2^{k-1} \log(1/2\epsilon),
\]

where inequality (*) follows from the lower bounds on \( \epsilon \), and \( n \) (for completeness we give the proof of (*) in Appendix A). We get that

\[
\text{bias}_w(G_h) \leq \frac{1}{2} \left(1 - \frac{2}{2^k}\right)^{|I|} \leq \frac{1}{2} \left(1 - \frac{1}{2^{k-1}}\right)^{2^{k-1} \log(1/2\epsilon)} \leq \epsilon.
\]

It is clear that the complexity of computing this encoding is polynomial in \( n \). This completes the proof of the Theorem.

\[ \Box \]

### 3.2 From heavy to light

In this section we prove a theorem that shows that in order to construct an \( \epsilon \)-biased generator it is sufficient to construct a generator whose output is almost unbiased w.r.t. "heavy" tests. The basic tool in proving this theorem is a linear transformation from \( \{0,1\}^n \) to itself, that sends all the non-zero vectors in the Hamming ball of radius \( a \cdot n \) (light vectors) to vectors of weight at least \( b \cdot n \) (heavy vectors). Stated differently, this linear transformation has the property that it takes any two vectors that are at distance at most \( a \cdot n \) and sends them to vectors at distance at least \( b \cdot n \). This definition immediately brings to mind error correcting codes, and indeed the construction of such transformations is based on the generating matrix of a suitable error correcting code.

**Definition 11** A linear transformation \( A : \{0,1\}^n \rightarrow \{0,1\}^n \) is \((a,b)\)-expanding if for every \( v \in \{0,1\}^n \) such that \( \text{wt}(v) \leq a \cdot n \) we have that \( \text{wt}(v^t A) = \text{wt}(A^t v) \geq b \cdot n \). We say that \( A \) is symmetric \((a,b)\)-expanding if in addition for every \( u \in \{0,1\}^n \) such that \( \text{wt}(u) \leq a \cdot n \) we have that \( \text{wt}(Au) \geq b \cdot n \).

We now show how to construct symmetric expanding linear transformations.
**Theorem 12** (Expanding transformations) Assume that there exists an explicit construction of a linear (self-dual) \([\frac{1}{2}, \delta]\)-code of block length \(2n\) over \(\text{GF}(2)\). Then there is an explicit (symmetric) \((a, \delta - a)\)-expanding transformation \(A : \{0,1\}^n \to \{0,1\}^n\) that can be constructed in the same time (up to an additive \(O(n^2)\) term) as the generating matrix of the underlying code.

**Proof** Let \(C\) be a \([\frac{1}{2}, \delta] \) code of block length \(2n\) (in particular the rate of \(C\) is \(n\)). Let \(G : \{0,1\}^n \to \{0,1\}^{2n}\) be the generating matrix of \(C\). As the rate is 1/2 we can assume w.l.o.g. that \(G\) has the following form \(G = \begin{pmatrix} I & A \end{pmatrix}\) where \(I\) is the \(n \times n\) identity matrix and \(A\) is an \(n \times n\) matrix. Let \(\bar{0} \neq w \in \{0,1\}^n\) be a vector of weight \(\leq an\). Then \(\delta n \leq \text{wt}(Gw) = \text{wt}(w) + \text{wt}(Aw) \leq an + \text{wt}(Aw)\). In particular \(\text{wt}(Aw) \geq \delta n - an\). Thus the matrix \(A^t\) is \((a, \delta - a)\)-expanding.

Let us now assume that \(C\) is also self-dual. It is easy to see that the matrix \(H = \begin{pmatrix} A^t & I \end{pmatrix}\) is a generating matrix for the dual code, and as \(C\) is self-dual we have that \(H\) is also a generating matrix for \(C\). As before we get that \(\delta n \leq \text{wt}(Hw) = \text{wt}(A^t w) + \text{wt}(w) \leq an + \text{wt}(A^t w)\). Hence, \(\text{wt}(A^t w) \geq \delta n - an\). Together with the above observation that \(\text{wt}(Aw) \geq \delta n - an\) we get that \(A\) is symmetric \((a, \delta - a)\)-expanding. \(\square\)

By applying Theorem 12 on the codes obtained from Theorem 3 we get the following corollary.

