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Constraint Satisfaction: A Personal Perspective

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Abstract

Attempts at classifying computational problems as polynomial time solvable, NP-complete, or belonging to a higher level in the polynomial hierarchy, face the difficulty of undecidability. These classes, including NP, admit a logic formulation. By suitably restricting the formulation, one finds the logic class MMSNP, or monotone monadic strict NP without inequality, as a largest class that seems to avoid diagonalization arguments. Representative of this logic class is the class CSP of constraint satisfaction problems. Both MMSNP and CSP admit generalizations via alternations of quantifiers corresponding to higher levels in the hierarchy. Examining CSP from a computational point of view, one finds that the polynomial time solvable problems that do not have the bounded width property of Datalog are group theoretic in nature. In general, closure properties of the constraints characterize the complexity of the problems. When one restricts the number of occurrences of each variable, the problems that are encountered relate to delta-matroid intersection. When such a restriction forbidding copying is introduced in the context of input-output constraints, one finds nonexpansive mappings as characterizing this restriction. Both delta-matroid intersection and nonexpansive network stability problems yield polynomial time algorithms. When the general approach to the classification of constraint satisfaction problems is restricted to graphs and digraphs, one finds that the chordality of graphs plays a crucial role both for the structure of allowed constraints and for the structure of an instance.

1 Computational Complexity and the Diagonalization Barrier

The early theory of computation focused on the existence of decision procedures. Gödel's incompleteness theorem indicates that certain properties of natural numbers cannot be decided. The approach is based on Cantor's diagonalization method, and produces functions that are not recursively enumerable. The same approach shows that the halting problem for a Turing machine is undecidable.

Later studies considered restricting the expressive power of Turing machines by limiting the allowed running time or space. This gives rise to the class P of problems that can be solved by a Turing machine in polynomial time, the class NP of problems that can be solved by a Turing machine in nondeterministic polynomial time, and the class PSPACE of problems that can be solved by a Turing machine in polynomial space. These classes have containments P$\subseteq$NP$\subseteq$PSPACE, and whether P=NP or even P=PSPACE remains open. Each of these classes contains problems that are computationally hardest, and thus complete for the classes. Under the assumption P$\neq$NP, Ladner [75] showed that there exist problems in NP that are neither in P nor NP-complete, again by means of diagonalization. It can be shown, under the same assumption, that whether a problem in NP is polynomial time solvable or NP-complete is undecidable, from the undecidablity of the halting problem.

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These undecidability results, obtained via diagonalization, suggest the question of how to determine, for more natural problems, whether they are polynomial time solvable or NP-complete. One can similarly consider a polynomial hierarchy of problems lying between P and PSPACE, where P is level 0, NP is level 1, the problems solvable by an NP machine that has access to an oracle for an NP-complete problem form level 2, and in general level i is given by NP machines that have access to an oracle for level $i-1$. All of these levels lie between P and PSPACE. One may then wish to classify problems in PSPACE as polynomial or complete for a corresponding level. See [67] for an introduction to the theory of NP-completeness.

2 The Logic Approach and What Lies Underneath the Diagonalization Barrier

Fagin [27] gave a logic characterization of the problems in NP, by showing that the class NP contains a problem on an input finite structure if and only if this problem can be expressed by a second-order formula with an existential second-order part. Since the problems in NP cannot be classified as polynomial or NP-complete, it is then natural to ask which restrictions of this logic define subclasses that contain only polynomial or NP-complete problems. Feder and Vardi [65] defined the logic class MMSNP, or monotone monadic strict NP without inequality, and conjectured that this class contains only polynomial or NP-complete problems. The class MMSNP consists of problems expressed by an existential second-order formula with a universal first-order part, whose existentially quantified relations are monadic, whose input relations appear monotonically (say with negative polarity), and which does not contain the equality or the inequality relations.

Feder and Vardi [65] also showed that if any one of the three restrictions of monotonicity, monadicity, and no inequalities, are dropped in the definition of MMSNP, then every problem in NP can be encoded via polynomial time equivalence in the corresponding class, and thus a dichotomy as polynomial or NP-complete is not possible. The approach is via a representation of Turing machines using Datalog of Hillebrand, Kanellakis, Mairson, and Vardi [72]. Thus in a sense MMSNP is a largest class for which a dichotomy might be possible.

