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Eisenberg-Gale Markets: Rationality, Strongly Polynomial Solvability, and Competition Monotonicity*

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Abstract

We study the structure of EG[2], the class of Eisenberg-Gale markets with two agents. We prove that all markets in this class are rational and they admit strongly polynomial algorithms whenever the polytope containing the set of feasible utilities of the two agents can be described via a combinatorial LP. This helps resolve positively the status of two markets left as open problems by [JV]: the capacity allocation market in a directed graph with two source-sink pairs and the network coding market in a directed network with two sources.

Our algorithms for solving the corresponding nonlinear convex programs are fundamentally different from those obtained by [JV]; whereas they use the primal-dual schema, we use a carefully constructed binary search.

We also settle a third open problem of [JV], that of determining whether the notion of competition monotonicity characterizes the class of SUA markets within UUA markets. We give a positive resolution of this problem as well.

1 Introduction

The classic Eisenberg-Gale convex program captures, as its optimal solution, equilibrium allocations for the linear case of Fisher's market equilibrium model [EG59, BS00]. Over the years, convex programs with the same basic structure were found for more general utility functions: scalable utilities [Eis61], Leontief utilities [CV04], Linear Substitution utilities [Ye] and homothetic utilities with productions [JVY05].

Interestingly enough, a program with the same structure as the Eisenberg-Gale program is used by Kelly [Kel97] in his seminal work giving a mathematical model for understanding TCP congestion control. Given a network (directed or undirected) with edge capacities specified and a set of source-sink pairs (agents), each with initial endowment of money specified, Kelly's program maximizes the total utility of the agents. Using KKT conditions, Kelly showed that the optimal allocation of flows and prices (which are the Lagrangian variables of his program) must satisfy the following equilibrium conditions:

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- Only saturated edges can have positive prices.
- All flows are sent along a minimum cost (with respect to the prices) path from source to sink.
- The money of each source-sink pair is fully spent.

A useful interpretation of edge prices in the context of TCP congestion control is either as probability of packet loss in TCP Reno or as queuing delay in TCP Vegas [TWHL05].

Continuous time algorithms, not having polynomial time guarantees, were developed for problems in Kelly's resource allocation model by several researchers, e.g., see [KMT98, WLLD05, Kel03], and this study eventually led to new protocols and practical impact, e.g., see the FAST project [WJLH07, WWCL06, CKLL05]. [KV] observed that the above-stated flow equilibrium problem generalizes the linear case of Fisher's model, and stated, "Continuous time algorithms similar to TCP are known, but insights from discrete algorithms may be provocative".

Recently, progress on this question was made by Jain and Vazirani [JV]. They defined the notion of *Eisenberg-Gale markets* which generalizes all the models stated in the first paragraph (see Section 2 for definitions), and studied the class of Eisenberg-Gale markets from the five viewpoints of solvability via strongly polynomial time algorithms, rationality, efficiency, fairness and competition monotonicity, and they found a surprisingly rich structure. They also stated a host of open problems whose resolution should lead to a deeper understanding not only of these markets but also of the issue of solvability of nonlinear convex programs via strongly polynomial algorithms. In this paper we investigate Eisenberg-Gale markets further and settle three open problems of [JV]. In particular, we dispel the belief that existence of underlying max-min relations is essential to obtaining strongly polynomial algorithms for these markets.

1.1 Rationality and Strongly polynomial solvability

A remarkable property of the Eisenberg-Gale program is that, despite its being nonlinear, it always has a rational solution if all the input parameters are rational. We will say that a market or a nonlinear program is *rational* if it has this property and *irrational* otherwise. Interestingly enough, rationality is not unique to the Eisenberg-Gale program. [JV] established rationality and strongly-polynomial solvability for the following cases of the above-stated flow equilibrium problem:

- There is only one source, though multiple sinks, and the network is directed or undirected.
- The network is undirected and has two source-sink pairs.

Both these settings support combinatorial max-flow min-cut theorems – due to Ford and Fulkerson [FF56] and Elias, Feinstien and Shannon [EFS56], and Hu [Hu63], respectively. In all other cases there is a gap between max-flow and min-cut, and computing the latter is **NP**-hard. Let us partition these into two:

- The graph is directed or undirected with three or more source-sink pairs. **NP**-hardness for this case was established in [DJP$^+$94], and [GJTV05] show that this case is irrational.

- The graph is directed and there are two source-sink pairs. This case was shown **NP**-hard in [GVY94], and [JV] left its rationality as an open problem.

[JV] also established the rationality of the following two markets related to broadcasting:

- Spanning trees in an undirected network with arbitrarily many sources.
- Branchings in a directed network with two sources.

Once again, max-min theorems of Nash-William and Tutte [NW61, Tut61], and [JV], respectively, played a central role in the algorithms. For the second market, for the case of three or more sources, [JV] established irrationality. They left the open problem of determining rationality of the network coding market with two sources; the network in this case is allowed to have Steiner nodes (see Section 2 for a definition).

To summarize, among the markets characterized so far, an important distinction between the rational and irrational markets was that combinatorial problems underlying the former satisfied max-min theorems, which were used critically to establish rationality, and those for the latter didn't. The two markets left open do not support max-min theorems. Surprisingly enough, despite this, both of them turn out to be rational. More generally, in this paper we show that all markets in EG[2], the class of Eisenberg-Gale markets with two agents, are rational. We also show that whenever the polytope containing the set of feasible utilities of the two agents can be described via a combinatorial LP, the market admits a strongly polynomial algorithm; both markets described above admit such LP's.

Our algorithm circumvents the lack of underlying max-min theorems by using the more general LP-Duality Theorem itself. This difference manifests itself in the algorithmic ideas needed in the two cases – whereas [JV] use the primal-dual schema and their algorithms can be viewed as ascending price auction, we use a carefully constructed binary search. The algorithms of [JV] are combinatorial whereas ours are not, ours require a subroutine for solving combinatorial LP's. The latter can be accomplished in strongly polynomial time using Tardos' algorithm [Tar86].

1.2 Competition Monotonicity

[JV] defined two abstract subclasses of Eisenberg-Gale markets – uniform utility allocation (UUA) markets and submodular utility allocation (SUA) markets, that are central to many of their results. The latter is properly contained in the former. A UUA market is defined via a valuation function on subsets of agents specifying a bound on the total utility obtainable by that subset of agents. If the valuation function is a polymatroid function, the market is said to be a SUA market. [JV] establish rationality and strongly polynomial solvability of SUA[1] markets. Further, they study the notions of efficiency and fairness in UUA markets, and show that each of these characterizes the class of SUA markets.

Among other structural properties investigated by [JV], they observe that the classic notion of weak gross substitutability is not always relevant to Eisenberg-Gale markets since they may

[1] In fact, some of the markets in the capacity allocation framework are SUA markets; this gives an alternate, albeit slower, algorithm for such markets.

not involve goods but instead deal directly with utilities. They define the relevant notion of competition monotonicity for these markets and show that when both notions are applicable, the former is weaker than the latter. A market satisfies *competition monotonicity* if increasing the money of one agent cannot lead to an increase in the utility of any other agent. Clearly, after the money of one agent is increased, the rest of the agents need to compete against a more powerful competitor and it is reasonable to expect that their utilities will not increase. They also showed that SUA markets satisfy competition monotonicity and left the open problem of determining whether this notion leads to a third characterization of SUA markets within UUA markets.

In the second part of our paper we give a positive resolution of this problem as well. For each UUA market which is not SUA, we show how to construct a pair of instances for which competition monotonicity fails. Since the valuation function is not submodular, there exists of a set of agents T and two agents $i, j \notin T$ which contradict submodularity: the marginal utility of agent j is greater for $T \cup i$ than T alone. To construct instances which contradict competition monotonicity a simple approach would be to construct two instances such that,

- The money of i is zero in the first and some huge amount in the second.
- The equilibrium utility of j in both the instances is its marginal utility for the sets T and $T \cup i$ respectively.

Unfortunately, such an easy approach does not work and we need a more careful construction. One of the key ideas in the construction is to achieve a delicate balance of *tight* sets: the total utility allocated to agents in the set equals the valuation of that set. We show that one can choose the sets T, i, j and an allocation which tightens $T, T \cup i, T \cup j$ with the following surprising properties:

- *No set* containing both i and j is tight.
- The intersection of *all* tight sets containing any one of these is non-empty.

Such a structure enables us to construct a second instance of a market where the money of j, remains the same, moneys of *all* other players *increase* and yet the equilibrium utility of j increases. This contradicts competition monotonicity. We refer the reader to Section 6 for details.

2 Definitions and Results

Jain and Vazirani [JV] define a class of abstract markets, called the Eisenberg-Gale or EG Markets.

Definition 2.1 EG Markets *An EG Market $\mathcal{M}$ with the set of buyers $[n]$ is such that the set of feasible utilities $\boldsymbol{u} \in \mathbf{R}_+^n$ for $\mathcal{M}$ is captured by a polytope $\mathcal{P}$ defined by linear equations of the form*

$$\begin{aligned}
&\forall\ j \in J, \sum_{i \in [n]} a_{ij} u_i + \sum_{k \in K} a'_{ij} t_k \leq b_j, \\
&\forall i \in [n], k \in K,\ u_i, t_k \geq 0,
\end{aligned}$$

such that it satisfies the following two conditions:

- Free disposal: *if* $\boldsymbol{u}$ *is feasible, then so is any other* $\boldsymbol{u}'$ *dominated by* $\boldsymbol{u}$.
- Utility Homogenity: *for all* $j \in J$*, if for some* $i \in [n]$*,* $a_{ij} > 0$ *then* $b_j = 0$.

The auxiliary variables t_k might be used for instance, to give a more efficient representation of the feasible region, or as a means to provide semantics for the market. An instance of $\mathcal{M}$ is given by the moneys $\boldsymbol{m}$ of the buyers. We say that a feasible utility $\boldsymbol{u}$ is an equilibrium allocation if there exist witness $\boldsymbol{t} \in \mathbf{R}_+^K$ and prices $\boldsymbol{p} \in \mathbf{R}_+^J$ such that

- $\forall\ i \in [n], m_i = rate(i)u_i$, where $rate(i) = \sum_j a_{ij}p_j$.
- $\forall\ j \in J, p_j > 0 \implies \sum_{i \in [n]} a_{ij}u_i + \sum_{k \in K} a'_{ij}t_k = b_j$.
- $\forall\ t \in K, t_k > 0 \implies \sum_{k \in K} a'_{ij}p_j = 0$, and $\sum_{k \in K} a'_{ij}p_j \geq 0$ otherwise.

Note that the equilibrium conditions are exactly equivalent to the KKT conditions for the following convex program:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} m_i \log u_i \\ \text{subject to} \quad & \forall j \in J,\ \sum_{i \in [n]} a_{ij}u_i + \sum_{k \in K} a'_{ij}t_k \leq b_j, \\ & \forall i \in [n], k \in K,\ u_i, t_k \geq 0. \end{aligned}$$

Such programs were first considered by Eisenberg and Gale [EG59] for the Fisher model of a market, with linear utilities. In the Fisher model of a market, there are buyers and divisible goods. An instance of a market is defined by the endowments of money for each of the buyers, their utilities for the goods, and the supplies of the goods. Prices are at an equilibrium if every buyer can be assigned a bundle of goods such that

- the bundle maximizes his utility subject to the constraint that it does not cost more than his endowment.
- the goods clear exactly.

