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Hyperbolic Polynomials Approach to Van der Waerden/Schrijver-Valiant like Conjectures : Sharper Bounds , Simpler Proofs and Algorithmic Applications

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Abstract

Let $p(x_1, ..., x_n) = p(X), X \in R^n$ be a homogeneous polynomial of degree n in n real variables , $e = (1, 1, .., 1) \in R^n$ be a vector of all ones . Such polynomial p is called e-hyperbolic if for all real vectors $X \in R^n$ the univariate polynomial equation $p(te - X) = 0$ has all real roots $\lambda_1(X) \geq ... \geq \lambda_n(X)$. The number of nonzero roots $|\{i : \lambda_i(X) \neq 0\}|$ is called $Rank_p(X)$. A e-hyperbolic polynomial p is called POS-hyperbolic if roots of vectors $X \in R^n_+$ with nonnegative coordinates are also nonnegative (the orthant R^n_+ belongs to the hyperbolic cone) and $p(e) > 0$. Below $\{e_1, ..., e_n\}$ stands for the canonical orthogonal basis in R^n.

The main results states that if $p(x_1, x_2, ..., x_n)$ is a POS-hyperbolic (homogeneous) polynomial of degree n , $Rank_p(e_i) = R_i$ and $p(x_1, x_2, ..., x_n) \geq \prod_{1 \leq i \leq n} x_i; x_i > 0, 1 \leq i \leq n$, then the following inequality holds

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \geq \prod_{1 \leq i \leq n} (\frac{G_i - 1}{G_i})^{G_i - 1} (G_i = \min(R_i, n + 1 - i)).$$

This theorem is a vast (and unifying) generalization of the van der Waerden conjecture on the permanents of doubly stochastic matrices as well as the Schrijver-Valiant conjecture on the number of perfect matchings in k-regular bipartite graphs . These two famous results correspond to the POS-hyperbolic polynomials being products of linear forms.

Our proof is relatively simple and "noncomputational" ; it actually slightly improves Schrijver's lower bound , and uses very basic (more or less centered around Rolle's theorem) properties of hyperbolic polynomials .

We present some important algorithmic applications of the result, including a polynomial time deterministic algorithm approximating the permanent of $n \times n$ nonnegative entry-wise matrices within a multiplicative factor $\frac{e^n}{n^m}$ for any fixed positive m . This paper introduces a new powerful "polynomial" technique , which allows as to simplify/unify famous and hard known results as well to prove new important theorems .

The paper is (almost) entirely self-contained , most of the proofs can be found in the **Appendices**.

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1 Introduction

An $n \times n$ matrix A is called doubly stochastic if it is nonnegative entry-wise and every column and row sum to one. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n. Let $\Lambda(k,n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k . We define the following subset of rational doubly stochastic matrices : $\Omega_{k,n} = \{k^{-1}A : A \in \Lambda(k,n)\}$. In a 1989 paper [5] R.B. Bapat defined the set D_n of doubly stochastic n-tuples of $n \times n$ matrices.
An n-tuple $\mathbf{A} = (A_1, \cdots, A_n)$ belongs to D_n iff $A_i \succeq 0$, i.e. A_i is a positive semi-definite matrix, $1 \leq i \leq n$; $trA_i = 1$ for $1 \leq i \leq n$; $\sum_{i=1}^{n} A_i = I$, where I, as usual, stands for the identity matrix. Recall that the permanent of a square matrix A is defined by

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A(i, \sigma(i)).$$

Let us consider an n-tuple $\mathbf{A} = (A_1, A_2, ...A_n)$, where $A_i = (A_i(k,l) : 1 \leq k,l \leq n)$ is a complex $n \times n$ matrix $(1 \leq i \leq n)$. Then $\det(\sum_{1 \leq i \leq n} t_i A_i)$ is a homogeneous polynomial of degree n in $t_1, t_2, \cdots, t_n$. The number

$$M(\mathbf{A}) := D(A_1, A_2, \cdots, A_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \det(t_1 A_1 + \cdots + t_n A_n) \tag{1}$$

is called the mixed discriminant of $A_1, A_2, \cdots, A_n$.
The permanent is a particular (diagonal) case of the mixed discriminant . I.e. define a multilinear polynomial $Mul_A(t_1, ..., t_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i,j) t_j$. Then $per(A) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} Mul_A(t_1, ..., t_n)$.

Let us recall two famous results and one recent result by the author.

1. **Van der Waerden Conjecture**
   The famous Van der Waerden Conjecture [4] states that $min_{A \in \Omega_n} D(A) = \frac{n!}{n^n}$ **(VDW-bound)** and the minimum is attained uniquely at the matrix J_n in which every entry equals $\frac{1}{n}$. Van der Waerden Conjecture was posed in 1926 and proved only in 1981 : D.I. Falikman proved in [13] the lower bound $\frac{n!}{n^n}$; the full conjecture , i.e. the uniqueness part , was proved by G.P. Egorychev in [12] .

2. **Schrijver-Valiant Conjecture**
   Define

   $$\lambda(k,n) = \min\{per(A) : A \in \Omega_{k,n}\} = k^{-n} \min\{per(A) : A \in \Lambda_{k,n}\}; \theta(k) = \lim_{n \to \infty} (\lambda(k,n))^{\frac{1}{n}}.$$

   It was proved in [34] that , using our notations , $\theta(k) \leq g(k) = (\frac{k-1}{k})^{k-1}$ and conjectured that $\theta(k) = g(k)$. Though the case of $k = 3$ was proved by M. Voorhoeve in 1979 [36] , this conjecture was settled only in 1998 [35] (17 years after the published proof of the Van der Waerden Conjecture). The main result of [35] is the following remarkable inequality : $\min\{per(A) : A \in \Omega_{k,n}\} \geq (\frac{k-1}{k})^{(k-1)n}$ **(Schrijver-bound)** .
   The proof in [35] is probably one of the most complicated and least understood in the theory of graphs .

3. **Bapat's Conjecture (Van der Waerden Conjecture for mixed discriminants)**
   One of the problems posed in [5] is to determine the minimum of mixed discriminants of doubly stochastic tuples : $min_{A \in D_n} D(A) =?$
   Quite naturally, R.V.Bapat conjectured that $min_{A \in D_n} D(A) = \frac{n!}{n^n}$ **(Bapat-bound)** and that it is attained uniquely at $\mathbf{J}_n =: (\frac{1}{n} I, ..., \frac{1}{n} I)$.
   In [5] this conjecture was formulated for real matrices. The author had proved it [31] for the complex case, i.e. when matrices A_i above are complex positive semidefinite and, thus, hermitian.

The **(VDW-bound)** is the simplest and most powerful bound on permanents and therefore among the simplest and most powerful general purpose bounds in combinatorics. Besides its many applications to the graph theory and combinatorics , **(VDW-bound)** has been recently used for deterministic approximations of permanents [17] .(Much more recent proof of **(Bapat-bound)** was actually motivated by the scaling algorithm [18] ,[19] to approximate mixed discriminants and mixed volumes.) It is easy to check that **(Schrijver-bound)** is implied by **(VDW-bound)** for $k \geq n$: $\frac{n!}{n^n} = \prod_{1 \leq k \leq n} (\frac{k-1}{k})^{k-1} > (\frac{k-1}{k})^{(k-1)n}$. Therefore , it was not clear whether the scaling algorithm in [17] gives better approximating exponent for sparse matrices : the "scaled" doubly stochastic matrix may have irrational entries even if the input matrix is boolean and **(Schrijver-bound)** is superior to the **(VDW-bound)** only on "very" rational sparse doubly stochastic matrices .
Since our generalized **(Schrijver-bounds)** (18),(19) depend only on the "sparsity" hence the scaling algorithm for permanents in [17] indeed gives better approximating exponent for sparse matrices (scaling algorithm for mixed discriminants in [18] ,[19] gives better approximating exponent for tuples of "small" rank PSD matrices).