The class CSP of constraint satisfaction problems asks whether an input structure maps to a given fixed structure known as the template via a homomorphism. For every fixed template, the corresponding CSP problem is also in MMSNP. Feder and Vardi [65] obtained a converse, showing that every problem in MMSNP has an equivalent problem in CSP, where the MMSNP problem reduces to the CSP problem by a deterministic polynomial time reduction, and the CSP problem reduces to the MMSNP problem by a randomized polynomial time reduction. The approach is based on a randomized construction of Erdös [25] of graphs that are simulataneously of high girth, thus containing no short cycle, and of high chromatic number, thus not colorable with a small number of colors.

Feder and Vardi [66] considered the logical properties of classes related to MMSNP, and showed that if a monadic existential second-order problem with universal first-order part has inequalities and is not monotone, but has a complement closed under homomorphisms, then inequalities may be removed and monotonicity enforced to obtain a description of the problem in MMSNP. However, whether the complement of the problem is closed under homomorphisms is undecidable. Both results are extended to binary and unbounded existential second-order logic by Ramsey-theoretic methods, and similar results on removing inequalities and enforcing monotonicity are obtained for the polynomial logic language Datalog.

Feder and Kolaitis [55] considered similar monadic logic classes for higher levels of the hierarchy up to PSPACE, defining the class MMSNP(Q, F), where a given or arbitrary quantifier alternation

Q extends the usual MMSNP=MMSNP($\exists, F$). Again, dropping monadicity, monotonicity, or no inequalities gives arbitrary problems within the corresponding level of the hierarchy. These classes contain problems CSP(Q, S) and CSP'(Q, S) extending the usual CSP=CSP($\exists, S$)=CSP'($\exists, S$). The classes CSP'(Q, S) differ from the classes CSP(Q, S) in that universally quantified variables are allowed to range over arbitrary subsets of the domain. These extensions of CSP ask whether a fixed template satisfies a quantified conjunction with alternation of quantifiers Q. When Q is arbitrary, such problems may be PSPACE-complete. The problems with a fixed alternation of quantifiers Q have a linear-time algorithm on instances of bounded treewidth by Courcelle's theorem for monadic second-order logic [16, 24].

3 Mathematical Group Theory Arising from the Logic through the Computational Approach

Schaefer [81] classified all problems in CSP with a Boolean template, containing two elements 0 and 1, as polynomial time solvable or NP-complete. This result contains six polynomial familes, namely 0-valid, 1-valid, Horn, dual-Horn, bijunctive (2-satisfiability), and linear (modulo 2). The problems that are not in these six families are NP-complete, containing in particular 3-satisfiability, one-in-three satisfiability, and not-all-equal satisfiability.

Schaefer [81] also considered Boolean CSP'(Q, S), with arbitrary quantification Q, and classified these problems as polynomial or PSPACE-complete. The classification of Boolean CSP(Q, S) as polynomial or PSPACE-complete was obtained by Creignou, Khanna, and Sudan [17], and by Dalmau [18]. Here the 0-valid and 1-valid cases that were polynomial for CSP but not in any of the other four polynomial cases become PSPACE-complete, while the other four polynomial cases remain polynomial and the NP-complete cases become PSPACE-complete. Feder and Kolaitis [55] obtained a classification analogous to this classification as polynomial or PSPACE-complete for CSP(Q, S) and CSP'(Q, S) in the case where Q is a fixed alternation of quantifiers, with the PSPACE-complete cases becoming complete for the corresponding level of the hierarchy.

The classification of Schaefer was obtained by means of closure functions f, namely functions f such that for each relation in the template, tuples satisfying the relation are mapped to tuples satisfing the relation under componentwise application of f. The closure functions for the Horn and dual-Horn cases can be viewed as special cases of either commutative conservative operations, satisfying $f(a, b) = f(b, a) \in \{a, b\}$, or semilattice operations, satisfying $f(a, a) = a$, $f(a, b) = f(b, a)$, and $f(a, f(b, c)) = f(f(a, b), c)$. The semilattice operations are special cases of set operations of all arities, which map subsets to chosen elements. It was shown by Feder and Vardi [65] that the existence of a set operation corresponds to the existence of a Datalog program of a special kind for the complement of the problem, so that the problem has bounded width. In fact, a problem in CSP has a set operation if and only if it has width one. This generalizes to extended width one with a closure function related to the set operation. The problems with a commutative conservative operation were shown polynomial of bounded width by Bulatov and Jeavons [13].

