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Hardness of asymptotic approximation for orthogonal rectangle packing and covering problems

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Abstract

Recently Bansal and Sviridenko [4] proved that for 2-dimensional Orthogonal Rectangle Bin Packing without rotations allowed there is no asymptotic PTAS, unless P = NP. We show that similar approximation hardness results hold for several rectangle packing and covering problems even if rotations by ninety degrees around the axes are allowed. Moreover, for some of these problems we provide explicit lower bounds on asymptotic approximation ratio of any polynomial time approximation algorithm.

1 Introduction

Packing and covering problems have many real-world applications in areas like job scheduling, container loading, and cutting objects out of a strip of material in such a way that the amount of material wasted is minimal. As these problems are usually NP-hard, an effort of many researchers is focused on designing polynomial time approximation algorithms and schemes for them. On the other hand, for some problems it is provably hard even to approximate the solution by an algorithm with asymptotic approximation ratio close to 1. In this paper we present approximation hardness results for orthogonal packing and covering problems of rectangles into bins in 2 and 3-dimensions, where ninety-degree rotations of rectangles around any of the axes are allowed. Our hardness results apply also to the most studied case with unit square bins (resp. a strip in 3D with unit square base). Notice that when rotations are allowed then the bin shape and/or its dependance on an input can play a role. Allowing rotations causes that these packing and covering problems are no longer invariant under scaling an input instance by different factors in different coordinate directions.

Notation and terminology

For convenience, we use the terminology related to bin packing problems also for covering problems. Throughout this paper we only consider offline versions of the problems. In all problems, the input consists of a list $\mathscr{L} = \{R^1, R^2, \dots, R^n\}$ of d-dimensional rectangles in the Euclidean space $\mathbb{R}^d$ and a d-dimensional rectangular bin $\mathbb{B}$. Each rectangle R^i is given with an (initial) *orientation* related to the coordinate axes and has its dimensions denoted as $(r_1^i, r_2^i, \dots, r_d^i)$. The

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first and the last dimension of R^i are called width and height and denoted by $w(R^i) := r^i_1$ and $h(R^i) := r^i_d$, respectively. For the bin $\mathbb{B}$, we use the notation $[0, b_1] \times [0, b_2] \times \cdots \times [0, b_d]$ (or $[0, b]^d$, if $b_1 = b_2 = \cdots = b_d = b$) instead of $(b_1, b_2, \ldots, b_d)$. In the strip versions of the problems we suppose that the last dimension of the bin $\mathbb{B}$ is unlimited and we call such bin $\mathbb{B} = (b_1, b_2, \ldots, b_{d-1}, \infty)$ a *strip*.

In the packing problems all rectangles of the list $\mathscr{L}$ need to be packed into bins without overlap; in covering problems rectangles can overlap but have to cover bins completely. The most interesting and well-studied version of these problems is the so called *orthogonal* version, where the edges of packed rectangles and bins are always parallel to the coordinate axes. In problems *without rotations* rectangles have to be placed into the bin with given orientation and a feasible solution is called *oriented packing* and *oriented covering*, respectively. In problems *with rotations allowed* rectangles to be placed may be *rotated around any of the axes by* 90° and a feasible solution is referred to as *r-packing* and *r-covering*, respectively. In the 3-dimensional case, if only rotations around the *z-axis* (the last one) are allowed, we call such packing (resp. covering) as *z-oriented.*

Definition 1 *Given a list $\mathscr{L}$ of d-dimensional rectangles and a d-dimensional bin $\mathbb{B} = (b_1, b_2, \ldots, b_d)$. The goal of the problems d-DIMENSIONAL BIN PACKING (d-BP) and d-DIMENSIONAL BIN PACKING WITH ROTATIONS (d-BP^r) is to find an oriented packing and an r-packing of all rectangles of $\mathscr{L}$ into the minimum number of bins $\mathbb{B}$, respectively.*

In the problems d-DIMENSIONAL BIN COVERING (d-BC) and d-DIMENSIONAL BIN COVERING WITH ROTATIONS (d-BC^r) one has to find an oriented covering and an r-covering, respectively, which covers completely the maximum number of bins $\mathbb{B}$ by rectangles from the list $\mathscr{L}$.

Definition 2 *In strip versions of the problems (defined for $d \geq 2$) a list $\mathscr{L}$ of d-dimensional rectangles and a d-dimensional strip $\mathbb{B} = (b_1, b_2, \ldots, b_{d-1}, \infty)$ are given. In the problems d-DIMENSIONAL STRIP PACKING (d-SP) and d-DIMENSIONAL STRIP PACKING WITH ROTATIONS (d-SP^r) one has to find an oriented packing and an r-packing, respectively, that minimizes h such that all rectangles of $\mathscr{L}$ are packed into the bin $(b_1, b_2, \ldots, b_{d-1}, h)$. In case $d = 3$, if only 90° rotations around the z-axis (the unlimited direction of the strip $\mathbb{B}$) are allowed, the problem is called z-ORIENTED 3-DIMENSIONAL STRIP PACKING.*

The goal of the d-DIMENSIONAL STRIP COVERING problem (d-SC) is to maximize h such that the part $(b_1, b_2, \ldots, b_{d-1}, h)$ of the strip $\mathbb{B}$ is completely covered. We also consider the z-ORIENTED 3-DIMENSIONAL STRIP COVERING problem (3-SC^z), where rectangles can be rotated by 90° around the z-axis.

It is well known that already for 1-BP (and hence for d-BP, for any $d \geq 1$) no polynomial time approximation algorithm can have an approximation ratio less than $\frac{3}{2}$, unless P = NP. This follows from the NP-hardness of the PARTITION problem defined as follows: for a given set of rational numbers with sum 2 one has to decide if it can be partitioned into two subsets whose elements add up to 1. For 2-BP even stronger inapproximability results are known. Given a collection of rectangles (in fact, squares), it is NP-hard to decide whether these can be packed into a single bin $[0, 1]^2$ or they require more bins [18]. Hence no $(2 - \varepsilon)$-approximation algorithm for 2-BP (resp. 2-BP^r) can exist, unless P = NP. This kind of approximation hardness results achieved on instances requiring only a very small number of bins are always present in bin packing and covering problems. To describe better the real difficulty of these problems, the *asymptotic approximation* *ratio* $\rho^\infty_{\mathcal{A}}$ has become the standard measure used to analyse the quality of an algorithm $\mathcal{A}$. For a minimization problem it is defined as

$$\rho^\infty_{\mathcal{A}} = \lim_{n\to\infty} \sup_I \left\{ \frac{\mathcal{A}(I)}{\mathrm{OPT}(I)} : \mathrm{OPT}(I) \geq n \right\},$$

where I ranges over the set of all problem instances, and $\mathcal{A}(I)$ (resp. $\mathrm{OPT}(I)$) denote the value of the solution returned by $\mathcal{A}$ (resp. the optimum value) for an input instance I. For a maximization problem, $\frac{\mathcal{A}(I)}{\mathrm{OPT}(I)}$ is replaced by $\frac{\mathrm{OPT}(I)}{\mathcal{A}(I)}$ so that always $\rho^\infty_{\mathcal{A}} \geq 1$. The notion of asymptotic approximation ratio allows us to ignore the anomalous behavior of the algorithm for instances with small optimal value. We say, that a problem admits an *asymptotic approximation scheme* (shortly, APTAS), if for any $\varepsilon > 0$ there is a polynomial time algorithm with an asymptotic approximation ratio less than $1 + \varepsilon$. For other optimization terminology we refer to Ausiello et al. [1].

Overview

For 1-BP, Fernandez de la Vega & Lueker [11] designed an APTAS. More precisely, for any positive integer k they provided a polynomial time algorithm $\mathcal{A}_k$ that uses at most $(1 + \frac{1}{k})\mathrm{OPT} + 1$ bins. Later, Karmarkar & Karp [16] gave a single algorithm with asymptotic approximation ratio 1 that uses $\mathrm{OPT} + O(1 + \log^2 \mathrm{OPT})$ bins. For the 2-BP problem Caprara [5] presented an algorithm with currently the best asymptotic approximation ratio 1.691. On the negative side, Bansal & Sviridenko [4] proved that there is no APTAS for 2-BP, unless P = NP. Interestingly, they provided an APTAS for a restricted version of d-BP in which the items and the bins are d-cubes; this result was independently obtained by Correa & Kenyon [8]. For 3-BP, Li & Cheng [19] and Csizik & van Vliet [10] designed algorithms with asymptotic ratio at most 4.84. Their algorithms generalize to the problem d-BP with asymptotic approximation ratio at most 1.691^d. For the problem 2-SP, the breakthrough result was obtained by Kenyon & Rémila [17] who gave an APTAS. For 3-SP, Miyazawa & Wakabayashi [20] presented an algorithm with asymptotic approximation ratio at most 2.67. On the other hand, it is easy to see that approximation hardness result for 2-BP implies that no APTAS for 3-SP can exist, unless P = NP (see Section 4 for more details).

When ninety-degree rotations are allowed, only weaker results are known. Some algorithms for the versions without rotations provide upper bounds on asymptotic approximation ratio for versions with rotations allowed as well. The results by Miyazawa & Wakabayashi [21] were the first ones where rotations are exploited in non-trivial way. Currently the best upper bounds on asymptotic approximation ratio for the problems 2-BP^r, 3-BP^r, 3-SP^r, and 3-SP^z, are $2 + \varepsilon$, 4.89, 2.76, and 2.64, respectively, see [22] and [13]. Very recently, Jansen & Stee found an APTAS for 2-SP^r ([13]).

Bin covering problems are in a sense dual problems to bin packing problems and approximation hardness results obtained for them are similar. Already for 1-BC (and hence for d-BC, for any $d \geq 1$) no polynomial time algorithm can achieve an approximation ratio less than 2, unless P = NP. This follows from the NP-hardness of the PARTITION problem similarly as for the 1-BP problem. Hence also for these problems an asymptotic polynomial time approximation scheme is the best what one can hope for. But many techniques used for bin packing problems do not seem to be adaptable for bin covering problems. Only a few results for bin covering problems are known; an APTAS for 1-BC ([9]) and non-existence of an APTAS for the 2-DIMENSIONAL VECTOR BIN COVERING problem [24] are among them.

Rectangle Packing and Covering without and with Rotations. When dealing with packing and covering problems *without rotation*, one can always assume that a bin $\mathbb{B}$ is a cube $[0,1]^d$ (resp., a base of a strip $\mathbb{B}$ is a unit $(d-1)$-cube), as the problems are invariant under *heterogeneous scaling*, i.e., the one which scales by different factors in different coordinate directions. However, the problems with rotations allowed are not invariant under such scalings. In this case we can scale homogeneously and rotate a bin and rectangles from an input to obtain an instance with the bin dimensions $(b_1, b_2, \ldots, b_d)$ satisfying $0 < b_1 \le b_2 \le \cdots \le b_{d-1} \le b_d = 1$. In general the bin dimensions b_i can depend on an instance of the problem. It is not clear if the restricted versions of the problem, when the bin $\mathbb{B}$ is uniform over all instances, or if $\mathbb{B}$ is even a unit cube, are easier to approximate than the general one. However, for such restricted versions of some problems algorithms with better asymptotic approximation ratio are known. For example, when a base of the strip in the problem 3-SP^z is a unit square, an algorithm with asymptotic approximation ratio at most 2.528 is known [21].

