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On the Approximation and Smoothed Complexity of Leontief Market Equilibria

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Abstract

We show that the problem of finding an ϵ-approximate Nash equilibrium of an $n \times n$ two-person games can be reduced to the computation of an $(\epsilon/n)^2$-approximate market equilibrium of a Leontief economy. Together with a recent result of Chen, Deng and Teng, this polynomial reduction implies that the Leontief market exchange problem does not have a fully polynomial-time approximation scheme, that is, there is no algorithm that can compute an ϵ-approximate market equilibrium in time polynomial in m, n, and $1/\epsilon$, unless $\mathbf{PPAD} \subseteq \mathbf{P}$, We also extend the analysis of our reduction to show, unless $\mathbf{PPAD} \subseteq \mathbf{RP}$, that the smoothed complexity of the Scarf's general fixed-point approximation algorithm (when applying to solve the approximate Leontief market exchange problem) or of any algorithm for computing an approximate market equilibrium of Leontief economies is not polynomial in n and $1/\sigma$, under Gaussian or uniform perturbations with magnitude σ.

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1 Introduction

A *Leontief economy* [23, 28] with m divisible goods (or commodities) and n traders is specified by a pair of non-negative $m \times n$ matrices[1] $\mathbf{E} = (e_{i,j})$ and $\mathbf{D} = (d_{i,j})$. After trading, the exchange of goods can also be expressed by a non-negative $m \times n$ matrix $\mathbf{X} = (x_{i,j})$.

- The matrix $\mathbf{E}$ is the *endowment matrix* of the traders, that is, $e_{i,j}$ is the amount of commodity i that trader j initially has. We can assume, without loss of generality, that there is one unit of each type of good. With this assumption, the sum of every row of $\mathbf{E}$ is equal to 1.

- The matrix $\mathbf{D}$ is the *demand matrix* or the *utility matrix*. It defines n utility functions $u_1, \ldots, u_n$, one for each trader. For each exchange $\mathbf{X}$ of goods, let $\mathbf{x}_j$ be the j^{th} column of $\mathbf{X}$, then
$$u_j(\mathbf{x}_j) = \min_i \left\{ \frac{x_{i,j}}{d_{i,j}} \right\}$$
is the *Leontief utility* of trader j.

The initial utility of trader j is $u_j(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the j^{th} column of $\mathbf{E}$. The *individual objective* of each trader is to maximize his or her utility. However, the utilities that these traders can achieve depend on the initial endowments, the individual utilities, and potentially, the (complex) process that they perform their exchanges.

In Walras' pure view of economics [26], the individual objectives of traders and their initial endowments enable the market to establish a price vector $\mathbf{p}$ of the goods in the market. Then the whole exchange can be conceptually characterized as: the traders sell their endowments – to obtain money or budgets – and individually optimize their objectives by buying the bundles of goods that maximize their utilities.

By selling the initial endowment $\mathbf{e}_j$, trader j obtains a budget of $\langle \mathbf{e}_j | \mathbf{p} \rangle$ amount, where $\langle \mathbf{e}_j | \mathbf{p} \rangle$ denotes the dot-product of these two vectors. The *optimal bundle* for u_j is a solution to the following mathematical program:

$$\max u_j(\mathbf{x}_j) \quad \text{subject to } \langle \mathbf{x}_j | \mathbf{p} \rangle \leq \langle \mathbf{e}_j | \mathbf{p} \rangle . \tag{1}$$

A solution to Equation (1) is referred to as an *optimal demand* of trader j under prices $\mathbf{p}$. The price vector $\mathbf{p}$ is a *Walrasian equilibrium*, an *Arrow-Debreu equilibrium*, or simply an *equilibrium* of the Leontief economy $(\mathbf{E}, \mathbf{D})$ if there exists optimal solution $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ to Equation (1) such that

$$\sum_j \mathbf{x}_j \leq \vec{\mathbf{1}}, \tag{2}$$

where $\vec{\mathbf{1}}$ is the m-dimensional column vector with all ones. This last constraint states that the traders' optimal demands can be met by the market.

In other words, an equilibrium price vector essentially allows each trader to make individual decision without considering others' utilities nor how they achieve their objectives.

[1] See Section 2 for the basic notations used in this paper.

The computation of a market equilibrium is a fundamental problem in modern economics [23] as Walrasian equilibria might provide useful information for the prediction of market trends, in the decision for future investments, and in the development of economic policies. So a central complexity question in Leontief market exchange problem is:

Question 1 (Polynomial Leontief?). *Is the problem of computing an equilibrium of a Leontief economy in* **P***?*

So far no polynomial-time algorithm has been found for this problem. In practice, one may be willing to relax the condition of equilibria and considers the computation of approximate market equilibria. For example, Scarf [22] developed a general algorithm for computing approximate fixed points and equilibria. Recently, Deng, Papadimitriou, Safra [8] proposed a notion of an approximate market equilibrium as a price vector that allows each trader to independently and approximately optimize her utilities.

For any $\epsilon \geq 0$, let $\mathrm{OPT}_{j,\epsilon}(\mathbf{p}, \mathbf{e}_j)$ be the set of ϵ-approximately optimal vectors in $\mathbb{R}_+^m$ for Equation (1), that is, the set of all $\mathbf{x}_j$ satisfying

$$\begin{aligned} \langle \mathbf{x}_j | \mathbf{p} \rangle &\leq (1+\epsilon) \langle \mathbf{e}_j | \mathbf{p} \rangle, \quad \text{and} \\ u_j(\mathbf{x}_j) &\geq (1-\epsilon) u(\mathbf{x}'_j), \quad \forall \mathbf{x}'_j : \langle \mathbf{x}'_j | \mathbf{p} \rangle \leq \langle \mathbf{e}_j | \mathbf{p} \rangle \end{aligned}$$

Then $\mathbf{p}$ is an *ϵ-approximate equilibrium*[2] of a Leontief economy $(\mathbf{E}, \mathbf{D})$ if there exists ϵ-approximately optimal solution $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ with $\mathbf{x}_j \in \mathrm{OPT}_{j,\epsilon}(\mathbf{p}, \mathbf{e}_j)$ such that

$$\sum_j \mathbf{x}_j \leq (1+\epsilon)\vec{\mathbf{1}}. \tag{3}$$

Question 2 (Fully Polynomial Approximate Leontief?). *Can an ϵ-approximate equilibrium of a Leontief economy with m goods and n traders be computed in time polynomial in m, n, and $1/\epsilon$?*

The combination of two recent results greatly dashed the hope for a positive answer to Question 1. Codenotti, Saberi, Varadarajan, and Ye [5] gave a polynomial-time reduction from two-person games to a special case of the Leontief economy. In a remarkable breakthrough, Chen and Deng [3] subsequently proved that the problem of finding a Nash equilibrium of a two-person game is **PPAD**-complete[3].