The closure function for the bijunctive case, or 2-satisfiability, is a majority operation, satisfying $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$, giving a problem of strict width 2 and thus describable by Datalog. More generally, one may consider near-unanimity operations of arity $k + 1$, where if k arguments of g are $x = a$ then the value of g is also $g = a$. Such problems have strict width k and are thus also describable by Datalog.

The last case of Schaefer's classification, namely linear equations modulo 2, cannot be solved via Datalog, that is, it does not have bounded width. In fact, it exhibits a property known as the ability to count, which is shown by Feder and Vardi [65] to imply that the problem does not have bounded width, and in fact cannot be solved by Datalog with successor by a reduction of Afrati, Cosmadakis, and Yannakakis [1] from superpolynomial monotone circuit lower bounds which are obtained from Razborov's monotone lower bound for matching [80]. Cases with the ability to count contain linear equations modulo p for any p. These cases are closed under the affine operation $h(a,b,c) = a - b + c$ for abelian groups. More generally, for any finite group, closure under the coset operation $h(a,b,c) = ab^{-1}c$ is shown polynomial by Feder and Vardi [65], who also introduce the notion of a near subgroup, and show that if in addition to all cosets, a non-near subgroup is allowed, then the problem becomes NP-complete. The polynomiality of the case with near subgroups is shown by Feder [31] based on group-theoretic characterizations of near subgroups of Aschbacher extending the work in [3]. Near subgroups have a closure function of the form $h(a,b) = abc$, where c must belong to the commutator group of the subgroup generated by a and b. In fact, the closure functions for near-subgroups and their cosets are special cases of Malt'sev operations, satisfying $h(a,a,b) = h(b,a,a) = b$. The polynomiality of the case with a Malt'sev operation was shown by Bulatov [9].

The characterization of near subgroups by Aschbacher suggests that these have a radical, then a solvable level consisting of an odd order level, a power of two level, and another odd order level, with the top level being a direct product of simple non-abelian groups. The algorithm of Feder for near subgroups solves the odd order part of the problem and the power of two part of the problem separately, needing for the former a linking of products of elements with products of their inverses, and for the latter a special case of the above conjectured characterization of near subgroups, without the second odd order level, for the case of near subgroups generated by 2-elements, as shown by Aschbacher. Feder [32] also studies properties of the closure function $h(a,b) = abc$ for near subgroups. It is shown that this operation can be chosen on reduced near subgroups to be a left loop if and only if the near subgroup is a strong near subgroup. Strong near subgroups are the cases that have only the power of two level and the second odd order level. In these cases, the left loop can also satisfy the automorphic inverse property $h(a,b)^{-1} = h(a)^{-1}h(b)^{-1}$, and may often be defined to give a left gyrogroup. All of these properties are achieved simultaneously for odd order groups by the operation $h(a,b) = (ab^2a)^{1/2}$ on twisted subgroups. Strong near subgroups are the only cases of near subgroups, or more generally, of twisted subgroups, that satisfy the Lagrange divisibility property for subgroups, so that the order of a strong near subgroup divides the order of any strong near subgroup containing it.

Back to the classification project for CSP, two results were obtained. Bulatov [10] classified constraint satisfaction problems with a three-element template, having elements 0,1,2, as polynomial or NP-complete. The polynomial cases combine the above closure functions. Bulatov [11] classified conservative constraint satisfaction problems with a template of arbitrary size, where a conservative template is one that includes arbitrary subsets, called lists, of the elements of the template. This classification depends on determining, for each pair of elements in the template, whether the corresponding constraints are closed for conservative commutative, majority, or Malt'sev operations, and combining operations that partially satisfy these three properties solves these cases in polynomial time. For CSP'(Q,S), where Q is an arbitrary quantification, Feder and Kolaitis [55] showed that these three cases, namely conservative commutative, near-unanimity, and Malt'sev, remain polynomial, extending the work of Börner, Krokhin, Bulatov, and Jeavons [7] and Chen [15]. This was used in [55] to show that conservative problems CSP'(Q,S) are either polynomial or complete for the corresponding level of the hierarchy. By contrast, the work in [15] and [55] shows that for semilattice operation, the complexity with alternations of quantifiers Q may go up the hierarchy, remaining polynomial only if the semilattice operation has a bottom element for CSP(Q,S), and only if the semilattice operation is a conservative commutative operation for

CSP'(Q, S). Otherwise, the problems become co-NP-complete for quantification forall-exists. Other classifications with problems of intermediate completeness levels can be found in [7, 19, 55], in particular the cases where the graphs of permutations are allowed constraints, giving polynomial cases, NP-complete cases, and cases complete for PSPACE or for the appropriate level with fixed quantification.