The linear utilities case of the Fisher model are EG markets. The feasible region of utilities in a Fisher market is defined by

$$\begin{aligned} \forall\ i \in [n]\,, u_i &\leq \sum_{j \in J} u_{ij}x_{ij}, \\ \forall\ j \in J\,, \sum_{i \in [n]} x_{ij} &\leq b_j, \end{aligned}$$

where J is the set of goods, the x_{ij}'s indicate the amount of good j allocated to buyer i, and b_j is the supply of good j. Also, at equilibrium, $\forall i \in [n], m_i = \sum_{j \in J} x_{ij} p_j$. It is easy to see that the equilibrium conditions for the Fisher model are equivalent to the equilibrium conditions as defined for Eisenberg Gale markets:

- $\forall\ i \in [n], m_i = \alpha_i u_i$, where $\alpha_i = rate(i)$.
- $\forall\ j \in J, p_j > 0 \implies \sum_{i \in [n]} x_{ij} = b_j$.
- $\forall\ i \in [n], j \in J, x_{ij} > 0 \implies p_j = \alpha_i u_{ij}$, and $p_j \geq \alpha_i u_{ij}$ otherwise.

It is already known that a Fisher model with any number of agents and linear utilities is rational. In fact, there is a strongly polynomial time algorithm for the Fisher model with linear utilities and 2 buyers [DPS02]. Obtaining a strongly polynomial time algorithm for any number of buyers is an open problem.

Definition 2.2 *EG[k] denotes the class of EG markets with k buyers.*

Definition 2.3 *Suppose that the inequalities defining the feasible polytope of utilities have rational co-efficients. If such a market is guaranteed to have rational prices whenever the moneys of the buyers are rational, then it is called a* rational *market.*

Theorem 2.1 *EG[2] markets are rational.*

Theorem 2.1 is proved in Section 3.

Consider an LP of the form $\max\{cx : Ax \leq b, x \geq 0\}$. A combinatorial LP is one in which the entries in A have binary encoding length polynomial in the dimension of A. Tardos [Tar86] gave a strongly polynomial algorithm for solving combinatorial LPs. A *strongly polynomial time algorithm* [GLS93] is one in which the running time of the algorithm is polynomial in the dimension of the input, i.e., the number of data points in the input. Arithmetic operations count as one unit of time. We extend the definition of a combinatorial LP to also mean a set of inequalities $Ax \leq b$ such that the entries in A have binary encoding length polynomial in the dimension of A.

Definition 2.4 *A market is* combinatorial *if the LP describing the feasible utilities is combinatorial.*

Theorem 2.2 *If an EG[2] market is combinatorial, then the equilibrium prices can be found in strongly polynomial time.*

Theorem 2.2 is proved in Section 4.

Definition 2.5 *([JV]) Suppose that $\boldsymbol{m}$ is the vector of moneys describing an instance of a market $\mathcal{M}$. Let the equilibrium utility allocation for this instance be $\boldsymbol{u}$. Define the vector $\boldsymbol{m}'$, as a function of $\boldsymbol{m}, i \in [n]$ and $\epsilon > 0$, as $m'_i = m_i + \epsilon$, and $m'_{i'} = m_{i'}$ for all $i' \neq i$. Let the equilibrium utility allocation for $\boldsymbol{m}'$ be $\boldsymbol{u}'$. $\mathcal{M}$ is* competition monotone *if for all $\boldsymbol{m}, i \in [n]$, $\epsilon > 0$, and $i' \neq i$, $u'_{i'} \leq u_{i'}$.*

The following theorem follows from Pareto optimality of the equilibrium allocation.

Theorem 2.3 *An EG[2] Market is competition monotone.*

Definition 2.6 *([JV]) A Uniform Utility Allocation (UUA) market is one in which the polytope $\mathcal{P}$ is of the form*

$$\mathcal{P} = \left\{ \boldsymbol{u} : \forall\ S \subseteq [n], \sum_{i \in S} u_i \leq v(S) \right\}.$$

In general, one need not have a constraint for all the subsets of $[n]$. A UUA market is defined by the valuation function $v(\cdot)$. Without loss of generality, one can assume that v satisfies the *covering property:* Let $S \subseteq [n]$, and let $\{x_T\}_{T \subseteq [n]}$ be a fractional cover of S. That is, for all $i \in S$, $\sum_{T: i \in T} x_T \geq 1$ and $x_T \geq 0$ for all $T \subseteq [n]$. Then $v(S) \leq \sum_{T \subseteq [n]} v(T) x_T$.

Definition 2.7 *([JV]) A UUA market is a Submodular Utility Allocation (SUA) market if the valuation function v is a polymatroid function.*

Jain and Vazirani [JV] proved that a SUA market is competition monotone. In this paper, we prove the converse.

Theorem 2.4 *A UUA market is Competition Monotone if and only if it is an SUA market.*

This theorem is proved in Section 6.

2.1 Capacity Allocation Markets

In this model of Kelly [Kel97], we are given a directed graph with capacities on the edges, which are the goods, a set of source-sink pairs, which are the buyers, and the endowment of money for each source-sink pair. At equilibrium, edges are priced, and feasible flows are allocated between the source-sink pairs such that

- Every source-sink pair sends flow on the cheapest path.
- Only those edges that are saturated are priced.
- All the moneys of the source-sink pairs are used up.

The set of feasible flows $\{f_i\}_{i \in I}$ for this market is defined by the following LP:

$$\begin{aligned}
&\forall\, i \in I, && f_i = \sum_{e=(s_i,v) \in E} f_i(e), \\
&\forall\, e \in E, && \sum_{i \in I} f_i(e) \leq c(e), \\
&\forall\, i \in I, \forall v \in V - \{s_i, t_i\}, && \sum_{e=(u,v) \in E} f_i(e) = \sum_{e=(v,w) \in E} f_i(e).
\end{aligned}$$

Since this LP is combinatorial, it follows from Theorem 2.2 that the equilibrium prices for such markets with 2 agents can be found in strongly polynomial time. This was one of the questions left open by [JV].

2.2 The Network Coding Market

A slight generalization of the framework of Kelly given in Section 2.1 is needed for defining the network coding market; we will now allow resources to be picked fractionally to construct objects.

We are given a directed graph $G = (V, E)$; E is the set of resources, with capacities $c : E \to \mathbf{R}_+$. The set V is partitioned into two sets, terminals and Steiner nodes, denoted T and R, respectively. A set $S \subseteq T$ is the set of sources with money $m_v,\ v \in S$ specified. Sources wish to broadcast messages to all terminals. Each source will, in general, use network coding to maximize its rate; however, we do not allow two different sources to jointly use network coding.

For $v \in S$, a *generalized branching rooted at* v picks a (fractional) subgraph of G; it is specified via a function $b : E \to \mathbf{R}_+$ satisfying:

- $\forall e \in E, b(e) \le c(e)$.
- $\forall u \in T$, a flow of one unit is possible in this subgraph from v to u.

We will pick generalized branchings pick fractionally. If generalized branchings $b_1, \ldots, b_k$ rooted at $v \in S$ are picked with weights $w_i,\ 1 \le i \le k$, then the utility derived by v, $u_v = w_1 + \ldots w_k$. By the theorem of Ahlswede, Cai, Li and Yeung [ACLY00], u_v is the rate at which v can broadcast messages to the terminals.

Generalized branchings rooted at vertices of S, $b_1, \ldots, b_k$ picked with weights $w_i,\ 1 \le i \le k$, are said to form a *feasible packing* for G if

$$\forall e \in E, w_1 b_1(e) + \ldots + w_k b_k(e) \le c(e).$$

Edge e is said to be *saturated* if this inequality holds with equality. Given prices p_e for $e \in E$, the price of generalized branching b is defined to be $\sum_{e \in E} b(e) p_e$.

The network coding market asks for a feasible packing of generalized branchings, together with weights, and prices on edges such that

- The generalized branchings rooted at each source are cheapest possible.
- Only saturated edges have positive prices.
- The money of each source is fully used up.

It can be shown that the feasible utilities of a Network Coding market can also be described by a combinatorial LP, and hence there exists a strongly polynomial time algorithm for network coding markets with 2 sources.

2.3 Projection of Polytopes

Suppose we eliminate the auxiliary variables $\boldsymbol{t}$ from the equations to get an equivalent formulation for the feasible region of utilities as

$$\mathcal{P}_u = \left\{ \boldsymbol{u} : \forall\, l \in L, \sum_{i \in [n]} \alpha_{il} u_i \leq \beta_l \right\}.$$

This should define the same market as before. However, the prices now correspond to the new constraints, which correspond to the facets of $\mathcal{P}_u$, indexed by L. We show that given the prices on the facets in L, one can find prices for the original constraints in J that form an equilibrium. Suppose the equilibrium price of facet $l \in L$ is q_l. Let $\boldsymbol{u}$ be the equilibrium utility, with $\boldsymbol{t}$ being its witness. Then $rate(i) = \sum_l \alpha_{il} q_l$. At equilibrium, $m_i = rate(i) u_i$ and $q_l > 0 \implies \sum_i \alpha_{il} u_i = \beta_l$. Now consider the following LP:

$$\begin{aligned} \text{maximize} \quad & \sum_i \alpha_{il} u_i \\ \text{subject to} \quad & \forall\, j \in J, \sum_{i \in [n]} a_{ij} u_i + \sum_{k \in K} a'_{ij} t_k \leq b_j. \\ & \forall i \in [n], k \in K,\ \ u_i, t_k \geq 0. \end{aligned}$$

For any l with $q_l > 0$ the optimal value of this LPs has to be β_l. In fact, $(\boldsymbol{u}, \boldsymbol{t})$ is an optimal solution. For each such l, consider any optimal solution $\boldsymbol{y}^l$ to the dual:

$$\begin{aligned} \text{minimize} \quad & \sum_j b_j y_j^l \\ \text{subject to} \quad & \forall\, i \in [n], \sum_j a_{ij} y_j^l \geq \alpha_{il}, \\ & \forall\, k \in K, \sum_j a'_{kj} y_j^l \geq 0, \\ & \forall\, j \in J, y_j^l \geq 0. \end{aligned}$$

$(\boldsymbol{u}, \boldsymbol{t})$ and $\boldsymbol{y}^l$ satisfy the complementary slackness conditions for the above pair of primal-dual programs:

$$\begin{aligned} y_j^l > 0 &\implies \sum_{i \in [n]} a_{ij} u_i + \sum_{k \in K} a'_{ij} t_k \leq b_j. \\ u_i > 0 &\implies \sum_j a_{ij} y_j^l = \alpha_{il}. \\ t_k > 0 &\implies \sum_j a'_{kj} y_j^l = 0. \end{aligned}$$

Let $p_j = \sum_l y_j^l q_l$. Using the feasibility and complementary slackness conditions above, one can show that p_j and $(\boldsymbol{u}, \boldsymbol{t})$ indeed satisfy the equilibrium conditions.

In general, a high dimensional polytope when projected onto the two dimensional plane blows up in the number of facets [Nem05]. We show that this is also the case when the flow polytope with 2 source-sink pairs is projected onto the plane spanned by f_1 and f_2. This result is in contrast it to the case of two commodity flow in *undirected* networks where by Hu's theorem, the corresponding polytope has at most three facets [Hu63].

Theorem 2.5 *There exists a network N whose flow polytope when projected onto the plane spanned by f_1 and f_2, gives a polytope with exponentially many facets.*

We provide a proof of this theorem in Section 5.