[2] One can of course define a stronger notion of approximate equilibria: $\mathbf{p}$ is an *ϵ-strictly approximate equilibrium* if there exists $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ such that

$$\begin{aligned} \langle \mathbf{x}_j | \mathbf{p} \rangle &\leq \langle \mathbf{e}_j | \mathbf{p} \rangle, \\ u_j(\mathbf{x}_j) &\geq (1-\epsilon) u(\mathbf{x}'_j), \quad \forall \mathbf{x}'_j : \langle \mathbf{x}'_j | \mathbf{p} \rangle \leq \langle \mathbf{e}_j | \mathbf{p} \rangle, \quad \textit{and} \\ \sum_j \mathbf{x}_j &\leq \vec{\mathbf{1}}. \end{aligned}$$

It is easy to show that every $(\epsilon/2)$-approximate Walrasian equilibrium $\mathbf{p}$ is an ϵ-strictly approximate Walrasian equilibrium, by dividing its associated exchange $\mathbf{X}$ by a factor of $(1-\epsilon/2)$. Therefore, in the remainder of this paper, we will, without loss of generality, stay with the less restrictive notion of approximation equilibria.

[3] We refer the readers who are not familiar with the complexity class **PPAD** to the paper by Papadimitriou [21] and also to the subsequent papers on the **PPAD**-completeness of normal games [6, 2, 7, 3, 4]. This class includes several important search problems such as various discrete fixed-point problems and the problem of finding a Nash equilibrium of r-person games for any fixed r.

In this paper, we show that there is no fully polynomial-time approximation scheme for Leontief economies unless **PPAD** $\subseteq$ **P**. Hence it is unlikely that Question 2 has a positive answer. By analyzing the numerical properties of the reduction of Condenotti, Saberi, Varadarajan, and Ye [5], we prove that the problem of finding an ϵ-approximate Nash equilibrium of an $n \times n$ two-person games can be reduced to the computation of an $(\epsilon/n)^2$-approximate equilibrium of a Leontief economy with $2n$ goods and $2n$ traders. This polynomial relationship between approximate Nash equilibria of bimatrix games and approximate Walrasian equilibria of Leontief economies is significant because it enables us to apply the recent result of Chen, Deng, and Teng [4] to show that finding an approximate market equilibrium with only $O(\log n)$-bits of precision is as hard as finding an exact market equilibrium, which in turn is as hard as finding a Nash equilibrium of a two-person game or a discrete Brouwer fixed point in the most general settings.

We also consider the smoothed complexity of the Leontief market exchange problem. In the smoothed model introduced by Spielman and Teng [24], an algorithm receives and solves a perturbed instances. The smoothed complexity of an algorithm is the maximum over its inputs of the expected running time of the algorithm under slight random perturbations of that input. The smoothed complexity is then measured as a function of both the input length and the magnitude σ of the perturbations. An algorithm has *smoothed polynomial-time complexity* if its smoothed measure is polynomial in n, the problem size, and in $1/\sigma$ [24, 25].

In the smoothed model for Leontief economies, we start with a pair of $m \times n$ matrices $\bar{\mathbf{E}} = (\bar{e}_{i,j})$ and $\bar{\mathbf{D}} = \left(\bar{d}_{i,j}\right)$ with $0 \leq \bar{d}_{i,j} \leq 2$ and $0 \leq \bar{e}_{i,j} \leq 1$. Suppose $\mathbf{E} = (e_{i,j})$ and $\mathbf{D} = (d_{i,j})$ are perturbations of $\bar{\mathbf{E}}$ and $\bar{\mathbf{D}}$ where $e_{i,j} = \max(0, \bar{e}_{i,j} + r^E_{i,j})$ and $d_{i,j} = \max(0, \bar{d}_{i,j} + r^D_{i,j})$, with $r^E_{i,j}$ and $r^D_{i,j}$ being chosen independently and uniformly from $[-\sigma, \sigma]$. The *smoothed complexity* of the Leontief exchange problem $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ is then measured by the expected complexity of finding an equilibrium of the Leontief economy $(\mathbf{E}, \mathbf{D})$.

The following has been an open question in the smoothed analysis of algorithms.

Question 3 (Smoothed Polynomial Leontief?). *Can an equilibrium of a Leontief economy be computed in smoothed time polynomial in m, n, and $1/\sigma$?*

Can an ϵ-equilibrium of a Leontief economy be computed in smoothed time polynomial in m, n, $1/\epsilon$ and $1/\sigma$?

A concrete open question has been whether the smoothed complexity of the classic Scarf's general fixed-point approximation algorithm [22] is polynomial for solving the Leontief market exchange problem.

By refining our analysis of the reduction from the two-person games to Leontief economies, we show it is unlikely that Scarf's algorithm has polynomial smoothed complexity for computing an approximate equilibrium of Leontief economies. In particular, we prove that, unless **PPAD** $\subseteq$ **RP**, the problem of finding an (approximate) equilibrium of a Leontief economy is not in smoothed polynomial time.

2 Notations

We will use bold lower-case Roman letters such as $\mathbf{x}$, $\mathbf{a}$, $\mathbf{b}_j$ to denote vectors. Whenever a vector, say $\mathbf{a} \in \mathbb{R}^n$ is present, its components will be denoted by lower-case Roman letters with subscripts, such as $a_1, \ldots, a_n$. Matrices are denoted by bold upper-case Roman letters such as $\mathbf{A}$ and scalars are usually denoted by lower-case Roman letters. The $(i,j)^{th}$ entry of a matrix $\mathbf{A}$ is denoted by $a_{i,j}$. We use $\mathbf{a}_i$ to denote the i^{th} column of $\mathbf{A}$.

We now enumerate some other notations that are used in this paper.

- $\mathbb{R}^m_+$: the set of m-dimensional vectors with non-negative real entries.
- $\mathbb{R}^{m\times n}_{[a:b]}$: the set of all $m \times n$ matrices with real entries between a and b. For example, $\mathbb{R}^{m\times n}_{[0:2]}$ is the set of non-negative matrices with entries at most 2.
- $\mathbb{P}^n$: the set of all vectors $\mathbf{x}$ in n dimensions such that $\sum_{i=1}^n x_i = 1$ and $x_i \geq 0$ for all $1 \leq i \leq n$.
- $\langle \mathbf{a}|\mathbf{b}\rangle$: the dot-product of two vectors in the same dimension.
- $\|\mathbf{x}\|_p$: the p-norm of vector $\mathbf{x}$, that is, $\left(\sum |x_i^p|\right)^{1/p}$ and $\|\mathbf{x}\|_\infty = \max_i |x_i|$.

3 Approximate Nash Equilibria of Two-Person Games

The non-zero-sum *two-person game* or the *bimatrix game* is a non-cooperative game between two players [18, 15, 16], the *row player* and the *column player*. If the row player has m pure strategies and the column player has n pure strategies, then their payoffs are given by a pair of $m \times n$ matrices $(\mathbf{A}, \mathbf{B})$.

A mixed row strategy is a vector $\mathbf{x} \in \mathbb{P}^m$ and a mixed column strategy is a vector $\mathbf{y} \in \mathbb{P}^n$. The expected payoffs to these two players are respectively $\mathbf{x}^\top \mathbf{A}\mathbf{y}$ and $\mathbf{x}^\top \mathbf{B}\mathbf{y}$. A *Nash equilibrium* is then a pair of vectors $(\mathbf{x}^* \in \mathbb{P}^m, \mathbf{y}^* \in \mathbb{P}^n)$ such that for all pairs of vectors $\mathbf{x} \in \mathbb{P}^m$ and $\mathbf{y} \in \mathbb{P}^n$,

$$(\mathbf{x}^*)^\top \mathbf{A}\mathbf{y}^* \geq \mathbf{x}^\top \mathbf{A}\mathbf{y}^* \quad \text{and} \quad (\mathbf{x}^*)^\top \mathbf{B}\mathbf{y}^* \geq (\mathbf{x}^*)^\top \mathbf{B}\mathbf{y}.$$

Every two-person game has at least one Nash equilibrium [18]. But in a recent breakthrough, Chen and Deng [3] proved that the problem of computing a Nash equilibrium of a two-person game is **PPAD**-complete.