1.1 Van der Waerden / Schrijver-Valiant like conjectures and homogeneous polynomials

Let $Hom(m, n)$ be the linear space of homogeneous polynomials $p(x), x \in R^m$ of degree n in m real varibles ; correspondingly $Hom_+(m, n) (Hom_{++}(m, n))$ be a subset of homogeneous polynomials $p(x), x \in R^m$ of degree n in m real varibles and nonnegative(positive) coefficients .

Definition 1.1:

1. Let $p \in Hom_+(n, n), p(x_1, ..., x_n) = \sum_{(r_1,...,r_n) \in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be a homogeneous polynomial of degree n in n real variables. Here $I_{m,n}$ stands for the set of vectors $r = (r_1, ..., r_m)$ with nonnegative integer components and $\sum_{1 \leq i \leq m} r_i = n$.
   The support of the polynomial $p(x_1, ..., x_n)$ as above is defined as $supp(p) = \{(r_1, ..., r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p .
   For a subset $A \subset \{1, ..., n\}$ we define $S_p(A) = \max_{(r_1,...,r_n) \in supp(p)} \sum_{i \in A} r_i$. Given a vector $(a_1, ..., a_n)$ with positive real coordinates , consider univariate polynomials $D_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{1 \leq j \leq n} a_j e_j), V_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{j \in A'} a_j e_j)$. Then $S_p(A)$ is equal to the degree of the polynomials $D_A, V_A(t)$:
$$S_p(A) = deg(D_A) = deg(V_A) \tag{2}$$

2. The following linear differential operator maps $Hom(n,n)$ onto $Hom(n-1,n-1)$:

$$p_{x_1}(x_2,...,x_n) = \frac{\partial}{\partial x_1} p(0,x_2,...,x_n).$$

We define $p_{x_i}, 2 \leq i \leq n$ in the same way for all polynomials $p \in Hom(n,n)$. Notice that

$$p(x_1,...,x_n) = x_i p_{x_i}(x_2,...,x_n) + q(x_1,...,x_n); q_{x_i} = 0. \tag{3}$$

The following inequality follows straight from the definition :

$$S_{p_{x_1}}(A) \leq \min(n-1, S_p(A)) : A \subset \{2,...,n\}, p \in Hom_+(n,n). \tag{4}$$

3. Consider $p \in Hom_+(n,n)$ We define the **Capacity** as

$$Cap(p) = \inf_{x_i>0, \prod_{1\leq i\leq n} x_i = 1} p(x_1,...,x_n).$$

It follows that if $p \in Hom_+(n,n)$ then

$$Cap(p) \geq \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0) \tag{5}$$

Notice that

$$\log(Cap(p)) = inf_{\sum_{1\leq i\leq n} y_i = 0} \log(p(e^{y_1},...,e^{y_n})),$$

and if $p \in Hom_+(n,n)$ then the functional $\log(p(e^{y_1},...,e^{y_n}))$ is convex .

4. Consider a stratified set of homogeneous polynomials : $F = \bigcup_{1\leq n<\infty} F_n$, where $F_n \in Hom_+(n,n)$. We call such set **VDW-FAMILY** if it satisfies the following properties :

   (a) If a polynomial $p \in F_n, n > 1$ then for all $1 \leq i \leq n$ the polynomials $p_{x_i} \in F_{n-1}$.

   (b)

$$Cap(p_{x_i}) \geq g(S_p(\{i\}))Cap(p) : p \in F_j, 1 \leq i \leq j; g(k) = (\frac{k-1}{k})^{k-1}, k \geq 1. \tag{6}$$

∎

Example 1.2: Let $A = \{A(i,j) : 1 \leq i \leq n\}$ be $n \times n$ matrix with nonnegative entries . Assume that $\sum_{1\leq j\leq n} A(i,j) > 0$ for all $1 \leq i \leq n$. Define the following homogeneous polynomial $Mul_A(t_1,...,t_n) = \prod_{1\leq i\leq n} \sum_{1\leq j\leq n} A(i,j)t_j$. Clearly , $Mul_A \in Hom_+(n,n)$ and $Mul_A \neq 0$. It is easy to check that $S_{Mul_A}(\{j\}) = |\{i : A(i,j) \neq 0\}|$ ($S_{Mul_A}(\{j\})$ is equal to the number of non-zero entries in the jth column of A) .
Notice that if $A \in \Lambda(k,n)$ (or $A \in \Omega(k,n)$) then $S_{Mul_A}(\{j\}) \leq k, 1 \leq j \leq n$.
More generally , consider a n-tuple $\mathbf{A} = (A_1, A_2, ...A_n)$, where the complex hermitian $n \times n$ matrices are positive semidefinite and $\sum_{1\leq i\leq n} A_i \succ 0$ (their sum is positive definite). Then the homogeneous polynomial $DET_{\mathbf{A}}(t_1,...,t_n) = \det(\sum_{1\leq i\leq n} t_i A_i) \in Hom_+(n,n)$ and $DET_{\mathbf{A}} \neq 0$. Similarly to polynomials Mul_A , we get that $S_{DET_{\mathbf{A}}}(\{j\}) = Rank(A_j), 1 \leq j \leq n$.

The Van Der Waerden conjecture on permanents as well as Bapat's conjecture on mixed discriminants can be equivalently stated in the following way (notice the absence of doubly stochasticity):

$$\frac{n!}{n^n} Cap(q) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} q(0,...,0) \leq Cap(q) \tag{7}$$

The van der Waerden conjecture on the permanents corresponds to polynomials $Mul_A \in Hom_+(n,n) : A \geq 0$, the Bapat's conjecture on mixed discriminants corresponds to $DET_{\mathbf{A}} \in Hom_+(n,n) : \mathbf{A} \succeq 0$. The connection between inequality (7) and the standard forms of the van der Waerden and Bapat's conjectures is established with the help of the scaling ([17] , [18] , [19]). Notice that the functional $\log(p(e^{y_1},...,e^{y_1}))$ is convex if $p \in Hom_+(n,n)$. Thus the inequality (7) allows a convex relaxation of the permanent of nonnegative matrices and the mixed discriminant of semidefinite tuples . This observation was implicit in [17] and crucial in [18] , [19] .

∎

1.2 The Main (polynomial) Idea

The following (meta)theorem describes the main idea of this paper .