3 Rationality of EG[2] Markets

The main results of this section are that EG markets with 2 agents are rational. Let the polytope of feasible utilities be

$$\mathcal{P} = \{x : Ax \leq b, x \geq 0\},$$

with $u_1 = x_1$ and $u_2 = x_2$ being the utilities of agents 1 and 2 respectively. Let c be a vector such that $c_1 = 1, c_2 = \alpha$, and $c_i = 0$ otherwise. This is defined so that $cx = u_1 + \alpha u_2$. Let $\mathcal{L}(\alpha) = \max\{cx : x \in \mathcal{P}\} = \min\{by : y \in \mathcal{D}\}$, where $\mathcal{D}$ is the dual polytope $\{y : A^T y \geq c, y \geq 0\}$. In particular, $\mathcal{L}(0) = \max\{u_1 : x \in \mathcal{P}\}$ and $\mathcal{L}(\infty) = \max\{u_2 : x \in \mathcal{P}\}$. Let the projection of $\mathcal{P}$ on (u_1, u_2) be

$$\mathcal{P}_u = \{(u_1, u_2) : u_2 \leq \beta_0, u_1 + \alpha_l u_2 \leq \beta_l, 1 \leq l \leq m, u_1 \leq \beta_{m+1}\}.$$

Observe that $\beta_l = \mathcal{L}(\alpha_l)$ for all $0 \leq l \leq m+1$ if we define $\alpha_0 = \infty$ and $\alpha_{m+1} = 0$. We may assume that we only consider facet inducing inequalities: for all $1 \leq l \leq m$, $u_1 + \alpha_l u_2 = \beta_l$ is a facet of $\mathcal{P}_u$. Call it facet l. Without loss of generality, assume that the α_l's and β_l's are strictly decreasing.

Definition 3.1 *Let $(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1})$ with $\alpha_l > \alpha_{l+1}$ be the* profile *of $\mathcal{P}_u$ which completely describes it.*

We recall the definition of a facet, for a polytope in two dimensions.

Definition 3.2 *$u_1 + \alpha u_2 = \beta$ is a facet of $\mathcal{P}_u$ iff it is a valid inequality for $\mathcal{P}_u$ and there exist two distinct feasible utilities (v_1, v_2) and (w_1, w_2) such that $v_1 + \alpha v_2 = \beta$ and $w_1 + \alpha w_2 = \beta$. In particular, $\mathcal{L}(\alpha) = \beta$.*

Two adjacent facets intersect at a point.

Definition 3.3 *Let the facets l and $l+1$ intersect at the point (u_1^l, u_2^l).*

Then the endpoints of the facet l are (u_1^{l-1}, u_2^{l-1}) and (u_1^l, u_2^l). Associate subintervals of $[0,1]$ to the facets as follows.

Definition 3.4

$$\forall 1 \leq l \leq m, I_l := \left[\frac{u_1^{l-1}}{\beta_l}, \frac{u_1^l}{\beta_l}\right].$$

$$\forall 1 \leq l \leq m, I_{l,l+1} := \left[\frac{u_1^l}{\beta_l}, \frac{u_1^l}{\beta_{l+1}}\right].$$

$$I_{0,1} :=:= \left[0, 1 - \frac{\alpha_1 \beta_0}{\beta_1}\right].$$

Since projection, using say the Fourier-Motzkin elimination, preserves rationality, α_l's are rational. Therefore β_l and the points (u_1^l, u_2^l) are also rational. Let p_l be the price of facet l, with p_0 being the price of the facet $u_2 \leq \beta_0$. The main idea is that $\frac{m_1}{m_1+m_2}$ falls in one of the intervals I_l or $I_{l,l+1}$, and in any case, the p_l's are rational in $m_1, m_2, \alpha_l, \beta_l, u_1^l$ and u_2^l's.

Lemma 3.1 *If* $\frac{m_1}{m_1+m_2} \in I_l, 1 \leq l \leq m$, *then* $p_l = \frac{m_1+m_2}{\beta_l}$ *(and 0 otherwise) is an equilibrium price.*

Proof: Define utilities

$$u_1^* := \frac{m_1 \beta_l}{m_1 + m_2} \text{ and } u_2^* := \frac{m_2 \beta_l}{\alpha_l (m_1 + m_2)}. \tag{1}$$

It is easy to check that $u_1^* + \alpha_l u_2^* = \beta_l$. So the equilibrium conditions are satisfied. Given that $\frac{m_1}{m_1+m_2} = \frac{u_1^*}{\beta_l}$ lies in I_l, it follows that $u_1^* \in \left[u_1^{l-1}, u_1^l\right]$. Hence (u_1^*, u_2^*) lies on facet l, and is feasible. ∎

Lemma 3.2 *If* $\frac{m_1}{m_1+m_2} \in I_{l,l+1}, 1 \leq l \leq m$, *then there exists an equilibrium price with only* p_{l+1} *and* p_l *having no-zero prices.*

Proof: The equilibrium utility allocation is (u_1^l, u_2^l). We want p_l and p_{l+1} that satisfy the following two equations. $m_1 = u_1^l(p_l + p_{l+1})$, and $m_2 = u_2^l(\alpha_l p_l + \alpha_{l+1} p_{l+1})$. Note that this system of two equations in two unknowns has a unique solution since they are linearly independent:

$$p_l = \frac{u_1^l m_2 - \alpha_{l+1} u_2^l m_1}{u_1^l u_2^l (\alpha_l - \alpha_{l+1})}$$

$$p_{l+1} = \frac{\alpha_j u_2^l m_1 - u_1^l m_2}{u_1^l u_2^l (\alpha_l - \alpha_{l+1})}$$

However the prices are positive exactly when $\frac{m_1}{m_2} \in \left[\frac{u_1^l}{\alpha_l u_2^l}, \frac{u_1^l}{\alpha_{l+1} u_2^l}\right]$, which happens when $\frac{m_1}{m_1+m_2}$ is in the interval $I_{l,l+1}$. ∎

$I \leq I'$ means interval I ends where I' begins. $I < I'$ means interval I ends before I' begins. $I \leq x$ means interval I ends before or at x. $x \leq I$ means interval I starts after or at x. We note the following for future reference.

Observation 3.1

$$I_l \leq I_{l,l+1} \leq I_{l+1}.$$

Proof of Theorem 2.1 Proof follows from noting that the intervals I_l, for $1 \leq l \leq m$, and $I_{l,l+1}$, for $0 \leq l \leq M$, cover the entire unit interval (Observation 3.1).

We did not say how to price the facet 0, which we need to do when $\frac{m_1}{m_1+m_2}$ falls in $I_{0,1}$. But by symmetry of choice between u_1 and u_2 it follows that we can price it accordingly. ∎

Rationality implies that solving the Eisenberg-Gale-type convex program using, say, the ellipsoid or interior point algorithms with a suitably good precision would give the exact equilibrium. But since the equilibrium utilities depend on m_1 and m_2 (see Equation (1) from the proof of Lemma 3.1), the precision required for such an algorithm would depend on m_1 and m_2. In the next section, we show an algorithm whose running time is independent of m_1, m_2. We further show that when applied to combinatorial markets, this algorithm runs in strongly polynomial time.

4 Algorithms for combinatorial EG[2] markets

4.1 Binary Search Algorithm

In this section we give a binary search algorithm for finding equilibrium prices. We also give a strongly polynomial time algorithm for finding the equilibrium prices in EG[2] markets that are combinatorial. The algorithm takes as input, the moneys of the buyers, m_1 and m_2, a description of the polytope $\mathcal{P}$, and two parameters, M and ϵ such that we are guaranteed that $M \geq \alpha_1$, and $\alpha_l - \alpha_{l+1} \geq 2\epsilon$ for all l.

We now describe the algorithm at a high level. The algorithm does a binary search on α. First, it finds the facets adjacent to α, say l and $l+1$ such that $\alpha \in [\alpha_l, \alpha_{l+1}]$, and their endpoints. Now, it checks if the equilibrium can be attained by pricing these two facets, using Lemmas 3.1 and 3.2. If yes, the algorithm outputs those prices and halts. Otherwise, the monotonicity of the intervals (Observation 3.1) allows us to restrict our attention to a smaller range.

input : $m_1, m_2, \mathcal{P}, M, \epsilon$.
$U \leftarrow M$;
$L \leftarrow 0$;
$\rho \leftarrow \frac{m_1}{m_1+m_2}$;
1 **repeat**
    $\alpha \leftarrow (U+L)/2$;
2   Find l such that $\alpha \in [\alpha_l, \alpha_{l+1}]$;
3   Find the endpoints of the facets l and $l+1$;
    if $\rho \in I_l \cup I_{l,l+1} \cup I_{l+1}$ **then**
        Assign prices to the facets l and $l+1$ as in Lemmas 3.1 and 3.2, and halt;
    else if $\rho < I_l$ **then**
        $L \leftarrow \alpha_l$;
    else
        $U \leftarrow \alpha_{l+1}$;
    end
until $U - L < \epsilon$;

Algorithm 1: The Binary Search Algorithm

The rest of the section describes how to implement Lines 2 and 3 in Algorithm 1. Let the entries of the matrix A be $A_{(i,j)} = a_{ij}$. Recall that $\mathcal{P} = \{x : Ax \leq b, x \geq 0\}$ and $\mathcal{D} = \{y : A^T y \geq c, y \geq 0\}$. Given any $x \in \mathcal{P}$, define the polytope $\mathcal{Q}(x)$ as the set of all vectors (y, α) that satisfy

$$\forall i\,, \sum_j a_{ij} y_j \leq c_i,$$
$$\forall j\,, y_j \geq 0.$$
$$\forall i : x_i > 0, \sum_j a_{ij} y_j = c_i,$$
$$\forall j : \sum_i a_{ij} x_i < b_j, y_j = 0.$$

Note that the first two constraints imply that $y \in \mathcal{D}$. The last two constraints imply that x and y satisfy the complementary slackness conditions. However, in $\mathcal{Q}(x)$, α is treated as a variable. The algorithm to find the facets adjacent to any given α makes use of Lemmas 4.1 and 4.2.

Lemma 4.1 *Let x be any feasible extension of (u_1^l, u_2^l), that is $x \in \mathcal{P}$, $x_1 = u_1^l$ and $x_2 = u_2^l$. Then $\alpha_l = \min\{\alpha : (y,\alpha) \in \mathcal{Q}(x)\}$, and $\alpha_{l+1} = \max\{\alpha : (y,\alpha) \in \mathcal{Q}(x)\}$.*

Lemma 4.2 *$\mathcal{L}(\alpha) = u_1^l + \alpha u_2^l$ if and only if $\alpha \in [\alpha_l, \alpha_{l+1}]$.*

Proof: Suppose $\alpha \in [\alpha_l, \alpha_{l+1}]$. Say $\alpha = \mu\alpha_l + (1-\mu)\alpha_{l+1}$, for some $0 \leq \mu \leq 1$. Let $\beta = \mu\beta_l + (1-\mu)\beta_{l+1}$. For all $(u_1, u_2) \in \mathcal{P}, u_1 + \alpha_l u_2 \leq \beta_l$ and $u_1 + \alpha_{l+1} u_2 \leq \beta_{l+1}$. By adding μ times the first equation to $1-\mu$ times the second one, we get $u_1 + \alpha u_2 \leq \beta$. Hence $\beta \geq \mathcal{L}(\alpha) \geq u_1^l + \alpha u_2^l = \beta$.

Suppose $\alpha \in [\alpha_k, \alpha_{k+1}]$, for some $k \neq l$. Let (v_1^k, v_2^k) be the intersection of facets k and $k+1$. Then by the first part, $\mathcal{L}(\alpha) = v_1^k + \alpha v_2^k$. If $\mathcal{L}(\alpha) = u_1^l + \alpha u_2^l$, then there are two distinct points maximizing $\mathcal{L}(\alpha)$ and by Definition 3.2, we get that $u_1 + \alpha u_2 \leq \mathcal{L}(\alpha)$ itself is a facet. In that case, either $\alpha = \alpha_l = \alpha_{k+1}$ or $\alpha = \alpha_k = \alpha_{l+1}$ and we are done. ∎

Lemma 4.3 *Let x be any feasible extension of (u_1^l, u_2^l), that is $x \in \mathcal{P}$, $x_1 = u_1^l$ and $x_2 = u_2^l$. Then $(y, \alpha) \in \mathcal{Q}(x)$ if and only if $\alpha \in [\alpha_l, \alpha_{l+1}]$.*

Proof: Suppose $(y, \alpha) \in \mathcal{Q}(x)$. Then $\mathcal{L}(\alpha) \geq u_1^l + \alpha u_2^l = \sum_i c_i x_i = \sum_i x_i \sum_j a_{ij} y_j = \sum_j y_j \sum_i a_{ij} x_i = \sum_j y_j b_j \geq \mathcal{L}(\alpha)$. So by Lemma 4.2, $\alpha \in [\alpha_l, \alpha_{l+1}]$.