**Corollary 13** For every \(0 < a < 0.0595\) and every large enough (independent of \(a\)) integer \(t\) there is an explicit symmetric \((a, 0.0595 - a)\)-expanding transformation of dimension \(189 \cdot 8^t\) that can be constructed in time polynomial in its dimension.

We now show that using expanding linear transformations we can transform any generator that is \(\epsilon\)-biased w.r.t. heavy tests to an \(\epsilon\)-biased generator.

**Theorem 14** Let \(A\) be an \((a, a)\)-expanding transformation of dimension \(n\). Let \(G^{(h)} : \{0,1\}^m \to \{0,1\}^n\) satisfy that there exists an \(0 < \epsilon\) such that for every \(w \in \{0,1\}^n\) with \(\text{wt}(w) \geq an\), \(\text{bias}_w(G^{(h)}) \leq \epsilon\). Define \(G^{(l)} : \{0,1\}^m \to \{0,1\}^n\) in the following way \(G^{(l)}(x) = A(G^{(h)}(x))\). Then for every \(w \in \{0,1\}^n\) with \(\text{wt}(w) \leq an\), \(\text{bias}_w(G^{(l)}) \leq \epsilon\). In particular, if we define \(G : \{0,1\}^{2m} \to \{0,1\}^n\) as

\[
G(x, y) = G^{(h)}(x) \oplus G^{(l)}(y)
\]

we get that \(\text{bias}(G) \leq \epsilon\). The map \(G^{(l)}\) can be constructed in time polynomial in the construction time of \(G^{(h)}\) and of \(A\), and hence so can \(G\).

**Proof** Let \(w \in \{0,1\}^n\) be such that \(\text{wt}(w) \leq an\). It is clear that

\[
\text{bias}_w(G^{(l)}) = \text{bias}_w(A(G^{(h)}(x))) = \frac{1}{2} - \mathbb{E}_{x \in \{0,1\}^n} \langle A(G^{(h)}), w \rangle
\]
As $A$ is $(a,a)$-expanding we have that $\text{wt}(A^tw) \geq an$. By the assumption on $G^{(h)}$ we get that $\text{bias}_{A^t w}(G^{(h)}) \leq \epsilon$ and so $\text{bias}_{w}(G^{(l)}) \leq \epsilon$. The claim regarding the bias of $G$ is an immediate corollary of Lemma 6. The claim regarding the construction time of $G$ is obvious.

We note that as $A$ is a linear transformation then the degree of $G$ (i.e. the maximal degree of its output bits viewed as polynomials, over $\text{GF}(2)$), in the input variables is at most the degree of $G^{(h)}$.

### 3.3 Proof of Theorem 1

The proof of Theorem 1 follows from Theorem 10, Corollary 13 and Theorem 14. Indeed, let $a = 0.0595/2$. Given two integers $0 < k, m$ and $\epsilon \geq \exp(\text{O}(\frac{am^{1+\frac{1}{2}}}{2a}))$, let $t$ be defined as the largest integer such that $189 \cdot 8^t \leq i_0(s_0)$, where $s_0, i_0$ are as in Theorem 10. We want $m$ to be large enough so that Theorem 3 will ensure the existence of $[\frac{1}{2}, 2a]$-codes of block length $189 \cdot 8^{t+1}$. Let $n = 189 \cdot 8^t$. Clearly $\frac{n}{s_0} < n \leq i_0(s_0)$. Using the notations of Theorem 10 we bound $n$ from below by

$$n > \frac{1}{8} i_0(s_0) > \left(\frac{\gamma m}{k}\right)^k = \Omega\left(\frac{m}{2^k \log(1/\epsilon)^k}\right).$$

By Corollary 13 we can construct in polynomial time an $(a,a)$-expanding transformation of dimension $n$. For our $k, m, n, a, \epsilon$ let $G^{(h)}$ be the generator obtained from theorem 10. Clearly $G^{(h)}$ satisfies the conditions of Theorem 14. Let $G$ be the generator obtained from Theorem 14. It is easy to verify that $G$ is the desired generator.