Theorem 1.3: *Let $F = \bigcup_{1 \leq n < \infty} F_n$ be a* **VDW-FAMILY** *and the homogeneous polynomial $p \in F_n$. Then the following inequality holds :*

$$\prod_{1 \leq i \leq n} g(\min(S_p(\{i\})), n+1-i)) Cap(p) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq Cap(p). \tag{8}$$

Proof: Our proof is by natural induction . Notice that the function $g(k) = (\frac{k-1}{k})^{k-1}$ is strictly decreasing on the semiinterval $[1, \infty)$ (we define $g(0) = 1$). The theorem is obviously true for $n = 1$. Suppose it is true for all $n \leq k < \infty$ and the polynomial $p \in F_{k+1}$. Since $F = \bigcup_{1 \leq n < \infty} F_n$ is a **VDW-FAMILY** hence $p_{x_{k+1}} \in F_k; Cap(p_{x_{k+1}}) \geq g(S_p(\{k+1\})) Cap(p)$. It follows from (obvious) inequality (4) that $S_{p_{x_{k+1}}}(\{i\}) \leq S_p(\{i\}), 1 \leq i \leq k$. Therefore , by induction , we get the needed inequality :

$$\frac{\partial^{k+1}}{\partial x_1 ... \partial x_{k+1}} p(0,...,0) = \frac{\partial}{\partial x_{k+1}} (\frac{\partial^k}{\partial x_1 ... \partial x_k} p_{x_{k+1}}(0,...,0)) \geq$$
$$\geq \prod_{1 \leq i \leq k} g(\min(S_{p_{x_{k+1}}}(\{i\})), n+1-i)) Cap(p_{x_{k+1}}) \geq$$
$$\geq \prod_{1 \leq i \leq k} g(\min(S_{p_{x_{k+1}}}(\{i\})), n+1-i)) Cap(p) g(S_p(\{k+1\})) \geq$$
$$\geq \prod_{1 \leq i \leq k+1} g(\min(S_p(\{i\})), n+1-i)) Cap(p).$$ ∎

Corollary 1.4:

1. *If the homogeneous polynomial $p \in F_n$ then*

$$\frac{n!}{n^n} Cap(p) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq Cap(p). \tag{9}$$

2. *If the homogeneous polynomial $p \in F_n$ and $S_p(\{i\})) \leq k \leq n, 1 \leq i \leq n$ then*

$$(\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k} Cap(p) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \leq Cap(p). \quad (10)$$

Proof: Both inequalities follow the main inequality (8) and from the next easily proved identity

$$\frac{n!}{n^n} = \prod_{1 \leq k \leq n} g(k). \quad (11)$$

▮

What is left now is to present a **VDW-FAMILY** which contains all polynomials $DET_{\mathbf{A}}$, where the n-tuple $\mathbf{A} = (A_1, ..., A_n)$ consists of positive semidefinite hermitian matrices (and thus contains all polynomials Mul_A , where A is $n \times n$ matrix with nonnegative entries). If such **VDW-FAMILY** set exists then the Van der Waerden , Bapat , Schrijver-Valiant conjectures would follow (without any extra work , see Example 1.2) from Theorem 1.3 and Corollary 1.4 . One of such **VDW-FAMILY** , consisting of POS-hyperbolic polynomials , is defined in the next section.

2 Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations [15], [8] ,[9] . It recently became "popular" in the optimization literature [11] ,[10],[38]. The paper [38] gives nice and concise introduction to the area (with much simplified proofs of the key theorems) .

Definition 2.1:

1. A homogeneous polynomial $p : C^m \rightarrow C$ of degree $n(p \in Hom(m, n))$ is called hyperbolic in the direction $e \in R^m$ (or e- hyperbolic) if $p(e) \neq 0$ and for each vector $X \in R^m$ the univariate (in λ) polynomial $p(X - \lambda e)$ has exactly n real roots counting their multiplicities.

2. Denote an ordered vector of roots of $p(x - \lambda e)$ as $\lambda(X) = (\lambda_n(X) \geq \lambda_{n-1}(X) \geq ... \lambda_1(X))$. Call $X \in R^m$ e-positive (e-nonnegative) if $\lambda_1(X) > 0$ ($\lambda_n(X) \geq 0$). We denote the closed set of e-nonnegative vectors as $N_e(p)$, and the open set of e-positive vectors as $C_e(p)$.

▮

Definition 2.2: Let $p : C^m \rightarrow C$ be a homogeneous polynomial of degree n in m variables. Following [23] , we define the p-mixed form of an n-vector tuple $\mathbf{X} = (X_1, .., X_n) : X_i \in C^m$ as

$$M_p(\mathbf{X}) =: M_p(X_1, .., X_n) = \frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} p(\sum_{1 \leq i \leq n} \alpha_i X_i) \quad (12)$$

The following polarization identity is well known

$$M_p(X_1,..,X_n) = 2^{-n} \sum_{b_i \in \{-1,1\}, 1 \le i \le n} p(\sum_{1 \le i \le n} b_i X_i) \prod_{1 \le i \le n} b_i \tag{13}$$

Associate with any vector $r = (r_1, ..., r_n) \in I_{n,n}$ an n-tuple of m-dimensional vectors $\mathbf{X}_r$ consisting of r_i copies of $x_i (1 \le i \le n)$. It follows from the Taylor's formula that

$$p(\sum_{1 \le i \le n} \alpha_i X_i) = \sum_{r \in I_{n,n}} \prod_{1 \le i \le n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \le i \le n} r_i!} \tag{14}$$

∎

We collected in the following proposition the properties of hyperbolic polynomials used in this paper .

Proposition 2.3: FACT 1 .

$$p(X) = p(e) \prod_{1 \le i \le n} \lambda_i(X). \tag{15}$$

FACT 2 . *If p is e-hyperbolic polynomial and p(e) is a real nonzero number then the coefficients of p are real ([8] , follows from (15) via the standard interpolation). If p is e-hyperbolic polynomial and $p(e) > 0$ then $p(X) > 0$ for all e-positive vectors $X \in C_e(p) \subset R^m$.*

FACT 3 . *Let $p \in Hom(m,n)$ be e-hyperbolic polynomial and $d \in C_e(p) \subset R^m$. Then p is also d- hyperbolic and $C_d(p) = C_e(p), N_d(p) = N_e(p)$. ([15] , [23] , very simple proof in [38] .)*

FACT 4 . *Let $p \in Hom(m,n)$ be e-hyperbolic polynomial . Then the polynomial $p_e(X) =: \frac{d}{dt} p(X + te)|_{(t=0)}; p_e \in Hom(m, n-1)$ is also e- hyperbolic and $C_e(p) \subset C_e(p_e)$ ([23] , [38] , Rolle's theorem).*

FACT 5 . *Let $p \in Hom(m,n)$. Then the p-mixed form $M_p(X_1,..,X_n)$ is linear in each vector argument $X_i \in C^m$. Let $p \in Hom(m,n)$ be e- hyperbolic and $p(e) > 0$. Then $M_p(X_1,..,X_n) > 0$ if the vectors $X_i \in R^m, 1 \le i \le n$ are e-positive ([23] , proved by induction using FACT 4) .*

We use in this paper the following sub-class of hyperbolic polynomials .

Definition 2.4: A polynomial $p \in Hom(m,n)$ is called POS-hyperbolic if $p(e) > 0, e = (1,1,...,1) \in R^m$; p is e-hyperbolic and the closed convex cone $N_e(p)$ contains the nonnegative orthant R_+^m . (In other words , all the roots of the univariate polynomial equation $p(X - te) = 0$ are nonnegative if the coordinates of the vector X are nonnegative real numbers.)
It follows from the identity (14) and **FACT 5** that POS-hyperbolic polynomials have nonnegative coefficients . ∎

Probably the best known example of a hyperbolic polynomial comes from the hyperbolic geometry : $p(x_0, ..., x_k) = x_0^2 - \sum_{1 \le i \le k} x_i^2$. This polynomial is hyperbolic in the direction $(1, 0, 0, ..., 0)$. Another "popular" hyperbolic polynomial is $\det(X)$ restricted on a linear real space of hermitian $n \times n$ matrices . In this case mixed forms are just mixed discriminants , hyperbolic direction is the identity matrix I , the corresponding closed hyperbolic cone of I-nonnegative vectors coincides with a closed convex cone of positive semidefinite matrices . Less known , but very interesting , hyperbolic polynomial is the Moore determinant $M\det(Y)$ restricted on a linear real space of hermitian quaternionic $n \times n$ matrices . (The Moore determinant is a particular case of the generic norms on Jordan Algebras.) The Moore determinant is , essentially , the Pfaffian (see the corresponding definitions and the theory in a very readable paper [40]) .
This paper benefits from the fact that as multilinear polynomials $Mul_A \in Hom_+(n, n)$: $A \ge 0, Ae > 0$, as well determinantal polynomials $DET_{\mathbf{A}} \in Hom_+(n, n)$: $\mathbf{A} \succeq 0, \sum_{1 \le i \le n} A_i \succ 0$ are POS-hyperbolic.