One can relax the condition of Nash equilibria and considers approximate Nash equilibria. There are two possible notions of approximation.

Definition 3.1 (Approximate Nash equilibria). *An* ϵ-approximate Nash equilibrium *of game* $(\mathbf{A}, \mathbf{B})$ *is a pair of mixed strategies* $(\mathbf{x}^*, \mathbf{y}^*)$*, such that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{P}^n$*,*

$$(\mathbf{x}^*)^\top \mathbf{A}\mathbf{y}^* \geq \mathbf{x}^\top \mathbf{A}\mathbf{y}^* - \epsilon \quad \textit{and} \quad (\mathbf{x}^*)^\top \mathbf{B}\mathbf{y}^* \geq (\mathbf{x}^*)^\top \mathbf{B}\mathbf{y} - \epsilon.$$

Definition 3.2 (Relatively Approximate Nash equilibria). *An* ϵ-relatively-approximate Nash equilibrium *of game* $(\mathbf{A}, \mathbf{B})$ *is a pair of mixed strategies* $(\mathbf{x}^*, \mathbf{y}^*)$*, such that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{P}^n$*,*

$$(\mathbf{x}^*)^\top \mathbf{A}\mathbf{y}^* \geq (1-\epsilon)\mathbf{x}^\top \mathbf{A}\mathbf{y}^* \quad \textit{and} \quad (\mathbf{x}^*)^\top \mathbf{B}\mathbf{y}^* \geq (1-\epsilon)(\mathbf{x}^*)^\top \mathbf{B}\mathbf{y}.$$

Note that the Nash equilibria and the relatively-approximate Nash equilibria of a two-person game $(\mathbf{A}, \mathbf{B})$ are invariant under positive scalings, i.e., the bimatrix game $(c_1\mathbf{A}, c_2\mathbf{B})$ has the same set of Nash equilibria and relatively-approximate Nash equilibria as the bimatrix game $(\mathbf{A}, \mathbf{B})$, as long as $c_1, c_2 > 0$. However, each ϵ-approximate Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $(\mathbf{A}, \mathbf{B})$ becomes a $c{\cdot}\epsilon$-approximate Nash equilibrium of the bimatrix game $(c\mathbf{A}, c\mathbf{B})$ for $c > 0$.

On the other hand, Nash equilibria and approximate Nash equilibria are invariant under shifting, that is, for any constants c_1 and c_2, the bimatrix game $(c_1 + \mathbf{A}, c_2 + \mathbf{B})$ has the same set of Nash equilibria and approximate Nash equilibria as the bimatrix game $(\mathbf{A}, \mathbf{B})$. However, shifting may not preserve the relatively-approximate Nash equilibria.

Thus, we often normalize the matrices $\mathbf{A}$ and $\mathbf{B}$ so that all their entries are between 0 and 1, or between -1 and 1, in order to study the complexity of approximate Nash equilibria [17, 4].

Recently, Chen, Deng, and Teng [4] proved the following result.

Theorem 3.3 (Chen-Deng-Teng). *The problem of computing an $1/n^6$-approximate Nash equilibrium of a normalized $n \times n$ two-person game is* **PPAD***-complete.*

As pointed out in [4], the 6 in the exponent of the above theorem can be replaced by any positive constant. One can easily derive the following corollary.

Corollary 3.4 (Relative approximation is also hard). *It remains* **PPAD***-complete to compute a $1/n^{\Theta(1)}$-relatively-approximate Nash equilibrium of an $n \times n$ two-person game.*

4 Leontief Market Equilibria: Approximation and Smoothed Complexity

In this and the next sections, we analyze a reduction π that transforms a two-person game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ into a Leontief economy $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ such that from each $(\epsilon/n)^2$-approximate Walrasian equilibrium of $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ we can construct an ϵ-relatively-approximate Nash equilibrium of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.

We will also consider the smoothed complexity of Leontief economies. To establish a hardness result for computing an (approximate) market equilibrium in the smoothed model, we will examine the relationship of Walrasian equilibria of a perturbed instance $(\mathbf{E}, \mathbf{D})$ of $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ and approximate Nash equilibria of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$. In particular, we show that if the magnitude of the perturbation is σ, then we can construct an $(\epsilon + n^{1.5}\sqrt{\sigma})$-relatively-approximate Nash equilibrium of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ from each $(\epsilon/n)^2$-approximate equilibria of $(\mathbf{E}, \mathbf{D})$.

Because the analysis needed for approximate Walrasian equilibria is a special case (with $\sigma = 0$) of the analysis for the smoothed model, we will write only one proof for this general case, and present it in the next section.

In this section, we discuss the reduction from two-person games to Leontief economies, connect the smoothed model with approximation, and present the main theorems of this paper.

4.1 Approximate Market Equilibria of Leontief Economy

We first introduce a form of approximate market equilibria that is easier for the analysis of the reduction between game equilibria and market equilibria.

Let $\mathbf{D}$ be the demand matrix and $\mathbf{E}$ be the endowment matrix of a Leontief economy with m goods and n traders. Given a price vector $\mathbf{p}$, trader j can obtain a budget of $\langle \mathbf{e}_j|\mathbf{p}\rangle$ by selling the endowment. By a simple variational argument, one can show that the optimal demand $\mathbf{x}_j$ with budget $\langle \mathbf{e}_j|\mathbf{p}\rangle$ satisfies $x_{i,j}/d_{i,j} = x_{i',j}/d_{i',j}$ for all i and i' with $d_{i,j} > 0$ and $d_{i',j} > 0$. Thus, under the price vector $\mathbf{p}$, the maximum utility that trader j can achieve is 0 if $\langle \mathbf{e}_j|\mathbf{p}\rangle = 0$, and $\langle \mathbf{e}_j|\mathbf{p}\rangle / \langle \mathbf{d}_j|\mathbf{p}\rangle$ otherwise. Moreover, in the latter case, $x_{i,j} = d_{i,j}\left(\langle \mathbf{e}_j|\mathbf{p}\rangle / \langle \mathbf{d}_j|\mathbf{p}\rangle\right)$. Let $\mathbf{u} = (u_1, \ldots, u_n)$ denote the vector of utilities of the traders. Then $\mathbf{p}$ is a Walrasian equilibrium price if

$$\mathbf{p} \geq \vec{\mathbf{0}}, \quad u_i = \frac{\langle \mathbf{e}_i|\mathbf{p}\rangle}{\langle \mathbf{d}_i|\mathbf{p}\rangle}, \quad \text{and} \quad \mathbf{D}\mathbf{u} \leq \vec{\mathbf{1}}. \tag{4}$$