Suppose $\alpha \in [\alpha_l, \alpha_{l+1}]$. Then by Lemma 4.2, $\mathcal{L}(\alpha) = u_1^l + \alpha u_2^l$. So x is an optimal primal solution satisfying $Ax \geq b, x \geq 0$, and $cx = \mathcal{L}(\alpha)$, Consider an optimal dual solution y such that $A^T y \geq c$, $y \geq 0$ and $by = \mathcal{L}(\alpha)$. Apply complementary slackness conditions to x and y to conclude that $(y, \alpha) \in \mathcal{Q}(x)$. ∎

Lemma 4.1 is an immediate corollary of this lemma. Now given α, one can find the facets adjacent to it, that is, l such that $\alpha \in [\alpha_l, \alpha_{l+1}]$. First find x that maximizes $cx = u_1 + \alpha u_2$ such that $x \in \mathcal{P}$. Then find $\alpha_l = \min\{\alpha : (y, \alpha) \in \mathcal{Q}(x)\}$, and $\alpha_{l+1} = \max\{\alpha : (y, \alpha) \in \mathcal{Q}(x)\}$. We now give a lemma that enables us to find the endpoints of a facet.

Lemma 4.4 $\mathcal{L}(\alpha_l + \epsilon) = u_1^{l-1} + (\alpha_l + \epsilon) u_2^{l-1}$ *and* $\mathcal{L}(\alpha_l - \epsilon) = u_1^l + (\alpha_l - \epsilon) u_2^l$.

Proof: (u_1^{l-1}, u_2^{l-1}) is the intersection of facets $l-1$ and l. Since $\alpha_{l-1} - \alpha_l \geq 2\epsilon$, $\alpha_l + \epsilon \in [\alpha_l, \alpha_{l+1}]$ and by Lemma 4.2, $\mathcal{L}(\alpha_l + \epsilon) = u_1^{l-1} + \alpha_l u_2^{l-1}$. The other part is identical. ∎

Let T be the time required to optimize any linear objective function over the polytopes $\mathcal{P}$ and $\mathcal{Q}(x)$.

Theorem 4.5 *The running time of the algorithm is* $O\left(T \log\left(\frac{M}{\epsilon}\right)\right)$.

Proof: The number of iterations of the repeat loop in Line 1 is bounded by $O\left(\log\left(\frac{M}{\epsilon}\right)\right)$. Line 2 can be done in $O(T)$ time as follows: first find x that maximizes $cx = u_1 + \alpha u_2$ such that $x \in \mathcal{P}$. Then find $\alpha_l = \min\{\alpha : (y, \alpha) \in \mathcal{Q}(x)\}$, and $\alpha_{l+1} = \max\{\alpha : (y, \alpha) \in \mathcal{Q}(x)\}$ (Lemma 4.1). Each of this takes time T. From Lemma 4.4, Line 3 can be done in $O(T)$ time too. Hence we are done. ∎

Theorem 4.6 *The algorithm always outputs the equilibrium prices.*

Proof: Suppose $\frac{m_1}{m_1+m_2} \in I_k \cup I_{k-1,k}$. Then $L \leq \alpha_k \leq U$ throughout the algorithm. This is true initially, since $M \geq \alpha_k \geq 0$. Suppose this is true at the beginning of an iteration. If in the iteration, $\rho < I_l$, then from Observation 3.1, $\alpha_k < \alpha_l$. Similarly, if $\rho > I_{l+1}$, then $\alpha_k > \alpha_{l+1}$. Hence the assertion is true at the end of the iteration too.

Suppose that at the end of an iteration, $U - L < \epsilon$. Note that after each iteration, either both U and L have a value equal to one of the α_l's, or one of them is 0 or M and the other has a value equal to an α_l. In either case, $U - L < \epsilon \implies U = L$, which should equal α_k by the first part. Hence we must have found the equilibrium prices in this iteration. ∎

4.2 Combinatorial Markets

In this section, we show that for combinatorial EG[2] markets, the equilibrium price can be found in strongly polynomial time. Let $\nu(\cdot)$ denote the binary encoding length.

Lemma 4.7 $\forall\, l, \nu(\alpha_l) = \nu(A)^{O(1)}$.

Proof: Note that $\mathcal{Q}(x)$ is described by the a_{ij}'s. Theorem follows from Lemma 4.1 and standard application of Cramer's rule. ■

Lemma 4.8 *One can find M and ϵ such that* $\log\left(\frac{M}{\epsilon}\right) = \nu(A)^{O(1)}$.

Proof: Let c be the constant in the $O(1)$ in Lemma 4.7. M can be chosen to be the largest integer with a binary encoding length $\nu(A)^c$. Clearly $\alpha_1 \leq M$. ϵ can then be chosen to be $1/(2M)$. α_l's have their denominators at most M and hence $\alpha_l - \alpha_{l+1} \geq 1/M = 2\epsilon$. ■

Theorem 2.2 follows from this lemma and Theorem 4.5. As a corollary, we get that there is a strongly polynomial time algorithm for the capacity allocation market in directed graphs with two source-sink pairs and the network coding market in a directed network with two sources.

5 An EG[2] market with exponentially many facets

In this section, we construct an EG[2] market defined by $\mathcal{P}$ such that the number of facets of the projection $\mathcal{P}_u$ is exponential in the number of constraints in $\mathcal{P}$. This proves Theorem 2.5. Consider a directed network $N(V, E, \mathbf{c})$ with two source-sink pairs (s_1, t_1) and (s_2, t_2). The polytope $\mathcal{P}(N)$ defining the feasible flows is

$$\begin{aligned}
&\forall\, i = 1, 2, && f_i = \sum_{e=(s_i,v)\in E} f_i(e), \\
&\forall\, e \in E, && f_1(e) + f_2(e) \leq c(e), \\
&\forall\, i = 1, 2, \forall v \in V - \{s_i, t_i\}, && \sum_{e=(u,v)\in E} f_i(e) = \sum_{e=(v,w)\in E} f_i(e),
\end{aligned}$$

where $f_1(e), f_2(e)$ correspond to the s_1–t_1 and s_2–t_2 flow on edge e. f_1 and f_2 correspond to the total flow of each kind. $\mathcal{P}_f(N)$ is the projection of $\mathcal{P}(N)$ on the variables f_1, f_2. Since $\mathcal{P}(N)$ can be described in polynomially (in say the number of edges) many inequalities, it has at most a polynomial number of facets. Recall that in general, $\mathcal{P}_f(N)$ has the following form

$$\mathcal{P}_f(N) = \{(f_1, f_2) : f_1 + \alpha_i f_2 \leq \beta_i \qquad \forall i \in I\}$$

where I^2 is the set of facet inducing inequalities of $\mathcal{P}_f(N)$ and each equality $f_1 + \alpha_i f_2 = \beta_i$ induces a facet. We may further assume that $\infty \geq \alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 0$. $\alpha_1 = \infty$ implies that the facet is of the form $f_2 \leq \beta$. $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ is called the *profile* of $\mathcal{P}_f(N)$.

[2] In this section, the facets are indexed by i and range from 1 to k.

Proof of Theorem 2.5 The construction involves two operations called *doubling* and *shifting*. Given any $(s_1, t_1), (s_2, t_2)$ network N with profile $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ with $\alpha_k \geq 1$, the doubling operation constructs a new network N' which has a constant number of edges more than N and has a profile $(\alpha_1, \alpha_2, \cdots, \alpha_k, \zeta_k, \zeta_{k-1}, \cdots, \zeta_1, 0)$. Thus it doubles the number of facets.

Given a network N with profile $(\alpha_1, \alpha_2, \cdots, \alpha_k)$, the shifting operation constructs a new network N' with a constant number more edges with profile $(\alpha_1 + 1, \alpha_2 + 1, \cdots, \alpha_k + 1)$. Thus we see that given any network with $\alpha_k \geq 1$, we can perform the doubling and the shifting operation to get a network with a constant number more edges, with least α greater than 1, and having double the number of facets. Thus performing this operation m times, starting on a seed network which had least $\alpha \geq 1$, we will get a network with $O(m)$ edges and 2^m facets, thus proving the theorem. ■

We now describe the two operations.

Doubling Operation:
Given a network N, Figure 1 shows how the network N' is constructed.

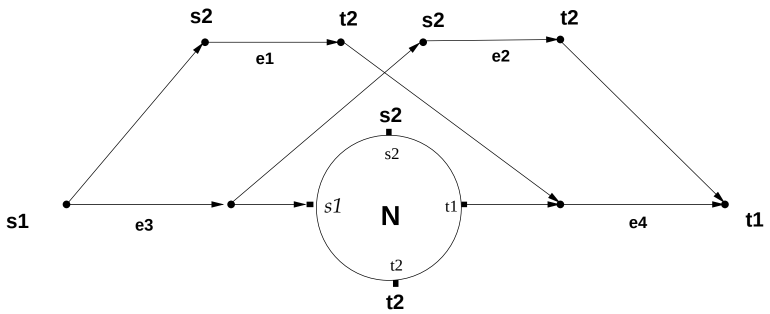

Figure 1: THE NETWORK N' OBTAINED FROM N. THE EDGES e_i HAVE A CAPACITY c, WHERE c IS THE MAXIMUM f_1 FLOW THAT IS FEASIBLE IN N.

Lemma 5.1 *Suppose the profile of N was $(\alpha_1, \cdots, \alpha_k)$ with $\alpha_k \geq 1$. Then the profile of N' is $(\alpha_1, \cdots, \alpha_k, \zeta_k, \cdots, \zeta_1, 0)$ where $\zeta_i = \frac{1}{2 - \frac{1}{\alpha_i}}$.*

Proof: We first sketch a reason why the number of facets double. Let us move from 'left to right' on the facets of N'. Initially suppose there was only s_2, t_2 flowing in N'. This flow would saturate edges e_1, e_2 and the edges in network N. Now the first facet of N' would correspond to the 'best way' we can increase f_1 while decreasing f_2, that is, we would decrease f_2 in such a way that we can send maximum amount of f_1. For example, if we decrease f_2 along edge e_1 by ϵ, then we can send ϵ of f_1 flow through the path e_1, e_4. This would give us a tradeoff of 1. But this may not be the best. Indeed, decreasing ϵ amount of f_2 flow through N, would allow us to increase the f_1 flow by $\alpha_1 \epsilon$. Thus we would send flow this way, until we cannot do so any more. Then we shall get a new facet. At this point again, decreasing f_2 flow through N and increasing f_1 flow through N would give us a tradeoff of $\alpha_2 > 1$. Thus we shall keep on doing this until we have no more f_2 flow through N, that is, we send maximum f_1 flow through N of value $\beta_k = c$. Note that this flow saturates e_3 and e_4 and at this point we cannot send any more f_1 flow through these.