2.1 POS-Hyperbolic polynomials form VDW-FAMILY

Let $q \in Hom_+(n, n)$ be a POS-hyperbolic polynomial. For a vector $X \in C^n$ we define the integer number $Rank_q(X)$ as the number of nonzero roots of the equation $q(X - te) = 0, e = (1, 1, ..., 1) = \sum_{1 \le i \le n} e_i$. It follows from the identity (2) that $Rank_q(\sum_{i \in A} e_i) = S_q(A), A \subset \{1, 2, ..., n\}$.

Theorem 2.5:

1. *Let $q \in Hom_+(n, n)$ be POS-hyperbolic polynomial . If $1 \le Rank_q(e_1) = k \le n$ then*

$$Cap(q_{x_1}) \ge g(k) Cap(q) \tag{16}$$

2. *Let $q(x_1, x_2, ..., x_n)$ be a POS-hyperbolic (homogeneous) polynomial of degree n . Then either the polynomial $q_{x_1} = 0$ or q_{x_1} is POS-hyperbolic . If $Cap(q) > 0$ then q_{x_1} is (nonzero) POS-hyperbolic .*

Corollary 2.6: *Let $PHP(n) \subset Hom_+(n, n)$ be a set of POS-hyperbolic polynomials of degree n in n variables ; define $PHP_+(n) = \{p \in PHP(n) : Cap(p) > 0\}$. Then as $\cup_{n \ge 1}(PHP(n) \cup \{0\})$ as well $\cup_{n \ge 1} PHP_+(n)$ is* **VDW-FAMILY** .

The second part of Theorem 2.5 is , up to minor modifications , well known (**FACT 4**; see , for instance, [23], [38]) and follows from Rolle's theorem . The main new results "responsible" for the first part of Theorem 2.5 are the next Lemma 2.7 and its Corollary 2.8.

Lemma 2.7:

1. *Let $c_1, ..., c_n$ be real numbers ; $0 \leq c_i \leq 1, 1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} c_i = n-1$. Define the following symmetric functions :*

$$S_n = \prod_{1 \leq i \leq n} c_i, S_{n-1} = \sum_{1 \leq i \leq n} \prod_{j \neq i} c_j.$$

   Then the following entropic inequality holds :

$$S_{n-1} - nS_n \geq e^{\sum_{1 \leq i \leq n} c_i \log(c_i)}.$$

2. *(Mini van der Waerden conjecture) Consider a doubly-stochastic $n \times n$ matrix $A = [a|b|...|b]$. I.e. A has $n-1$ columns equal to the column vector b , and one column equal to the column vector a . Let $a = (a_1, ..., a_n)^T : a_i \geq 0, \sum_{1 \leq i \leq n} a_i = 1$; $b = (b_1, ..., b_n)^T : b_i = \frac{1-a_i}{n-1}, 1 \leq i \leq n$. Then the permanent $Per(A) \geq \frac{n!}{n^n}$.*

Corollary 2.8: *Consider an univariate polynomial $R(t) = \sum_{0 \leq i \leq n} d_i t^i = \prod_{1 \leq i \leq n}(a_i t + b_i)$, where $a_i, b_i \geq 0$. If for some positive real number C the inequality $R(t) \geq Ct$ holds for all $t \geq 0$ then*

$$d_1 = \frac{\partial}{\partial t} R(0) \geq C((\frac{n-1}{n})^{n-1}) \tag{17}$$

The inequality (17) is attained only on the polynomials $R(t) = A(t+a)^n : A, a > 0$.

2.2 Newton Inequalities , Alternative Proof of Corollary 2.8 , Volume Polynomials

Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real coefficients . If such polynomial R has all real roots then its coefficients satisfy the following Newton's inequalities :

$$NIs : d_i^2 \geq d_{i-1} d_{i+1} \frac{\binom{n}{i}^2}{\binom{n}{i-1}\binom{n}{i+1}} : 1 \leq i \leq n-1.$$

The following weak Newton's inequalities $WNIs$ follow from NIs if the coefficients are non-negative:

$$WNIs : d_i d_0^{i-1} \leq \frac{d_1}{n}^i \binom{n}{i} : 2 \leq i \leq n.$$

Lemma 2.9: *Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real nonnegative coefficients satisfying weak Newton's inequalities $WNIs$ .If for some positive real number C the inequality $R(t) \geq Ct$ holds for all $t \geq 0$ then*

$$d_1 \geq C((\frac{n-1}{n})^{n-1}).$$

Proof: If $d_0 = 0$ then $d_1 \geq C > C((\frac{n-1}{n})^{n-1})$. Thus we can assume that $d_0 = 1$. It follows from weak Newton's inequalities $WNIs$ that

$$d_i \leq (\frac{d_1}{n})^i \binom{n}{i} : 2 \leq i \leq n.$$

Therefore for nonnegative values of $t \geq 0$ we get the inequality

$$R(t) \leq 1 + (\frac{d_1 t}{n}) \binom{n}{1} + (\frac{d_1 t}{n})^2 \binom{n}{2} + ...(\frac{d_1 t}{n})^n \binom{n}{n} = (1 + \frac{d_1 t}{n})^n.$$

Which gives the inequality $(1 + \frac{d_1 t}{n})^n \geq Ct$. The inequality $d_1 \geq C((\frac{n-1}{n})^{n-1})$ follows now easily . ∎

Remark 2.10: The Newton Inequalities are not sufficient for the real rootedness . The classical example is provided by some univariate volume polynomials $R(t) = Vol(tC_1 + C_2)$, where C_1, C_2 are convex compact sets (see , for instance , [23]) . In this case the Newton Inequalities follow from the celebrated Alexandrov-Fenchel Inequalities.
Using Lemma 2.9 , the Alexandrov-Fenchel Inequalities and a bit of extra work allows to extend the results of this paper , i.e. Theorem 1.3 ,to the multivariate volume polynomials $Vol(t_1C_1 + t_2C_2 + ... + t_nC_n)$, where $C_1, C_2, ..., C_n$ are convex compact subsets of R^n . In other words , there exists a **VDW-FAMILY** which contains all such volume polynomials $Vol(t_1C_1 + t_2C_2 + ... + t_nC_n)$.
This extension leads to a randomized poly-time algorithm to approximate the mixed volume $M(C_1, ..., C_n) = \frac{\partial^n}{\partial t_1 ... \partial t_n} Vol(t_1C_1 + t_2C_2 + ... + t_nC_n)$ within exponential factor e^n . The algorithm is pretty much the same as in Theorem 4.7 in this paper , the randomization is needed to evaluate the oracle , i.e. to evaluate $Vol(t_1C_1 + t_2C_2 + ... + t_nC_n)$.
The best current appoximation factor is $n^{O(n)}$ ([32] , [33] (randomized) ; [18] , [19] (deterministic) . ∎

Definition 2.11: Let $p \in Hom_+(n,n), p(x_1, ..., x_n) = \sum_{(r_1,...,r_n) \in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be a homogeneous polynomial with nonnegative coefficients of degree n in n real variables. Call such polynomial **AF-Polynomial** if the following Alexandrov-Fenchel Inequalities hold :

$$M_p(X_1, X_2, X_3, ..., X_n)^2 \geq M_p(X_1, X_1, X_3, ..., X_n) M_p(X_2, X_2, X_3, ..., X_n) : X_1, ..., X_n \in R^n_+.$$

(The p-mixed form $M_p(X_1, X_2, X_3, ..., X_n)$ is defined in Definition 2.2 (formula (12) .)
We denote as $AF(n)$ a set of all **AF-Polynomial** of degree n in n real variables and define $AF_+(n) : \{p \in AF(n) : Cap(p) > 0\}$. We denote as $Vol(n)$ a set of polynomials $Vol(t_1C_1 + t_2C_2 + ... + t_nC_n)$, where $C_1, C_2, ..., C_n$ are convex compact subsets of R^n .