In the remainder of this paper, we will refer to a pair of vectors $(\mathbf{u}, \mathbf{p})$ that satisfies Equation (4) as an *equilibrium* of the Leontief economy $(\mathbf{E}, \mathbf{D})$. Then, an *ϵ-approximate equilibrium* of the Leontief economy $(\mathbf{E}, \mathbf{D})$ is a pair of utility and price vectors $(\mathbf{u}, \mathbf{p})$ satisfying:

$$\begin{cases} u_i & \geq & (1-\epsilon)\langle \mathbf{e}_i|\mathbf{p}\rangle / \langle \mathbf{d}_i|\mathbf{p}\rangle, \forall i. & \text{— All traders are approximately satisfied.} \\ u_i & \leq & (1+\epsilon)\langle \mathbf{e}_i|\mathbf{p}\rangle / \langle \mathbf{d}_i|\mathbf{p}\rangle, \forall i. & \text{— Budget constraints approximately hold.} \\ \mathbf{D}\mathbf{u} & \leq & (1+\epsilon)\cdot \vec{\mathbf{1}}. & \text{— The demands approximately meet the supply.} \end{cases}$$

Note that if $(\mathbf{u}, \mathbf{p})$ is an equilibrium of $(\mathbf{E}, \mathbf{D})$, so is $(\mathbf{u}, \alpha\mathbf{p})$ for every $\alpha > 0$. Similarly, if $(\mathbf{u}, \mathbf{p})$ is an ϵ-equilibrium of $(\mathbf{E}, \mathbf{D})$, so is $(\mathbf{u}, \alpha\mathbf{p})$ for every $\alpha > 0$. Thus, we can normalize $\mathbf{p}$ so that $\|\mathbf{p}\|_1 = 1$. In addition, for approximate equilibria, we assume without loss of generality that $\mathbf{u}$ and $\mathbf{p}$ are strictly positive to avoid division-by-zero since a small perturbation of an approximate equilibrium is still a good approximate equilibrium.

4.2 Reduction from NASH to LEONTIEF

Let $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ be a two-person game in which each player has n strategies. Below we assume $\bar{\mathbf{A}} \in \mathbb{R}^{n\times n}_{[1,2]}$ and $\bar{\mathbf{B}} \in \mathbb{R}^{n\times n}_{[1,2]}$. We use the reduction introduced by Codenotti, Saberi, Varadarajan, and Ye [5] to map a bimatrix game to a Leontief economy. This reduction constructs a Leontief economy with $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ where the endowment matrix is simply $\bar{\mathbf{E}} = \mathbf{I}_{2n}$, the $(2n) \times (2n)$ identity matrix and the utility matrix is given by

$$\bar{\mathbf{D}} = \begin{pmatrix} 0 & \bar{\mathbf{A}} \\ \bar{\mathbf{B}} & 0 \end{pmatrix}.$$

$(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ is a special form of Leontief exchange economies [28, 5]. It has $2n$ goods and $2n$ traders. The j^{th} trader comes to the market with one unit of the j^{th}-good. In addition, the traders are divided into two groups $\mathcal{M} = \{1, 2, ..., n\}$ and $\mathcal{N} = \{n+1, ..., 2n\}$. Traders in $\mathcal{M}$ only interests in the goods associated with traders in $\mathcal{N}$ and vice versa.

Codenotti *et al* [5] prove that there is a one-to-one correspondence between Nash equilibria of the two person game $(\mathbf{A}, \mathbf{B})$ and market equilibria of Leontief economy $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$. It thus follows from the theorem of Nash [19, 18], that the Leontief economy $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ has at least one equilibrium.

We will prove the following extension of their result in the next section.

Lemma 4.1 (Approximation of Games and Markets). *For any bimatrix game* $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$*, let* $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$*. Let* $(\mathbf{u}, \mathbf{w})$ *be an* ϵ*-approximate equilibrium of* $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ *and assume* $\mathbf{u} = (\mathbf{x}^\top, \mathbf{y}^\top)^\top$ *and* $\mathbf{w} = (\mathbf{p}^\top, \mathbf{q}^\top)^\top$*. Then,* $(\mathbf{x}, \mathbf{y})$ *is an* $O(n\sqrt{\epsilon})$*-relatively-approximate Nash equilibrium for* $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$*.*

Lemma 4.1 enables us to prove one of the main results of this paper.

Theorem 4.2 (Market Approximation is Likely Hard). *The problem of finding a* $1/n^{\Theta(1)}$*-approximate equilibrium of a Leontief economy with* n *goods and* n *traders is* **PPAD***-hard.*

Therefore, if **PPAD** *is not in* **P***, then there is no algorithm for finding an* ϵ*-equilibrium of Leontief economies in time polynomial in* n *and* $1/\epsilon$*.*

Proof. Apply Lemma 4.1 with $\epsilon = n^{-h}$ for a sufficiently large constant h and Corollary 3.4. □

4.3 The Smoothed Complexity of Market Equilibria

In the smoothed analysis of the Leontief market exchange problem, we assume that entries of the endowment and utility matrices is subject to slight random perturbations.

Consider an economy with $\left(\bar{\mathbf{E}} \in \mathbb{R}^{n\times n}_{[0,2]}, \bar{\mathbf{D}} \in \mathbb{R}^{n\times n}_{[0,1]}\right)$. For a $\sigma > 0$, a perturbed economy is defined by a pair of random matrices $\left(\mathbf{\Delta}^E, \mathbf{\Delta}^D\right)$ where $\delta^E_{i,j}$ and $\delta^D_{i,j}$ are independent random variables of magnitude σ. The common two perturbation models are the uniform perturbation and Gaussian perturbation. In the *uniform perturbation* with magnitude σ, a random variable is chosen uniformly from the interval $[-\sigma, \sigma]$. In the *Gaussian perturbation* with variance σ^2, a random variable δ is chosen with density

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\delta^2/2\sigma^2}.$$

Let $\mathbf{D} = \max\left(\bar{\mathbf{D}} + \mathbf{\Delta}^D, \mathbf{0}\right)$ and let $\mathbf{E} = \max\left(\bar{\mathbf{E}} + \mathbf{\Delta}^E, \mathbf{0}\right)$. Although we can renormalize $\mathbf{E}$ so that the sum of each row is equal to 1, we choose not to do so in favor of a simpler presentation. The perturbed game is then given by $(\mathbf{E}, \mathbf{D})$.

Following Spielman and Teng [24], the smoothed complexity of an algorithm W for the Leontief economy is defined as following: Let $T_W(\mathbf{E}, \mathbf{D})$ be the complexity of algorithm W for solving a market economy defined by $(\mathbf{E}, \mathbf{D})$. Then, the smoothed complexity of algorithm W under perturbations $N_\sigma()$ of magnitude σ is

$$\text{Smoothed}_W[n, \sigma] = \max_{\bar{\mathbf{D}} \in \mathbb{R}^{n\times n}_{[0,2]}, \bar{\mathbf{E}} \in \mathbb{R}^{n\times n}_{[0,1]}} \mathrm{E}_{\mathbf{E} \leftarrow N_\sigma(\bar{\mathbf{E}}), \mathbf{D} \leftarrow N_\sigma(\bar{\mathbf{D}})} \left[T_W(\mathbf{E}, \mathbf{D})\right],$$

where we use $\mathbf{E} \leftarrow N_\sigma(\bar{\mathbf{E}})$ to denote that $\mathbf{E}$ is a perturbation of $\bar{\mathbf{E}}$ according to $N_\sigma(\bar{\mathbf{E}})$.