### 4 Construction of $\epsilon$-biased good codes

The construction of $\epsilon$-biased good codes follows the same lines as the construction of low-degree $\epsilon$-biased generators. The main difference is that we don’t have to keep the degree low but rather make sure that the generator outputs a good code. Thus, we will need a generator for heavy tests that outputs a good code and a way of transforming this generator in to a truly unbiased generator that also outputs a good code. The difference from the proof of Theorem 1 is in two points. First we will need a different construction of a generator $G^{(h)}$ for heavy tests (compare to Theorem 10). Then we will need a construction of a symmetric expanding transformation (see Definition 11) that will enable us to transform this generator to an $\epsilon$-biased good code (compare to Theorem 14).
As before we start by giving a construction of a good code that is also unbiased w.r.t. heavy tests. We then show a general way of transforming codes that are unbiased w.r.t. heavy tests to \(e\)-biased codes.

### 4.1 Construction of codes that are unbiased w.r.t. heavy tests

**Theorem 15** Let \(0 < a < 1\) be a constant. Let \(n\) be an integer. For \(an/48 \leq M < an/6\) let \(\hat{C}\) be an \([M, m, d]\) binary linear error correcting code that has a polynomial time encoding algorithm (i.e. its generating matrix can be computed in polynomial time) and a polynomial time decoding algorithm that can correct \(\alpha > 0\) fraction of errors. Denote \(\hat{R} = m/M\) and \(\hat{\delta} = d/M\). Then there is a polynomial time constructible map \(G(h) : \{0, 1\}^m \rightarrow \{0, 1\}^n\) such that its image, \(\hat{C} = G(h) \{0, 1\}^m\) has the following properties.

- \(\hat{C}\) is an \([R, \delta]\)-code of block length \(n\) for \(R > \frac{a}{48} \hat{R}\) and \(\delta \geq \frac{a}{48} \hat{\delta}\).
- The weight of every \(v \in \hat{C}\) is bounded from above by \(\text{wt}(v) \leq an/3\).
- \(\hat{C}\) has a polynomial time encoding algorithm and a polynomial time decoding algorithm that can correct from \(\alpha \geq \frac{a}{48} \hat{\alpha}\) fraction of errors.
- For every \(w \in \{0, 1\}^n\) such that \(\text{wt}(w) > an\) we have that \(\text{bias}_w(\hat{C}) \leq \exp(-O(n))\).

**Proof** We start with a sketch of the proof. The proof is similar in nature to the proof of Theorem 10. As before we partition the input string to roughly \(m/k\) subsets of size \(k\), for some \(k\). For each of the subsets we define \(2^{k-2}\) distinct output bits that correspond to the values of \(2^{k-2}\) linearly independent polynomials, in \(k\) variables, evaluated on the input bits that belong to the subset. In this way we get roughly \(2^{k-2}m/k\) output bits. We then concatenate (in the sense of string concatenation) to these output bits the encoding of the input bits w.r.t. the code \(\hat{C}\). This defines the map \(G(h)\). It remains to show that \(G(h)\) has the required properties. Indeed, the first \(2^{k-2}m/k\) output bits will assure us that the weight of each output word is not too large and that (as in the proof of Theorem 10) the output has a small bias. The concatenation of \(\hat{C}\) ensures that the distance between any two output words is linearly large which gives us the required decoding property. As before, in order to be completely accurate we have to handle the case that for our \(k\) (that will be later specified) \(m/k\) is not an integer, and the case where \(2^{k-2}m/k > n - M\). We now give the formal proof.