The next facet of $\mathcal{P}_f(N')$ will be obtained as follows. We decrease f_2 flow on both e_1 and e_2, say by ϵ. We also *decrease* the f_1 flow through N by ϵ. Since this frees e_3, e_4, we can send 2ϵ f_1 flow through e_1, e_4 and e_3, e_2. Moreover, since we decreased the f_1 flow in N by ϵ, we can increase the f_2 flow in N by $\frac{1}{\alpha_k}\epsilon$. Thus, we decrease the total f_2 flow by $(2 - \frac{1}{\alpha_k})\epsilon$ and increase f_1 flow by ϵ. Thus the tradeoff is $\frac{1}{2-\frac{1}{\alpha_k}} = \zeta_k$. We keep on doing this till at the end we send $2\beta_k$, f_1 flow through the paths e_1, e_4 and e_3, e_2, and we send β_1, f_2 flow through N. The last facet corresponds to just decreasing the f_2 flow through N to zero without changing the f_1 flow.

Thus as we move from left to right on the profile of N', we moved from left to right and then again right to left on that of N. Hence the number of facets of N' is twice that of N.

Recall that $\mathcal{L}_N(\alpha)$ is the optimum value of the following LP. For $i = 1, 2$, letting $\mathbf{P_i}$ denote the set of paths from s_i to t_i.

$$
\begin{aligned}
\text{maximize} \quad & f_1 + \alpha f_2 \\
\text{subject to} \quad & f_1 = \sum_{P \in \mathbf{P_1}} f_1^P \\
& f_2 = \sum_{P \in \mathbf{P_2}} f_2^P \\
& \sum_{P: e \in P} f_1^P + f_2^P \leq \mathbf{c}(e) & \forall e \in E \\
& f_1, f_2, f_1^P, f_2^P \geq 0 & \forall P \in \mathbf{P_1} \cup \mathbf{P_2}
\end{aligned}
$$

Let $\mathcal{D}_N(\alpha)$ denote the dual of this LP:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c(e) x_e \\
\text{subject to} \quad & \sum_{e \in P} x_e \geq 1 & \forall P \in \mathbf{P_1} \\
& \sum_{e \in P} x_e \geq \alpha & \forall P \in \mathbf{P_2} \\
& x_e \geq 0 & \forall e \in E
\end{aligned}
$$

Further recall that by Definition 3.2, $f_1 + \alpha f_2 = \beta$ is a facet of $\mathcal{P}_f(N)$ iff it is a valid inequality for $\mathcal{P}_f(N)$ and there exist two distinct feasible flows (g_1, g_2) and (h_1, h_2) such that $g_1 + \alpha g_2 = \beta$ and $h_1 + \alpha h_2 = \beta$. Also recall that $\mathcal{L}_N(\alpha_i) = \beta_i$. We now formally prove the lemma by giving for each α_i (and ζ_j) two feasible flows on the facet and a cut of value $\mathcal{L}_{N'}(\alpha_i)$ (and $\mathcal{L}_{N'}(\zeta_j)$). Moreover we shall see that last feasible f_1 flow for the last facet is the maximum f_1 flow, and hence these will be the only facets.

Since $(\alpha_1, \cdots, \alpha_k)$ is the profile of N, there are feasible flows (f_1^i, f_2^i) for each $i = 1..k$, such that both (f_1^i, f_2^i) and (f_1^{i+1}, f_2^{i+1}) satisfy $f_1 + \alpha_i f_2 = \mathcal{L}_N(\alpha_i)$. Moreover there exists a solution x_e^i to $\mathcal{D}_N(\alpha_i)$ of value $\mathcal{L}_N(\alpha_i)$. We shall now describe the feasible flows (g_1^i, g_2^i) in N' and also dual solutions y_e^i which account for all the facets of N'.

Let c be the maximum f_1 flow that can be sent in N. Note that $c = \mathcal{L}_N(\alpha_k)$ For $i = 1, .., k+1$, that is the first half of the facets, we have

$$\begin{aligned} g_1^i &= f_1^i \\ g_2^i &= f_2^i + 2c \\ y_e^i &= x_e^i \qquad\qquad \forall e \in E[N] \\ y_{e_1}^i &= y_{e_2}^i = \alpha_i \\ y_{e_3}^i &= y_{e_4}^i = 0 \end{aligned}$$

Claim 1 *g_1^i, g_2^i is realizable in N' for all $i = 1, .., k+1$.*

Proof: Realization of g_1^i: Send f_1^i flow through the path e_3, N, e_4.
Realization of g_2^i: Send f_2^i flow through the path N and c through both e_1 and e_2.
The realization is feasible because (f_1^i, f_2^i) is feasible in N. ■

Claim 2 *For $i = 1, .., k$, $\{y_e^i\}_{e \in E[N']}$ forms a feasible solution to the dual program $\mathcal{D}_{N'}(\alpha_i)$. In fact, this is the optimum solution, since (g_1^i, g_1^i) and (g_1^{i+1}, g_2^{i+1}) both satisfy $g_1 + \alpha_i g_2 = \mathcal{L}_{N'}(\alpha_i) = \sum_{e \in E[N']} \mathbf{c}(e) y_e^i$. Thus α_i is in the profile of N' for all $i = 1, .., k$.*

Proof: Every s_1, t_1 path P passes through N or uses the edge e_1 or e_2. In the first case, $\sum_{e \in P} y_e^i \geq 1$ because $\sum_{e \in P \in \mathbf{P_1}(N)} x_e^i \geq 1$. In the second case, feasibility is ensured by the fact that $y_{e_1}^i = y_{e_2}^i = \alpha_i \geq \alpha_k \geq 1$.

Every s_2, t_2 path also either passes through N or uses e_1 or e_2 and so $\sum_{e \in P} y_e^i \geq \alpha_i$ for all such paths. Hence $\{y_e^i\}_{e \in E[N']}$ is feasible for all $i = 1, .., k$.
Cost of this solution

$$\sum_{e \in E[N']} c(e) y_e^i = \sum_{e \in E[N]} c(e) x_e^i + 2c\alpha_i = \mathcal{L}_N(\alpha_i) + 2c\alpha_i$$

Moreover

$$\begin{aligned} g_1^i + \alpha_i g_2^i &= f_1^i + \alpha_i (f_2^i + 2c) \\ &= (f_1^i + \alpha_i f_2^i) + 2c\alpha_i \\ &= \mathcal{L}_N(\alpha_i) + 2c\alpha_i \\ &= \sum_{e \in E[N']} c(e) y_e^i \end{aligned}$$

Similarly $g_1^{i+1} + \alpha_i g_2^{i+1} = (f_1^{i+1} + \alpha_i f_2^{i+1}) + 2c\alpha_i = \mathcal{L}_{N'}(\alpha_i)$. ■

We describe the next half of the facets. For $i = k+2, .., 2k+1$, let $j = 2k - i + 2$. The flows are

$$
\begin{aligned}
g_1^i &= 2c - f_1^j \\
g_2^i &= 2f_1^j + f_2^j \\
y_e^i &= (2\zeta_j - 1)x_e^j && \forall e \in E[N] \\
y_{e_1}^i &= y_{e_2}^i = \zeta_j \\
y_{e_3}^i &= y_{e_4}^i = 1 - \zeta_j
\end{aligned}
$$

In subsequent claims, we show that for each i we get a facet $g_1 + \zeta_j g_2 \leq \mathcal{L}_{N'}(\zeta_j)$ by showing that (g_1^i, g_2^i) and (g_1^{i-1}, g_2^{i-1}) lie on it and exhibiting a feasible solution of $\mathcal{D}_{N'}(\zeta_j)$ of value $\mathcal{L}_{N'}(\zeta_j)$. Note that $g_1^{2k+1} = 2c$ which is the maximum s_1, t_1 flow since $\{e_1, e_3\}$ form a minimum cut of value $2c$. This shows that the next facet has to be the 0-facet since we cannot increase f_1 any more. Thus the profile of N' is as claimed and lemma 5.1 is proved. ■

Claim 3 *g_1^i, g_2^i is realizable in N' for all $i = k+2, .., 2k+1$.*

Proof: Realization of g_1^i: Send f_1^j through e_3, N, e_4 and $c - f_1^j$ through the paths e_1, e_4 and e_3, e_2.
Realization of g_2^i: Send f_2^j through N, f_1^j through both e_1 and e_2.
Realization is feasible because f_1^j, f_2^j is feasible through N, and the edges $e_i, i = 1, 2, 3, 4$ carry total flow c and is also feasible. ■

Claim 4 *For $i = k+2, .., 2k+1$, $\{y_e^i\}_{e \in E[N']}$ forms a feasible solution to the dual program $\mathcal{D}_{N'}(\zeta_j)$. In fact, this is the optimum solution, since (g_1^i, g_1^i) and (g_1^{i-1}, g_2^{i-1}) both satisfy $g_1 + \zeta_j g_2 = \mathcal{L}_{N'}(\zeta_j) = \sum_{e \in E[N']} c(e) y_e^i$. Thus ζ_j is in the profile of N' for all $i = k+2, .., 2k+1$, that is $j = k, .., 1$.*

Proof: Every s_1, t_1 path passes through N and e_3, e_4 or uses the pairs of edges e_1, e_4 or e_3, e_2. In the first case, $\sum_{e \in P} y_e^i \geq 1$ because

$$
\begin{aligned}
\sum_{e \in P} y_e^i &= y_{e_3}^i + y_{e_4}^i + \sum_{e \in P \in \mathbf{P_1}(N)} (2\zeta_j - 1)x_e^j \\
&\geq 2(1 - \zeta_j) + 2\zeta_j - 1 \\
&= 1
\end{aligned}
$$

In the second case, $\displaystyle\sum_{e \in \{e_1, e_4\} \text{ or } \{e_3, e_2\}} y_e^i = \zeta_j + 1 - \zeta_j = 1.$

Every s_2, t_2 path passes through N,e_1 or e_2. In the latter two cases we have by definition $\sum_{e \in P} y_e^i = \zeta_j$. For every path passing through N we have

$$
\begin{aligned}
\sum_{e \in P} y_e^i &= \sum_{e \in P \in \mathbf{P_2}(N)} (2\zeta_j - 1) x_e^j \\
&\geq (2\zeta_j - 1)\alpha_j \\
&= (2\frac{1}{2 - \frac{1}{\alpha_j}} - 1)\alpha_j \\
&= \frac{1}{2 - \frac{1}{\alpha_j}} = \zeta_j
\end{aligned}
$$

Thus this forms a feasible solution to $\mathcal{D}_{N'}(\zeta_j)$. Cost of this solution

$$
\sum_{e \in E[N']} c(e) y_e^i = \sum_{e \in E[N]} (2\zeta_j - 1) c(e) x_e^j + 2c(1 - \zeta_j) + 2c\zeta_j = (2\zeta_j - 1) L(\alpha_j) + 2c
$$

Now

$$
\begin{aligned}
g_1^i + \zeta_j g_2^i &= (2c - f_1^j) + \zeta_j (2 f_1^j + f_2^j) \\
&= (2\zeta_j - 1) f_1^j + \zeta_j f_2^j + 2c \\
&= (2\zeta_j - 1) f_1^j + (2\zeta_j - 1)\alpha_j f_2^j + 2c \qquad \because \zeta_j = (2\zeta_j - 1)\alpha_j \\
&= (2\zeta_j - 1)(f_1^j + \alpha_j f_2^j) + 2c \\
&= \sum_{e \in E[N']} c(e) y_e^i
\end{aligned}
$$

Similarly

$$
\begin{aligned}
g_1^{i-1} + \zeta_j g_2^{i-1} &= (2c - f_1^{j+1}) + \zeta_j (2 f_1^{j+1} + f_2^{j+1}) \\
&= (2\zeta_j - 1)(f_1^{j+1} + \alpha_j f_2^{j+1}) + 2c \qquad \text{as before} \\
&= (2\zeta_j - 1)\mathcal{L}_N(\alpha_j) + 2c
\end{aligned}
$$

Note that we need to treat the case $i = k + 2$ separately. But $g_1^{k+1} = f_1^{k+1} = 2c - f^{2k-(k+1)+2}$ since $c = f_1^{k+1}$, and $g_1^{k+1} = f_2^{k+1} + 2c = f_2^{2k-(k+1)+2} + 2 f_1^{2k-(k+1)+2}$. Thus the two definitions are consistent and hence the above is true for $i = k + 2$ as well. This proves the claim. ∎

Shifting Operation:
Given a network N, Figure 2 shows how to construct the stretched network N'.