Using heterogeneous scaling one can show that d-BP can be viewed as a particular case of general d-BP^r with highly excentric instances. Let a list $\mathscr{L} = (R^1, R^2, \ldots, R^n)$ of rectangles with dimensions $R^i = (r^i_1, r^i_2, \ldots, r^i_d)$, $i = 1, 2, \ldots, n$, and a bin $\mathbb{B} = (b_1, b_2, \ldots, b_d)$ be an instance of d-BP. We can find positive scaling factors λ_1, λ_2, …, λ_d and use scaling $(x_1, x_2, \ldots, x_d) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_d x_d)$ to map any R^i of $\mathscr{L}$ to $\widetilde{R}^i = (\widetilde{r}^i_1, \widetilde{r}^i_2, \ldots, \widetilde{r}^i_d)$, and the bin $\mathbb{B}$ to $\widetilde{\mathbb{B}} = (\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_d)$, so that for each $p = 2, 3, \ldots, d$, the minimum $\min\{\widetilde{r}^i_j : 1 \le i \le n, 1 \le j < p\} > \widetilde{b}_p$. It is easy to see that the only way how a (rescaled) rectangle $\widetilde{R}^i$ can fit in the (rescaled) bin $\widetilde{\mathbb{B}}$, even if ninety-degree rotations are allowed, is that $\widetilde{R}^i$ is not rotated. Similarly, 3-SP can be handled as a particular case of 3-SP^r or 3-SP^z. In such a way a heterogeneous scaling can be used to reduce oriented packing problems to the ones with ninety-degree rotations allowed. Thus, for problems 2-BP^r, 3-SP^r, and 3-SP^z without any restriction on the bin $\mathbb{B}$, non-existence of an APTAS easily follows from results by Bansal & Sviridenko [4] for 2-BP (see Sections 3 and 4 for more details). For example, using a suitable heterogeneous scaling one can derive for any fixed $a \in (0, \frac{1}{2})$, an NP-hard gap type result for 2-dimensional Bin Packing with Rotations on instances with a rectangular bin $\mathbb{B} = (a, 1)$ and with all rectangles being a small perturbation of $(\frac{a}{4}, \frac{1}{2})$. However, for the most interesting case of 2-BP^r with a *square* bin $\mathbb{B}$, one can hardly obtain hardness results in such a way.

To the best knowledge of authors, no approximation hardness results were known prior this work for rectangle packing problems with rotations allowed in the most studied case of a unit square bin (resp., a unit square base of a strip), and for rectangle covering problems at all.

Main results

In this paper we prove approximation hardness results for bin and strip variants of rectangle covering and packing problems, where ninety-degree rotations are allowed. Our results apply also to the most studied restricted case with unit square bins (resp. a strip with unit square base). More precisely, we prove non-existence of an APTAS (unless P = NP) for 2-dimensional Bin Packing with Rotations into unit square bins (Section 3), 3-dimensional Strip Packing with Rotations, and z-oriented 3-dimensional Strip Packing (Section 4) into a strip with unit square base. The methods allow to provide, for each of problems studied, an explicit lower bound on asymptotic approximation ratio of any polynomial time approximation algorithm (unless P = NP). For example, we provide a lower bound $1 + \frac{1}{3792}$ for 2-Dimensional Bin Packing with Rotations, and $1 + \frac{1}{2196}$ for the same problem without rotations.

In Section 5 we develop methods suitable for covering counterparts of packing problems, and prove similar approximation hardness results for both variants of them: *without* and *with* rotations allowed.

We prove also non-existence of an APTAS for a related 3-dimensional problem where the goal is to pack the maximum number of rectangles from a given collection into a *single* cube bin (Section 6).

2 General technique

In this section we demonstrate the general technique used for the reduction from a bounded Max-3DM to geometric bin packing and covering problems. First, recall the definition of the Maximum 3-Dimensional Matching problem.

Definition 3 *Given three sets X, Y, Z, and a set $T \subseteq X \times Y \times Z$. Without loss of generality we assume that X, Y, and Z are pairwise disjoint, and any element of $X \cup Y \cup Z$ occurs in at least one triple in T. A matching in T is a subset $M \subseteq T$ such that no two ordered triples in M agree in any coordinate. The goal of the* Maximum 3-Dimensional Matching *problem (shortly,* Max-3DM*) is to find a matching in T of maximum cardinality. The k-*bounded Max-3DM *problem is the problem restricted to instances of* 3DM, *in which each element of $X \cup Y \cup Z$ occurs at most k times in T.*

Kann [15] showed that the 3-bounded Max-3DM problem is Max SNP-complete (hence also APX-complete). Thus, using the PCP-theorem, the existence of a PTAS for it would imply that P = NP. Petrank [23] proved a refined approximation hardness result that an NP-hard gap occurs also on instances with perfect matching.

Theorem A. [23] *There is an absolute constant $\varepsilon > 0$ such that for instances $T \subseteq X \times Y \times Z$ of 3-bounded* Max-3DM *with $|X| = |Y| = |Z|$ $(:= q)$ and with the optimum value $\mathrm{OPT}(T)$ it is* NP*-hard to decide of whether $\mathrm{OPT}(T) = q$ or $\mathrm{OPT}(T) < (1 - \varepsilon)q$.*

Unfortunately, the estimates that are implicit in his proof provide only extremely small $\varepsilon > 0$. To achieve explicit inapproximability results it is more convenient to use the following NP-hard gap type result for 2-bounded instances of Max-3DM ([7]).

Theorem B. [7] *There are instances $T \subseteq X \times Y \times Z$ of 2-bounded* Max-3DM *with $|X| = |Y| = |Z|(:= q)$ and every element of $X \cup Y \cup Z$ occurring in exactly 2 triples in T such that it is* NP*-hard to distinguish between instances with $\mathrm{OPT}(T) > 0.979338843q$ and $\mathrm{OPT}(T) < 0.9690082645q$.*

These approximation hardness results for a bounded Max-3DM suit well as a starting point to inapproximability results for various (multidimensional) packing, covering, and scheduling problems, see e.g., [24], [6], and [4]. All these results build on the ideas used for the first time in the reduction from Max-3DM to the 4-Partition problem. We will recall ideas of such reduction and sketch the proof of APX-completeness for the 2-dimensional Vector Bin Packing and Covering problems (see [24] and [6] for more details) to provide first explicit inapproximability results for both problems.

Definition 4 *The goal of the* 2-dimensional Vector Bin Packing *(resp.,* Covering*) problem is to partition a given list of vectors in $[0, 1]^2$ into the minimum (resp. the maximum) number of subsets such that in every subset the sum of all vectors is at most (resp. at least) one in every coordinate.*

The $\mathscr{M}$ reduction. ([24]) Given an instance $T \subseteq X \times Y \times Z$ of MAX-3DM, one can create for each element from $X \cup Y \cup Z \cup T$ a vector in $[0,1]^2$ with both coordinates being rational and from $(\frac{1}{5}, \frac{1}{3})$ such that (i) the sum of any subset of 3 vectors is at most one in every coordinate; (ii) the sum of any subset of 5 vectors is at least one in every coordinate; (iii) the sum of 4 vectors is at most one in every coordinate if and only if this sum is at least one in every coordinate if and only if the vectors correspond to $x \in X$, $y \in Y$, $z \in Z$, and $t \in T$ for some triple $t = (x, y, z) \in T$.

The way how the NP-hard gap of MAX-3DM is preserved provides a lower bound on an asymptotic ratio for the problems MAXIMUM 2-DIMENSIONAL VECTOR BIN PACKING and MINIMUM 2-DIMENSIONAL VECTOR BIN COVERING.

Theorem 1 *It is* NP-*hard to achieve an asymptotic approximation ratio* 1.00256 ($> \frac{391}{390}$) *for the* MAXIMUM 2-DIMENSIONAL VECTOR BIN PACKING *problem and* 1.0017307 ($> \frac{579}{578}$) *for the* MINIMUM 2-DIMENSIONAL VECTOR BIN COVERING *problem.*

Proof. Let $T \subseteq X \times Y \times Z$ be an instance of MAX-3DM with the optimal value OPT(T) and denote $s := |X \cup Y \cup Z| + |T|$. As follows from the $\mathscr{M}$ reduction above, the optimum $\mathrm{OPT}'(f(T))$ for an instance $f(T)$ of MAXIMUM 2-DIMENSIONAL VECTOR BIN PACKING obtained from T can be expressed as $\mathrm{OPT}'(f(T)) = \lceil \frac{s - \mathrm{OPT}(T)}{3} \rceil$. Now using an explicit NP-hard gap result for MAX-3DM (Theorem B) we obtain that for produced instances of $s = 5q$ vectors it is NP-hard to decide of whether $\mathrm{OPT}'(f(T)) > 1.343663912q$ or $\mathrm{OPT}'(f(T)) < 1.340220386q$. Consequently, it is NP-hard to achieve an asymptotic approximation ratio 1.00256 ($> 1 + \frac{1}{390}$) for the MAXIMUM 2-DIMENSIONAL VECTOR BIN PACKING problem.

Similarly, if $\mathrm{OPT}''(f(T))$ is the optimum for MINIMUM 2-DIMENSIONAL VECTOR BIN COVERING, we obtain that $\mathrm{OPT}''(f(T)) = \lfloor \frac{s + \mathrm{OPT}(T)}{5} \rfloor$. Hence, it is NP-hard to decide of whether $\mathrm{OPT}''(f(T)) < 1.193801653q$ or $\mathrm{OPT}''(f(T)) > 1.195867769q$. Consequently, it is NP-hard to achieve an asymptotic approximation ratio 1.0017307 ($> 1 + \frac{1}{578}$) for the MINIMUM 2-DIMENSIONAL VECTOR BIN COVERING problem. □

Remark. It is easy to see, that the $\mathscr{M}$ reduction is an L-reduction from a bounded version of MAX-3DM, which is known to be APX-complete, to the 2-DIMENSIONAL VECTOR BIN PACKING (resp. COVERING) problem. Consequently, MAXIMUM 2-DIMENSIONAL VECTOR BIN PACKING and MINIMUM 2-DIMENSIONAL VECTOR BIN COVERING are APX-complete (the fact, that they belong to APX, is easy to prove).

Remark. For 2-DIMENSIONAL BIN PACKING (without rotations) Bansal & Sviridenko [4] (see also [3]) recently showed that there is no APTAS for it, unless P = NP. They also claim to prove Max SNP-hardness (and APX-hardness) of the problem. However, such result does not follow from their proof. The presented reduction from 3-bounded MAX-3DM to 2-BP is not an L-reduction (or an approximation preserving reduction), but it is a *gap preserving reduction*. It only preserves the NP-hard gap guaranteed for 3-bounded MAX-3DM on instances with perfect matching by the Petrank's result. Instances T of 3-bounded MAX-3DM with perfect matching are transformed to instances that can be packed into $|T|$ unit square bins. On the other hand, any instance T with the optimum of size $< (1 - \varepsilon)q$ is transformed to the one that requires more than $(1 + \frac{\varepsilon}{33})|T|$ unit square bins. Consequently, to achieve an asymptotic approximation ratio less than $1 + \frac{\varepsilon}{33}$ for the 2-BP problem is NP-hard.

3 2-dimensional Bin Packing with Rotations

In this section we build on ideas from [4] and introduce a general parametrised version of the gap preserving reduction from bounded MAX-3DM to 2-DIMENSIONAL BIN PACKING. We show that with properly chosen parameters this reduction can be used to obtain approximation hardness results for the problem 2-DIMENSIONAL BIN PACKING WITH ROTATIONS and with unit square bin $\mathbb{B}$.