Let \(k\) be the smallest integer satisfying \([m/k]2^{k-2} \geq n - M\). It is clear that \(k\) is a constant depending only on \(a\) and \(\hat{R}\). Let \(i_0 = [m/k]\), \(i_0 = [(n - M)/i_0]\) and \(i_1 = n - M - i_0 \cdot t_0\). Clearly \(i_1 < i_0\). Let \(x = (x_1, \ldots, x_m)\) be our input. Partition the first \(k[m/k] = ki_0\) bits to \(i_0\) sets of size \(k\). The \(i\)-th set in the partition is \(B_i = \{x_{(i-1)k+1}, \ldots, x_{ik}\}\). For each of the \(B_i\)'s we define \(t_0\) output bits, and for the first \(i_1\) sets we define an additional output bit. This gives a total of \(i_0t_0 + i_1 = n - M\) output bits.
Denote with \( \{ \chi_0, \ldots, \chi_{2^{k-2}-1} \} \) the following characteristic functions in \( k \) variables:

\[
\chi_j(y_1, \ldots, y_k) = 1 \leftrightarrow (y_1 = 1) \wedge (y_2 = 1) \wedge \left( \sum_{i=3}^{k} y_i 2^{i-1} = j \right).
\]

Equivalently, let \( j_0, \ldots, j_{k-3} \) be the binary representation of \( j \) when \( j_0 \) is the LSB and \( j_{k-3} \) is the MSB (i.e. \( j = \sum_{i=0}^{k-3} j_i 2^i \)). We have that

\[
\chi_j(y_1, \ldots, y_k) = y_1 \cdot y_2 \cdot \prod_{i=3}^{k} (y_i - j_{i-3} + 1) \mod 2.
\]

It is easy to see that the \( \chi_j \)-s are linearly independent, and that on every input at most one of the \( \chi_j \)-s is non-zero. Note that the degree of every monomial of \( \chi_j \) is at least 2 and at most \( k \).

We denote with \( G_i \) the set of output bits corresponding to \( B_i \) and with \( g_{i,j} \in G_i \) the output bit corresponding to \( \chi_j \). Namely,

\[
g_{i,j}(x_1, \ldots, x_m) = \chi_j(x_{k-(i-1)+1}, \ldots, x_{k-1}).
\]

With these notations we have that

\[
\forall 1 \leq i \leq i_1 \quad G_i = (g_{i,j})_{j=0,\ldots,i_0},
\]

\[
\forall i_1 < i \leq i_0 \quad G_i = (g_{i,j})_{j=0,\ldots,i_0-1}.
\]

The length of the output is clearly \( i_1(t_0 + 1) + (i_0 - i_1)t_0 = n - M \). Denote with \( G_C : \{0,1\}^m \to \{0,1\}^M \) the generating matrix of the code \( \hat{C} \). In particular, the encoding of the vector \( x = (x_1, \ldots, x_m) \) is \( G_C \cdot x \). We now define the map \( G^{(h)} \):

\[
G^{(h)} = (G_1, \ldots, G_{i_0}, G_C),
\]

that is, on an input \( x \) we first have \( n - M \) output bits that come from \( G_1, \ldots, G_{i_0} \), and the last \( M \) bits are the encoding of \( x \) w.r.t. to the code \( \hat{C} \). Clearly the output length is \( n \). Let \( C = G^{(h)}(\{0,1\}^m) \) be the image of \( G^{(h)} \). We show that \( C \) has the required properties.

The rate of \( C \) is \( m \) and thus its relative rate \( R \) is

\[
R = m/n \geq m/(48M/a) = \frac{a}{48} \hat{R}.
\]

It is clear that \( C \) contains \( \hat{C} \) as its last \( M \) bits and so the minimal distance of \( C \) is at least \( \hat{\delta} M \). Thus the relative minimal distance of \( C \) is:

\[
\delta \geq \hat{\delta} M/n \geq \frac{a}{48} \hat{\delta}.
\]

In order to bound the weight of every \( v \in C \), we recall that in every \( G_i \) at most one of the output bits is non-zero. Thus the total weight of \( v \in C \) is bounded from above by \( i_0 + M \). We get that

\[
\forall v \in C, \quad \text{wt}(v) \leq i_0 + M = \lfloor m/k \rfloor + M \leq M/k + M \leq 2M \leq an/3.
\]
To show the error correcting property we note that if the total number of errors is $\hat{a}M$, then in particular the last $M$ bits of $C$ contain at most $\hat{a}M$ errors. As the last $M$ bits of $C$ correspond to a codeword in $\hat{C}$ we can use the decoding algorithm of $\hat{C}$ to obtain the original message. Thus we can fix at least $\hat{a}M \geq \frac{a}{2^k} \hat{a}n$ errors.

