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Scaled Dimension of Individual Strings

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Abstract

We define a new discrete version of scaled dimension and we find connections between the scaled dimension of a string and its Kolmogorov complexity and predictability. We give a new characterization of constructive scaled dimension by Kolmogorov complexity, and prove a new result about scaled dimension and prediction.

1 Introduction

Effective fractal dimension, defined by Lutz (2003) [10], allow us to study the fractal structure of many sets of interest in computational complexity. Furthermore, many connections have been found since then between effective fractal dimension and other topics in computational complexity like Kolmogorov complexity [12], [11] and prediction [2], [3].

In 2004, scaled dimension was introduced by Hitchcock et al [5] as a natural hierarchy of "rescale" effective fractal dimensions. The main objective was to overcome limitations of effective fractal dimension for investigating complexity classes. For example classes such as the Boolean circuit-size complexity classes SIZE($2^{\alpha n}$) and SIZE($2^{n^{\alpha}}$) have trivial dimensions, and the definition of scaled dimension make it possible to quantify the difference between those classes. Connections between Kolmogorov Complexity and scaled dimension were found in [6].

The definition of effective fractal dimension is based in a characterization of classical Hausdorff dimension in the Cantor space $\mathbf{C}$ in terms of gales (s-gales). Intuitively, we regard a s-gale d as an strategy for betting on the successive bits of a sequence $S \in \mathbf{C}$ and the parameter s gives us an idea about the fairness on the gambling game. Scaled dimension is defined using scaled gales (s^g-gales), intuitively, d is an strategy for betting on a sequence but the fairness on the gambling game depends on the s and on the scale g.

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In [11], Lutz uses supertermgales, which are supergale-like functions that bet on the terminations of (finite, binary) strings as well on their successive bits, to define a discrete version of constructive dimension (an special case of effective fractal dimension). Lutz then characterizes the dimension of a finite string in terms of its Kolmogorov complexity. We generalize those results by defining a new discrete version of constructive scaled dimension (section 3). The main result of this section states that the scaled dimension of an infinite sequence is characterized by the scaled dimension of its prefixes (Theorem 3.6). Now, when we obtain characterizations of the scaled dimension of individual strings in terms of Kolmogorov complexity or prediction (section 4), we can obtain results in constructive scaled dimension, just applying the results to the prefixes of a sequence.

With this method, we obtain a new characterization of the i^{th} -order scaled constructive dimension in terms of Kolmogorov Complexity extending the results in [6].

Furthermore, we define the concept of termpredictor by adding the ability to predict the end of an unknown finite string to the standard on-line prediction algorithms. That is, a termpredictor guesses the next character as well as the termination point of a finite string.

We show that the scaled constructive dimension of sets of sequences can be bounded in terms of the log-loss of constructive termpredictors. This extends partially the characterization that Hitchcock obtains in [3] for resource-bounded dimension to the cases of scaled and constructive dimension.

2 Preliminaries

A string is a finite, binary string $w \in \{0,1\}^*$. We write $|w|$ for the length of a string and λ for the empty string. The Cantor space $\mathbf{C}$ is the set of all infinite binary sequences. If $w \in \{0,1\}^*$ and $x \in \{0,1\}^* \cup \mathbf{C}$, $w \sqsubseteq x$ means that w is a prefix of x. For $0 \leq i \leq j$, we write $x[i \ldots j]$ for the string consisting of the i-th through the j-th bits of x.
The set of all terminated binary strings and prefixes thereof is the set

$$\mathcal{T} = \{0,1\}^* \cup \{0,1\}^* \Box$$

where we use the symbol $\Box$ to mark the end of a string.
Definition. Let $f : D \to \mathbb{R}$ be a function where D is some discrete domain such as $\mathbb{N}, \{0,1\}^*, \mathcal{T}$, etc.