The Bin Packing reduction. Let $\mathcal{T}$ be an infinite set of instances (ordered triples) T of MAX-3DM with the optimum value $\mathrm{OPT}(T)$, such that for some efficiently computable function $\alpha(T) < \beta(T)$ it is NP-hard to decide of whether $\mathrm{OPT}(T) \geq \beta(T)$, or $\mathrm{OPT}(T) < \alpha(T)$. For a fixed instance $T \in \mathcal{T}$ let $X := \Pi_1(T)$, $Y := \Pi_2(T)$, and $Z := \Pi_3(T)$, where $\Pi_i(T) = \{p_i : (p_1, p_2, p_3) \in T\}$ for $i = 1, 2, 3$, and X, Y, Z are pairwise disjoint sets. The objects in X, Y, Z, and T will be denoted as $\{x_i : 1 \leq i \leq |X|\}$, $\{y_j : 1 \leq j \leq |Y|\}$, $\{z_k : 1 \leq k \leq |Z|\}$, and $\{t_l : 1 \leq l \leq |T|\}$, respectively. Of course, any $t_l \in T$ is of the form $t_l = (x_i, y_j, z_k) \in X \times Y \times Z$. Let $n = |X| + |Y| + |Z|$, $q = \max\{|X|, |Y|, |Z|\}$, and $r = 32q$. The reduction has several parameters: a gap location β, $\delta \in \left(0, \frac{1}{500}\right]$, and $p \in \left[\frac{1}{4} + 9\delta, \frac{1}{2} - 20\delta\right]$.

We first define an integer for each object in X, Y, Z, and T as follows:

$$\begin{aligned} x'_i &= ir^3 + i^2 r + 1, \quad \text{for } 1 \leq i \leq |X|, \\ y'_j &= jr^6 + j^2 r^4 + 2, \quad \text{for } 1 \leq j \leq |Y|, \\ z'_k &= kr^9 + k^2 r^7 + 4, \quad \text{for } 1 \leq k \leq |Z|. \end{aligned}$$

For each triple $t_l = (x_i, y_j, z_k) \in T$ we define an integer $t'_l = r^{10} - x'_i - y'_j - z'_k + 15$. Put $c = \frac{r^{10}+15}{\delta}$ and observe that $0 < x'_i, y'_j, z'_k < \frac{\delta c}{10}$ for all i, j, k, and $t'_l + x'_i + y'_j + z'_k = c\delta$ whenever $t_l = (x_i, y_j, z_k) \in T$.

For each $x_i \in X$ (resp., $y_j \in Y$ and $z_k \in Z$) we define a pair of rectangles $A_{X,i}$, $A'_{X,i}$ (resp., $A_{Y,j}$, $A'_{Y,j}$ and $A_{Z,k}$, $A'_{Z,k}$) with width about $\frac{1}{4}$ and with heights about $\frac{1}{2} + p$ and $\frac{1}{2} - p$ as follows:

$$\begin{aligned} A_{X,i} &= \left(\frac{1}{4} - 4\delta + \frac{x'_i}{c}, \frac{1}{2} + p + 4\delta - \frac{x'_i}{c}\right) \text{ and } A'_{X,i} = \left(\frac{1}{4} + 4\delta - \frac{x'_i}{c}, \frac{1}{2} - p - 4\delta + \frac{x'_i}{c}\right), \\ A_{Y,j} &= \left(\frac{1}{4} - 3\delta + \frac{y'_j}{c}, \frac{1}{2} + p + 3\delta - \frac{y'_j}{c}\right) \text{ and } A'_{Y,j} = \left(\frac{1}{4} + 3\delta - \frac{y'_j}{c}, \frac{1}{2} - p - 3\delta + \frac{y'_j}{c}\right), \\ A_{Z,k} &= \left(\frac{1}{4} - 2\delta + \frac{z'_k}{c}, \frac{1}{2} + p + 2\delta - \frac{z'_k}{c}\right) \text{ and } A'_{Z,k} = \left(\frac{1}{4} + 2\delta - \frac{z'_k}{c}, \frac{1}{2} - p - 2\delta + \frac{z'_k}{c}\right). \end{aligned}$$

For each $t_l \in T$ we define two rectangles B_l and B'_l such that

$$B_l = \left(\frac{1}{4} + 8\delta + \frac{t'_l}{c}, \frac{1}{2} + p + \delta - \frac{t'_l}{c}\right) \quad \text{and} \quad B'_l = \left(\frac{1}{4} - 8\delta - \frac{t'_l}{c}, \frac{1}{2} - p - \delta + \frac{t'_l}{c}\right).$$

Let $\mathscr{A}_X = \{A_{X,1}, A_{X,2}, \ldots, A_{X,|X|}\}$, $\mathscr{A}'_X = \{A'_{X,1}, A'_{X,2}, \ldots, A'_{X,|X|}\}$ and define sets of rectangles $\mathscr{A}_Y$, $\mathscr{A}'_Y$, $\mathscr{A}_Z$, and $\mathscr{A}'_Z$ analogously. Put $\mathscr{A} = \mathscr{A}_X \cup \mathscr{A}_Y \cup \mathscr{A}_Z$ and $\mathscr{A}' = \mathscr{A}'_X \cup \mathscr{A}'_Y \cup \mathscr{A}'_Z$. Similarly, let $\mathscr{B} = \{B_1, B_2, \ldots, B_{|T|}\}$ and $\mathscr{B}' = \{B'_1, B'_2, \ldots, B'_{|T|}\}$. We define also $\mathscr{D}$ to be a collection of $|T| + n - 4\beta(T)$ dummy rectangles of the same size $\left(\frac{3}{4} - 10\delta, 1\right)$.

The collection of rectangles $\mathscr{A} \cup \mathscr{A}' \cup \mathscr{B} \cup \mathscr{B}' \cup \mathscr{D}$, together with a square bin $\mathbb{B} = [0, 1]^2$ is now viewed as an instance of the 2-BP^r problem and denoted by $f(T)$. Our aim is to relate the optimum value $\mathrm{OPT}'(f(T))$ of 2-BP^r for an instance $f(T)$ to $\mathrm{OPT}(T)$. The dimensions of rectangles and dummy rectangles are chosen such that if $\mathrm{OPT}(T) \geq \beta(T)$, the rectangles can be packed into bins such that their number is within a factor $(1+O(\delta))$ of the total area of rectangles. On the other hand, if $\mathrm{OPT}(T) < \frac{\alpha(T)}{\gamma}$ for a constant $\gamma > 1$, then the number of bins needed to pack all rectangles of $f(T)$ is larger than the total area of rectangles by a constant factor $\gamma' > 1$ independent of δ for $\delta > 0$ small enough.

Remark. The reduction given by Bansal & Sviridenko ([4]) can be viewed as a particular case of the Bin Packing reduction with $\delta = \frac{1}{500}$, $p = 0$, a set $\mathcal{T}$ of instances $T \subseteq X \times Y \times Z$ of 3-bounded MAX-3DM with $|X| = |Y| = |Z| = q$, and a gap location $\beta(T) = q$.

Now we prove that for a proper choice of the parameter p, $p \in \left[\frac{1}{4} + 9\delta, \frac{1}{2} - 20\delta\right]$, using rotations doesn't help to produce better packings than are those without rotations. This fact allows us to use the Bin Packing reduction also as a gap preserving reduction to the problem BIN PACKING WITH ROTATIONS into a unit square bin.

Definition 5 *We say that a set of rectangles packed in a bin is* iso-oriented *if all rectangles of the set are placed with the same orientation, i.e., either all are in their initial orientations or all are rotated by ninety degrees.*

Lemma 1 *(i) For every r-packing of $f(T)$ the rectangles from $\mathscr{A} \cup \mathscr{B}$ contained in the same bin are iso-oriented.*

(ii) For every r-packing of $f(T)$ if a bin contains exactly 4 rectangles from $\mathscr{A} \cup \mathscr{B}$, then all rectangles from $\mathscr{A} \cup \mathscr{B} \cup \mathscr{A}' \cup \mathscr{B}'$ packed in this bin are iso-oriented.

Proof. (i) It easily follows from the fact that any rectangle from $\mathscr{A} \cup \mathscr{B}$ has width $> \frac{1}{4} - 4\delta$ and height $> \frac{1}{2} + p \geq \frac{3}{4} + 9\delta$.

(ii) By part (i), all 4 rectangles from $\mathscr{A} \cup \mathscr{B}$ contained in the same bin are iso-oriented. We can assume that they are in the initial position. As height of each of them is $> \frac{1}{2} + p$, any line in y-direction (i.e., parallel to y-axis) intersects the interior of at most one rectangle from $\mathscr{A} \cup \mathscr{B}$. Moreover, the sum of widths of those 4 rectangles is $> 1 - 16\delta$. Consequently, if another rectangle A (rotated, or not) is packed in this bin, then some line in y-direction intersects interiors of both, A and one of those rectangles from $\mathscr{A} \cup \mathscr{B}$. It easily follows that A is in its initial position as well, as rotated A would be too hight to fit. Consequently, all rectangles in the bin are iso-oriented. □

For oriented packings, some results from the reduction of Bansal & Sviridenko ([4]) are preserved to the general situation with the parameter p introduced. We use some important properties of their reduction with the obvious modification to the general case. The proofs of Lemmas 3 and 4 given in [4] work in this case as well, as widths of rectangles are the same in both situations.

Definition 6 ([4]) *We say that two rectangles A and A' from $\mathscr{A} \cup \mathscr{A}' \cup \mathscr{B} \cup \mathscr{B}'$ are* buddies *if $\{A, A'\}$ correspond to a pair of rectangles for a single element from X, Y, Z or T, e.g., $\{A, A'\} = \{A_{X,i}, A'_{X,i}\}$ for some $x_i \in X$ and similarly for other sets Y, Z, and T.*

Observation 1 *For any rectangle, $A \in \mathscr{A}$ implies $w(A) + h(A) = \frac{3}{4} + p$, $A' \in \mathscr{A}'$ implies $w(A') + h(A') = \frac{3}{4} - p$, $B \in \mathscr{B}$ implies $w(B) + h(B) = \frac{3}{4} + p + 9\delta$, and $B' \in \mathscr{B}'$ implies $w(B') + h(B') = \frac{3}{4} - p - 9\delta$.*

Observation 2 *For any two rectangles A, A' in $\mathscr{A} \cup \mathscr{A}' \cup \mathscr{B} \cup \mathscr{B}'$, $h(A) + h(A') = 1$ if and only if A and A' are buddies.*

Lemma 2 *Consider a bin containing exactly 4 rectangles from $\mathscr{A} \cup \mathscr{B}$ for an oriented packing of $f(T)$. Then the bin contains at most 8 rectangles and if it contains exactly 8 rectangles, then for any $h \in \left[4\delta, \frac{1}{2} - p - 4\delta\right]$, each rectangle intersects exactly one of lines $L_1 = \{(x, y) : y = h\}$ and $L_2 = \{(x, y) : y = 1 - h\}$.*