It remains to show that $C$ is $\epsilon$-biased w.r.t. words of weight $\geq an$. Indeed let $\text{wt}(w) \geq an$. As in the proof of Theorem 10 we partition $w$ to $i_0 + 1$ disjoint sets $w = (w_1, \ldots, w_{i_0}, w_{i_0}^C)$, where the number of bits in $w_i$ is the same as the number of output bits in $G_i$. We also write $w_i = (w_{i,0}, \ldots, w_{i|G_i|-1})$. Since $\text{wt}(w) \geq an$ we have that the supports of at least

$$\left\lceil \frac{an - M}{2^k - 2} \right\rceil \geq \left\lceil \frac{a - a/6}{2^k - 2}n \right\rceil$$

of $w_1, \ldots, w_{i_0}$ are not empty. As in Theorem 10 we get that

$$\langle w, G(h) \rangle = \left( \bigoplus_{i=1}^{i_0} \langle w_i, G_i \rangle \right) \oplus \langle w_{i_0}^C, G_{i_0}^C \rangle = \left( \bigoplus_{i=1}^{i_0} \left( \sum_{g_{i,j} \in G_i} w_{i,j} x_{i,j}(B_i) \right) \right) \oplus \langle w_{i_0}^C, G_{i_0}^C \rangle$$

$$= \left( \bigoplus_{i=1}^{i_0} p_i(B_i) \right) \oplus \langle w_{i_0}^C, G_{i_0}^C \rangle,$$

where $p_i(B_i)$ is a polynomial over $\text{GF}(2)$ in the variables of $B_i$. As $G_{i_0}^C$ is a linear function in $\{x_1, \ldots, x_m\}$ we have that

$$\langle w_{i_0}^C, G_{i_0}^C \rangle = \sum_{i=1}^{i_0} \ell_i(B_i),$$

where the $\ell_i$’s are linear functions. We thus have that

$$\langle w, G(h) \rangle = \left( \bigoplus_{i=1}^{i_0} p_i(B_i) \right) \oplus \left( \sum_{i=1}^{i_0} \ell_i(B_i) \right) = \left( \bigoplus_{i=1}^{i_0} \tilde{p}_i(B_i) \right),$$

where each $\tilde{p}_i$ is a polynomial over $\text{GF}(2)$ in the variables of $B_i$.

Denote with $I$ the set of indices for which $p_i \neq 0$. As each $p_i$ is a sum of linearly independent polynomials we have that the size of $I$ is equal to the number of non empty $w_i$’s. Since $\tilde{p}_i = p_i + \ell_i$ and each monomial of $p_i$ has degree at least 2 we get that if $p_i \neq 0$ then $\tilde{p}_i \neq 0$ (because $p_i$ and $\ell_i$ cannot cancel each other). We conclude that at least $\left\lceil \frac{a - a/6}{2^k - 2}n \right\rceil$ of the $p_i$’s are non-zero polynomials (of degree at most $k$). As the sets $B_i$ are disjoint the polynomials $\tilde{p}_i(B_i)$ for $i \in I$, viewed as random variables in the input bits, are independent random variables. By the Schwartz-Zippel lemma (Lemma 9), we get that the bias of each $p_i$, for $i \in I$ is at most $\frac{1}{2} - \frac{1}{2^k}$, and so by Lemma 7 we get that

$$\text{bias}_w(G_h) \leq \frac{1}{2} \left( 1 - \frac{2}{2^k} \right)^{|I|} \leq \frac{1}{2} \left( 1 - \frac{1}{2^k - 1} \right)^{\left\lceil \frac{a - a/6}{2^k - 2}n \right\rceil} = \exp \left( -O \left( \frac{a}{2^k n} \right) \right) = \exp \left( -O \left( n \right) \right).$$

\qed
4.2 From heavy to light: keeping the distance large

For the purpose of constructing \( \varepsilon \)-biased codes we shall need the more powerful notion of a symmetric expanding transformation. We will also require that the transformation has an efficient decoding algorithm (in some special sense).