1. f is computable if there is a computable function $\hat{f} : D \times \mathbb{N} \to \mathbb{Q}$ such that for all $(w, n) \in D \times \mathbb{N}$, $|\hat{f}(w, n) - f(w)| \leq 2^{-n}$.

2. f is lower semicomputable if there is a computable function $\hat{f} : D \times \mathbb{N} \to \mathbb{Q}$ such that

   (a) for all $(w, n) \in D \times \mathbb{N}$, $\hat{f}(w, n) \leq \hat{f}(w, n+1) < f(w)$, and

   (b) for all $w \in D$, $\lim_{n \to \infty} \hat{f}(w, n) = f(w)$.

Definition.

1. A subprobability measure on $\{0,1\}^*$ is a function $p : \{0,1\}^* \to [0,1]$ such that

$$\sum_{w \in \{0,1\}^*} p(w) \leq 1$$

2. A subprobability measure on $\{0,1\}^*$ is constructive if it is lower semicomputable.

3. A subprobability measure p on $\{0,1\}^*$ is optimal constructive if for every constructive subprobability measure p' there is a real constant $\alpha > 0$ such that, for all $w \in \{0,1\}^*$, $p(w) > \alpha p'(w)$.

Theorem 2.1 *(Levin [13]) There exists an optimal constructive subprobability measure* $\mathbf{m}$ *on* $\{0,1\}^*$.

The following theorem is the well-know characterization by Levin [7], [8] and Chaitin [1] of Kolmogorov complexity in terms of $\mathbf{m}$. Further details may be found in [9].

Theorem 2.2 *There is a constant $c \in \mathbb{N}$ such that for all $w \in \{0,1\}^*$,*

$$\left| K(w) - \log \frac{1}{\mathbf{m}(w)} \right| \leq c$$

Definition. A scale is a continuous function $g : H \times [0, \infty) \to \mathbb{R}$ with the following properties.

1. $H = (a, \infty)$ for some $a \in \mathbb{R} \cup \{-\infty\}$.

2. $g(m, 1) = m$ for all $m \in H$.

3. $g(m, 0) = g(m', 0) \geq 0$ for all $m, m' \in H$.

4. For every sufficiently large $m \in H$, the function $s \mapsto g(m,s)$ is nonnegative and strictly increasing.

5. For all $s' > s \geq 0$, $\lim_{m\to\infty}[g(m,s') - g(m,s)] = \infty$.

Definition. A smooth scale is a computable scale function $g : H \times [0,\infty) \to \mathbb{R}$ such that verifies

1. g is differentiable in the second coordinate and $\frac{\partial g}{\partial s}(m,\cdot)$ are strictly increasing for all $m \in H$.

2. $\frac{\partial g}{\partial s}(m,0) \to \infty$ as $m \to \infty$

3. $\triangle g(m,s') - \triangle g(m,s) > 0$ for all $m \in H,\ s' > s$.

For each scale $g : H \times [0,\infty) \to \mathbb{R}$, we define $\Delta g : H \times [0,\infty) \to \mathbb{R}$ by

$$\Delta g(m,s) = g(m+1,s) - g(m,s)$$

The following important family of smooth scales is used in the definition of the i^{th}-order dimension.

Definition. We define $g_i : H_i \times [0,\infty) \to \mathbb{R}$ by the recursion on $i \in \mathbb{N}$ as follows:

$$g_0(m,s) = ms.$$

$$g_{i+1}(m,s) = 2^{g_i(\log m,s)}$$

The domain of g_i is of the form $H_i = (a_i,\infty)$, where $a_0 = -\infty$ and $a_{i+1} = 2^{a_i}$.