Proof. Assume that an oriented packing (i.e., without rotations) of a square bin $[0, 1]^2$ contains exactly 4 rectangles from $\mathscr{A} \cup \mathscr{B}$, and some rectangles from $\mathscr{A}' \cup \mathscr{B}'$. The projections of rectangles from $\mathscr{A} \cup \mathscr{B}$ on the x-axis cannot overlap (rectangles are too high) and hence, less than 16δ of the length of $[0, 1]$ can be uncovered by them. As width of each rectangle is roughly $\frac{1}{4}$, these projections are only small perturbations of intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Now consider a rectangle A' from $\mathscr{A}' \cup \mathscr{B}'$ packed in this unit square bin. Its projection on the x-axis has to overlap with at least one rectangle A from $\mathscr{A} \cup \mathscr{B}$. As height of A is larger than $\frac{1}{2} + p$, A' is either completely above the line $\{(x, y) : y = \frac{1}{2} + p\}$, or below the line $\{(x, y) : y = \frac{1}{2} - p\}$. It is also easy to see that no line in y-direction can intersect 3 rectangles. Hence, if such line intersects interiors of two distinct rectangles from $\mathscr{A}' \cup \mathscr{B}'$, then one is located completely above the line $\{(x, y) : y = \frac{1}{2} + p\}$ and another one is below the line $\{(x, y) : y = \frac{1}{2} - p\}$. Moreover, the total overlap of projections in y-direction of rectangles from $\mathscr{A}' \cup \mathscr{B}'$ in the bin $[0, 1]^2$ is less than 16δ. As width of each rectangle from $\mathscr{A}' \cup \mathscr{B}'$ is roughly $\frac{1}{4}$, there are at most 4 rectangles from them in the bin. If there are exactly 4 such rectangles, their projections on the x-axis are again small perturbations of intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Assume now that there are exactly 4 rectangles from $\mathscr{A}' \cup \mathscr{B}'$ in a bin $[0, 1]^2$, let A be one of four rectangles from $\mathscr{A} \cup \mathscr{B}$ in this bin. Clearly, A has its projection on the x-axis overlapping with that of a rectangle from $\mathscr{A}' \cup \mathscr{B}'$, say A'. As height of A is $> \frac{1}{2} + p$ and height of A' is $> \frac{1}{2} - p - 4\delta$, it easily follows that whenever $h \in \left[4\delta, \frac{1}{2} - p - 4\delta\right]$, each of both rectangles intersects exactly one of lines $L_1 = \{(x, y) : y = h\}$ and $L_2 = \{(x, y) : y = 1 - h\}$. The rest follows from the fact that the projection of each rectangle from $\mathscr{A}' \cup \mathscr{B}'$ on the x-axis overlaps with the projection of some rectangle from $\mathscr{A} \cup \mathscr{B}$. □

Lemma 3 *For any rectangles A_1, A_2, $A_3 \in \mathscr{A}$ and $B \in \mathscr{B}$, $w(A_1) + w(A_2) + w(A_3) + w(B) = 1$ if and only if $\{A_1, A_2, A_3, B\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l\}$ for some integers i, j, k, and l such that $t_l = (x_i, y_j, z_k) \in T$. A similar statement holds also for rectangles A'_1, A'_2, $A'_3 \in \mathscr{A}'$, $B' \in \mathscr{B}'$.*

Lemma 4 *Let A_1, A_2, A_3, $A_4 \in \mathscr{A} \cup \mathscr{A}'$ be such that no two of them are buddies. Then $\sum_{i=1}^{4} w(A_i) \neq 1$.*

Definition 7 *Given an r-packing of a bin by some rectangles from $f(T)$. The bin is called* well-packed*, if it contains exactly 4 rectangles from $\mathscr{A} \cup \mathscr{B}$ and 4 rectangles from $\mathscr{A}' \cup \mathscr{B}'$.*

Now the crucial fact is, that for any choice of the parameter p from the interval $\left[\frac{1}{4} + 9\delta, \frac{1}{2} - 20\delta\right]$, we can characterize the structure of well-packed bins for packing with rotations allowed similarly as it has been done in [4] for oriented packings.

Lemma 5 *A bin is well-packed if and only if it contains the rectangles $A_{X,i}$, $A_{Y,j}$, $A_{Z,k}$, B_l, $A'_{X,i}$, $A'_{Y,j}$, $A'_{Z,k}$, B'_l, for some $t_l = (x_i, y_j, z_k) \in T$.*

Proof. The previous lemmas and observations show that we can argue essentially in the same way as Bansal & Sviridenko [4].

(I) The 8-tuple of rectangles corresponding to a triple as above can be packed in a square bin $\mathbb{B} = [0,1]^2$ even without rotations. Starting from the bottom left corner of the bin $\mathbb{B}$ and moving to the right, each of rectangles $A_{X,i}$, $A_{Y,j}$, $A_{Z,k}$, and B_l is placed such that it touches the bottom of the bin $\mathbb{B}$. As $w(A_{X,i}) + w(A_{Y,j}) + w(A_{Z,k}) + w(B_l) = 1$ (Lemma 3), the rectangles can be packed in this way. The rectangles $A'_{X,i}$, $A'_{Y,j}$, $A'_{Z,k}$, and B'_l can be placed in the remaining gaps starting from the top left corner of the bin $\mathbb{B}$ and moving towards the right touching the top of the bin. Clearly, it is possible as

$$\begin{aligned}
&w(A'_{X,i}) + w(A'_{Y,j}) + w(A'_{Z,k}) + w(B'_l) = 1,\\
&h(A_{X,i}) + h(A'_{X,i}) = h(A_{Y,j}) + h(A'_{Y,j}) = h(A_{Z,k}) + h(A'_{Z,k}) = h(B_l) + h(B'_l) = 1,\\
&h(A_{X,i}) > h(A_{Y,j}) > h(A_{Z,k}) > h(B_l),\\
&w(A_{X,i}) < w(A'_{X,i}), w(A_{Y,j}) < w(A'_{Y,j}), \text{ and } w(A_{Z,k}) < w(A'_{Z,k}).
\end{aligned}$$

(II) Now we show that any well-packed bin contains rectangles that correspond to a triple in T. Due to Lemma 1(ii), the rectangles are iso-oriented. We can assume that they are all in the initial position in a well-packed bin; the case when all are rotated by 90° could be discussed similarly. Fix $h \in \left[4\delta, \frac{1}{2} - p - 4\delta\right]$ and consider the lines $L_1 = \{(x,y) : y = h\}$ and $L_2 = \{(x,y) : y = 1-h\}$. Due to Lemma 2, each rectangle must intersect exactly one of the lines L_1 and L_2. Moreover, as any rectangle has width larger than $\frac{1}{5}$, each of lines L_1 and L_2 intersects exactly 4 rectangles. Let $\{A_1, A_2, A_3, A_4\}$ denote the rectangles that intersect L_1 such that A_i is to the left of A_j for $i < j$. Similarly, let $\{A_5, A_6, A_7, A_8\}$ denote the rectangles that intersect L_2 in the left to right order. Thus, we have that

$$\sum_{i=1}^{4} w(A_i) \leq 1, \tag{1}$$

$$\sum_{i=1}^{4} w(A_{i+4}) \leq 1. \tag{2}$$

Observe that for each $i = 1,2,3,4$ the rectangle A_i must overlap with A_{i+4} in the x-coordinate. Thus, we have that

$$h(A_i) + h(A_{i+4}) \leq 1 \quad \text{for } i = 1,2,3,4. \tag{3}$$

From (3) it follows that, for each $i = 1,2,3,4$, at most one of A_i, A_{i+4} belongs to $\mathscr{A} \cup \mathscr{B}$. Consequently, for each $i = 1,2,3,4$ exactly one of A_i, A_{i+4} is from $\mathscr{A} \cup \mathscr{B}$ and another one is from $\mathscr{A}' \cup \mathscr{B}'$. Using these facts, we can use the same arguments as in [4]:

(i) First observe that *at most* 1 *from rectangles* $\{A_1, \ldots, A_8\}$ *belongs to* $\mathscr{B}$. Indeed, if $k \geq 2$ of them belong to $\mathscr{B}$ and $4-k$ belong to $\mathscr{A}$, then the sum of widths of these rectangles from $\mathscr{A} \cup \mathscr{B}$ would be > 1, a contradiction with the fact that any line in y-direction intersects at most 1 rectangle from $\mathscr{A} \cup \mathscr{B}$.

(ii) *If no rectangle from* $\{A_1, \ldots, A_8\}$ *belongs to* $\mathscr{B}$*, than the same is true for* $\mathscr{B}'$. The height of any rectangle in $\mathscr{B}'$ is larger then $\frac{1}{2} - p - \delta$ so such rectangle cannot form a pair $\{A_i, A_{i+4}\}$ with a rectangle from $\mathscr{A}$. Thus, in this case four rectangles belong to $\mathscr{A}$ and four to $\mathscr{A}'$. Using

Observation 1 we get $\sum_{i=1}^{8}(w(A_i) + h(A_i)) = 6$, thus it must be the case that each of (1), (2) and (3) must hold with equality. By Observation 2, A_i and A_{i+4} are buddies for each $i = 1, 2, 3, 4$. In particular, no two rectangles among A_1, A_2, A_3, and A_4 are buddies. Now Lemma 4 contradicts with $\sum_{i=1}^{4} w(A_i) = 1$ that has been observed earlier. Thus this case is impossible.

So, necessarily *exactly one of rectangles* $\{A_1, A_2, \ldots, A_8\}$ *belongs to* $\mathscr{B}$, say B_l.

(iii)As, due to (3), no pair $\{A_i, A_{i+4}\}$ can contain a rectangle from $\mathscr{B}'$ and a rectangle from $\mathscr{A}$, there can be *at most one rectangle from* $\mathscr{B}'$. But if there are no rectangles from $\mathscr{B}'$, then the sum of widths of all 8 rectangles would be > 2, a contradiction.

Consequently, there is *exactly* 1 *rectangle from* $\mathscr{B}'$, 1 *from* $\mathscr{B}$, 3 *from* $\mathscr{A}$, *and* 3 *from* $\mathscr{A}'$. Using Observation 1 we get $\sum_{i=1}^{8}(w(A_i)+h(A_i)) = 6$, thus each of (1), (2), and (3) holds with equality. In particular, for each $i = 1, 2, 3, 4$, A_i and A_{i+4} are buddies due to Observation 2. Let $m \in \{1, 2\}$ be such that B_l intersects the line L_m. Let A_{m_1}, A_{m_2}, A_{m_3} denote the other three rectangles (from $\mathscr{A} \cup \mathscr{A}'$) which are also intersected by L_m. Thus we have that $w(A_{m_1})+w(A_{m_2})+w(A_{m_3})+w(B_l) = 1$. None of A_{m_1}, A_{m_2}, A_{m_3} can lie in $\mathscr{A}'$ because otherwise $w(A_{m_1})+w(A_{m_2})+w(A_{m_3})+w(B_l) > (\frac{1}{4} + 8\delta) + (\frac{1}{4} + \delta) + 2(\frac{1}{4} - 4\delta) = 1 + \delta$, a contradiction. Hence $\{A_{m_1}, A_{m_2}, A_{m_3}\} \subseteq \mathscr{A}$, and using Lemma 3 we get that $\{A_{m_1}, A_{m_2}, A_{m_3}\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}\}$ for integers i, j, k such that $t_l = (x_i, y_j, z_k)$, where t_l is the corresponding triple for the rectangle B_l. This completes the proof. $\square$

Now we can prove the main theorem of this section

Theorem 2 *There is a constant* $\rho > 1$ *such that it is* NP*-hard to approximate* 2-DIMENSIONAL BIN PACKING WITH ROTATIONS *into unit square bins with an asymptotic approximation ratio less than* ρ.

Proof. Recall that the Bin Packing reduction f started from a set $\mathcal{T}$ of instances of MAX-3DM such that for $T \in \mathcal{T}$ it is NP-hard to decide of whether $\text{OPT}(T) \geq \beta(T)$, or $\text{OPT}(T) < \alpha(T)$.

(a) Assume first that $T \in \mathcal{T}$ is such that $\text{OPT}(T) \geq \beta(T)$. We will show that the corresponding instance $f(T)$ of the 2-BP^r problem has its optimum $\text{OPT}'(f(T))$ of size at most $|T| + n - 3\beta(T)$. Consider a matching M in T consisting of $\beta(T)$ triples. For each triple $t_l = (x_i, y_j, z_k) \in M$ we create a well-packed bin with rectangles $\{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l, A'_{X,i}, A'_{Y,j}, A'_{Z,k}, B'_l\}$ packed.

For each $t_l \in T \setminus M$ we can put B_l and B'_l along with a dummy rectangle into a bin; in this way we use $|T| - \beta(T)$ dummy rectangles.