Suppose the profile of N was $(\alpha_1, \cdots, \alpha_k)$.

Lemma 5.2 *The profile for N' is $(\alpha_1 + 1, \cdots, \alpha_k + 1)$.*

Proof: We sketch a proof as in Lemma 5.1. When we move from left to right on the profile of N', and we decrease ϵ s_2, t_2 flow, then we can increase $\alpha_1 \epsilon$ s_1, t_1 flow in the network N, and ϵ flow on the edge e. Thus the net tradeoff is $\alpha_1 + 1$. Hence we get the profile as claimed.

Once again, for each $i = 1, \ldots, k$, let there be feasible flows (f_1^i, f_2^i), and feasible dual solutions $\{x_e^i\}_{e \in E[N]}$ of value $\mathcal{L}_N(\alpha_i)$. Moreover (f_1^i, f_2^i) and (f_1^{i+1}, f_2^{i+1}) lie on the facet $f_1 + \alpha_i f_2 = \mathcal{L}_N(\alpha_i)$.

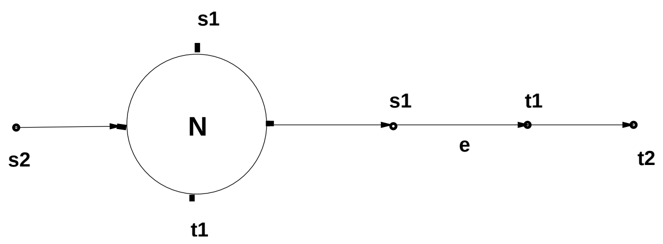

Figure 2: THE NETWORK N' OBTAINED FROM N. THE EDGE e HAS A CAPACITY c, WHERE c IS THE MAXIMUM f_2 FLOW THAT IS FEASIBLE IN N.

To prove the lemma we need to show feasible (g_1^i, g_2^i) and feasible duals $\{y_e^i\}_{e \in E[N']}$ of value $\mathcal{L}_N(\alpha_i + 1)$.

Let c be the max f_2 flow feasible in N. Note $c = \mathcal{L}_N(\alpha_1)$

$$
\begin{aligned}
g_1^i &= f_1^i + c - f_2^i \\
g_2^i &= f_2^i \\
y_e^i &= x_e^i \qquad \forall e \in E[N] \\
y_e^i &= 1
\end{aligned}
$$

Claim 5 *g_1^i, g_2^i is realizable in N' for all i. $\{y_e^i\}_{e \in E[N']}$ form a feasible solution of value $\mathcal{L}_{N'}(\alpha_i + 1)$, and (g_1^i, g_2^i) and $g_1^{i+1}, g_2^{i+1})$ lie on the facet $g_1 + (\alpha_i + 1)g_2 = \mathcal{L}_{N'}(\alpha_i + 1)$.*

Proof: To realize g_1^i send f_1^i through N and $c - f_2^i$ through e. Realize g_2^i by sending f_2^i through N and e.

The shortest s_1, t_1 paths P are either paths in N or the path $\{e\}$. In any case $\sum_{e \in P} y_e^i \geq 1$. Any s_2, t_2 path has to be a path in N concatenated with e. Thus $\sum_{e \in P \in \mathbf{P_2}(N')} y_e^i = \sum_{e \in P \in \mathbf{P_2}(N)} x_e^i + 1 \geq \alpha_i + 1$.
Thus the y_e^i form a feasible solution of value

$$\sum_{e \in E[N']} c(e) y_e^i = \sum_{e \in E[N]} c(e) x_e^i + c = \mathcal{L}_N(\alpha_i) + c$$

The claim is proved by noticing that $g_1^i + (\alpha_i + 1)g_2^i = f_2^i + c - f_2^i + f_2^i + \alpha_i f_2^i = \mathcal{L}_N(\alpha_i) + c$. The same is true for $g_1^{i+1} + (\alpha_i + 1)g_2^{i+1}$ ■

To finish the proof of the lemma, note that $g_2^k = f_2^k = 0$. Hence we have enumerated all the facets. ■

6 Competition Monotonicity and Submodularity

In this section we prove Theorem 2.4, by showing that a UUA market that is competition monotone is an SUA market. Suppose that the valuation function of a UUA market is not submodular. Then we give two instances of moneys of players, such that in the second instance, moneys of no player decreases and there is a particular player j whose equilibrium utility allocation increases although his money remains the same. This implies that one can construct a sequence of instances such that

- each subsequent instance has the same moneys as the previous one for all except one player,
- the money of the exceptional player (not j) increases,
- for some consecutive pair of instances, the utility of j increases.

This contradicts competition monotonicity, and so the theorem follows.

Let S be the smallest set which contradicts submodularity, in the sense, there exists an element $j \notin S$ and $T \subseteq S$ such that $v(S \cup j) - v(S) > v(T \cup j) - v(T)$. Choose $T \subseteq S$ such that $v(T \cup j) - v(T)$ is minimum over all subsets of S.

Observation 6.1 *We may assume $|T| = |S| - 1$. By minimality of S we have for all $T \subseteq T' \subsetneq S$, $v(T \cup j) - v(T) \geq v(T' \cup j) - v(T')$. Thus a superset of T which is a strict subset of S only has smaller marginal for j. Let $S = T \cup i$.*

Observation 6.2 *By minimality of S, we have that any set of cardinality at most $|S|$ is submodular[3]. Thus $S = T \cup i$ and $T \cup j$ are submodular sets.*

Observation 6.3 *If v satisfies the covering property, then for any set T, there is a feasible allocation vector $\boldsymbol{u}$ that tightens T, i.e., $\boldsymbol{u}(T) = v(T)$.*

Proof Sketch: In order to prove Theorem 2.4, we need to construct two instances of moneys so that in the second instance moneys of each player is at least their moneys in the first instance. Moreover there is a player whose money remains the same but his equilibrium utility strictly increases. This player will be the player j described above. The other players who have moneys will be those in the set S.

Since v satisfies the covering property, we know there exists a feasible allocation vector $\boldsymbol{u}$ which tightens T. Given any such $\boldsymbol{u}$ we can define the family of tight subsets of T as $\mathcal{F} = \{Z \subseteq T : \boldsymbol{u}(Z) = v(Z)\}$. Note that $\mathcal{F}$ is nonempty since $T \in \mathcal{F}$. Choose $\boldsymbol{u}$ so that $|\mathcal{F}|$ is as small as possible. We shall now sketch the two instances.

In the first instance, $\boldsymbol{u}(T)$ will be the utility function described above, and $\boldsymbol{u}(j) = v(T \cup j) - v(T)$. We prove that this is feasible. Moreover, we shall arrange the moneys in such a way that when we price the tight set $T \cup j$, we utilize all the money.

Let $\boldsymbol{u}(i) = v(T \cup i) - v(T)$. Note that now the set $S = T \cup i$ is tight. Indeed we prove this utility allocation is feasible. Moreover, the tight sets have the following properties. No set containing both i and j are tight. There is an element w which is in all the tight sets which contain either i or j. This property allows us to increase the utilities of i and j by suitably small amount, and decrease the utility of w by the same amount and still retain feasibility. These shall be the utilities in the second instance.

[3]Actually we should say that the valuation function restricted to S is submodular, but we shall abuse notation and call a set submodular to mean the same.

Note that the set $T \cup j$ remains tight. Also note that the set S is also tight. These are the sets that are priced. Thus every element other than j is priced by two sets while j is priced by one. This allows us to increase the moneys of all other players keeping that of j the same and finding suitable prices so as to make this allocation the equilibrium allocation. But the utility of j increases, and this shall contradict competition monotonicity.

We now state lemmas which claim the feasibility of the utility functions described above. Subsequently we describe the instances in greater detail.

Lemma 6.1 *Let $\boldsymbol{u}$ be the utility allocation which tightens T and keeps the number of tight subsets of T to a minimum. Extend it to $\boldsymbol{u}(i) = v(T \cup i) - v(T), \boldsymbol{u}(j) = v(T \cup j) - v(T)$. $\boldsymbol{u}$ still remains feasible. Moreover, no set containing both i and j becomes tight.*

Lemma 6.2 *Given the $\boldsymbol{u}$ described above, any tight set which contains either i or j also contains an element w so that $\boldsymbol{u}(w) > 0$.*

Proof of Theorem 2.4
Define

$$\epsilon_i := \min_{Z \subseteq T : Z \cup i \text{ not tight}} (v(Z \cup i) - \boldsymbol{u}(Z \cup i))$$
$$\epsilon_j := \min_{Z \subseteq T : Z \cup j \text{ not tight}} (v(Z \cup j) - \boldsymbol{u}(Z \cup j))$$
$$\epsilon_{ij} := \min_{Z \subseteq T} \frac{v(Z \cup i \cup j) - \boldsymbol{u}(Z \cup i \cup j)}{2}$$

Note $\epsilon_i, \epsilon_j > 0$, and by Lemma 6.1, $\epsilon_{ij} > 0$. Choose $\epsilon := \min(\epsilon_i, \epsilon_j, \epsilon_{ij}, \boldsymbol{u}(w)/2)$. Again $\epsilon > 0$ by Lemma 6.2. Let $\boldsymbol{u}'$ be a utility function defined as follows:
$\boldsymbol{u}'(i) = \boldsymbol{u}(i) + \epsilon$, $\boldsymbol{u}'(j) = \boldsymbol{u}(j) + \epsilon$, $\boldsymbol{u}'(w) = \boldsymbol{u}(w) - \epsilon$ and $\boldsymbol{u}'(k) = \boldsymbol{u}(k)$ otherwise.

Claim 6 *$\boldsymbol{u}'$ is a feasible allocation. Moreover, S and $T \cup j$ are tight under this allocation.*

Proof: Any subset of T is feasible since the utility of no player in T increases. By choice of ϵ, no set containing i or j becomes infeasible. Similarly no set of the form $Z \cup i$ or $Z \cup j$ which was not tight under $\boldsymbol{u}$ becomes infeasible. If a set containing i or j was tight, then it must contain w by Lemma 6.2, and since we decrease the utility of w, the set remains tight. Thus, in particular, S and $T \cup j$ remain tight. ■

We shall now construct the two instances. m_1, m_2 will be the moneys of players in the two instances.

Instance 1
$m_1(k) = \boldsymbol{u}(k)$ for all $k \in T$.
$m_1(j) = \boldsymbol{u}(j)$.
$m_1(k) = 0$ for all players not in $T \cup j$.
$\boldsymbol{u}$ restricted to $T \cup j$ is the feasible utility allocation. $T \cup j$ is a tight set.
$p(T \cup j) = 1$. Its easy to see that this is the equilibrium utility allocation.

Instance 2
Let $p' := \frac{\boldsymbol{u}(j)}{\boldsymbol{u}'(j)}$. The moneys in this instance are as follows:
$m_2(j) = m_1(j)$.
$m_2(k) = (2 + p')\boldsymbol{u}'(k)$ for all $k \in T$.
$m_2(i) = 2\boldsymbol{u}'(i)$.
The feasible utility allocation is $\boldsymbol{u}'$. By the above claim, S and $T \cup j$ are tight. We price these sets as follows: $p(S) = 2$, $p(T \cup j) = p'$.