Definition. Let $g : H \times [0,\infty)$ be a scale function. Denote by $f^m : [g(m,0),\infty] \to [0,\infty)$ the inverse of $g(m,.)$, that is the function defined as $f^m(x) = y$ if $g(m,y) = x$. This function is well define since $g(m,.)$ is strictly increasing. For the family $\{g_i\}$ we denote by f_i^m the inverse of $g_i(m,.)$ and

$$f_i^m(x) = \frac{\log(\log(.\overset{i}{\vdots}.\log(x)..))}{\log(\log(.\overset{i}{\vdots}.(\log(m+1))..))}$$

3 Scaled Dimension of Individual Strings

In this section we first introduce scaled termgales and scaled supertermgales, which are a generalization of termgales introduced by Lutz in [11]. Next, we show the existence of optimal constructive scaled supertermgales that allows us to give a universal definition of the scaled dimension of a string.

Definition. For $s \in [0,\infty)$ and $g : H \times [0,\infty) \to [0,\infty)$ a scale function,

1. An s^g-supertermgale is a function $d_g : \mathcal{T} \to [0,\infty)$ such that

   (a) $d_g(w) \leq 1$ for $|w| \notin H$.

   (b) For all $w \in \{0,1\}^*$ with $|w| \in H$,

$$d_g(w) \geq 2^{-\triangle g(|w|,s)}[d_g(w0) + d_g(w1) + d_g(w\Box)] \qquad (1)$$

2. An s^g-termgale is an s^g-supertermgale that satisfies (1) with equality for all $w \in \{0,1\}^*$ with $|w| \in H$.

An s^g-termgale is a strategy for betting on the successive bits of a binary string and also on the point at which the string terminates. The fairness of the gambling game depends on the s and on the scale function g.

Observation 3.1 *Let $g : H \times [0,\infty) \to \mathbb{R}$ be a scale, $d_g, d'_g : \mathcal{T} \to [0,\infty)$ and $s, s' \in [0,\infty)$. If*

$$2^{-g(|w|,s)} d_g(w) = 2^{-g(|w|,s')} d'_g(w)$$

for all $w \in \mathcal{T}$ with $|w| \in H$, then d_g is an s^g-supertermgale (s^g-termgale) if and only if d'_g is an $s'^g -$ supertermgale (s'^g-termgale).

Thanks to this observation, a 0^g-supertermgale (termgale) determines a whole family of s^g-supertermgales (termgales).
Definition. For $g : H \times [0,\infty) \to [0,\infty)$ a constructive scale function,

1. A g-supertermgale is a family $d_g = \{d_g^s \mid s \in [0,\infty)\}$ such that each d_g^s is an s^g-supertermgale and

$$2^{-g(|w|,s)} d_g^s(w) = 2^{-g(|w|,s')} d_g^{s'}(w)$$

   for all $s, s' \in [0,\infty)$, $w \in \mathcal{T}, |w| \in H$.

2. A g-termgale is a g-supertermgale where each d_g^s is an s^g-termgale for all $s \in [0,\infty)$.

3. A g-supertermgale d_g is constructive if d_g^0 is constructive.

4. A constructive g-supertermgale $\tilde{d}_g$ is optimal if for every constructive g-supertermgale d_g there is a constant $\alpha > 0$ such that for all $s \in [0,\infty)$ and $w \in \{0,1\}^*$ with $|w| \in H$,

$$\tilde{d}_g^{\,s}(w\Box) > \alpha \, d_g^s(w\Box)$$

5. The g-supertermgale induced by a subprobability measure p on $\{0,1\}^*$ is the family $d_g[p] = \{d_g^s[p] \mid s \in [0,\infty)\}$, where each $d_g^s[p]$ is defined by
$$d_g^s[p](w) = 2^{g(|w|,s)} \sum_{\substack{x \in \{0,1\}^* \\ w \sqsubseteq x\Box}} p(x)$$
for all $w \in \mathcal{T}$ with $|w| \in H$.