For each of $n - 3\beta(T)$ elements in $X \cup Y \cup Z$ that are not covered by M, we put in a bin the corresponding buddies A and A' along with one dummy rectangle. The rest of dummy rectangles is used in this way and all rectangles from $f(T)$ are packed into $|T| + n - 3\beta(T)$ bins.

(b) Assume now that $T \in \mathcal{T}$ satisfies $\text{OPT}(T) < \alpha(T)$. Our aim is to estimate $\text{OPT}'(f(T))$ from below. Consider for an instance $f(T)$ any feasible solution of 2-BP^r. There will be exactly $N_d = |T| + n - 4\beta(T)$ bins with dummy rectangles, each of them can contain at most one rectangle from $\mathscr{A} \cup \mathscr{B}$. Let us consider now bins without dummy rectangles. If such bin is not well-packed then it either contains at most 3 rectangles from $\mathscr{A} \cup \mathscr{B}$ or else it contains at most 3 rectangles from $\mathscr{A}' \cup \mathscr{B}'$. Let N_g denote the number of well-packed bins. Among the bins without dummy rectangles which are not well-packed, let N_{b_2} denote the number of bins with at most 3 rectangles from $\mathscr{A} \cup \mathscr{B}$, and let N_{b_1} denote the number of the rest rectangles (i.e., N_{b_1} is the number of bins with 4 rectangles from $\mathscr{A} \cup \mathscr{B}$, but with at most 3 rectangles from $\mathscr{A}' \cup \mathscr{B}'$).

Since all $|T| + n$ rectangles from $\mathscr{A} \cup \mathscr{B}$ have to be packed, we have the constraint that

$$4N_g + 4N_{b_1} + 3N_{b_2} + N_d \geq |T| + n,$$

or equivalently

$$4N_g + 4N_{b_1} + 3N_{b_2} \geq 4\beta(T). \tag{4}$$

With the choice of parameter $p = \frac{1}{4} + 9\delta$ and assuming $\delta \in (0, \frac{1}{500}]$ as small as we need, rectangles from $\mathscr{A} \cup \mathscr{B}$ are roughly $(\frac{1}{4}, \frac{3}{4})$ each, and those from $\mathscr{A}' \cup \mathscr{B}'$ are roughly $(\frac{1}{4}, \frac{1}{4})$ each. In what follows we will count rectangles from $\mathscr{A} \cup \mathscr{B}$ with weight 3, and those from $\mathscr{A}' \cup \mathscr{B}'$ with weight 1 each. Easy area's estimate shows that the total weight of rectangles packed to a unit square bin cannot exceed 16. Further, any bin containing a dummy rectangle can contain rectangles from $\mathscr{A} \cup \mathscr{B} \cup \mathscr{A}' \cup \mathscr{B}'$ of weight at most 4. Observe that each of N_{b_1} bins contains rectangles of weight at most 15. Hence the second constraint derived from the fact that all rectangles have to be packed reads as follows:

$$16N_g + 15N_{b_1} + 16N_{b_2} + 4N_d \geq 4(|T| + n).$$

Using $N_d = |T| + n - 4\beta(T)$ and adding the constraint (4) to the last one we get

$$20N_g + 19N_{b_1} + 19N_{b_2} \geq 20\beta(T).$$

Since the set of well-packed bins corresponds to a feasible solution for a matching (by Lemma 5), $N_g < \alpha(T)$. Thus, assuming $\mathrm{OPT}(T) < \alpha(T)$ we get

$$\mathrm{OPT}'(f(T)) > N_g + N_{b_1} + N_{b_2} + N_d \geq \frac{20}{19}\beta(T) - \frac{1}{19}N_g + N_d > |T| + n - 3\beta(T) + \frac{1}{19}(\beta(T) - \alpha(T)).$$

It easily follows that our reduction f is a gap preserving reduction assuming that we started from $(\alpha(T), \beta(T))$-gap version of the bounded Max-3DM problem.

Now suppose that for a fixed constant ρ, $1 < \rho < 1 + \frac{1}{19}\frac{\beta(T)-\alpha(T)}{|T|+n-3\beta(T)}$, there exists a polynomial time algorithm $\mathcal{A}_\rho$ and a constant C such that for instances $f(T)$ if $\mathrm{OPT}'(f(T)) > C$, then $\mathcal{A}_\rho \leq \rho\mathrm{OPT}'(f(T))$. Thus, for any corresponding instance T of Max-3DM we could distinguish whether $\mathrm{OPT}(T) \geq \beta(T)$, or $\mathrm{OPT}(T) < \alpha(T)$, which is an NP-hard problem. Hence, it is NP-hard to achieve an asymptotic approximation ratio $\leq \rho$ for the problem 2-dimensional Bin Packing with Rotations into unit square bins. □

Using the NP-hard gap result from Theorem B we can obtain an explicit lower bound $1 + \frac{1}{3792}$ on asymptotic approximation ratio of any polynomial time approximation algorithm for 2-dimensional Bin Packing with Rotations into unit square bins. For the same problem *without* rotations our method provides a lower bound $1 + \frac{1}{2196}$.

4 3-dimensional Strip Packing problems

In this section we apply the approximation hardness results for 2-dimensional bin packing problems to obtain similar hardness results for variants of 3-dimensional strip packing problems restricted to instances in which the strip have a unit square base.

Definition 8 *Let a list of 2-dimensional rectangles $\mathscr{L} = \{(r_1^1, r_2^1), (r_1^2, r_2^2), \ldots, (r_1^n, r_2^n)\}$ with a bin $\mathbb{B} = (b_1, b_2)$ be an instance of the 2-dimensional Bin Packing problem (possibly with rotations).*

For a fixed parameter $t > 0$ we define an instance of the 3-DIMENSIONAL STRIP PACKING *problem (possibly with rotations) as a list of* 3*-dimensional rectangles* $\mathscr{L}_t = \{(r_1^i, r_2^i, t) : 1 \le i \le n\}$ *with a strip* (b_1, b_2, ∞).

The optimum for all three variants of 3-dimensional strip packing problems 3-SP, 3-SP^r, and 3-SP^z, for the instance $\mathscr{L}_t$ can be expressed using the optimum for the instance $\mathscr{L}$ of the 2-dimensional bin packing problem (possibly with rotations) as follows.

Lemma 6 *If* $\mathrm{OPT}(\mathscr{L})$ *denote the optimum of an instance* $\mathscr{L}$ *for* 2-BP *and* $\mathrm{OPT}'(\mathscr{L}_t)$ *the optimum of the corresponding* 3*-dimensional instance* $\mathscr{L}_t$ *for* 3-SP*, then* $\mathrm{OPT}'(\mathscr{L}_t) = t \cdot \mathrm{OPT}(\mathscr{L})$*. The same statement holds also for the optimum of an instance* $\mathscr{L}$ *for* 2-BP^r *and the optimum of the corresponding* 3*-dimensional instance* $\mathscr{L}_t$ *for* 3-SP^z*, resp.* 3-SP^r *(in case of* 3-SP^r *we assume that* $t > \max\{b_1, b_2\}$*).*

Proof. (i) Consider a packing of $\mathscr{L}$ into $\mathrm{OPT}(\mathscr{L})$ bins (b_1, b_2). It generates the strip packing of $\mathscr{L}_t$ into a strip (b_1, b_2, ∞) with height $t \cdot \mathrm{OPT}(\mathscr{L})$. Hence $\mathrm{OPT}'(\mathscr{L}_t) \le t \cdot \mathrm{OPT}(\mathscr{L})$. Now assume, that $\mathscr{L}_t$ can be packed into the strip (b_1, b_2, ∞) with height $h < t \cdot \mathrm{OPT}(\mathscr{L})$. If $s = \min\{t, t \cdot \mathrm{OPT}(\mathscr{L}) - h\}$, then planes $\{z = t - s\}$, $\{z = 2t - s\}$, ..., $\{z = t \cdot (\mathrm{OPT}(\mathscr{L}) - 1) - s\}$ intersect interiors or touch bottom of all rectangles from the list $\mathscr{L}$. These plane cuts determine packing of $\mathscr{L}$ into $\mathrm{OPT}(\mathscr{L}) - 1$ bins (b_1, b_2), a contradiction that completes the proof.

The statement for 3-SP^z can be proved in the same way. Moreover, if $t > \max\{b_1, b_2\}$, then any r-packing of $\mathscr{L}_t$-rectangles into the strip (b_1, b_2, ∞) has to be z-oriented. □

Using Lemma 6 it is easy to see that non-existence of an APTAS for 2-BP ([4]) implies non-existence of APTAS for the 3-SP problem, unless P = NP. Moreover, using a heterogeneous scaling one can obtain some inapproximability results also for 3-SP^z and 3-SP^r already from hardness results for 2-BP. In this way one can obtain approximation hardness results, for example, for instances of 3-SP^z and 3-SP^r with a strip $(b, 1, \infty)$ for any fixed $b \in (0, \frac{1}{2})$. But for a strip with a square base we can rely on the approximation hardness result shown in the previous section for 2-BP^r with square bin.

Theorem 3 *There is no* APTAS *for any of* 3*-dimensional strip packing problems* 3-SP*,* 3-SP^z*, and* 3-SP^r *on instances with the strip* $(1, 1, \infty)$*, unless* P = NP.

Proof. The result for 3-SP follows directly from [4] by Lemma 6. Similarly, Theorem 2 together with Lemma 6 imply that no APTAS can exist, unless P = NP, for 3-SP^z and 3-SP^r on instances with the strip $(1, 1, \infty)$. □

Remark. In fact, all achieved approximation hardness results are of "NP-hard gap type", and they are obtained on instances with quite uniform structure. Namely,

- for 2-BP all rectangles are small perturbations of $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{3}{4}, 1\right)$,
- for 2-BP^r all rectangles are small perturbations of $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\left(\frac{1}{4}, \frac{1}{4}\right)$, and $\left(\frac{3}{4}, 1\right)$,
- for 3-SP all rectangles are small perturbations of $\left(\frac{1}{4}, \frac{1}{2}, 1\right)$ and $\left(\frac{3}{4}, 1, 1\right)$,
- for 3-SP^z and 3-SP^r all rectangles are small perturbations of $\left(\frac{1}{4}, \frac{3}{4}, 2\right)$, $\left(\frac{1}{4}, \frac{1}{4}, 2\right)$, and $\left(\frac{3}{4}, 1, 2\right)$.

5 Bin Covering Problems

In this section we prove that for many rectangle covering problems no APTAS exist, unless P = NP, similarly as for their packing counterparts.

5.1 2-dimensional Bin Covering

In this section we provide approximation hardness results for both versions of the 2-dimensional Bin Covering problems, without, resp. with ninety-degree rotations allowed. Our gap preserving reduction from the bounded Max-3DM problem is similar to the Bin Packing Reduction presented for packing problems in Section 3; the only difference is in values of some parameters and dimensions of rectangles. Even if our analysis has many similarities with that given for packings, the case of coverings appears to be technically more complicated to handle. One of reasons, that makes the case of packings easier, is that we have à priori an upper bound on number of rectangles used for a single bin. For covering problems no such upper bound is available and we have to deal with a variety of possibilities how a bin can be covered.