Notice that $PHP(n) \subset AF(n)$ (POS-Hyperbolic polynomials are **AF-Polynomials**) *[23] ; $Vol(n) \subset AF(n)$, this inclusion is just a restatement of the celebrated Alexandrov-Fenchel Inequalities for mixed volumes [1], [2].*

∎

Theorem 2.12: *as* $\cup_{n\geq 1} AF(n)$ *as well* $\cup_{n\geq 1} AF_+(n)$ *is* **VDW-FAMILY** .

Proof: The definition of the **VDW-FAMILY** consists of two properties (see Part 4 of Definition 1.1) . The property (a) follows from the definition of **AF-Polynomials** :

$$p_{x_1}(x_2,...,x_n) = \frac{\partial}{\partial x_1} p(0,x_2,...,x_n) = ((n-1)!)^{-1} M_p(e_1,Y,Y,...,Y); Y = (0,x_2,...,x_n).$$

The property (b) follows from Lemma 2.9 . Indeed , if $X,Y \in R^n_+$ and p is a **AF-Polynomial** then the coefficients of the univariate polynomial $p(tX+Y)$ are nonnegative and satisfy the Newton inequalities NIs . ∎

3 Harvest

Corollary 2.6 allows to "plug-in" POS-hyperbolic polynomials to Theorem 1.3 . The most spectacular application is the following generalization of **(Schrijver-bound)** .

Theorem 3.1: *Let* $A = \{A(i,j) : 1 \leq i,j \leq n\}$ *be a matrix with nonnegative entries . Define* $C_j = Card(\{i : A(i,j) \neq 0\}), 1 \leq j \leq n$. *I.e.* C_j *is the number of non-zero entries in the* j*th column of* A . *Then*

$$per(A) \geq \prod_{1\leq j\leq n} g(\min(C_j, n+1-j)) Cap(Mul_A). \tag{18}$$

If $C_j \leq k, 1 \leq j \leq n$ *then*

$$per(A) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k} Cap(Mul_A). \tag{19}$$

(Recall that if A *is doubly stochastic then* $Cap(Mul_A) = 1$ *.)*

Remark 3.2: The lower bound (18) can be viewed as a **NONREGULAR** generalization of **(Schrijver-bound)** ; it "interpolates" between **(VDW-bound)** ($C_j = n$) and the sparse case ($C_j << n$) . The lower bound (19) is actually sharper than **(Schrijver-bound)** : $\frac{k!}{k^k} = \prod_{1\leq j\leq k} g(j) > g(k)^K = (\frac{k-1}{k})^{(k-1)(k)}$. ∎

4 Algorithmic Applications

Suppose that a POS-hyperbolic polynomial

$$p(x_1,...,x_n) = \sum_{\sum_{1\leq i\leq n} r_i = n} a_{(r_1,...,r_n)} \prod_{1\leq i\leq n} x_i^{r_i}$$

has nonnegative integer coefficients and is given as an oracle . I.e. we don't have a list coefficients , but can evaluate $p(x_1,...,x_n)$ on rational inputs .

A deterministic polynomial-time oracle algorithm is any algorithm which evaluates the given polynomial $p(.)$ at a number of rational vectors $q^{(i)} = (q_1^{(i)}, ..., q_n^{(i)})$ which is polynomial in n and $\log(p(1,1,..,1))$; these rational vectors $q^{(i)}$ are required to have bit-wise complexity which is polynomial in n and $\log(p(1,1,..,1))$; and the number of additional auxilary arithmetic computations is also polynomial in n and $\log(p(1,1,..,1))$.
If the number of oracle calls (evaluations of the given polynomial $p(.)$) , the number of additional auxilary arithmetic computations and bit-wise complexity of the rational input vectors $q^{(i)}$ are all polynomial in n (no dependence on $\log(p(1,1,..,1))$) then such algorithm is called deterministic strongly polynomial-time oracle algorithm.

The following result was proved in [39].

Theorem 4.1:

1. *Let $p \in Hom_+(n,n)$ be POS-hyperbolic polynomial . Then the function $Rank_p(\sum_{i \in A} e_i) = S_p(A)$ is submodular , i.e. $S_p(A \cup B) \leq S_p(A) + S_p(B) - S_p(A \cap B) : A, B \subset \{1,2,...,n\}$.*
2. *Consider a nonnegative integer vector $r = (r_1, ..., r_n), \sum_{1 \leq i \leq n} r_i = n$. Then $r \in supp(p)$ iff $r(S) = \sum_{i \in S} r_i \leq S_p(S) : S \subset \{1,2,...,n\}$.*

Corollary 4.2: *Let $p \in Hom_+(n,n)$ be POS-hyperbolic polynomial. Associate with this polynomial p the following bounded convex polytope :*

$$SUB_p = \{(x_1, ..., x_n) : \sum_{i \in S} x_i \leq S_p(S) : S \subset \{1,2,...,n\}; \sum_{1 \leq i \leq n} x_i = n; x_i \geq 0, 1 \leq i \leq n\}$$

Then SUB_p is equal to the Newton polytope of p , i.e. $SUB_p = CO(supp(p)$.

Proof: The inclusion $CO(supp(p)) \subset SUB_p$ follows from the definition of the function S_p . Since the function $S_p : 2^{\{1,2,...,n\}} \rightarrow \{0,1,2,...,n\}$ is submodular and integer valued hence the extreme points of the polytope SUB_p are integer nonnegative vectors [14] . Using the second part of Theorem 4.1 , we conclude that all the extreme points of the polytope SUB_p belong to $supp(p)$. It follows from the Krein-Milman Theorem that $SUB_p \subset CO(supp(p))$. ∎

Corollary 4.3: *Given POS-hyperbolic polynomial $p \in Hom_+(n,n)$ as an oracle , there exists strongly polynomial-time oracle algorithm for the membership problem as for $supp(p)$ as well for the Newton polytope $CO(supp(p))$.*

Proof: Let $X = (x_1, ..., x_n)$ be a vector with real nonnegative coordinates , $sum_{1 \leq i \leq n} x_i = n$. Consider a function $G_X(S) = S_p(S) - \sum_{i \in S} x_i, S \subset \{1,2,...,n\}$. Then G_X is submodular and $X \in CO(supp(p))$ iff $\min_{S \subset \{1,2,...,n\}} G_X(S) \geq 0$ ($X \in supp(p)$ iff $\min_{S \subset \{1,2,...,n\}} G_X(S) \geq 0$ and X is integer). In view on the recent results on the minimization of submodular functions we only need to prove that there exists a strogly polynomial-time oracle algorithm to compute $G_X(S)$. Computing $\sum_{i \in S} x_i$ is easy . And $S_p(S) = deg(D_A)$, where the univariate polynomial $D_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{1 \leq j \leq n} e_j)$. Clearly we can compute the degree $deg(D_A)$ via the standard interpolation , which amounts to at most $n+1$ evaluations of p and $O(n \log(n)^2$ arithmetic operations . ∎