A result of Jeavons [73] shows that closure functions completely define the complexity of CSP problems, and a result of Börner, Krokhin, Bulatov, and Jeavons [7] shows that surjective closure functions completely define the complexity of CSP(Q, S) problems when Q consists of all quantifications.

Malt'sev functions give also the only cases of the problem of counting satisfying assignments for a CSP problem that may not be #P-complete, as shown by Bulatov and Dalmau [12].

For NP-complete CSP problems, both the problem of reducing the exponential complexity of algorithms [23, 59, 82] and the MAX CSP problem of maximizing the number of constraints satisfied [2, 17, 21, 62, 83] have been studied. Most problems in CSP are NP-complete, as shown by Luczak and Nešetřil [77].

4 Restricting Fanout Gives Rise to the Interaction of 2-Satisfiability with Delta-Matroids

Feder [30] studied Boolean constraint satisfaction problems from CSP in the cases where the constants 0 and 1 are present, and every variable is only allowed to appear twice in the constraints. Feder showed that the only new potential polynomial cases other than Horn, dual-Horn, 2-satisfiability, and linear, from Schaefer's classification, are delta-matroid partity cases, the remaining cases remaining NP-complete. For bipartite Boolean constraint satisfaction, where there are two sets of constraints in an instance, the only new problems are delta-matroid intersection cases. Of these, when the delta-matroids contain neither the equal nor the less-than-or-equal relations, the delta-matroid intersection problems become matroid intersection problems and are thus polynomial.

Feder and Ford [34] generalized the study of bipartite Boolean constraint satisfaction with constants to the case where there are two sets of allowed constraints and each variable is allowed to occur only once in each set. This gives rise to a new polynomial case other than delta-matroid intersection, namely cases where the constraints are upward (resp. downward for the other side) closed delta-matroids, having neither equal nor less-than-or-equal, intersected with downward (resp. upward) closed 2-satisfiability instance, with an appropriate interaction between 2-satisfiability clauses and the delta-matroid. The remaining delta-matroid intersection cases are those where neither delta-matroid contains equality, or neither contains inequality, or one contains neither equality nor inequality and the other is arbitrary, which are all polynomial. This leaves out the case where both delta-matroids contain equality, so that both allowed constraint sets may be assumed to be the same and give a delta-matroid parity problem, which remains open. All of these polynomial cases, combinations of Schaefer and delta-matroid intersection cases, are polynomial even when the two constraint sides are described by an oracle. For delta-matroid parity cases, with equality, one observes by contrast that there are polynomial cases that become exponential in the oracle model, including coindependent delta-matroids that generalize to families of zebra delta-matroids. Previously known polynomial cases include linear [68] and local [20] delta-matroid parity problems, which give rise to zebra-linear and zebra-local cases.

For the k-partite case of Boolean constraint satisfaction with $k \geq 3$, where variables appear k times and in k allowed sets of constraint types, a full classification as polynomial or NP-complete is obtained, with the polynomial cases remaining polynomial with an oracle. The phenomenon of bounding the number of occurrences was also studied by Feder and Hell [37], who observed that all NP-complete cases of conservative (list) CSP problems seem to become NP-complete at three occurrences per variable, and showed this for the case of templates that are graphs in Feder, Hell, and Huang [47], and for multiple Boolean domains or for multiple N-free digraphs [37]. By contrast, for retraction problems, the degrees may need to increase more to obtain NP-completeness [48, 49], obtaining at intermediate degrees tree-like structures for negative instances along the lines of the result for coloring graphs whose degrees do not exceed list size of Erdős, Rubin, and Taylor [26, 49].

Another problem relating to matroid parity, namely finding a Hamiltonian cycle in a graph as studied for instance in [60, 61], corresponds closely to the case of two occurrences per variable. Thus for the Barnette conjecture [63], one finds spanning tree parity playing a crucial role, while the long cyles in [60] involve graph matching and flow. Delta-matroid parity and intersection also play a role in variations of the H-matching problem involving a given homomorphism to a graph H [64].