Theorem 3.2 *If $\tilde{p}$ is an optimal constructive subprobability measure on $\{0,1\}^*$ and $g : H \times [0,\infty) \to [0,\infty)$ is a constructive scale function then $d_g[\tilde{p}]$ is an optimal constructive g-supertermgale.*

Corollary 3.3 *For every $g : H \times [0,\infty] \to [0,\infty)$ constructive scale function, there exists an optimal constructive g-supertermgale.*

Definition. Let $g : H \times [0,\infty] \to [0,\infty)$ be a scale function and $w \in \{0,1\}^*$ with $|w| \in H$. If d_g is a constructive g-supertermgale, then the g-dimension of w relative to d_g is
$$\dim_{d_g}(w) = \inf\{s \in [0,\infty) \mid d_g^s(w\Box) > 1\}$$

The next two results prepare the definition of g-dimension of a string.

Proposition 3.4 *Let $g : H \times [0,\infty) \to [0,\infty)$ be a smooth scale function. If $\tilde{d}_g$ is an optimal constructive g-supertermgale and d_g is a constructive g-supertermgale, there exists $C > 0$ such that*
$$\dim_{\tilde{d}_g}(w) \leq \dim_{d_g}(w) + \frac{C}{\frac{\partial g}{\partial s}(|w|+1, \dim_{d_g}(w))}$$
for all $|w| \in \{0,1\}^*$ ($|w|$ enough large).

Corollary 3.5 *Let $g : H \times [0,\infty) \to [0,\infty)$ be a smooth scale function. If $\tilde{d}_{g1}$ and $\tilde{d}_{g2}$ are optimal constructive g-supertermgales, then there is a constant $C > 0$ such that for all $w \in \{0,1\}^*$ ($|w|$ enough large).*
$$|\dim_{\tilde{d}_{g_1}}(w) - \dim_{\tilde{d}_{g_2}}(w)| \leq \frac{C}{\frac{\partial g}{\partial s}(|w|+1, s_0)}$$
where $s_0 = \min\{\dim_{\tilde{d}_{g_1}}(w), \dim_{\tilde{d}_{g_2}}(w)\}$.

As g is a smooth scale function, $\frac{\partial g}{\partial s}(m, 0) \to +\infty$ as $m \to \infty$, and Corollary 3.5 says that if we base our definition of g-dimension on an optimal constructive g-supertermgale $\tilde{d}_g$, then the particular choice of $\tilde{d}_g$ has negligible impact on the dimension $\dim_{\tilde{d}_g}(w)$.

We fix an optimal constructive g-supertermgale $d_{g\square}$ and define the g-dimensions of finite strings as follows.

Definition. Let $g : H \times [0, \infty) \to [0, \infty)$ be a smooth scale function. The g-dimension of a string $w \in \{0, 1\}^*$ with $|w| \in H$ is

$$\dim_g(w) = \dim_{d_{g\square}}(w)$$

3.1 Scaled dimension of strings and sequences

Resource-bounded scaled dimension of sequences in the Cantor space was defined in [5] as a generalization of resource-bounded dimension. In that definition scaled gales were used.

Definition. Let $g : H \times [0, \infty) \to \mathbb{R}$ be a scale function, and let $s \in [0, \infty)$.

1. An s^g-supergale is a function $d : \{0, 1\}^* \to [0, \infty)$ such that for all $w \in \{0, 1\}^*$ with $|w| \in H$,

$$d(w) \geq 2^{-\Delta g(|w|, s)}[d(w0) + d(w1)]$$

2. We say that an s^g-supergale d succeeds on a sequence $S \in \mathbf{C}$ if

$$\limsup_n d(S[0 \ldots n-1]) = \infty$$

3. The success set of an s^g-supergale d is $S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}$.

Definition. Let g be a scale function and $X \subseteq C$

1. $\widehat{\mathcal{G}}(X)$ is the set of all $s \in [0, \infty)$ such that there is an s^g-supergale d for which $X \subseteq S^\infty[d]$.

2. $\widehat{\mathcal{G}}_{\text{constr}}(X)$ is the set of all $s \in [0, \infty)$ such that there is a lower semi-computable s^g-supergale d for which $X \subseteq S^\infty[d]$.