Remark 4.4: Consider $p \in Hom_+(4,4), p = x_1x_2x_3x_4 + x_2^2x_4^2$. Then $S_p(\{1,2,3\}) = 3, S_p(\{1,2\}) = 2, S_p(\{2,3\}) = 2, S_p(\{2\}) = 2$ and therefore the function S_p is not submodular . It is easy to see that S_q is submodular for all $q \in Hom_+(3,3)$. On the other hand , consider $p \in Hom_+(3,3), p = x_1x_2^2 + x_2x_3^2 + x_3x_1^2$. One can check that $SUB_p \neq CO(supp(p)$. The proof in [39] of submodularity of S_p for POS-hyperbolic polynomials p is based on the proved LAX conjecture ([24] , [3], [37]), very nonelementary result .
It was proved in [39] that , unless $\mathbf{P} = \mathbf{NP}$, there is no deterministic polynomial-time oracle algorithm to check if $(1,1,...,1) \in supp(p)$ for integer polynomials $p \in Hom_+(n,n)$. ▮

Definition 4.5: A homogeneous polynomial $q \in Hom_+(n,n)$ is called doubly-stochastic if $\frac{\partial}{\partial x_i} q(1,1,...,1) = 1$ for all $1 \leq i \leq n$. The doubly-stochastic defect of the polynomial q is defined as
$DS(q) = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial x_i} q(1,1,...,1) - 1)^2$.
A polynomial $q \in Hom_+(n,n)$ is called scalable if there exists a positive vector $\beta = (\beta_1, ..., \beta_n)$ such that the scaled polynomial $q_\beta(x_1,...,x_n)) = q(\sum_{1 \leq i \leq n} \beta_i x_i)$ is doubly-stochastic . (It is easy to see that $q \in Hom_+(n,n)$ is scalable iff the infimum $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} q(x_1,...,x_n) = Cap(q)$ is attained .) A polynomial $q \in Hom_+(n,n)$ is called indecomposable if infimum $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} q(x_1,...,x_n) = Cap(q)$ is attained and unique . (Theorem D.1 in Appendix D "justifies" , in the POS-hyperbolic case , our notion of indecomposability.) ▮

Theorem 4.6: *A POS-hyperbolic polynomial $q \in Hom_+(n,n)$ is indecomposable if and only if the following two equivalent conditions hold :*
Condition 1. $\frac{\partial^n}{\partial x_i \partial x_j \prod_{m \neq (i,j)} \partial x_m} q(0,...,0) > 0 : 1 \leq i \neq j \leq n$.
Condition 2. $Rank_p(\sum_{i \in A} e_i) = S_p(A) > |A| : A \subset \{1,2,...,n\}, 1 \leq |A| < n$.

The following theorem combines the algorithm and its analysis from [19] (see section 4 [19])in and Theorem . Similarly to [19] , we use the ellipsoid method to approximate $\min_{\sum_{1 \leq i \leq n} \alpha_i = 0} \log(p(e^{\alpha_1},...,e^{\alpha_n})$. The starting ball is centered at 0 and has the radius $n^{\frac{1}{2}} \log(2p(1,1,...,1))$. To run the ellipsoid method we need to compute the gradient of $\log(p(e^{\alpha_1},...,e^{\alpha_n})$; since p is homogeneous polynomial of degree n hence we can compute the gradient by n standard univariate interpolations. These univariate interpolations amount to $O(n^2)$ oracle calls and $O(n^2)$ arithmetic operations . The ellipsoid updating also requires $O(n^2)$ arithmetic operations .

Theorem 4.7: *There exists a deterministic polynomial-time oracle algorithm which computes for given as an oracle indecomposable POS-hyperbolic polynomial $p(x_1,...,x_n)$ a number $F(p)$ satisfying the inequality*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq F(p) \leq 2(\prod_{1 \leq i \leq n} g(\min(S_p(\{i\})), n+1-i)))^{-1} \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq$$

$$\leq 2\frac{n^n}{n!} \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0).$$

Theorem 4.7 can be (slightly) improved . I.e. it can be applied to the polynomial

$$p_k(x_{k+1}, ..., x_n) = \frac{\partial^k}{\partial x_1 ... \partial x_k} p(0, .., 0, x_{k+1}, ..., x_n).$$

Notice that the polynomial p_k is a homogeneous polynomial of degree $n-k$ in $n-k$ variables . It follows from Theorem 2.5 that if $p = p_0$ is POS-hyperbolic and $Cap(p) > 0$ then for all $0 \leq k \leq n$ the polynomials p_k are also POS-hyperbolic and $Cap(p_k) > 0$. Also , if $p = p_0$ is indecomposable then p_k is indecomposable as well (Theorem 4.6).
The trick is that if $k = m \log_2(n)$ then (using the polarizational formula (13)) the polynomial p_k can be evaluated using $O(n^{m+1})$ oracle calls of the (original) polynomial $p = p_0$. This observations allows to decrease the worst case multiplicative factor in Theorem 4.7 from e^n to $\frac{e^n}{n^m}$ for any fixed m . If the polynomial $p = p_0$ can be explicitly evaluated in deterministic polynomial time , this observation results in deterministic polynomial time algorithms to approximate $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0)$ within multiplicative factor $\frac{e^n}{n^m}$ for any fixed m . Which is an improvement of results in [17] (permanents , p is a multilinear polynomial) and in [18] , [19] (mixed discriminants, p is a determinantal polynomial) .

Remark 4.8: Let A be an $n \times n$ matrix with nonnegative entries , $S \subset \{1, 2, ..., n\}, |S| = m \log_2(n)$. Assume , modulo polynomial time preprocessing , that A is fully indecomposable [17] . Using the Laplace expansion for permanents , we get that $per(A) = \sum_{|T|=n-|S|} per(A_{S,T}) per(A_{S',T'})$. This suggest the following deterministic algorithm : compute exactly the permanents $per(A_{S,T})$ of "small" matrices $A_{S,T}$ and run the algorithm from [17] to approximate with the multiplicative factor $\frac{e^n}{n^m}$ the permanents of "large" matrices $A_{S',T'}$. This algorithm achieves the multiplicative factor $\frac{e^n}{n^m}$, but it runs in quasi-polynomial time . Our approach is to apply Theorem 4.7 to indecomposable POS-hyperbolic polynomial $\sum_{|T|=n-|S|} per(A_{S,T}) Mul_{A_{S',T'}}$, which can be evaluated in deterministic $Poly(n)$-time. Our new "hyperbolic" **(VDW-bound)** (9) allows multiplicative factor $\frac{e^n}{n^m}$. We can use the same trick for sparse matrices using our new "hyperbolic" **(Schrijver-bounds)** (8),(10). ∎

5 Conclusion and Acknowledgements

Univariate polynomials with nonnegative real roots appear quite often in modern combinatorics , especially in the context of integer polytopes . The closest to our approach is the class of univariate *rook* polynomials [28] . We discovered in this paper rather unexpected and very likely far-reaching connections between hyperbolic multivariate polynomials and many classical combinatorial and algorithmic problems .
The main "spring" of our approach is that the class of POS-hyperbolic polynomials is large enough to allow the easy induction . The reader might be surprised by the absence of Alexandrov-Fenchel inequalities and other ingredients of proofs in [13] , [12] , [31]. In fact , the clearest (in our opinion) proof of the Alexandrov-Fenchel inequalities for mixed discriminants is in A.G. Khovanskii' 1984 paper [23] . The Khovanskii' proof is based on the similar induction (via partial differentiations) to the one used in this paper . In a way , the Alexandrov-Fenchel inequalities are "hidden" in our proof .