**Definition 16** An \((a, b)\)-expanding transformation \( A \) of dimension \( n \) can decode from \( \alpha \) fraction of errors if there exists a decoding algorithm \( D \) such that for any two vectors \( v, err \in B^n \), satisfying \( \text{wt}(v) \leq an \) and \( \text{wt}(err) \leq an \), we have that \( D(Av + err) = v \).

**Theorem 17** (Symmetric expanding transformations) For any \( [\frac{1}{2}, \delta] \) self-dual code \( C \) of block length \( 2n \), that has a polynomial time decoding algorithm that can handle \( \alpha \) fraction of errors, and for every \( 0 < a < \delta \), there is an explicit symmetric \((a, \delta - a)\)-expanding transformation \( A \) of dimension \( n \), that has a polynomial time algorithm for decoding from \( 2\alpha - \) a fraction of errors. Moreover, the constructing time of \( A \) is the same (up to \( \pm O(n^3) \)) as the time for constructing the generating matrix of the code, and the running time of the decoding algorithm is the same (up to \( \pm O(n) \)) as the running time of the decoding algorithm of \( C \).

**Proof** Let \( G \) be the generating matrix of \( C \). W.l.o.g we can assume that \( G = \begin{pmatrix} I & A \end{pmatrix} \). The proof of Theorem 12 shows that \( A \) is symmetric \((a, \delta - a)\)-expanding. Thus, it remains to prove the decoding property. Given a vector of the form \( A(v) + err \), where \( \text{wt}(v) \leq an \), and \( \text{wt}(err) \leq (2\alpha - a)n \), consider the \( 2n \) dimensional vector \((\vec{0}, A(v) + err)\), where \( \vec{0} \) is the \( n \)-th dimensional zero vector. The distance of this vector from the vector \((v, A(v))\) is at most \( an + (2\alpha - a)n = 2an \). As \((v, A(v))\) belongs to the image of \( G \) (and hence belongs to \( C \)) we can apply the given decoding algorithm for \( C \) (that can correct \( \alpha \cdot 2n \) errors) on the word \((\vec{0}, A(v) + err)\) to get the word \((v, A(v))\) from which we get \( v \). The claim regarding the running time is obvious. \( \square \)

Applying Theorem 17 on the codes obtained from Theorem 3 we get the following corollary.

**Corollary 18** For every \( 0 < a < 0.0595 \) and every large enough integer \( t \) there is an explicit symmetric \((a, 0.0595 - a)\)-expanding transformation \( A \), of dimension \( n = 189 \cdot 8^t \), that can be constructed in polynomial time (in \( n \)), and that has a polynomial time decoding algorithm that can correct \( 0.0595 - \) a fraction of errors.

We now show that by using symmetric expanding transformations we can transform a good code that is \( \varepsilon \)-biased w.r.t. heavy tests to a good code that is \( \varepsilon \)-biased.

**Theorem 19** Let \( A \) be a symmetric \((a, a)\)-expanding transformation of dimension \( n \) that can correct \( \beta \) fraction of errors. Let \( G^{(h)} : \{0, 1\}^m \to \{0, 1\}^n \) be a mapping whose image is a code \( C \subset \{0, 1\}^n \) with the following properties.

- \( C \) is an \([R, \delta]\) code of block length \( n \).
- For every word \( v \in C \) we have that \( \text{wt}(v) \leq \Delta n < an/2 \).
\begin{itemize}
  \item $C$ has a polynomial time decoding algorithm that can correct $\alpha$ fraction of errors.
  \item For every $w \in \{0,1\}^n$ such that $\text{wt}(w) \geq an$, $\text{bias}_w(C) \leq \epsilon$.
\end{itemize}

Let $G^{(l)}(x) = A(G^{(h)}(x))$. Finally let $G : \{0,1\}^{2n} \rightarrow \{0,1\}^n$ be the following generator $G(x,y) = G^{(h)}(x) \oplus G^{(l)}(y)$. Let $\tilde{C} \subset \{0,1\}^n$ be the image of $G$ (note that as sets we have that $\tilde{C} = C \oplus A(C)$). Then $\tilde{C}$ has the following properties.