An algorithm W for computing Walrasian equilibria has *polynomial smoothed time complexity* if for all $0 < \sigma < 1$ and for all positive integer n, there exist positive constants c, k_1 and k_2 such that

$$\text{Smoothed}_W[n, \sigma] \le c \cdot n^{k_1} \sigma^{-k_2}.$$

The Leontief exchange economy is in *smoothed polynomial time* if there exists an algorithm W with polynomial smoothed time-complexity for computing a Walrasian equilibrium.

To relate the complexity of finding an approximate Nash equilibrium of two-person games with the smoothed complexity of Leontief economies, we examine the equilibria of perturbations of the reduction presented in the last subsection. In the remainder of this subsection, we will focus on the smoothed complexity under uniform perturbations with magnitude σ. One can similarly extend the results to Gaussian perturbation with standard deviation σ.

Let $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ be a two-person game in which each player has n strategies. Let $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$. Let $\left(\mathbf{\Delta}^E, \mathbf{\Delta}^D\right)$ be a pair of perturbation matrices with entries drawn uniformly at random from $[-\sigma, \sigma]$. The perturbed game is then given by $\mathbf{E} = \max\left(\bar{\mathbf{E}} + \mathbf{\Delta}^E, \mathbf{0}\right)$ and $\mathbf{D} = \max\left(\bar{\mathbf{D}} + \mathbf{\Delta}^D, \mathbf{0}\right)$.

Let $\Pi_\sigma(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ be the set of all $(\mathbf{E}, \mathbf{D})$ that can be obtained by perturbing $\pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ with magnitude σ. Note that the off-diagonal entries of $\mathbf{E}$ are between 0 and σ, while the diagonal entries are between $1-\sigma$ and $1+\sigma$.

In the next section, we will prove the following lemma.

Lemma 4.3 (Approximation of Games and Perturbed Markets). *Let $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ be a bimatrix game with $\bar{\mathbf{A}}, \bar{\mathbf{B}} \in \mathbb{R}^{n\times n}_{[1,2]}$. For any $0 < \sigma < 1/(8n)$, let $(\mathbf{E}, \mathbf{D}) \in \Pi_\sigma(\bar{\mathbf{A}}, \bar{\mathbf{B}})$. Let $(\mathbf{u}, \mathbf{w})$ be an ϵ-approximate equilibrium of $(\mathbf{E}, \mathbf{D})$ and assume $\mathbf{u} = (\mathbf{x}^\top, \mathbf{y}^\top)^\top$ and $\mathbf{w} = (\mathbf{p}^\top, \mathbf{q}^\top)^\top$. Then, $(\mathbf{x}, \mathbf{y})$ is an $O(n\sqrt{\epsilon} + n^{1.5}\sqrt{\sigma})$-relatively-approximate Nash equilibrium for $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.*

We now follow the scheme outlined in [25] and used in [4] to use perturbations as a probabilistic polynomial reduction from the approximation problem of two-person games to market equilibrium problem over perturbed Leontief economies.

Lemma 4.4 (Smoothed Leontief and Approximate Nash). *If the problem of computing an equilibrium of a Leontief economy is in smoothed polynomial time under uniform perturbations, then for any $0 < \epsilon' < 1$, there exists a randomized algorithm for computing an ϵ'-approximate Nash equilibrium in expected time polynomial in n and $1/\epsilon'$.*

Proof. Suppose W is an algorithm with polynomial smoothed complexity for computing a equilibrium of a Leontief economy. Let $T_W(\mathbf{E}, \mathbf{D})$ be the complexity of algorithm W for solving the market problem defined by $(\mathbf{E}, \mathbf{D})$. Let $N_\sigma()$ denotes the uniform perturbation with magnitude σ. Then there exists constants c, k_1 and k_2 such that for all $0 < \sigma < 1$,

$$\max_{\bar{\mathbf{E}} \in \mathbb{R}^{n\times n}_{[0,1]}, \bar{\mathbf{D}} \in \mathbb{R}^{n\times n}_{[0,2]},} \mathrm{E}_{\mathbf{E} \leftarrow N_\sigma(\bar{\mathbf{E}}), \mathbf{D} \leftarrow N_\sigma(\bar{\mathbf{D}})} \left[T_W(\mathbf{E}, \mathbf{D})\right] \leq c \cdot n^{k_1} \sigma^{-k_2}.$$

Consider a bimatrix game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ with $\bar{\mathbf{A}}, \bar{\mathbf{B}} \in \mathbb{R}^{n\times n}_{[1,2]}$. For each $(\mathbf{E}, \mathbf{D}) \in \Pi_\sigma(\bar{\mathbf{A}}, \bar{\mathbf{B}})$, by Lemma 4.3, by setting $\epsilon = 0$ and $\sigma = O(\epsilon'/n^3)$, we can obtain an ϵ'-approximate Nash equilibrium of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ in polynomial time from an equilibrium of $(\mathbf{E}, \mathbf{D})$.

Now given the algorithm W with polynomial smoothed time-complexity, we can apply the following randomized algorithm with the help of uniform perturbations to find an ϵ-approximate Nash equilibrium of game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$:

Algorithm `ApproximateNashFromSmoothedLeontief` $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$

1. Let $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.
2. Randomly choose a pair of perturbation matrices $\left(\mathbf{\Delta}^E, \mathbf{\Delta}^D\right)$ of magnitude σ.
3. Let $\mathbf{D} = \max\left(\bar{\mathbf{D}} + \mathbf{\Delta}^D, \mathbf{0}\right)$ and let $\mathbf{E} = \max\left(\bar{\mathbf{E}} + \mathbf{\Delta}^E, \mathbf{0}\right)$.
4. Apply algorithm W to find an equilibrium $(\mathbf{u}, \mathbf{w})$ of $(\mathbf{E}, \mathbf{D})$.
5. Apply Lemma 4.3 to compute an approximate Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.

The expected time complexity of `ApproximateNashFromSmoothedLeontief` is bounded from above by the smoothed complexity of W when the magnitude perturbations is ϵ'/n^3 and hence is at most $c \cdot n^{k_1+3k_2}(\epsilon')^{-k_2}$. □

We can use this randomized reduction to prove the second main result of this paper.

Theorem 4.5 (Hardness of Smoothed Leontief Economies). *Unless* **PPAD** $\subset$ **RP**, *the problem of computing an equilibrium of a Leontief economy is not in smoothed polynomial time, under uniform or Gaussian perturbations.*

Proof. Setting $\epsilon' = n^{-h}$ for a sufficiently large constant and apply Lemma 4.4 and Corollary 3.4. □

5 The Approximation Analysis

In this section, we prove Lemma 4.3. Let us first recall all the matrices that will be involved: We start with two matrices $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ of the bimatrix game. We then obtain the two matrices $(\bar{\mathbf{E}}, \bar{\mathbf{D}}) = \pi(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ of the associated Leontief economy, where $\bar{\mathbf{E}} = \mathbf{I}_{2n}$ and

$$\bar{\mathbf{D}} = \begin{pmatrix} 0 & \bar{\mathbf{A}} \\ \bar{\mathbf{B}} & 0 \end{pmatrix}.$$

We then perturb $(\bar{\mathbf{E}}, \bar{\mathbf{D}})$ to obtain $(\mathbf{E}, \mathbf{D})$. We can write $\mathbf{D}$ as:

$$\mathbf{D} = \begin{pmatrix} \mathbf{Z} & \mathbf{A} \\ \mathbf{B} & \mathbf{N} \end{pmatrix}$$

where for all $\forall i, j,\ z_{ij}, n_{ij} \in [0, \sigma]$ and $a_{i,j} - \bar{a}_{ij}, b_{i,j} - \bar{b}_{i,j} \in [-\sigma, \sigma]$, Note also because $\bar{\mathbf{A}}, \bar{\mathbf{B}} \in \mathbb{R}^{n\times n}_{[1,2]}$ and $0 < \sigma < 1$, $\mathbf{A}$ and $\mathbf{B}$ are uniform perturbations with magnitude σ of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$, respectively. Moreover, $z_{i,j}$ and $n_{i,j}$ are 0 with probability 1/2 and otherwise, they are uniformly chosen from $[0, \sigma]$.