Let us summarize the main ideas of our approach :
IDEA 1 . To facilitate the induction we deal not with doubly stochastic matrices/tuples/polynomials but rather with the **CAPACITY** of homogeneous polynomials with nonnegative coefficients .
IDEA 2 . The notion of the **VDW-FAMILY** allowed to reformulate Van der Waerden / Schrijver-Valiant/Bapat conjectures in terms of homogeneous polynomials with nonnegative coefficients .
IDEA 3 . The notion from the theory of linear PDE , POS-hyperbolic polynomials , happened to give the needed , i.e. containing multilinear and determinantal polynomials , **VDW-FAMILY** . Corollary 2.8 , a particularly easy case of the Van der Waerden Conjecture , was the final strike .

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A Proof of Lemma 2.7

Proof:

1. Doing simple "algebra" we get that

$$S_{n-1} - nS_n = \prod_{1 \leq i \leq n} c_i (\sum_{1 \leq i \leq n} \frac{1-c_i}{c_i}).$$

Notice that $0 \leq 1 - c_i \leq 1$ and $\sum_{1 \leq i \leq n}(1-c_i) = 1$. Using the concavity of the logarithm we get that

$$\log(S_{n-1} - nS_n) \geq \sum_{1 \leq i \leq n} \log(c_i) + \sum_{1 \leq i \leq n} (1-c_i)\log(\frac{1}{c_i}) = \sum_{1 \leq i \leq n} c_i \log(c_i). \quad (20)$$

2.

$$per(A) = \frac{(n-1)!}{(n-1)^{n-1}} \sum_{1 \leq i \leq n} a_i \prod_{j \neq i} (1-a_j).$$

Define $c_i = 1 - a_i$. Then $0 \leq 1 - c_i \leq 1$, $\sum_{1 \leq i \leq n} c_i = n-1$ and the permanent

$$per(A) = \frac{(n-1)!}{(n-1)^{n-1}}(S_{n-1} - nS_n).$$

It is easy to prove and well known that

$$\min_{0 \leq 1-c_i \leq 1; \sum_{1 \leq i \leq n} c_i = n-1} \sum_{1 \leq i \leq n} c_i \log(c_i) = \sum_{1 \leq i \leq n} \frac{n-1}{n} \log(\frac{n-1}{n}) = \log((\frac{n-1}{n})^{n-1}).$$

Using the entropic inequality (20) from the first part we get the following equality

$$\min_{0 \leq 1-c_i \leq 1; \sum_{1 \leq i \leq n} c_i = n-1} S_{n-1} - nS_n = (\frac{n-1}{n})^{n-1}.$$

Which gives the needed inequality

$$Per(A) \geq \frac{(n-1)!}{(n-1)^{n-1}} (\frac{n-1}{n})^{n-1} = \frac{n!}{n^n}. \quad (21)$$

It is easy to see (strict concavity of $\sum_{1 \leq i \leq n} c_i \log(c_i)$) that the last inequality is strict unless $A(i,j) = \frac{1}{n}; 1 \leq i,j \leq n$.

∎

B Proof of Corollary 2.8

Proof: We can assume WLOG that $a_i > 0, 1 \leq i \leq n$.If $R(0) = 0$ then clearly $d_1 = R'(0) \geq C \geq C((\frac{n-1}{n})^{n-1})$. Let $R(0) > 0$, i.e. $b_i > 0, 1 \leq i \leq n$.
Associate with polynomial $R(t) = \prod_{1 \leq i \leq n}(a_i t + b_i)$ the following matrix with positive entries $A = [a|c|...|c]$, where $a = (a_1, ..., a_n)^T, c = \frac{1}{n-1}(b_1, ..., b_n)^T$. The condition $R(t) \geq Ct, \forall t \geq 0$ is

equivalent to the inequality $Cap(Mul_A) \geq C$. And $per(A) = \frac{(n-1)!}{(n-1)^{n-1}} d_1$. Since A has positive entries hence there exist two positive diagonal matrices D_1, D_2 such that $A = D_1 B D_2$, where the matrix B is doubly stochastic and $B = [f|d|...|d]$ (Sinkhorn Diagonal Scaling , see Lemma 3.6 in [19] in a more general setting). Since the matrix B is doubly stochastic hence $Cap(Mul_B) = 1$. Thus $Cap(Mul_A) = \det(D_1 D_2) Cap(Mul_B) = \det(D_1 D_2)$ and $per(A) = \det(D_1 D_2) per(B)$. Therefore , we get from inequality (21) that $per(A) \geq \frac{n!}{n^n} Cap(Mul_A) \geq \frac{n!}{n^n} C$.

Finally ,it follows that

$$d_1 = (\frac{(n-1)!}{(n-1)^{n-1}})^{-1} Per(A) \geq (\frac{(n-1)!}{(n-1)^{n-1}})^{-1} (\frac{n!}{n^n} C) = C((\frac{n-1}{n})^{n-1}).$$

∎

C Proof of Theorem 2.5

We need the following simple result .

Proposition C.1: *Let $p(X)$ be e-hyperbolic (homogeneous) polynomial of degree n , $p(e) > 0$. Consider two e-nonnegative vectors $Z, Y \in N_e(p)$ such that $Z + Y \in C_e(p)$, i.e. $Z + Y$ is e-positive . Then*

$$p(tZ + Y) = \prod_{1 \leq i \leq n} (a_i t + b_i); a_i, b_i \geq 0, a_i + b_i > 0, 1 \leq i \leq n. \tag{22}$$

Proof: As the vector $Z + Y = D$ is e-positive hence $p(Z + Y) > 0$ (**FACT 1**) ,the polynomial p is $Z + Y$-hyperbolic (**FACT 3**) and any e-positive (e-nonnegative) is also $Z + Y$-positive($Z + Y$-nonnegative) (**FACT 3**) . Doing simple algebra , we get that $p(tZ + Y) = p((t-1)Z + D)$.

Let $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be nonnegative roots of the equation $p(Z - xD) = 0$. Since $D - X = Y \in N_e(p) = N_D(p)$ hence $\lambda_n \leq 1$. Therefore

$$p(tZ + Y) = p((t-1)Z + D) = p(D) \prod_{1 \leq i \leq n} (t\lambda_i + (1 - \lambda_i))$$

We can put $a_i = (p(Z + Y))\lambda_i \geq 0, b_i = (p(Z + Y))(1 - \lambda_i) \geq 0$. ∎

C.1 Proof of the first part of Theorem 2.5 - inequality (16)

Proof: Let $q \in Hom_+(n, n)$ be POS-hyperbolic polynomial and $1 \leq Rank_q(e_1) = S_q(\{1\}) = k$. Fix positive real numbers $(x_2, ..., x_n)$ such that $\prod_{2 \leq i \leq n} x_i = 1$. Define the following two real n-dimensional vectors with nonnegative coordinates : $Z = (1, 0, 0, ..., 0), Y = (0, x_2, ..., x_n)$. The vector $Z + Y$ is e-positive . Consider the next univariate polynomial $R(t) = q(tZ + Y)$. It follows from Proposition C.1 that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = \sum_{0 \leq i \leq n} d_i t^i,$$

where $a_i, b_i \geq 0$ and $q_{x_1}(x_2, ..., x_n) = d_1$ and the cardinality $|\{i : a_i > 0\}| = k$ (see also equality (2)). In other words the degree $deg(R) = k$
We get straight from the definition of $Cap(q)$ that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = p(t, x_2, ..., x_n) \geq Cap(q) t \prod_{2 \leq i \leq n} x_i = tCap(q).$$

Using Corollary 2.8 , we get that

$$q_{x_1}(x_2, ..., x_n) = d_1 \geq ((\frac{k-1}{k})^{k-1}) Cap(q).$$

In other words , that $Cap(r) \geq (\frac{k-1}{k})^{k-1}) Cap(q)$. ∎

C.2 Proof of the second part

The second part of Theorem 2.5 is an easy modification of **FACT 4**. We need only to consider the case $q_{x_1} \neq 0$. We need the following well known fact .