For counting problems, one may approximate the number of bases of a balanced matroid with a randomized procedure as shown by Feder and Mihail [58], or approximate the number of matchings in a bipartite graph, a special case of matroid intersection, see Jerrum, Sinclair, and Vigoda [74].

5 Chordal Aspects of the Source and Destination

Feder and Vardi [65] showed that every CSP problem has a polynomial time equivalent problem in a subclass of CSP, where the subclass may be either the graph retract problems (also known as one-or-all list or extensibility problems for graphs), the digraph homomorphism problems (even for digraphs with five levels), or the partial order retract problems (or partial order extensibility problems, even for partial orders with just three levels). A similar equivalence to the case of two unary function problems was shown by Feder, Madelaine, and Stewart [56].

This suggests looking for subclasses to obtain a classification, even in the case of graphs or digraphs. Hell and Nešetřil [70] showed that the graph homomorphism problem (or H-coloring problem) is polynomial if H is bipartite and NP-complete otherwise. For digraphs where each vertex has at least one incoming edge and one outgoing edge, a similar result was conjectured by Bang-Jensen and Hell [5]. Homomorphisms to oriented cycles were classified by Feder [29], while a classification for oriented trees remains open, see Hell, Nešetřil, and Zhu [71]. A related problem with acyclic homomorphism, where the pre-image of a vertex must be an acyclic digraph, was classified by Feder, Hell and Mohar [52]. Partial order retract problems with just two levels were classified by Pratt and Tiuryn [79].

Feder and Hell [36] considered two problems, namely the connected list reflexive graph homomorphism and the one-or-same list reflexive graph homomorphism problems, and showed that both problems are polynomial if the template H is a chordal graph, and NP-complete otherwise. Similar questions in the case of reflexive digraphs were studied by Feder, Hell, and Huang [46], obtaining results related to chordal properties of the reflexive digraph H, and in particular a dichotomy for the one-or-same list digraph homomorphism problem on partial orders.

The list homomorphism problems for graphs was classified fully by Feder, Hell, and Huang as polynomial or NP-complete, with the polynomial cases given by interval graphs in the reflexive case [36], complements of circular arc graphs in the bipartite case [43], and bi-arc graphs in general [44]. A similar conjecture in terms of bi-interval graphs was suggested for list reflexive digraph homomorphism in [45].

In all of these cases, the families found relate to the case where the template H is related to perfect or chordal graphs. For the result in [36] with chordal graphs for connected lists, the problems turn out to have both width one and bounded strict width, that is both a set function and a near unanimity function. The advantage of set functions over near unanimity functions is that the complexity of the corresponding algorithm has a fixed exponent, as opposed to an exponent growing with the strict width. Thight upper bounds of the form $n-\Theta(\sqrt{n})$ on the strict width in the case of chordal graphs were obtained by Brewster, Feder, Hell, Huang, and MacGillivray [6]. It is not known whether all problems of bounded strict width can be solved by algorithms having a fixed exponent, except for special cases. Here Larose and Zádori [76] showed that the reflexive partial order retract problems of bounded strict width have also width one, have only a finite number of minimal forbidden instances, and are dismantlable. The same results relating bounded strict width and width one, minimal forbidden subgraphs, and dismantlability, in the cases of reflexive graph retract and irreflexive graph retract problems, were shown by Feder and Hell [39], who also gave classes of graphs of bounded strict width in terms of chordal extensible graphs, related to chordal and chordal bipartite graphs.

We have seen cases where the complexity depends on restricting the template to chordal graphs. This suggests studying the case where the input is restricted to chordal graph. The simplest such case is where the input graph is a clique. For this problem, if the template is not fixed, the problem of finding large cliques is hard to approximate. The problem of coloring a graph with few colors is also hard to approximate. A combined problem that asks for decomposing a graph into large cliques, in the context of graph compression, is studied in [57].