Now, let $(\mathbf{u}, \mathbf{w})$ be an ϵ-approximate equilibrium of $(\mathbf{E}, \mathbf{D})$ and assume $\mathbf{u} = (\mathbf{x}^\top, \mathbf{y}^\top)^\top$ and $\mathbf{w} = (\mathbf{p}^\top, \mathbf{q}^\top)^\top$, where all vectors are column vectors.

By the definition of ϵ-approximate market equilibrium, we have:

$$\begin{cases} \mathbf{Zx} + \mathbf{Ay} & \leq & (1+\epsilon) \cdot \vec{\mathbf{1}} \\ \mathbf{Bx} + \mathbf{Ny} & \leq & (1+\epsilon) \cdot \vec{\mathbf{1}} \\ (1-\epsilon)\mathbf{E}^\top \mathbf{w} & \leq & \mathrm{diag}(\mathbf{u})\mathbf{D}^\top \mathbf{w} \leq (1+\epsilon)\mathbf{E}^\top \mathbf{w}, \end{cases} \tag{5}$$

where diag($\mathbf{u}$) is the diagonal matrix whose diagonal is $\mathbf{u}$. Since the demand functions are homogeneous with respect to the price vector $\mathbf{w}$, we assume without loss of generality that $\|\mathbf{w}\|_1 = \|\mathbf{p}\|_1 + \|\mathbf{q}\|_1 = 1$.

We will prove $(\mathbf{x}, \mathbf{y})$ is an $O\left(n\sqrt{\epsilon} + n^{1.5}\sqrt{\sigma}\right)$-relatively-approximate Nash equilibrium of the two-person game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$. To this end, we first prove the following three properties of the approximate equilibrium $(\mathbf{u}, \mathbf{w})$. To simplify the presentation of our proofs, we will not aim at the best possible constants. Instead, we will make some crude approximations in our bounds to ensure the resulting formula are simple enough for readers.

Property 1 (Approximate Price Symmetry). *If* $\|\mathbf{w}\|_1 = 1$*,* $0 < \epsilon < 1/2$*, and* $0 < \sigma < 1/(2n)$*, then*

$$\frac{1 - \epsilon - 4n\sigma}{2 - 4n\sigma} \leq \|\mathbf{p}\|_1, \|\mathbf{q}\|_1 \leq \frac{1+\epsilon}{2 - 4n\sigma}.$$

Proof. Recall $\mathbf{u} = (\mathbf{x}^\top, \mathbf{y}^\top)^\top$ and $\mathbf{w} = (\mathbf{p}^\top, \mathbf{q}^\top)^\top$. By (5) and the fact that the diagonal entries of $\mathbf{E}$ are at least $1 - \sigma$, we have

$$\begin{aligned}(1-\epsilon)(1-\sigma)\|\mathbf{p}\|_1 &\leq \vec{\mathbf{1}}^\top (\mathrm{diag}(\mathbf{x})\,\mathbf{Z}^\top \mathbf{p} + \mathrm{diag}(\mathbf{x})\,\mathbf{B}^\top \mathbf{q}) = (\mathbf{Z}\mathbf{x})^\top \mathbf{p} + (\mathbf{B}\mathbf{x})^\top \mathbf{q} \\ &\leq 3n\sigma \|\mathbf{p}\|_1 + (1+\epsilon)\|\mathbf{q}\|_1,\end{aligned}$$

where last inequality follows from $(\mathbf{B}\mathbf{x}) \leq (1+\epsilon)\vec{\mathbf{1}}$ (obtained from Equation (5)), its simple consequence $x_i \leq (1+\epsilon)/(1-\sigma) \leq 3, \forall i$ (because entries of $\mathbf{B}$ are between $1-\sigma$ and $2+\sigma$), and the fact that entries of $\mathbf{Z}$ are between 0 and σ.

Applying, $\|\mathbf{q}\|_1 = \|\mathbf{w}\|_1 - \|\mathbf{p}\|_1 = 1 - \|\mathbf{p}\|_1$ to the inequality, we have

$$\|\mathbf{p}\|_1 \leq \frac{1+\epsilon}{(1-\epsilon)(1-\sigma) + (1+\epsilon) - 3n\sigma} \leq \frac{1+\epsilon}{2 - 4n\sigma}.$$

Thus,

$$\|\mathbf{q}\|_1 = 1 - \|\mathbf{p}\|_1 \geq \frac{1 - \epsilon - 4n\sigma}{2 - 4n\sigma}.$$

We can similarly prove the other direction. □

Property 2 (Approximate Utility Symmetry). *If* $\|\mathbf{w}\|_1 = 1$*,* $0 < \epsilon < 1/2$*, and* $0 < \sigma < 1/(8n)$*, then*

$$\frac{(1-\epsilon)(1-\sigma)(1-\epsilon-4n\sigma)}{(1+\epsilon)(2+2\sigma)} \leq \|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \leq \frac{(1+\epsilon)^2 + n\sigma(1+\epsilon)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}.$$

Proof. By our assumption on the payoff matrices of the two-person games, $1 \leq \bar{a}_{ij}, \bar{b}_{ij} \leq 2$, for all $1 \leq i, j, \leq n$. Thus, $1 - \sigma \leq a_{ij}, b_{ij} \leq 2 + \sigma$. By (5) and the fact the diagonal entries of $\mathbf{E}$ is at least $1 - \sigma$, we have

$$x_i \geq \frac{(1-\epsilon)(1-\sigma)p_i}{\langle \mathbf{b}_i | \mathbf{q}\rangle + \langle \mathbf{z}_i | \mathbf{p}\rangle} \geq \frac{(1-\epsilon)(1-\sigma)p_i}{(2+\sigma)\|\mathbf{q}\|_1 + \sigma\|\mathbf{p}\|_1} \geq \frac{(1-\epsilon)(1-\sigma)(2-4n\sigma)p_i}{(2+2\sigma)(1+\epsilon)},$$

where the last inequality follows from Property 1. Summing it up, we obtain,

$$\|\mathbf{x}\|_1 \geq \frac{(1-\epsilon)(1-\sigma)(2-4n\sigma)}{(2+2\sigma)(1+\epsilon)}\|\mathbf{p}\|_1 \geq \frac{(1-\epsilon)(1-\sigma)(1-\epsilon-4n\sigma)}{(1+\epsilon)(2+2\sigma)},$$

where again, we use Property 1 in the last inequality.