Claim 7 *$\boldsymbol{u}'$ is an equilibrium utility allocation for Instance 2.*

Proof: We need to show $m_2(k) = rate(k) \cdot \boldsymbol{u}'(k)$ for all $k \in T \cup i \cup j$. $rate(j) = p' = \frac{\boldsymbol{u}(j)}{\boldsymbol{u}'(j)}$. Thus $rate(j) \cdot \boldsymbol{u}'(j) = \boldsymbol{u}(j) = m_1(j) = m_2(j)$.
$rate(k) = 2 + p'$ for all $k \in T$, and thus $rate(k) \cdot \boldsymbol{u}'(k) = m_2(k)$.
$rate(i) = 2$ and thus $rate(i) \cdot \boldsymbol{u}'(i) = m_2(i)$. ■

Claim 8 *$m_2(k) \geq m_1(k)$ for all k.*

Proof: Note that $m_2(j) = m_1(j)$. For all players $k \in T - w$, $m_2(k) = (2 + p')\boldsymbol{u}'(k) \geq 2\boldsymbol{u}(k) \geq m(k)$.
For player w we have, $m_2(w) = (2 + p')\boldsymbol{u}'(w) \geq 2(\boldsymbol{u}(w) - \epsilon) \geq 2\boldsymbol{u}(w)/2 = m_1(w)$, where the last inequality is because $\epsilon \leq \boldsymbol{u}(w)/2$. ■

This proves Theorem 2.4.

6.1 Proof of lemma 6.1

Recall $\boldsymbol{u}$ is a feasible utility allocation which tightens T. $\mathcal{F} = \{Z \subseteq T : \boldsymbol{u}(Z) = v(Z)\}$. $\boldsymbol{u}$ is chosen so as to minimize $|\mathcal{F}|$. $\boldsymbol{u}$ is extended to a utility allocation for $T \cup i \cup j$ by defining $\boldsymbol{u}(i) = v(T \cup i) - v(T)$ and $\boldsymbol{u}(j) = v(T \cup j) - v(T)$.

Lemma 6.3 *$\boldsymbol{u}$ is feasible over $T \cup i$ and $T \cup j$. Moreover, if $Z \cup i$ or $Z \cup j$ is tight then Z is tight.*

Proof: Let $Z \subset T$ be such that $\boldsymbol{u}(Z \cup i) \geq v(Z \cup i)$. That is, $\boldsymbol{u}(Z) + v(T \cup i) - v(T) \geq v(Z \cup i)$. Since $\boldsymbol{u}$ is feasible over T, we have $\boldsymbol{u}(Z) \leq v(Z)$, with equality iff $Z \in \mathcal{F}$. Thus we get, $v(T \cup i) - v(T) \geq v(Z \cup i) - v(Z)$ with the inequality being strict if either $\boldsymbol{u}(Z \cup i) > v(Z \cup i)$, or $Z \notin \mathcal{F}$. Since that cannot be by submodularity of $T \cup i$, we have that $\boldsymbol{u}(Z \cup i) = v(Z \cup i)$ and $Z \in \mathcal{F}$. The same result holds even for j. ■

Define $\mathcal{F}_i := \{Z \subseteq T : \boldsymbol{u}(Z \cup i) = v(Z \cup i)\}$ and $\mathcal{F}_j := \{Z \subseteq T : \boldsymbol{u}(Z \cup j) = v(Z \cup j)\}$. Note that $\mathcal{F}_i$ and $\mathcal{F}_j$ are nonempty as they contain T. Moreover, by Lemma 6.3, $\mathcal{F}_i, \mathcal{F}_j \subseteq \mathcal{F}$. Also let $\mathcal{F}_{ij} := \{Z \subseteq T : \boldsymbol{u}(Z \cup i \cup j) \geq v(Z \cup i \cup j)\}$. Note that Lemma 6.1 is equivalent to showing that $\mathcal{F}_{ij}$ is empty, which is done in Lemma 6.6. But before that we need to prove a few structural facts about $\mathcal{F}, \mathcal{F}_i$ and $\mathcal{F}_j$.

Lemma 6.4 *$\mathcal{F}, \mathcal{F}_i$ and $\mathcal{F}_j$ are closed under union and intersection.*

Proof: Let $Z_1, Z_2 \in \mathcal{F}$. Thus $\boldsymbol{u}(Z_1) = v(Z_1)$ and $\boldsymbol{u}(Z_2) = v(Z_2)$. Now since the utility function of a set is just the sum over its elements, we have

$$\boldsymbol{u}(Z_1 \cup Z_2) + \boldsymbol{u}(Z_1 \cap Z_2) = \boldsymbol{u}(Z_1) + \boldsymbol{u}(Z_2)$$

Since $\boldsymbol{u}$ is feasible we have $v(Z_1 \cup Z_2) + v(Z_1 \cap Z_2) \geq \boldsymbol{u}(Z_1 \cup Z_2) + \boldsymbol{u}(Z_1 \cap Z_2)$ with equality occuring only if $Z_1 \cup Z_2, Z_1 \cap Z_2$ are in $\mathcal{F}$. Moreover since T is submodular we get,

$$v(Z_1) + v(Z_2) \geq v(Z_1 \cup Z_2) + v(Z_1 \cap Z_2)$$

Thus we have

$$v(Z_1) + v(Z_2) \geq \boldsymbol{u}(Z_1) + \boldsymbol{u}(Z_2) = v(Z_1) + v(Z_2)$$

Thus all inequalities should be equality, and thus $Z_1 \cup Z_2, Z_1 \cap Z_2 \in \mathcal{F}$

We now give a similar proof for $\mathcal{F}_i$; we skip the proof for $\mathcal{F}_j$. Let $Z_1, Z_2 \in \mathcal{F}_i$. Thus we get

$$v(Z_1 \cup i) - v(Z_1) = v(Z_2 \cup i) - v(Z_2) = v(T \cup i) - v(T)$$

Note that $\boldsymbol{u}(Z_1 \cup Z_2 \cup i) + \boldsymbol{u}((Z_1 \cap Z_2) \cap i) = \boldsymbol{u}(Z_1 \cup i) + \boldsymbol{u}(Z_2 \cup i)$ as in the proof for $\mathcal{F}$. Thus we have by feasibility of $\boldsymbol{u}$ on $T \cup i$ and the fact that $Z_1, Z_2 \in \mathcal{F}_i$,

$$v(Z_1 \cup Z_2 \cup i) + v((Z_1 \cap Z_2) \cap i) \geq v(Z_1 \cup i) + v(Z_2 \cup i)$$

with equality holding iff $Z_1 \cup Z_2, Z_1 \cap Z_2 \in \mathcal{F}_i$. But then if the above inequality doesn't hold with equality we shall contradict the submodularity of $T \cup i$. ∎

Lemma 6.5 *$\mathcal{F}$ is closed under complementation.*

Proof: Choose $Z \in \mathcal{F}$. If $T \setminus Z \notin \mathcal{F}$, then we shall modify the utility allocation so as $Z \notin \mathcal{F}$ and no new set comes in $\mathcal{F}$ thus contradicting minimality of $\mathcal{F}$. Note that, $\boldsymbol{u}(T \setminus Z) = v(T) - v(Z)$. If $v(Z) = 0$, then we shall have $v(T \setminus Z) \geq \boldsymbol{u}(T \setminus Z) = v(T) \geq v(T \setminus Z)$, and thus $T \setminus Z \in \mathcal{F}$.

If an element of $T \setminus Z$ were to be in no tight set (apart from T), then we could increase its utility by a small enough amount so that no new set becomes tight. Decreasing the utility of any element in Z (note that there are elements in Z having positive utility) by the same amount would slacken Z and still keep T tight. Thus we may assume all elements of $T \setminus Z$ lie in tight sets which are strict subsets of T.

We shall now show that $T \setminus Z$ is a union of tight sets and is hence tight by the previous fact. This is obviously true if the tight sets containing the elements of $T \setminus Z$ are subsets of $T \setminus Z$. We may assume that is not the case. Thus there is an element $x \in T \setminus Z$, such that the smallest tight set A containing x intersects Z. Let $z \in A \cap Z$. Once again, if $\boldsymbol{u}(A \cap Z) = 0$, then $v(A \cap T \setminus Z) \geq \boldsymbol{u}(A \cap T \setminus Z) = v(A) \geq v(A \cap T \setminus Z)$ implying $A \cap T \setminus Z$ is tight contradicting the minimality of A. Thus we may assume $\boldsymbol{u}(z) > 0$.

Note that all tight sets containing x must be supersets of A, otherwise we would have a smaller tight set by intersecting that set with A. Now as above, increase the utility of x by a suitably small amount that the only sets which become infeasible are the tight sets containing x (and thus A too), and no new set becomes tight. Since A intersects Z, decrease the utility z by the same amount. Under the new utility, T remains tight, no new set became tight, and Z became untight. Hence we have a contradiction by minimality of $|\mathcal{F}|$. ∎

Lemma 6.6 *$\mathcal{F}_{ij}$ is empty, that is no set containing both i, j becomes tight.*

Proof: The idea is this. Suppose a set $Z \cup i \cup j$ became tight. Think of the marginal of $\{i, j\}$ on Z as adding marginal of i on Z and then j on $Z \cup i$. Since $Z \subset T$ and T is submodular, the marginal of i on Z is greater than that of i on T. Moreover the marginal of adding j on $Z \cup i$ is greater than the marginal of adding j on T, by choice of T. Thus marginal of adding i, j on Z is greater than $\boldsymbol{u}(i) + \boldsymbol{u}(j)$. Thus, $Z \cup i \cup j$ cannot be infeasible. In fact, if Z weren't tight to start with, $Z \cup i \cup j$ cannot be tight. If Z were tight, $T \setminus Z$ is tight, and then we show that $v(Z \cup i \cup j)$ is large by noting that $S \cup j = (Z \cup i \cup j) \cup T \setminus Z$, and $v(S \cup j)$ is large by definition. We now write the above more carefully.

Suppose $Z \in \mathcal{F}_{ij}$. Thus we have

$$v(Z \cup i \cup j) \leq \boldsymbol{u}(Z \cup i \cup\ j) = \boldsymbol{u}(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T)$$

We have two cases. Suppose $Z \in \mathcal{F}$. Then we have $v(Z) + v(T \setminus Z) = v(T)$. Moreover, we have $T \cup i \cup j = (Z \cup i \cup j) \cup (T \setminus Z)$. Thus we get

$$v(T \cup i \cup j) \leq v(Z \cup i \cup j) + v(T) - v(Z)$$

The two equations above imply

$$v(T \cup i \cup j) \leq v(T \cup i) + v(T \cup j) - v(T)$$

But this is false since we know $v(T \cup i \cup j) - v(T \cup i) > v(T \cup j) - v(T)$ by definition.
Thus assume $Z \notin \mathcal{F}$. Then we have $\boldsymbol{u}(Z) < v(Z)$ implying

$$\begin{aligned} v(Z \cup i \cup j) &< v(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T) \\ v(Z \cup i \cup j) - v(Z) &< v(T \cup i) - v(T) + v(T \cup j) - v(T) \\ v(Z \cup i \cup j) - v(Z \cup i) + v(Z \cup i) - v(Z) &< v(T \cup i) - v(T) + v(T \cup j) - v(T) \end{aligned}$$

Now we know $v(Z \cup i) - v(Z) \geq v(T \cup i) - v(T)$ by submodularity of $T \cup i$. Thus we get

$$v(Z \cup i \cup j) - v(Z \cup i) < v(T \cup j) - v(T)$$

which contradicts the choice of T as the set minimizing $v(T \cup j) - v(T)$ among all subsets of S. ∎

6.2 Proof of Lemma 6.2

In Lemma 6.7, we show there exist tight sets T_i which is in every set of $\mathcal{F}_i$, and T_j which is in every set of $\mathcal{F}_j$. In Lemma 6.8, we show $v(T_i \cap T_j) > 0$. Thus the intersection, which is also a tight set, is nonempty and there is an element $w \in T_i \cap T_j$ with $\boldsymbol{u}(w) > 0$. This proves Lemma 6.2.