The case where the input is a clique appears in another context, namely the case where the template has several binary functions, describing digraphs, and each pair of input vertices must be related by a digraph, thus describing a colored clique or tournament. The simplest case of such a problem consists of two binary symmetric relations, describing edges and non-edges in a graph, and was first studied by Feder, Hell, Klein, and Motwani [50], who classified all such problems as polynomial or NP-complete on three-element templates, and as quasi-polynomial, of complexity $n^{O(\log n)}$, or NP-complete, on four-element templates. For this problem, known as the list graph partition problem, the quasi-polynomial cases were shown polynomial on four-element templates, see Cameron, Eschen, Hoang, and Sritharan [14], with the exception of a single problem that remained quasi-polynomial. The complexity of this open four-element problem was improved to $n^{O(\log n/\log\log n)}$ by Feder, Hell, Kral and Sgall [54], who derived the result from a more general class of problems on k-element domains having complexity $n^{O(\log k \log n/\log\log n)}$. In the digraph partition problems, the three-element domain cases were classified as polynomial or NP-complete by Feder, Hell, and Tucker-Nally [53]. For an arbitrary collection of binary relations and instances that are tournaments, a classification as quasi-polynomial of complexity $n^{O(\log n)}$ or NP-complete was obtained by Feder and Hell [37]; this result generalizes to the case where for some set of values k, all variables are k-wise constrained by relations of arity k. The cases with a three-element template that are quasi-polynomial have complexity $n^{O(\log n/\log\log n)}$ by the result of [54].

One may consider list graph partition problems on instances that are chordal. For these, NP-completeness is possible even in the absence of lists, and several families were shown polynomial, or giving corresponding dichotomies, by Feder, Hell, Klein, Nogueira, and Protti [51]. For partition problems on perfect graphs, several problems having a finite number of minimal forbidden subgraphs were obtained by Feder and Hell [38], and similarly for point determining graphs [40]. Graph list partition problems also give examples of problems of polynomial complexity where the complexity increases with the size of the domain. For instance, the k-bisplit graphs can be recognized in time $O(n^{2k+1})$ for fixed k [54], with these problems becoming NP-complete for unbounded k as shown by Branstädt, Hammer, Bang Le, and Lozin [8]. Families of instances including cographs and interval graphs have been studied as well [42], showing in particular that monadic logic with arbitrary monadic second order quantifier alternation, negation, and inequality has a finite number of obstructions and thus a polynomial time algorithm for cographs [41].

6 The Role of Nonexpansiveness when the Transformation Itself is not Known

We have mentioned the Boolean constraint satisfaction problem in the case where each variable occurs twice, once in each of two sets of allowed constraints. That problem involves the study of delta-matroid intersection. We may consider also the Boolean constraint satisfaction problem where the constraints are functions from several inputs to several outputs, and each variable occurs once as an input and once as an output. Mayr and Subramanian [78, 85] showed that this problem, known as the network stability problem, is polynomial in the cases of monotone and of linear networks, and otherwise NP-complete except for the case of adjacency-preserving, or nonexpansive, networks. This last case, corresponding to the absence of copying, was shown polynomial for scatter-free networks, which include the stable marriage problem for the case of comparators [84], and is in turn equivalent to the lexicographically first bipartite matching problem [84], and to the telephone connection problem [35]. The general nonexpansive network stability problem was shown polynomial by Feder [28], completing the classification. In fact, Feder considered three problems, namely circuit value, network stability, and network convergence, which are in general complete for P, NP, and PSPACE respectively, and showed that these three problems all reduce to each other while staying within P in the case of nonexpansive functions.

The case of nonexpansive functions asks for a fixed point of a homomorphism from a reflexive hypercube to itself. More generally, Feder gave algorithms and showed the above equivalence for fixed points of nonexpansive mappings on Cartesian products of given graphs given by an oracle. These problems always have fixed sub-products, generalizing the fixed cubes of Bandelt and Vel [4], and have a characterization by means of a generalization of 2-satisfiability for the set of all fixed points. The polynomiality of the network convergence problem on products of graphs depends on a characterization of isometric embeddings of graphs of Graham and Winkler [69] that is extended in [28]. Feder [33] generalized the problem to finding common fixed points of several nonexpansive mappings for cartesian products of given families of graphs, and showed that this problem is polynomial if the fixed points for the products of two such graphs have a common majority function, and is NP-complete otherwise.

For products of paths that may be exponentially long, the problem has both an NP and a co-NP algorithm that accesses the oracle, since fixed cubes can be used to certify the existence or absence of fixed points, but is not known to have a polynomial time algorithm. If a super-polynomial oracle lower bound were obtained, this could be considered as evidence that P$\neq$NP$\cap$co-NP. On the other hand, in the framework of multiparty communication complexity, where many parties have disjoint parts of the input, but these parts are uncorrelated, Dolev and Feder [22], showed that the communication complexity and number of parties accessed by a deterministic algorithm are polynomial in the corresponding values for a nondeterministic algorithm solving the problem and a nondeterministic algorithm solving the complementary problem.

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