On the other hand, from (5) we have

$$\begin{aligned}
\mathbf{x}_i &\leq \frac{(1+\epsilon)\langle \mathbf{e}_i|\mathbf{w}\rangle}{\langle \mathbf{b}_i|\mathbf{q}\rangle + \langle \mathbf{z}_i|\mathbf{p}\rangle} \leq \frac{(1+\epsilon)(p_i+\sigma)}{\langle \mathbf{b}_i|\mathbf{q}\rangle} \qquad \text{(using } \|\mathbf{w}\|_1 = 1 \text{ and the property of } \mathbf{e}_i\text{)} \\
&\leq \frac{(1+\epsilon)(p_i+\sigma)}{(1-\sigma)\|\mathbf{q}\|_1} \leq \frac{(1+\epsilon)(p_i+\sigma)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}. \qquad (6)
\end{aligned}$$

Summing it up, we obtain,

$$\begin{aligned}
\|\mathbf{x}\|_1 &\leq \frac{(1+\epsilon)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}\|\mathbf{p}\|_1 + \frac{n\sigma(1+\epsilon)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)} \\
&\leq \frac{(1+\epsilon)^2 + n\sigma(1+\epsilon)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}.
\end{aligned}$$

We can similarly prove the bound for $\|\mathbf{y}\|_1$. □

Property 3 (Utility Upper Bound). *Let* $\mathbf{s} = \mathbf{Z}\mathbf{x} + \mathbf{A}\mathbf{y}$ *and* $\mathbf{t} = \mathbf{B}\mathbf{p} + \mathbf{N}\mathbf{y}$. *Let* $\lambda = \max\{\epsilon, n\sigma\}$. *Under the same assumption as Property 2, if* $s_i \leq (1+\epsilon)(1-\sigma) - \sqrt{\lambda}$,

$$x_i \leq \frac{(1+\epsilon)(2-4n\sigma)(5\sqrt{\lambda}+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}.$$

Similarly, if $t_i \leq (1+\epsilon)(1-\sigma) - \sqrt{\lambda}$, *then*

$$y_i \leq \frac{(1+\epsilon)(2-4n\sigma)(5\sqrt{\lambda}+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}.$$

Proof. By (5), we have

$$\begin{aligned}
(1-\epsilon)(1-\sigma)\|\mathbf{w}\|_1 &\leq (1-\epsilon)\vec{\mathbf{1}}^\top\mathbf{E}^\top\mathbf{w} \leq \mathbf{u}^\top\mathbf{D}^\top\mathbf{w} = \langle \mathbf{s}|\mathbf{p}\rangle + \langle \mathbf{t}|\mathbf{q}\rangle \\
&= \sum_{j\neq i} s_j p_j + \langle \mathbf{t}|\mathbf{y}\rangle + s_i p_i \\
&\leq (1+\epsilon)\vec{\mathbf{1}}^\top\mathbf{E}^\top\mathbf{w} - (1+\epsilon)\langle \mathbf{e}_i|\mathbf{w}\rangle + s_i p_i \\
&\leq (1+\epsilon)(1+n\sigma)\|\mathbf{w}\|_1 - (1+\epsilon)(1-\sigma)p_i + s_i p_i,
\end{aligned}$$

where the inequality immediately after the second equation follows from the last inequality of (5).

Thus, $[(1+\epsilon)(1-\sigma) - s_i]p_i \leq (1+\epsilon)(1+n\sigma) - (1+\epsilon)(1-\sigma) \leq 2\epsilon + 3n\sigma$. Consequently, if $(1+\epsilon)(1-\sigma) - s_i \geq \sqrt{\lambda}$, then $p_i \leq (2\epsilon + 3n\sigma)/\sqrt{\lambda} \leq 5\sqrt{\lambda}$. By Equation (6)

$$x_i \leq \frac{(1+\epsilon)(2-4n\sigma)(p_i+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)} \leq \frac{(1+\epsilon)(2-4n\sigma)(5\sqrt{\lambda}+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)}.$$

We can similarly establish the bound for y_i. □

We now use these three properties to prove Lemma 4.3.

Proof. **[of Lemma 4.3]** In order to prove that $(\mathbf{x}, \mathbf{y})$ is a δ-relatively approximate Nash equilibrium for $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$, it is sufficient to establish:

$$\begin{cases} \mathbf{x}^\top \bar{\mathbf{A}} \mathbf{y} \geq (1-\delta) \max_{\|\tilde{\mathbf{x}}\|_1 = \|\mathbf{x}\|_1} \tilde{\mathbf{x}}^\top \bar{\mathbf{A}} \mathbf{y} \\ \mathbf{y}^\top \bar{\mathbf{B}}^\top \mathbf{x} \geq (1-\delta) \max_{\|\tilde{\mathbf{y}}\|_1 = \|\mathbf{y}\|_1} \tilde{\mathbf{y}}^\top \bar{\mathbf{B}}^\top \mathbf{x}. \end{cases} \tag{7}$$

Let $\mathbf{s} = \mathbf{Z}\mathbf{x} + \mathbf{A}\mathbf{y}$. We observe,

$$\begin{aligned} \mathbf{x}^\top \bar{\mathbf{A}} \mathbf{y} &= \mathbf{x}^\top \left(\mathbf{A}\mathbf{y} + \mathbf{Z}\mathbf{x} - \mathbf{Z}\mathbf{x} + (\bar{\mathbf{A}} - \mathbf{A})\mathbf{y} \right) = \mathbf{x}^\top \mathbf{s} - \mathbf{x}^\top \left(\mathbf{Z}\mathbf{x} + \left(\mathbf{A} - \bar{\mathbf{A}} \right) \mathbf{y} \right) \\ &\geq \mathbf{x}^\top \mathbf{s} - \|\mathbf{x}\|_1 \left(\|\mathbf{Z}\mathbf{x}\|_\infty + \left\| (\mathbf{A} - \bar{\mathbf{A}})\mathbf{y} \right\|_\infty \right) \geq \mathbf{x}^\top \mathbf{s} - \sigma \|\mathbf{x}\|_1 \left(\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1 \right) \\ &\geq \mathbf{x}^\top \mathbf{s} - \frac{2\sigma(1+\epsilon)^2 + 2n\sigma^2(1+\epsilon)(2-4n\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)} \|\mathbf{x}\|_1 \\ &= \mathbf{x}^\top \mathbf{s} - O(\sigma) \|\mathbf{x}\|_1, \end{aligned}$$

where the last inequality follows from Property 2.