Lemma 6.7 *There exists sets $T_i, T_j \in \mathcal{F}$ such that T_i is a subset of all sets in $\mathcal{F}_i$ and T_j is a subset of all sets in $\mathcal{F}_j$.*

Proof: We prove for $\mathcal{F}_j$, the proof for $\mathcal{F}_i$ is similar. Since $\mathcal{F}_j$ is closed under intersection by Lemma 6.4, it suffices to show that $\mathcal{F}_j$ contains no two disjoint sets. If $\mathcal{F}_j$ doesn't have any disjoint sets, then the minimal set of $\mathcal{F}_j$ would be T_j. For if not, then there is a set which doesn't contain T_j but intersects it, implying that the intersection is smaller than T_j.

Suppose A and B are disjoint sets in $\mathcal{F}_j$. But then since $A \cup j$ and $B \cup j$ are tight, their intersection j would also be tight from Fact 1. Thus $\boldsymbol{u}$ tightens both S and j implying it tightens $S \cup j$ because $v(S \cup j) \geq \boldsymbol{u}(S \cup j) = \boldsymbol{u}(S) + \boldsymbol{u}(j) = v(S) + v(j) \geq v(S \cup j)$. But $v(S \cup j) > v(S) + v(T \cup j) - v(T) = \boldsymbol{u}(S) + \boldsymbol{u}(j)$ by definition, and thus we have a contradiction. ■

Lemma 6.8 *T_i and T_j cannot be disjoint. In fact, $v(T_i \cap T_j) > 0$*

Proof: The idea is similar to the proof of last lemma. If T_i and T_j were disjoint, then $T_i \cup i$ and $T_j \cup j$ are disjoint tight sets. Moreover, since T_i, T_j are tight sets, the set $T \setminus (T_i \cup T_j)$ is also a tight set via Lemma 6.5. Since $S \cup j$ is a disjoint union of the tight sets $(T_i \cup i), (T_j \cup j), T \setminus (T_i \cup T_j)$, $S \cup j$ also should be tight as in the last proof. This is a contradiction as above. We now write this carefully.

Since T_i and T_j are in $\mathcal{F}$, we have $v(T_i \cup T_j) = v(T_i) + v(T_j) - v(T_i \cap T_j)$. Thus if $v(T_i \cap T_j) = 0$, then we have

$$v(T_i \cup T_j) = v(T_i) + v(T_j)$$

Also, since $T_i \cup T_j \in \mathcal{F}$, we have by Lemma 6.5, that $T \setminus (T_i \cup T_j) \in \mathcal{F}$. Thus we get

$$v(T_i) + v(T_j) + v(T \setminus (T_i \cup T_j) = v(T)$$

Now using the fact that $T_i \in \mathcal{F}_i$ and $T_j \in \mathcal{F}_j$, we get

$$\begin{aligned} v(T_i \cup i) + v(T_j \cup j) &= v(T_i) + v(T \cup i) - v(T) + v(T_j) + v(T \cup j) - v(T) \\ &= v(T \cup i) + v(T \cup j) - v(T) - v(T \setminus (T_i \cup T_j)) \end{aligned}$$

Thus we get

$$v(T_i \cup i) + v(T_j \cup j) + v(T \setminus (T_i \cup T_j)) = v(T \cup i) + v(T \cup j) - v(T)$$

Note that $(T_i \cup i) \cup (T_j \cup j) \cup T \setminus (T_i \cup T_j) = T \cup i \cup j$. Thus by covering property we have LHS of the last equation is at least $v(T \cup i \cup j)$. But the RHS is strictly less than $v(T \cup i \cup j)$, for that's the way $T \cup i \cup j$ is defined. ■

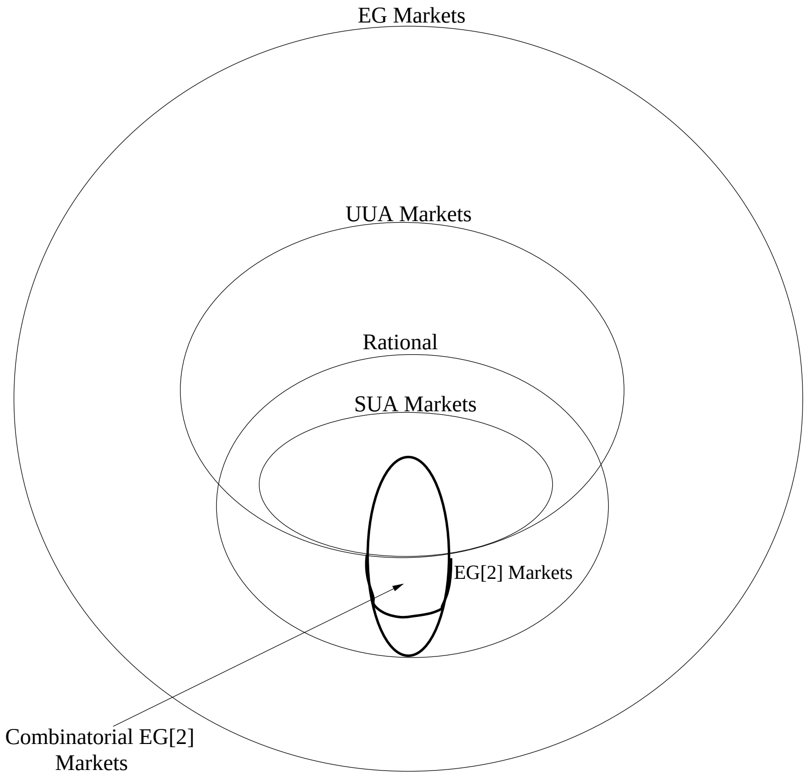

Figure 3: Venn diagram of markets. Sets in bold are introduced in current paper.

7 Discussion

Figure 3 gives a Venn diagram of markets defined by [JV], together with the two classes identified in this paper shown in bold. The latter two classes are EG[2] and Combinatorial EG[2] markets.

Our resolution of three open problems from [JV] raises the following new questions:

1. We have shown in this paper that competition monotonicity characterizes SUA markets within the class of UUA markets and that all EG[2] markets satisfy competition monotonicity. Is it possible to characterize the set of markets satisfying competition monotonicity within the class of EG markets? Perhaps a suitable generalization of the notion of a polymatroid function may be useful for this.

2. We have shown that for an EG[2] market, whenever the polytope containing the set of feasible utilities of the two agents can be described via a combinatorial LP, the market admits a strongly polynomial algorithm. The restriction of Fisher's linear utilities market to two agents is an EG[2] market which does not admit a combinatorial LP; however it does admit a strongly polynomial algorithm, since Deng, Papadimitriou and Safra [DPS02] have shown that Fisher's linear utilities market on a bounded number of agents always has a strongly polynomial algorithm. Do all markets in EG[2] admit strongly polynomial algorithms? Alternatively, can some evidence be given to establish the contrary?

References

[ACLY00] R. Ahlswede, N. Cai, S. Y. R. Li, and R. W. Yeung. Network information flow. *IEEE Trans. on Information Theory*, 46:1204–1216, 2000.

[BS00] W. C. Brainard and H. E. Scarf. How to compute equilibrium prices in 1891. *Cowles Foundation Discussion Paper*, (1270), 2000.

[CKLL05] J-Y. Choi, K. Koo, J. S. Lee, and S. H. Low. Global stability of fast tcp in single-link single-source network. *IEEE CDC*, December 2005.

[CV04] B. Codenotti and K. Varadarajan. Efficient computation of equilibrium prices for markets with Leontief utilities. In *Proc. 31st International Colloquium on Automata, Languages, and Programming*, Lecture Notes in Computer Science, 2004.

[DJP+94] E. Dahlhaus, D.S. Johnson, C.H. Papadimitriou, P.D. Seymour, and M. Yannakakis. The complexity of multiterminal cuts. *SIAM Journal on Computing*, 23:864–894, 1994.

[DPS02] Xiaotie Deng, Christos Papadimitriou, and Shmuel Safra. On the complexity of equilibria. In *Proceedings of ACM Symposium on Theory of Computing*, 2002.

[EFS56] P. Elias, A. Feinstein, and C. E. Shannon. Note on maximum flow through a network. *IRE Transactions on Information Theory IT-2*, pages 117–199, 1956.

[EG59] E. Eisenberg and D. Gale. Consensus of subjective probabilities: the Pari-Mutuel method. *The Annals of Mathematical Statistics*, 30:165–168, 1959.

[Eis61] E. Eisenberg. Aggregation of utility functions. *Management Sciences*, 7:337–350, 1961.

[FF56] L. R. Jr. Ford and D. R. Fulkerson. Maximal flow through a network. *Canadian Journal of Mathematics*, pages 99–404, 1956.

[GJTV05] D. Garg, K. Jain, K. Talwar, and V. Vazirani. A primal-dual algorithm for computing Fisher equilibrium in absence of gross substitutability property. In *Proceedings of wine*, 2005.

[GLS93] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Second edition. Springer-Verlag, Berlin, 1993.

[GVY94] N. Garg, V.V. Vazirani, and M. Yannakakis. Multiway cuts in directed and node weighted graphs. In *Proc. 21st International Colloquium on Automata, Languages, and Programming*, volume 820 of *Lecture Notes in Computer Science*, pages 487–498. Springer-Verlag, Berlin, 1994.

[Hu63] T.C. Hu. Multicommodity network flows. *Operations Research*, 14:344–360, 1963.

[JV] K. Jain and V. V. Vazirani. Combinatorial solvability in resource allocation markets, with applications to efficiency and fairness. Manuscript, 2005 available from `http://www.cc.gatech.edu/~vazirani/`.

[JVY05] K. Jain, V. V. Vazirani, and Y. Ye. Market equilibrium for homothetic, quasi-concave utilities and economies of scale in productio n. In *Proceedings, SODA*, 2005.

[Kel97] F. P. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8:33–37, 1997.

[Kel03] F. P. Kelly. Fariness and stability of end-to-end congestion control. *European Journal of Control*, 9:159–176, 2003.

[KMT98] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan. Rate control in communication networks. *Journal of Operational Research Society*, 49:237–252, 1998.

[KV] F. P. Kelly and V. V. Vazirani. Rate control as a market equilibrium. Manuscript, 2002.

[Nem05] A. Nemirovski. Lectures on modern convex optimization. 2005.

[NW61] C. S. J. A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. London Math Soc*, 36:445–450, 1961.

[Tar86] E. Tardos. A strongly polynomial algorithm to solve combinatorial linear programs. *Oper. Res.*, 34(2):250–256, 1986.

[Tut61] W. T. Tutte. On the problem of decomposing a graph into n connected factors. *J. London Math Soc*, 36:221–230, 1961.

[TWHL05] K. A. Tang, J. Wang, S. Hegde, and S. H. Low. Equilibrium and fairness of networks shared by tcp reno and fast. *Telecommunications Systems special issue on High Speed Transport Protocols*, 2005.

[WJLH07] D. X. Wei, C. Jin, S. H. Low, and S. Hegde. Fast TCP: motivation, architecture, algorithms, performance. *IEEE/ACM Transactions on Networking*, (to appear 2007).

[WLLD05] J. Wang, L. Li, S. H. Low, and J. C. Doyle. Cross-layer optimization in TCP/IP networks. *IEEE/ACM Trans. on Networking*, 13:582–268, 2005.

[WWCL06] J. Wang, D. X. Wei, J-Y. Choi, and S. H. Low. Modeling and stability of fast tcp. *IMA Volumes in Mathematics and its Applications, Volume 143: Wireless Communications*, 2006.

[Ye] Y. Ye. Note on exchange market equilibria with leontief's utility: Freedom of pricing leads to rationality. In *WINE 2005.*

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