The Bin Covering reduction. Let $\mathcal{T}$ be an infinite set of instances (ordered triples) T of Max-3DM with the optimum value OPT(T), such that for some efficiently computable functions $\alpha(T) < \beta(T)$ it is NP-hard to decide for $T \in \mathcal{T}$ of whether $\mathrm{OPT}(T) \geq \beta(T)$, or $\mathrm{OPT}(T) < \alpha(T)$. For an instance $T \in \mathcal{T}$ we define in the same way as in the Bin Packing reduction the pairwise disjoint sets X, Y, Z, and for each object in X, Y, Z, and T integers x'_i for $1 \leq i \leq |X|$, y'_j for $1 \leq j \leq |Y|$, z'_k for $1 \leq k \leq |Z|$, t'_l for $1 \leq l \leq |T|$. The reduction has several parameters: a gap location β, constants $\delta \in \big(0, \frac{1}{2000}\big]$, and $p \in \big[0, \frac{7}{16}\big]$. To simplify some considerations, instead of a general parameter p we can think about an explicit value of p, namely

- $p = 0$ for 2-dimensional Bin Covering (without rotations), and
- $p = \frac{3}{8}$ for 2-dimensional Bin Covering With Rotations.

As many considerations work well for both values of the parameter p, we will study them in parallel.

Similarly as in the Bin Packing reduction let $n = |X| + |Y| + |Z|$, $q = \max\{|X|, |Y|, |Z|\}$, $r = 32q$, $c = \frac{r^{10}+15}{\delta}$ and observe that $0 < x'_i, y'_j, z'_k < \frac{\delta c}{10}$ for all i, j, k, and $t'_l + x'_1 + y'_j + z'_k = c\delta$ whenever $t_l = (x_i, y_j, z_k) \in T$.

For each $x_i \in X$ (resp., $y_j \in Y$ and $z_k \in Z$) we define a pair of rectangles $A_{X,i}$, $A'_{X,i}$ (resp. $A_{Y,j}$, $A'_{Y,j}$ and $A_{Z,k}$, $A'_{Z,k}$) as follows:

$$A_{X,i} = \Big(\frac{1}{4} - 4\delta + \frac{x'_i}{c}, \frac{1}{2} - \delta - \frac{x'_i}{c} - p\Big) \text{ and } A'_{X,i} = \Big(\frac{1}{4} + 4\delta - \frac{x'_i}{c}, \frac{1}{2} + \delta + \frac{x'_i}{c} + p\Big),$$

$$A_{Y,j} = \Big(\frac{1}{4} - 3\delta + \frac{y'_j}{c}, \frac{1}{2} - 2\delta - \frac{y'_j}{c} - p\Big) \text{ and } A'_{Y,j} = \Big(\frac{1}{4} + 3\delta - \frac{y'_j}{c}, \frac{1}{2} + 2\delta + \frac{y'_j}{c} + p\Big),$$

$$A_{Z,k} = \Big(\frac{1}{4} - 2\delta + \frac{z'_k}{c}, \frac{1}{2} - 3\delta - \frac{z'_k}{c} - p\Big) \text{ and } A'_{Z,k} = \Big(\frac{1}{4} + 2\delta - \frac{z'_k}{c}, \frac{1}{2} + 3\delta + \frac{z'_k}{c} + p\Big).$$

For each $t_l \in T$ we define a pair of rectangles B_l and B'_l such that

$$B_l = \Big(\frac{1}{4} + 8\delta + \frac{t'_l}{c}, \frac{1}{2} - p - \frac{t'_l}{c}\Big) \quad \text{and} \quad B'_l = \Big(\frac{1}{4} - 8\delta - \frac{t'_l}{c}, \frac{1}{2} + p + \frac{t'_l}{c}\Big).$$

Let $\mathscr{A}_X = \{A_{X,1}, A_{X,2}, \dots, A_{X,|X|}\}$, $\mathscr{A}'_X = \{A'_{X,1}, A'_{X,2}, \dots, A'_{X,|X|}\}$ and let sets $\mathscr{A}_Y$, $\mathscr{A}'_Y$, $\mathscr{A}_Z$, and $\mathscr{A}'_Z$ be defined analogously. Put $\mathscr{A} = \mathscr{A}_X \cup \mathscr{A}_Y \cup \mathscr{A}_Z$ and $\mathscr{A}' = \mathscr{A}'_X \cup \mathscr{A}'_Y \cup \mathscr{A}'_Z$. Similarly, let

$\mathscr{B} = \{B_1, B_2, \ldots, B_{|T|}\}$ and $\mathscr{B}' = \{B'_1, B'_2, \ldots, B'_{|T|}\}$. Define $\mathscr{D}$ to be a collection of $|T|+n-4\beta(T)$ dummy rectangles of the same size $\left(\frac{3}{4}+9\delta, 1\right)$.

Now the collection of rectangles $\mathscr{A} \cup \mathscr{A}' \cup \mathscr{B} \cup \mathscr{B}' \cup \mathscr{D}$ together with a square bin $\mathbb{B} = [0,1]^2$ is viewed as an instance of the problem 2-BC, resp. 2-BC^r, and denoted by $f(T)$. Our aim is to relate the optimum value $\mathrm{OPT}'(f(T))$ of the problem 2-BC, resp. 2-BC^r, for an instance $f(T)$, to the optimum value $\mathrm{OPT}(T)$.

Remark. It is easy to see that if a rectangle A from $f(T)$ partially cover a bin $\mathbb{B}$, one can shift A to a new position A' such that $A \cap \mathbb{B} \subseteq A' \subseteq \mathbb{B}$. Hence we can confine ourselves to coverings, in which any rectangle partially covering a bin, is completely contained in this bin.

Some results proved for the Bin Packing reduction can be easily modified for the Bin Covering reduction and their consequences derived above apply without any change. In particular, the corresponding rectangles have the same width in both reductions.

Observation 1' *For any rectangle,* $A \in \mathscr{A}$ *implies* $w(A)+h(A) = \frac{3}{4} - 5\delta - p$, $A' \in \mathscr{A}'$ *implies* $w(A')+h(A') = \frac{3}{4}+5\delta+p$, $B \in \mathscr{B}$ *implies* $w(B)+h(B) = \frac{3}{4}+8\delta-p$, *and* $B' \in \mathscr{B}'$ *implies* $w(B')+h(B') = \frac{3}{4}-8\delta+p$.

One can observe that Observation 2, Lemma 3, and Lemma 4 apply as well.

Definition 9 *Given a covering of a bin by some rectangles of* $f(T)$. *The bin is called* well-covered *if it is covered by exactly* 8 *rectangles from which* 4 *rectangles are from* $\mathscr{A} \cup \mathscr{B}$, 4 *are from* $\mathscr{A}' \cup \mathscr{B}'$, *and the numbers of rectangles from* $\mathscr{B}$ *is the same as those from* $\mathscr{B}'$.

Analysis. Consider an oriented covering of the unit square bin by exactly 8 rectangles from $\mathscr{A} \cup \mathscr{B} \cup \mathscr{A}' \cup \mathscr{B}'$. Then no two of points $[0,0]$, $[\frac{1}{3},0]$, $[\frac{2}{3},0]$, $[1,0]$, $[0,1]$, $[\frac{1}{3},1]$, $[\frac{2}{3},1]$, and $[1,1]$ belong to the same rectangle. Let A_1, A_2, A_3, $\ldots$, A_8 denote, respectively, the rectangles that contain these points. Clearly, $A_1 \cup A_2 \cup A_3 \cup A_4$ covers $\{(x,0) : x \in [0,1]\}$, hence

$$\sum_{i=1}^{4} w(A_i) \geq 1. \tag{1'}$$

Similarly, $A_5 \cup A_6 \cup A_7 \cup A_8$ covers $\{(x,1) : x \in [0,1]\}$, hence

$$\sum_{i=1}^{4} w(A_{i+4}) \geq 1. \tag{2'}$$

For each $i = 1,2,3,4$, $(A_i \cup A_{i+4}) \cap [0,1]^2$ is a small (depending on δ) perturbation of $[\frac{i-1}{4}, \frac{i}{4}] \times [0,1]$. In particular, the segment $\{(\frac{i-1}{3}, y) : y \in [0,1]\}$ is covered by $A_i \cup A_{i+4}$ only and hence

$$h(A_i) + h(A_{i+4}) \geq 1 \quad \text{for } i = 1,2,3,4. \tag{3'}$$

Consequently, for each $i \in \{1,2,3,4\}$ at least one of rectangles A_i, A_{i+4} belongs to $\mathscr{A}' \cup \mathscr{B}'$. As height of any rectangle from $\mathscr{A} \cup \mathscr{B}$ is less than $\frac{1}{2}$, the segment $\{(x, \frac{1}{2}) : x \in [0,1]\}$ is covered by rectangles from $\mathscr{A}' \cup \mathscr{B}'$. Thus

$$\sum_{\substack{i=1 \\ A_i \in \mathscr{A}' \cup \mathscr{B}'}}^{8} w(A_i) \geq 1. \tag{5}$$

From now on we additionally assume that from the 8 rectangles $A_1, A_2, \ldots, A_8$ covering the bin $\mathbb{B}$ exactly 4 belong to $\mathscr{A} \cup \mathscr{B}$ and 4 belong to $\mathscr{A}' \cup \mathscr{B}'$. From (3') it easily follows that for each $i \in \{1,2,3,4\}$ exactly one of rectangles A_i, A_{i+4} belongs to $A \cup B$, and one to $A' \cup B'$. Inspecting the range of heights of rectangles in $\mathscr{A}_X$, $\mathscr{A}_Y$, $\mathscr{A}_Z$, $\mathscr{B}$, $\mathscr{A}'_X$, $\mathscr{A}'_Y$, $\mathscr{A}'_Z$, $\mathscr{B}'$ leads to more restrictions on possible combinations in pairs A_i, A_{i+4}. We will employ an observation that if one of A_i, A_{i+4} belongs to $\mathscr{B}'$ then the another one belongs to $\mathscr{B}$. In particular,

$$|\mathscr{B} \cap \{A_i : 1 \leq i \leq 8\}| \geq |\mathscr{B}' \cap \{A_i : 1 \leq i \leq 8\}|. \tag{6}$$

We can observe further that

$$|\mathscr{B}' \cap \{A_i : 1 \leq i \leq 8\}| \leq 1. \tag{7}$$

To show that, let $j := |\mathscr{B}' \cap \{A_i : 1 \leq i \leq 8\}|$. Thus j rectangles from $A_1, \ldots, A_8$ belong to $\mathscr{B}'$ and $(4-j)$ belong to $\mathscr{A}'$, thus $\sum_{i=1, A_i \in \mathscr{A}' \cup \mathscr{B}'}^{8} w(A_i) < 1 + 16\delta - 12j\delta$. Due to (5), only $j = 0$ or $j = 1$ is possible for an integer j.

The next lemma allows to prove that for the choice $p = \frac{3}{8}$ of the parameter in the Bin Packing reduction the situation for r-coverings reduces to the one for oriented coverings. Namely, even if rotations are allowed it is not advantageous to use them.

Lemma 7 *For the parameter $p = \frac{3}{8}$, if an r-covering of a bin consists of 8 rectangles such that 4 of them are from $\mathscr{A} \cup \mathscr{B}$ and 4 from $\mathscr{A}' \cup \mathscr{B}'$, then all 8 rectangles are iso-oriented (i.e., either all in the initial position or rotated by $90°$ in the same way). In particular, rectangles covering well-covered bins are always iso-oriented.*

Proof. It is easy to observe that the total area of 4 rectangles from $\mathscr{A} \cup \mathscr{B}$ and 4 rectangles from $\mathscr{A}' \cup \mathscr{B}'$ is at most $1 + 25\delta$. Hence, if these rectangles cover a unit square bin, only small overlap of rectangles is possible. Namely, the area in the bin with multiple covering is at most 25δ. So, it is sufficient to show that if rectangles are not iso-oriented, the overlap is larger.