Let $\lambda = \max(\epsilon, n\sigma)$. By Property 3, we can estimate the lower bound of $\langle \mathbf{x} | \mathbf{s} \rangle$:

$$\begin{aligned} \langle \mathbf{x} | \mathbf{s} \rangle &= \sum_{i=1}^{n} x_i s_i \geq \sum_{i: s_i > (1+\epsilon)(1-\sigma) - \sqrt{\lambda}} x_i s_i \\ &\geq \left[(1+\epsilon)(1-\sigma) - \sqrt{\lambda} \right] \left(\|\mathbf{x}\|_1 - n \frac{(1+\epsilon)(2-4n\sigma)(5\sqrt{\lambda}+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma)} \right) \\ &\geq \|\mathbf{x}\|_1 \left(1 - O\left(\sqrt{\lambda} \right) \right) \left(1 - n \frac{(1+\epsilon)(2-4n\sigma)(5\sqrt{\lambda}+\sigma)}{(1-\sigma)(1-\epsilon-4n\sigma) \|\mathbf{x}\|_1} \right) \\ &= \|\mathbf{x}\|_1 \left[1 - O\left(n\sqrt{\lambda} + n\sigma \right) \right]. \end{aligned}$$

On the other hand, by (5), we have $\mathbf{A}\mathbf{y} \leq (1+\epsilon)\vec{\mathbf{1}}$ and hence

$$\max_{\|\tilde{\mathbf{x}}\|_1 = \|\mathbf{x}\|_1} \tilde{\mathbf{x}}^\top \bar{\mathbf{A}} \mathbf{y} = \|\mathbf{x}\|_1 \left\| \bar{\mathbf{A}} \mathbf{y} \right\|_\infty \leq (1 + \epsilon + \sigma \|\mathbf{y}\|_1) \|\mathbf{x}\|_1 \leq (1 + \epsilon + O(\sigma)) \|\mathbf{x}\|_1.$$

Therefore,

$$\begin{aligned} \mathbf{x}^\top \bar{\mathbf{A}} \mathbf{y} &\geq \mathbf{x}^\top \mathbf{s} - O(\sigma) \|\mathbf{x}\|_1 \\ &\geq \|\mathbf{x}\|_1 \left[(1 - O\left(n\sqrt{\lambda} + n\sigma \right) - O(\sigma) \right] \\ &\geq \frac{1}{1+\epsilon+O(\sigma)} \left[1 - O\left(n\sqrt{\lambda} + n\sigma \right) \right] \max_{\|\tilde{\mathbf{x}}\|_1 = \|\mathbf{x}\|_1} \tilde{\mathbf{x}}^\top \bar{\mathbf{A}} \mathbf{y} \\ &= \left(1 - O\left(n\sqrt{\lambda} + n\sigma \right) \right) \max_{\|\tilde{\mathbf{x}}\|_1 = \|\mathbf{x}\|_1} \tilde{\mathbf{x}}^\top \bar{\mathbf{A}} \mathbf{y}. \end{aligned}$$

We can similarly prove

$$\mathbf{y}^\top \bar{\mathbf{B}}^\top \mathbf{x} = \left(1 - O\left(n\sqrt{\lambda} + n\sigma \right) \right) \max_{\|\tilde{\mathbf{y}}\|_1 = \|\mathbf{y}\|_1} \tilde{\mathbf{y}}^\top \bar{\mathbf{B}}^\top \mathbf{x}.$$

We then use the inequalities $\sqrt{\lambda} = \sqrt{\max(\epsilon, n\sigma)} \leq \sqrt{\epsilon + n\sigma} \leq \sqrt{\epsilon} + \sqrt{n\sigma}$ and $\sigma \leq \sqrt{\sigma}$ to complete the proof.

□

6 Remarks and Open Questions

In this section, we briefly summarize some remarkable algorithmic accomplishments in the computation of (approximate) market equilibria obtained prior to this work. We then present some open questions motivated by these and our new results.

6.1 General Market Exchange Problems and Algorithmic Results

In our paper, we have focused on Leontief market exchange economies. Various other market economies have been considered in the literature [23, 9].

An instance of a general market exchange economy with n traders and m goods is given by the endowment matrix $\mathbf{E}$ together with n utilities functions $v_1, \ldots, v_n$. Then an *equilibrium price vector* is a vector $\mathbf{p}$ satisfying

$$\exists\, \mathbf{X} \in \mathbb{R}^{m\times n}, \mathbf{X}\vec{\mathbf{1}} \leq \mathbf{E}\vec{\mathbf{1}},\ \mathbf{X}^\top \mathbf{p} \leq \mathbf{E}^\top \mathbf{p}, \text{ and } u_j(\mathbf{x}_j) = \max\left\{u_j(\mathbf{x}'_j) : \left\langle \mathbf{x}'_j | \mathbf{p}\right\rangle \leq \left\langle \mathbf{e}_j | \mathbf{p}\right\rangle\right\}.$$

The pairs $(\mathbf{X}, \mathbf{p})$ and $([u_1(\mathbf{x}_1), ..., u_n(\mathbf{x}_n)], \mathbf{p})$ is as also referred to as an equilibrium of the exchange market $(\mathbf{E}, (u_1, \ldots, u_n))$.

The celebrated theorem of Arrow and Debreu [1] states that if all utility functions are concave, then the exchange economy has an equilibrium. Moreover, if these functions are strictly concave, then for each equilibrium price vector $\mathbf{p}$, its associated exchange $\mathbf{X}$ or utilities is unique.

A popular family of utility functions is the CES (standing for Constant Elasticity of Substitution) utility functions. It is specified by an $m \times n$ demand matrix $\mathbf{D}$. The utility functions are then defined with the help of an additional parameter $\rho \in (-\infty, 1] \setminus \{0\}$:

$$u_j^{(\rho)}(\mathbf{x}_j) = \left(\sum_{i=1}^{m} d_{ij} x_{ij}^{\rho}\right)^{\frac{1}{\rho}}$$

As $\rho \to -\infty$, CES utilities become the Leontief utilities. When $\rho = 1$, the utility functions are linear functions.

Remarkablely, an (approximate) equilibrium of an exchange economy with linear utilities functions can be found in polynomial time [20, 12, 27, 13]. In fact, Ye shows that an ϵ-equilibrium of such an market with n traders and n goods can be found in $O(n^4 \log(1/\epsilon))$ time [27]. If data is given as rational numbers of L-bits, then an exact equilibrium can be found in $O(n^4 L)$ time.

A closely related market exchange model is the Fisher's model [23]. In this model, there are two types of traders: *producers* and *consumers*. Each consumer comes to the market with a budget and a utility function. Each producer comes to the market with an endowment of goods and will sell them to the consumers for money. An equilibrium is a price vector $\mathbf{p}$ for goods so that if each consumer spends all her budget to maximize her utilities, then the market clears, i.e., at the end of the exchange, all producers sold out.

Even more remarkablely, an (approximate) equilibrium in a Fisher's economy with any CES utilities can be found in polynomial time [11, 27, 28, 10, 14].

6.2 Open Questions

Our results as well as the combination of Codenotti, Saberi, Varadarajan, and Ye [5] and Chen and Deng [3] demonstrate that exchange economies with Leontief utility functions are fundamentally different from economies with linear utility functions. In Leontief economies, not only finding an exact equilibrium is likely hard, but finding an approximate equilibrium is just as hard.

Although, we prove that the computation of an $O(1/n^{\Theta(1)})$-approximate equilibrium of Leontief economies is **PPAD**-hard. our hardness result does not cover the case when ϵ is a constant between 0 and 1. The following are two optimistic conjectures.

Conjecture 4 (PTAS Approximate LEONTIEF). *There is an algorithm to find an ϵ-approximate equilibrium of a Leontief economy in time $O(n^{k+\epsilon^{-c}})$ for some positive constants c and k.*

Conjecture 5 (Smoothed LEONTIEF: Constant Perturbations). *There is an algorithm to find an equilibrium of a Leontief economy with smoothed time complexity $O(n^{k+\sigma^{-c}})$ under perturbations with magnitude σ, for some positive constants c and k.*

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