Fact C.2: Consider a sequence of univariate polynomials of the same degree n : $P_k(t) = \sum_{0 \leq i \leq n} a_{i,k} t^i$. suppose that $\lim_{k \to \infty} a_{i,k} = a_i, 0 \leq i \leq n$ and $a_n \neq 0$.
Define $P(t) = \sum_{0 \leq i \leq n} a_i t^i$. Then roots of P_k converge to roots of P . In particular if roots of all polynomials P_k are real then also roots of P are real ; if roots of all polynomials P_k are real nonnegative numbers then also roots of P are real nonnegative numbers .

It follows from Definition 2.2 and the Taylor's formula (14) that

$$q_d(X) = \frac{d}{dt} q(X + td)|_{(t=0)} = ((n-1)!)^{-1} M_q(d, X, X, ..., X) : d, X \in R^n \tag{23}$$

Notice that $q_{x_1}(x_2, ..., x_n) = q_{e_1}(0, x_2, ..., x_n)$. Consider the following perturbed univariate polynomials $P_\epsilon(t) = \sum_{0 \leq i \leq n-1} a_{\epsilon,i} t^i$:

$$P_\epsilon(t) = ((n-1)!)^{-1} M_q(e_1 + \epsilon e, Y - t((e - e_1) + \epsilon e), ..., Y - t((e - e_1) + \epsilon e)) :$$
$$e = \sum_{1 \leq i \leq n} e_i, Y = (x_2, ..., x_n) \in R^{n-1}, \epsilon > 0.$$

We get by a direct inspection that

$$\lim_{\epsilon \to \infty} P_\epsilon(t) = ((n-1)!)^{-1} M_q(e_1, Y - t(e - e_1), ..., Y - t(e - e_1)) = q_{x_1}(x_2 - t, ..., x_n - t);$$

$$\lim_{\epsilon \to \infty} a_{\epsilon,n-1} = (-1)^{n-1} q_{x_1}(1, 1, ..., 1).$$

As $q_{x_1} \in M_+(n-1, n-1)$ and $q_{x_1} \neq 0$ hence $q_{x_1}(1, 1, ..., 1) > 0$. Therefore $\lim_{\epsilon \to \infty} a_{\epsilon,n-1} \neq 0$. Since the polynomial $q \in Hom_+(n, n)$ is POS-hyperbolic hence it follows from **FACT 3** that all the roots of the equation $P_\epsilon(t) = 0$ are real ; if $x_i \geq 0, 2 \leq i \leq n$ then all the roots are nonnegative . We conclude, using Fact C.2 , that

1. If $Y = (x_2, ..., x_n) \in R^{n-1}$ then all the roots of the equation $q_{x_1}(Y - t(e - e_1)) = 0$ are real and $q_{x_1}(e - e_1) > 0$.

2. If $x_i \geq 0, 2 \leq i \leq n$ then all the roots are nonnegative .

3. Since $q_{x_1} \in M_+(n-1, n-1)$ and $q_{x_1} \neq 0$ hence $q_{x_1}(x_2, ..., x_n) > 0$ if $x_i > 0, 2 \leq i \leq n$. It follows from the equality (15) that all the roots of the equation $q_{x_1}(Y - t(e - e_1)) = 0$ are positive if $x_i > 0, 2 \leq i \leq n$.

4. The polynomial $q_{x_1} \in M_+(n-1, n-1)$ is POS-hyperbolic .

D Proof of Theorem 4.6

Proof: Let $q \in Hom_+(n, n)$ be POS-hyperbolic polynomial and a pair of indeces $(i \neq j) \subset \{1, 2, ..., n\}$. Define the following integer vectors $r^{(i,j)} = e + e_i - e_j$. **Condition 1** states that $r^{(i,j)} \in supp(q)$ for all such pairs . The equivalence of **Condition 1** and **Condition 2** follows from the second part of Theorem 4.1.
The fact that **Condition 1** implies indecomposability is valid for all polynomials in $Hom_+(n, n)$ and is proved in [19].
Suppose that there exists a positive vector $\beta = (\beta_1, ..., \beta_n), \prod_{1 \leq i \leq n} \beta_i = 1$ such that

$$q(\beta_1, ..., \beta_n) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} q(x_1, ..., x_n) = Cap(q).$$

Then the polynomial $Q(x_1, ..., x_n)) = \frac{1}{q(\beta_1, ..., \beta_n)} q(\sum_{1 \leq i \leq n} \beta_i x_i)$ is doubly-stochastic . Notice that the $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} q(x_1, ..., x_n)$ is attained and unique if and only if the $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} Q(x_1, ..., x_n)$ is attained and unique ; it follows from the Euler's identity that $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} Q(x_1, ..., x_n) = 1 = Q(e)$.
Assume that **Condition 2** does not hold : there exists a subset $A \subset \{1, 2, ..., n\}, 1 \leq |A| = m < n$ such that $Rank_q(\sum_{i \in A} e_i) = Rank_Q(\sum_{i \in A} e_i) = m$. Define $e_A = \sum_{i \in A} e_i, e_{A'} = \sum_{j \in A'} e_j = e - e_A$. Let $\lambda_n \geq ... \geq \lambda_{n-m+1} \geq 0 \geq ... \geq 0$ be the ordered roots of the equation $Q(e_A - te) = 0$. Since $e_A = e - e_{A'}$ and the vector $e_{A'}$ is e-nonnegative hence $0 \leq \lambda_i \leq 1, 1 \leq i \leq n$. As the polynomial $Q \in Hom_+(n, n)$ is doubly-stochastic hence (see [16] , [39])

$$\sum_{1 \leq i \leq n} \lambda_i = \sum_{n-m+1 \leq i \leq n} \lambda_i = m = |A|.$$

Therefore $\lambda_i = 1, n - m + 1 \leq i \leq n$ and $\lambda_j = 0, 1 \leq j \leq n - m$. It follows from the identity (15) that $Q(ae_A + be_{A'}) = 1 = Q(e)$ iff $a^m b^{n-m} = 1$, which proves the non-uniqueness of $\inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} Q(x_1, ..., x_n)$. ∎

The following result is proved very similarly to the previous proof , the only new ingredient is the subadditivity of $Rank_q(X), X \in R^n$. It "justifies" the notion of "indecomposibility of POS-hyperbolic polynomials" .

Theorem D.1: *Let $q \in Hom_+(n,n)$ be POS-hyperbolic polynomial . Supposed that $\inf_{x_i>0, \prod_{1 \leq i \leq n} x_i = 1} q(x_1,...,x_n)$ is attained and $Rank_q(\sum_{1 \leq i \leq m} e_i) = m, m < n$. Then the polynomial q can be decomposed in the following way :*

$$q(x_1,..,x_m,...,x_n) = q_1(x_1,..,x_m)q_2(x_{m+1},...,x_n) : q_1 \in Hom_+(m,m), q_2 \in Hom_+(n-m,n-m) \tag{24}$$

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