Observe first that rectangles from $\mathscr{A}' \cup \mathscr{B}'$ are iso-oriented. If two rectangles from $\mathscr{A}' \cup \mathscr{B}'$ contained in the unit square bin are *not* iso-oriented, then they overlap at least in a square with size $\frac{1}{8} - 8\delta$. Thus, the overlap area is larger than $\frac{1}{64} - 2\delta$ and hence, due to our choice of δ, larger than 25δ. This is a contradiction, showing that all 4 rectangles from $\mathscr{A}' \cup \mathscr{B}'$ are iso-oriented.

We can assume from now on that these 4 rectangles are in its original position, the case when all are rotated could be discussed similarly. To prove that all 8 rectangles are iso-oriented, assume on contrary that a rectangle A from them is rotated. As any segment $\{y\} \times [0,1]$ ($y \in [0,1]$) clearly intersects a rectangle from $\mathscr{A}' \cup \mathscr{B}'$, it is easy to see that A intersects the union of $\mathscr{A}' \cup \mathscr{B}'$ at least in a square with size $\frac{1}{8} - 4\delta$. Hence the overlap is again larger than 25δ, a contradiction that completes the proof. □

In the following lemma we will characterize the structure of well-covered bins similarly as for well-packed bins.

Lemma 8 *Let $p = \frac{3}{8}$ and suppose that an 8-tuple of rectangles cover a bin. Then a bin is well-covered if and only if the 8-tuple consists of rectangles $A_{X,i}$, $A_{Y,j}$, $A_{Z,k}$, B_l, $A'_{X,i}$, $A'_{Y,j}$, $A'_{Z,k}$, and B'_l, for some $t_l = (x_i, y_j, z_k) \in T$.*

Proof. (I) First we show that 8 rectangles that correspond to a triple $t_l = (x_i, y_j, z_k) \in T$ as described above can cover a unit square bin even without rotations.

Starting from the bottom left corner of the bin and moving towards the right, each of rectangles $A'_{Z,k}$, $A'_{Y,j}$, $A'_{X,i}$, and B'_l, (in this order) is placed such that it touches the bottom of the bin $\mathbb{B}$ and the previous rectangle. In this way these 4 rectangles cover more than the lower half of the bin, as the sum of their widths is exactly 1 and height of each is greater than $\frac{1}{2}$. The remaining rectangles $A_{Z,k}$, $A_{Y,j}$, $A_{X,i}$, and B_l will be placed in this order starting from the top left corner of the bin and moving towards to the right, such that each rectangle touches the top of the bin and the previous rectangle. Clearly, these 4 rectangles cover the gap left in the bin after the first 4 rectangles were placed, as

$$\begin{aligned}
&w(A_{Z,k}) + w(A_{Y,j}) + w(A_{X,i}) + w(B_l) = 1,\\
&h(A_{Z,k}) + h(A'_{Z,k}) = h(A_{Y,j}) + h(A'_{Y,j}) = h(A_{X,i}) + h(A'_{X,i}) = h(B_l) + h(B'_l) = 1,\\
&h(A'_{Z,k}) > h(A'_{Y,j}) > h(A'_{X,i}) > h(B'_l),\\
&w(A_{Z,k}) < w(A'_{Z,k}), w(A_{Y,j}) < w(A'_{Y,j}), \text{ and } w(A_{X,i}) < w(A'_{X,i}).
\end{aligned}$$

(II) Now we show that any well-covered bin is covered by 8-tuple of rectangles that correspond to some triple from T as described above. Due to Lemma 7, the rectangles are iso-oriented. We will assume that they are all in the initial position in the bin, the case when all are rotated by 90° can be discussed similarly. Let A_1, A_2, ..., A_8 denote rectangles of the covering that cover the points $[0,0]$, $[\frac{1}{3},0]$, $[\frac{2}{3},0]$, $[1,0]$, $[0,1]$, $[\frac{1}{3},1]$, $[\frac{2}{3},1]$, and $[1,1]$, respectively. Recall that from the analysis above (1'), (2'), and (3') hold.

As $|\mathscr{B}\cap\{A_i : 1 \le i \le 8\}| = |\mathscr{B}'\cap\{A_i : 1 \le i \le 8\}|$, by Observation 1' $\sum_{i=1}^{8}(w(A_i)+h(A_i)) = 6$, and hence each of relations in (1'), (2'), and (3') must hold with equality. By Observation 2, for each $i \in \{1,2,3,4\}$, A_i and A_{i+4} are buddies. In particular, no two rectangles among A_1, A_2, A_3, A_4 are buddies. Now we conclude from $\sum_{i=1}^{4} w(A_i) = 1$ and Lemma 4 that necessarily $|\mathscr{B}\cap\{A_i : 1 \le i \le 8\}| = |\mathscr{B}'\cap\{A_i : 1 \le i \le 8\}| > 0$.

Thus (7) implies that $|\mathscr{B}\cap\{A_i : 1 \le i \le 8\}| = |\mathscr{B}'\cap\{A_i : 1 \le i \le 8\}| = 1$. Hence $\{A_1, A_2, \ldots, A_8\}$ consists of buddies $B_l \in \mathscr{B}$, $B'_l \in \mathscr{B}'$ for some $l \in \{1,2,\ldots,|T|\}$, and from 6 rectangles (3 pairs of buddies) from $\mathscr{A}\cup\mathscr{A}'$. Let us assume that $B'_l \in \{A_5, A_6, A_7, A_8\}$ (i.e., $B_l \in \{A_1, A_2, A_3, A_4\}$); the opposite case can be discussed, due to the symmetry, in a similar way.

Recall that $w(B'_l) < \frac{1}{4} - 8\delta$, $\sum_{i=5}^{8} w(A_i) = 1$, and $\{A_5, A_6, A_7, A_8\} \setminus \{B'_l\} \subseteq \mathscr{A}\cup\mathscr{A}'$. If some of A_5, A_6, A_7, A_8 belongs to $\mathscr{A}$, we easily get

$$\sum_{i=5}^{8} w(A_i) < (\frac{1}{4} - 8\delta) + (\frac{1}{4} - \delta) + 2(\frac{1}{4} + 4\delta) = 1 - \delta,$$

a contradiction. Thus $\{A_5, A_6, A_7, A_8\} \setminus \{B'_l\} \subseteq \mathscr{A}'$, consequently $\{A_1, A_2, A_3, A_4\} \setminus \{B_l\} \subseteq \mathscr{A}$. Recalling $\sum_{i=1}^{4} w(A_i) = 1$, Lemma 3 implies that $\{A_1, A_2, A_3, A_4\} = \{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l\}$, for some i, j, k, l such that $(x_i, y_j, z_k) = t_l$. This completes the proof. □

Building on the ideas above, we can prove the main theorem for covering problems

Theorem 4 *Unless* P = NP*, there are no* APTAS *for the problems* 2-DIMENSIONAL BIN COVERING *and* 2-DIMENSIONAL BIN COVERING WITH ROTATIONS.

Proof. The Bin Covering reduction f starts from a set $\mathcal{T}$ of instances of MAX-3DM such that for an instance $T \in \mathcal{T}$ it is NP-hard to decide of whether $\mathrm{OPT}(T) \geq \beta(T)$, or $\mathrm{OPT}(T) < \alpha(T)$.

a) Assume first that $T \in \mathcal{T}$ satisfies $\mathrm{OPT}(T) \geq \beta(T)$. We will show that the corresponding instance $f(T)$ of the problem 2-BC (resp., 2-BC^r) has its optimum at least $|T| + n - 3\beta(T)$. Let a matching M in T consist of $\beta(T)$ triples. For each triple $t_l = (x_i, y_j, z_k) \in M$, the corresponding 8 rectangles $\{A_{X,i}, A_{Y,j}, A_{Z,k}, B_l, A'_{X,i}, A'_{Y,j}, A'_{Z,k}, B'_l\}$ will cover one (well-covered) bin. For each $t_l \in T \setminus M$ we take B_l, B'_l, and a dummy rectangle to cover a bin. In this way we use $|T| - \beta(T)$ dummy rectangles.

For each of $n - 3\beta(T)$ elements in $X \cup Y \cup Z$ that are not contained in triples of M and for remaining $n - 3\beta(T)$ dummy rectangles we take the corresponding buddies $A, A' \in \mathscr{A} \cup \mathscr{A}'$ along with a dummy rectangle to cover a bin. Hence, we have covered $\beta(T) + (|T| - \beta(T)) + (n - 3\beta(T)) = |T| + n - 3\beta(T)$ bins in total.

b) Assume now that $T \in \mathcal{T}$ satisfies $\mathrm{OPT}(T) < \alpha(T)$. Our aim is to estimate $\mathrm{OPT}'(f(T))$ for the problem 2-BC (resp., 2-BC^r) from above. We will do these computations to some details for the case of the problem without rotations, hence we assume $p = 0$ in what follows. Consider any optimal solution to the 2-BC problem for an instance $f(T)$, namely the solution that covers $\mathrm{OPT}'(f(T))$ bins. To simplify some considerations we first normalize the solution without decreasing the number of covered bins as follows:

(i) If rectangles A_1, A_2, …, A_j cover a bin, then no proper subset of them can cover the bin.

(ii) To ensure (i), some of rectangles (the *rest*) can be left unused, but it is impossible to cover a bin using only the rest of rectangles. One can ensure that no dummy rectangle is in the rest. The rest can contain at most 9 rectangles from $\mathscr{A}' \cup \mathscr{B}'$, and at most 14 in total.

(iii) *Each bin is covered using at most one dummy rectangle.* It is easy to see that in the optimal solution we have more than $|T| + n - 4\beta(T)$ bins covered. Thus if two dummy rectangles are used to cover the same bin, we can take another bin that is covered without dummy rectangle and easily change the covering of these two bins such that each of them uses one of dummy rectangles.

Now we fix one normalized solution. The $N_d = |T| + n - 4\beta(T)$ bins that use a dummy rectangle in their covering, are called D-bins; the remaining covered bins are termed non-D-bins. Each of D-bins uses except a dummy rectangle 2 or 3 other rectangles. These 2 or 3 other rectangles (except a dummy rectangle) used for covering of D-bins can be of different kinds. In our counting arguments we will consider the following particular cases of D-bins according to what non-dummy rectangles they use:

- N_{d_1} use only rectangles from $\mathscr{A} \cup \mathscr{B}$, necessarily 3 such rectangles,
- N_{d_4} use 1 rectangles from $\mathscr{B}'$ and 2 rectangles from $\mathscr{A}$,
- N'_{d_0} use 2 rectangles from $\mathscr{A}' \cup \mathscr{B}'$, and from that N_{d_2} use 2 rectangles from $\mathscr{B}'$ and N_{d_3} use 1 rectangle from $\mathscr{B}'$ and 1 rectangle from $\mathscr{A}'$.

Let N_{d_0} be the number of D-bins which use 2 non-dummy rectangles in its covering, then the remaining number $N_d - N_{d_0}$ of D-bins use necessary 3 such rectangles. Clearly, $N_{d_1} + N_{d_4} \leq N_d - N_{d_0}$ and $N_{d_2} + N_{d_3} \leq N'_{d_0}$.

Next we describe how non-D-bins are distinguished. Any such bin uses at least 8 rectangles in its covering (as we deal with the case without rotations). Moreover, if a bin uses exactly 8 rectangles, at least 4 of them are from $\mathscr{A}' \cup \mathscr{B}'$. Let N_{b_1} be the number of bins that use at least 9 rectangles, N_{b_2} be the number of bins that use 8 rectangles, but at least 5 of them is from $\mathscr{A}' \cup \mathscr{B}'$, and N_g be the number of well-covered bins. Denote by N_b the number of remaining non-D-bins, that means bins covered by 4 rectangles from $\mathscr{A} \cup \mathscr{B}$ and 4 rectangles from $\mathscr{A}' \cup \mathscr{B}'$ such that among them there is strictly more rectangles from $\mathscr{B}$ than from $\mathscr{B}'$ (use (6) for this argument).