\begin{itemize}
  \item $\tilde{C}$ has relative rate $2R$ and relative minimal distance at least $\min(\delta, a - 2\Delta)$.
  \item $\tilde{C}$ has a polynomial time decoding algorithm that can correct $\min(\beta - \Delta, \alpha)$ fraction of errors.
  \item $\text{bias}(\tilde{C}) \leq \epsilon$ (and hence $\text{bias}(G) \leq \epsilon$).
\end{itemize}

\textbf{Proof} We prove the properties of $\tilde{C}$ one by one. It is clear that the relative rate of $\tilde{C}$ is $2R$. Let $v, u$ be two different code words in $\tilde{C}$. Then we can write $v = v_1 + Av_2$ and $u = u_1 + Au_2$ for $v_1, v_2, u_1, u_2 \in C$. It is clear that $\text{dist}(u, v) = \text{dist}(u_1 + Au_2, v_1 + Av_2) = \text{dist}(u_1 - v_1, A(v_2 - u_2))$. As $u_1, v_1, u_2, v_2 \in C$ we have that $\text{wt}(u_1 - v_1) \leq \text{wt}(u_1) + \text{wt}(v_1) \leq 2\Delta n < an$ and similarly that $\text{wt}(v_2 - u_2) \leq 2\Delta n < an$. We analyze two cases.

\textbf{Case} $v_2 \neq u_2$: As $A$ is symmetric $(a,a)$-expanding we have that $\text{wt}(A(v_2 - u_2)) \geq an$. Thus, $\text{dist}(u_1 - v_1, A(v_2 - u_2)) \geq (a - 2\Delta)n$ (this is a trivial bound on the distance of a vector of weight at most $2\Delta$ and a vector of weight at least $a$). It follows that $\text{dist}(u, v) \geq (a - 2\Delta)n$.

\textbf{Case} $v_2 = u_2$: In this case $\text{dist}(u, v) = \text{dist}(u_1, v_1) \geq \delta n$, as $C$ has minimal distance $\geq \delta n$ (note that we must have that $v_1 \neq u_1$).

Combining the two cases we get that the relative distance of $\tilde{C}$ is at least $\min(a - 2\Delta, \delta)$.

We now show a decoding algorithm for $\tilde{C}$. Let $u$ be a codeword in $\tilde{C}$, and let $\text{err} \in \{0,1\}^n$ be an error vector of weight $\text{wt}(\text{err}) \leq \min(\alpha, \beta - \Delta) \cdot n$. Let $v_1, v_2 \in C$ be such that $u = v_1 + Av_2$. We now show how to recover $v_1, v_2$ from the corrupted word $u + \text{err}$. Let $\tilde{\text{err}} = \text{err} + v_1$. As $\text{wt}(\tilde{\text{err}}) \leq (\beta - \Delta)n$ and $\text{wt}(v_1) \leq \Delta n$ we have that $\text{wt}(\tilde{\text{err}}) \leq \beta n$. By our assumption $A$ can decode from $\beta$ fraction of errors, thus $A$ can recover the value of $v_2$ from the input $Av_2 + \tilde{\text{err}}$. As $u + \text{err} = Av_2 + (v_1 + \text{err}) = Av_2 + \tilde{\text{err}}$, we can recover $v_2$ from the input $u + \text{err}$. Given $v_2$ and the word $u + \text{err}$ we can get the vector $v_1 + \text{err}$ as $v_1 + \text{err} = u + \text{err} - Av_2$. Since $\text{wt}(\text{err}) \leq an$ we can use the decoding algorithm of $C$ to get $v_1$ from the input $v_1 + \text{err}$. Clearly the running time of this decoding algorithm is polynomial whenever the running time of the decoding algorithms for $A$ and $C$ are polynomial. This shows the decoding property.

Finally we notice that as a direct consequence of Theorem 14 we get that $G$ is an $\epsilon$-biased generator (equivalently, that $\tilde{C}$ is an $\epsilon$-biased set).
4.3 Proof of Theorem 2

In order to prove Theorem 2, we apply Theorem 15 on the codes promised by Theorem 3, to get a good code that is unbiased w.r.t. heavy tests. Then we apply Theorem 19 on this code to obtain a good code that is $\epsilon$-biased. Details follow.