3. The constructive g-scaled dimension of X is $\text{cdim}_g(X) = \inf \widehat{\mathcal{G}}_{\text{constr}}(X)$.

4. The constructive g-scaled dimension of a sequence $S \in \mathbf{C}$ is $\dim_g(S) = \text{cdim}_g(\{S\})$.

The main result of this section states that the constructive scaled dimension of a sequence is characterized by the scaled dimension of its prefixes in the following way,

Theorem 3.6 *Let $g : H \times [0, \infty) \to [0, \infty)$ be a smooth scale function and $S \in C$,*

$$\dim_g(S) = \liminf_n \dim_g(S[0 \ldots n-1])$$

In [4] Hitchcock shows that constructive gales and constructive supergales are interchangeable in order to define constructive Hausdorff dimension. In this spirit, the next lemma relates constructive scaled dimension of finite strings $\dim_g(w)$, that uses optimal constructive supertermgales, and constructive scaled dimension with just constructive termgales involved.

Lemma 3.7 *Let g be a smooth scale function and $w \in \{0,1\}^*$, then*

$$\dim_g(w) \geq \inf\{\dim_d(w) \mid d \ \ \textit{constructive g-termgale}\}$$

Such inequality has a remarkable aplication for infinite strings, namely the following characterization of constructive scaled dimension just using constructive termgales.

Corollary 3.8 *Let $S \in \mathbf{C}$ and g smooth scale function,*

$$dim_g(S) = \liminf_n \mathcal{D}_g(S0 \ldots n-1])$$

where $\mathcal{D}_g(w) = \inf\{dim_d(w) \mid d \text{ constructive } g\text{-termgale}\}$

4 Kolmogorov Complexity and Log-loss prediction

4.1 Scaled dimension and Kolmogorov Complexity

In [6] the authors give an exact characterization of computable and space-bounded scaled dimension of a sequence in terms of (time and space-bounded) Kolmogorov complexity .

Theorem 4.1 *[6]. Let $S \in \mathbf{C}$*

1. *For all $i \in \mathbb{N}$*

$$\dim^{(i)}_{\text{comp}}(S) = \inf_{t \in \text{comp}} \liminf_n f_i^n(K^{t(n)}(S[0 \ldots n-1]))$$

2. For all $i, j \in \mathbb{N}$ with $i < j$

$$\dim^{(i)}_{\mathrm{p}_j\mathrm{space}}(S) = \inf_{t \in \mathrm{p}_j\mathrm{space}} \liminf_n f_i^n(KS^{t(n)}(S[0 \ldots n-1]))$$

In this section we obtain the relationship between the scaled dimension of a finite string and its Kolmogorov complexity, and this result allow us to give a new characterization for constructive scaled dimension of an infinite sequence, extending theorem 4.1.

Theorem 4.2 *Let $g : H \times [0, \infty) \to [0, \infty)$ be a smooth scale function. Then there exists a constant $c > 0$ such that for all $w \in \{0,1\}^*$ ($|w|$ enough large),*

$$\left| f^{|w|+1}(K(w)) - dim_g(w) \right| \leq \frac{c}{\frac{\partial g}{\partial s}(|w|+1, 0)}$$

Corollary 4.3 *Let $S \in \mathbf{C}$ and g smooth scale function,*

$$\dim_g(S[0..n-1]) = \lim_n f^{n+1}(K(S[0..n-1]))$$

Example 4.4 *For the family g_i, $i \in \mathbb{N}$,*

$$\dim^{(i)}(S) = \liminf_n f_i^n(K(S[0 \ldots n-1]))$$

In the particular case of $i = 0$ we have the result of constructive dimension obtained by Mayordomo in [12].

$$\dim(S) = \liminf_n \frac{K(S[0..n-1])}{n+1}$$

4.2 Scaled dimension and Prediction.

Consider predicting the symbols of an unknown finite string. Then, given a prefix of this string, the next character could be 0, 1 or may be, the string doesn't have any characters. A termpredictor Π gives us an estimation of the probability of each of these cases.