Now we are ready to derive some constraints on these numbers. The first one describes how all $2(n + |T|)$ non-dummy rectangles can be distributed among various kinds of bins, hence

$$8N_g + 8N_b + 9N_{b_1} + 8N_{b_2} + 3N_d - N_{d_0} \le 2(n + |T|).$$

Equivalently, as $N_d = |T| + n - 4\beta(T)$,

$$8N_g + 8N_b + 9N_{b_1} + 8N_{b_2} \le 8\beta(T) + N_{d_0} - N_d. \tag{8}$$

The second one describes how all $(n + |T|)$ rectangles from $\mathscr{A}' \cup \mathscr{B}'$ can be distributed:

$$4N_g + 4N_b + 5N_{b_2} + (N_d - N_{d_1}) + N'_{d_0} \le n + |T|.$$

Equivalently,

$$4N_g + 4N_b + 5N_{b_2} \le 4\beta(T) + N_{d_1} - N'_{d_0}. \tag{9}$$

Multiplying (8) by 5 and adding (9) we get

$$44N_g + 44N_b + 45N_{b_1} + 45N_{b_2} \le 44\beta(T) - 5N_d + 5N_{d_0} + N_{d_1} - N'_{d_0}.$$

We will rewrite the last relation in two different ways:

$$\begin{aligned} &44(N_g + N_b + N_{b_1} + N_{b_2} + N_d) \\ &\qquad \le 44(|T| + n - 3\beta(T)) - 5N_d + 5N_{d_0} + N_{d_1} - N'_{d_0} - N_{b_1} - N_{b_2}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &45(N_g + N_b + N_{b_1} + N_{b_2} + N_d) \\ &\qquad \le 45(|T| + n - 3\beta(T)) - (\beta(T) - N_g) + N_b - 5N_d + 5N_{d_0} + N_{d_1} - N'_{d_0}. \end{aligned} \tag{11}$$

To be able to handle the term N_b, we now derive one more constraint.

Consider from now on a fixed normalized solution. Let S denote either a set of rectangles that cover a single bin in this solution, or the set of unused rectangles. Let $\mathscr{S}$ be the collection of all such sets S. For each set $S \in \mathscr{S}$ we define $\varphi(S)$ to be the number $\varphi(S) = |S \cap \mathscr{B}| - |S \cap \mathscr{B}'|$. As $|\mathscr{B}| = |\mathscr{B}'|$ and elements of $\mathscr{S}$ define a partition of all rectangles, $\sum_{S \in \mathscr{S}} \varphi(S) = 0$ follows. If S corresponds to a well-covered bin then $\varphi(S) = 0$, and at least for N_b non-D-bins we have $\varphi(S) \ge 1$. On the remaining sets S we will use an estimate of $\varphi(S)$ from below.

Clearly, $\varphi(S) \ge -10$ always holds, as S cannot contain more than 10 rectangles from $\mathscr{B}'$. One can obtain slightly better estimate $\varphi(S) \ge -8$ for N_{b_2} bins, $\varphi(S) \ge -9$ if S is the set of unused rectangles. As described above, $\varphi(S) = -2$ for N_{d_2} D-bins, $\varphi(S) = -1$ for $(N_{d_3} + N_{d_4})$ D-bins, and $\varphi(S) \ge 0$ for remaining D-bins. Hence, we easily get

$$0 = \sum_{S \in \mathscr{S}} \varphi(S) \ge N_b - 10N_{b_1} - 8N_{b_2} - 9 - 2N_{d_2} - N_{d_3} - N_{d_4},$$

which implies

$$N_b \le 10N_{b_1} + 8N_{b_2} + 9 + 2N_{d_2} + N_{d_3} + N_{d_4}. \tag{12}$$

Clearly, the number of covered bins in our normalized solution, $\mathrm{OPT}'(f(T))$, is the sum $N_g + N_b + N_{b_1} + N_{b_2} + N_d$. To obtain an upper estimate on $\mathrm{OPT}'(f(T))$ we will distinguish the following two cases, where $v := \frac{1}{485}\frac{\beta(T)-\alpha(T)-9}{|T|+n-3\beta(T)}$ is a suitable constant.

(I) Suppose that $5(N_d - N_{d_0}) - N_{d_1} + N'_{d_0} + N_{b_1} + N_{b_2} < 44v(|T| + n - 3\beta(T))$. As $N_{d_1} + N_{d_4} \leq N_d - N_{d_0}$ and $N_{d_2} + N_{d_3} \leq N'_{d_0}$, we get from (12) an easy estimate

$$N_b < 9 + 440v(|T| + n - 3\beta(T)).$$

Using this in (11) and recalling that our assumption $\mathrm{OPT}(T) < \alpha(T)$ implies, due to Lemma 8, that $N_g < \alpha(T)$, we get

$$\begin{aligned} \mathrm{OPT}'(f(T)) &< (|T| + n - 3\beta(T))\left(1 + \frac{440}{45}v\right) - \frac{1}{45}(\beta(T) - \alpha(T)) + \frac{1}{5} \\ &\leq (1 - v)(|T| + n - 3\beta(T)). \end{aligned} \tag{13}$$

(II) If $5(N_d - N_{d_0}) - N_{d_1} + N'_{d_0} + N_{b_1} + N_{b_2} \geq 44v(|T| + n - 3\beta(T))$, then we get directly from (10)

$$\mathrm{OPT}'(f(T)) \leq (1 - v)(|T| + n - 3\beta(T)). \tag{14}$$

From Theorem B it follows that the term $v := v(q) = \frac{1}{485}\frac{\beta(T)-\alpha(T)-9}{|T|+n-3\beta(T)}$, where $n = 3q$ and $|T| = 2q$, is bounded from below by a positive constant ε_0 for all sufficiently large instances $T \in \mathcal{T}$. Hence it is NP-hard to distinguish of whether $\mathrm{OPT}'(f(T)) \geq (|T| + n - 3\beta(T))$ or $\mathrm{OPT}'(f(T)) \leq (1-\varepsilon_0)(|T|+n-3\beta(T))$. Consequently, there is no APTAS for the 2-DIMENSIONAL BIN COVERING problem.

For the problem 2-BC^r we can keep fixed $p = \frac{3}{8}$ and continue similarly. The crucial fact is Lemma 7 and its proof showing that for instances which we produce, using rotations doesn't help to cover significantly more bins than if rotations are not used. The only technical complication in our counting arguments is that non-dummy rectangles are now of two distinct types. Each of them is a small perturbation of either $(\frac{1}{4}, \frac{1}{2} + p)$ rectangle, or $(\frac{1}{4}, \frac{1}{2} - p)$ rectangle. Hence in our derivation of constraints (like (8), (9), and (12)) on various kinds of covered bins it is more natural to work, instead of the number of rectangles of certain type, with the number weighted by the area of this type of rectangles. Similarly as for the packing problems, this approach leads to the same kind of approximation hardness results for bin covering problems with rotations, as was described above in details for bin covering problems without rotations. □

5.2 3-dimensional Strip Covering

In the same way as for packings, the 2-DIMENSIONAL COVERING problem can be seen as particular case of the 3-DIMENSIONAL STRIP COVERING problem. The transformation described in Section 4 for strip packings problems has similar properties for strip covering problems as well. In this way, 2-BC can be regarded as a particular case of 3-SC, and 2-BC^r with the bin (a, b) reduces to a particular case of 3-SP^z, with the strip (a, b, ∞). For an integer $t > 0$ a transformation transforming an instance $\mathscr{L}$ of 2-BC (resp., 2-BC^r) to an instance $\mathscr{L}_t$ of 3-SC (resp., 3-SC^z) essentially preserves an optimum value, namely the ratio between the optimum values for 3-SC (resp., 3-SC^z) and 2-BC (resp., 2-BC^r) is exactly t. The proof is very similar to the one given above for packings. Thus approximation hardness results for 2-BC (resp., 2-BC^r) with the bin

$(1,1)$ derived in the previous section translates to the same approximation hardness results for 3-SC (resp., 3-SC^z) with the strip $(1,1,\infty)$.

We can summarize these results as follows

Theorem 5 *There are no* APTAS *for strip covering problems* 3-SC *and* 3-SC^z *on instances with the strip* $(1,1,\infty)$*, unless* P = NP*.*

6 Maximum Rectangle Packing Problem

Another rectangle bin packing problem well studied in the literature (e.g., [14], [2]) is the following:

Definition 10 *Given a collection of d-dimensional rectangles together with a d-dimensional rectangular bin $\mathbb{B}$. The goal of the* Maximum *d*-dimensional Rectangle Packing *problem is to pack the maximum number of rectangles from the collection into a* single *bin $\mathbb{B}$.*

Other variants of this problem are studied as well, e.g., each of rectangles can be associated with weight, and the goal is to maximize the total weight of packed rectangles. In some variants ninety-degree rotations of rectangles can be allowed. The problem is motivated by scheduling parallel jobs with a common due date to maximize the profit of jobs completed by the due date, where each job can require several processors which are allocated on a line. It has also applications in the advertisement placement problem, see [12] for more details. But even in the most simple case, 2-dimensional unweighted case without rotations, only a $(2+\varepsilon)$-approximation algorithm is known [14]. The question of whether there is an APTAS is open. However, already in 3-dimensional case, the problem can be settled in the negative.

Theorem 6 *Unless* P = NP*, there is no* APTAS *for the* Maximum 3-dimensional Rectangle Packing *problem with unit cube bin. The same is true also for z-oriented packings. For r-packings the same hardness result holds for a bin* $(1,1,b)$*, where* $b \in \left(0, \frac{1}{4}\right)$*.*

Proof. For oriented packings (i.e., without rotations) we can use the hardness result for 3-SP with the strip $(1,1,\infty)$ from Theorem 3: there is a constant $\rho > 1$ and an infinite family $\mathcal{F}$ of instances of the 3-SP problem with the strip $(1,1,\infty)$, such that for some computable function $\alpha : \mathcal{F} \to \mathbb{N}$ it is NP-hard to distinguish for $\mathscr{L} \in \mathcal{F}$ of whether $\mathrm{OPT}(\mathscr{L}) \leq \alpha(\mathscr{L})$, or $\mathrm{OPT}(\mathscr{L}) > \rho \cdot \alpha(\mathscr{L})$. Moreover, each rectangle in $\mathscr{L}$ is a small perturbation of either $\left(\frac{1}{4}, \frac{1}{2}, 1\right)$ or $\left(\frac{3}{4}, 1, 1\right)$. For any $\mathscr{L} \in \mathcal{F}$ denote by $\mathscr{L}'$ rescaled copy of $\mathscr{L}$ by a factor $1/\alpha(\mathscr{L})$ in the direction of the z-axis. Now it is NP-hard for $\mathscr{L}'$ and for the 3-SP problem with the strip $(1,1,\infty)$ to decide if $\mathrm{OPT}(\mathscr{L}') \leq 1$, or $\mathrm{OPT}(\mathscr{L}') > \rho$. In the former case all rectangles of $\mathscr{L}'$ can be packed into a unit cube bin. In the latter one we easily obtain that less than $|\mathscr{L}'| - \lfloor(\rho-1)\alpha(T)\rfloor$ can be packed into this bin.

For z-oriented packings we can use the same arguments starting from NP-hard gap of the problem 2-BP^r with the bin $(1,1)$ instead.

For r-packings we scale to obtain the strip $(1,1,b)$, $b \in \left(0, \frac{1}{4}\right)$. The special uniform structure of instances in our hardness result for 2-BP^r imply that all r-packings for such rescaled instances are, in fact, z-oriented packings. Thus the results follow as above. □

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