Let $a = 0.0595/2$. Given a positive integer $n = 189 \cdot 8^t$, let $s$ be the largest integer such that $\frac{189}{4} 8^{s+1} \leq an/6$. Let $m = 189 \cdot 8^s$, then clearly $m > an/96$. We want $t$ (and so $n$) to be large enough so that Theorem 3 will guarantee the existence of a $[\frac{1}{2}, 2a]$-code of block length $\frac{189}{4} 8^{s+1}$, that we denote $\hat{C}$. In other words, $\hat{C}$ is a self-dual $[\frac{1}{2}, 2a]$-code of rate $m$ and block length $2m$. Recall that $\hat{C}$ has a decoding algorithm that can correct at least $\hat{\alpha} \triangleq a$ fraction of errors. By applying Theorem 15 on the code $\hat{C}$ we obtain a generator $G^{(h)} : \{0, 1\}^m \to \{0, 1\}^n$ that is (almost) unbiased against tests of weight at least $an$ and whose image is a code $C$ with the following parameters: The relative rate of $C$ is $m/n \geq \frac{a}{96}$; the relative distance of $C$ is at least $\frac{a^2}{24}$; the weight of every codeword of $C$ is at most $\frac{a^2}{3}$; $C$ has a decoding algorithm that can fix a fraction $\frac{a^2}{48}$ of errors; and the bias of $C$ against words of weight at least $an$ is at most $\epsilon = \exp(-O(n))$.

Now we apply Theorem 19 on the generator $G^{(h)}$, where $A$ be the symmetric $(a, a)$-expanding transformation of dimension $n$ guaranteed by Corollary 18. Recall that $A$ has a decoding algorithm that can fix a fraction of $a$ errors. We obtain a generator $G : \{0, 1\}^{2m} \to \{0, 1\}^n$ whose image is a code $\tilde{C}$ with the following parameters. The relative rate of $\tilde{C}$ is $2m/n > \frac{a}{48}$; the relative minimal distance of $\tilde{C}$ is at least $\min[a^2/24, a/3] = a^2/24$; $\tilde{C}$ has a decoding algorithm that can fix a $\min(2a/3, a^2/48) = a^2/48$ fraction of errors; and the bias of $\tilde{C}$ (and hence of $G$) is at most $\epsilon = \exp(-O(n/a))$. As the constructions of $A$ and of $\tilde{C}$, as well as their decoding algorithms, run in polynomial time, we get that $C$ and hence $\tilde{C}$ have polynomial time encoding and decoding algorithms. This completes the proof of Theorem 2. □

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References


A Missing details from the proof of Equation (1)

We first notice that from the definition of $\gamma, \epsilon$ it follows (after a short calculation) that

$$2k < \gamma m. \quad (2)$$

Turning back to inequality (*) from Equation 1, we see that it is enough to show that

$$n \geq \left( \left\lfloor \frac{n}{i_0} \right\rfloor + 1 \right) (i_0 - 1). \quad (3)$$

We have that

$$\left( \left\lfloor \frac{n}{i_0} \right\rfloor + 1 \right) (i_0 - 1) \leq \left( \frac{n}{i_0} + 1 \right) (i_0 - 1) = n - \frac{n}{i_0} + i_0 - 1.$$

Thus, Equation (3) will follow if we prove that

$$\frac{n}{i_0} \geq \frac{1}{8} \left( \left\lfloor \gamma m \right\rfloor \right) > i_0. \quad (4)$$

From equation (2) we get that

$$\frac{1}{8} \left( \left\lfloor \gamma m \right\rfloor \right) > \frac{1}{8} \left( \frac{\gamma m}{k} \right)^k \quad (5)$$

and

$$i_0 = \left\lfloor \frac{m}{\gamma m} \right\rfloor \leq \frac{m}{\gamma m} \leq \frac{m}{\gamma m - 1} \leq \frac{2}{\gamma}. \quad (6)$$

Thus, Equation (4) follow if we show that

$$\frac{1}{8} \left( \frac{\gamma m}{k} \right)^k > \frac{2}{\gamma}$$

which follows immediately from the definition of $\gamma$ and $\epsilon.$