Definition. A function $\Pi : \{0,1\}^* \times \{0, 1, \Box\} \to [0,1]$ is a termpredictor if

$$\Pi(w, 0) + \Pi(w, 1) + \Pi(w, \Box) = 1$$

We interpret $\Pi(w, a)$ as the Π's estimation of the likelihood that there is a bit a following the string (if $a = 0$ or 1) or there is not bit following the string (if $a = \Box$).

The next lemma establishes a correspondence between termpredictors and g-termgales.

Lemma 4.5 *Let g be a smooth scale function.*

1. *Let Π be a termpredictor, define $\forall s \in [0,\infty)$, $d^s_{\Pi,g} : \mathcal{T} \to [0,\infty)$ by*

$$\begin{aligned} d^s_{\Pi,g}(w) &= 1 & \text{if } |w| \notin H \\ d^s_{\Pi,g}(w) &= 2^{g(|w|,s)} \prod_{i=0}^{|w|-1} \Pi(w[0 \ldots i-1], w[i]) & \text{if } |w| \in H \end{aligned}$$

Then, $d_{\Pi,g}$ is a g-termgale.

2. *Let d_g be a g-termgale, then for $s \in [0,\infty)$ define $\Pi_{d_g} : \{0,1\}^* \times \{0,1,\Box\} \to [0,1]$ by*

$$\begin{aligned} \Pi_{d_g}(w,a) &= 2^{-\Delta g(|w|,s)} \frac{d_g(wa)}{d_g(w)} & \text{if } d_g(w) \neq 0 \\ \Pi_{d_g}(w,a) &= \frac{1}{3} & \text{if } d_g(w) = 0 \end{aligned}$$

Π_{d_g} is a termpredictor and this definition doesn't depends on s.

3. *$d_{\Pi_{d_g},g} = d_g$ and $\Pi_{d_{\Pi,g}} = \Pi$.*

In order to define the performance of a termpredictor, we will consider (as in [3]) the sum of its "loss" on each individual symbol (including $\Box$).
Definition. For $w \in \mathcal{T}$ and Π termpredictor we define the log-loss

$$\mathcal{L}^{\log}_{\Pi}(w) = \sum_{i=0}^{|w|-1} \log \frac{1}{\Pi(w[0 \ldots i-1], w[i])}$$

Theorem 4.6 *Let g be a smooth scale function, let d_g be a constructive g-termgale and $w \in \{0,1\}^*$ with $|w| \in H$ then*

$$\dim_{d_g}(w) = f^{|w|+1}(\mathcal{L}^{\log}_{\Pi_{d_g}}(w\Box))$$

In particular if d is a simple termgale and $w \in \{0,1\}^*$ then

$$\dim_d(w) = \frac{\mathcal{L}^{\log}_{\Pi_d}(w\Box)}{|w|+1}$$

Unfortunately, there are no existence of optimal constructive termgales (or optimal constructive termpredictors) and we can not prove an equality of this kind for the definition of scaled dimension of a string. But we have the following result for infinite sequences as a consequence of Proposition 3.4 and Theorem 4.6.

Theorem 4.7 *Let g be a smooth scale function and $S \in \mathbf{C}$,*

$$\dim^g(S) \leq \inf\{\mathcal{L}^{\log}_{\Pi,g}(S) \mid \Pi \textit{ is a constructive termpredictor }\}$$

where

$$\mathcal{L}^{\log}_{\Pi,g}(S) = \liminf_n f^{n+1}(\mathcal{L}^{\log}_{\Pi}(S[0 \ldots n-1]\Box))$$

The next result extend partially the characterization that Hitchcock obtains in [3] for resource-bounded dimension to the cases of scaled and constructive dimension.

Theorem 4.8 *Let $S \in \mathbf{C}$ and let g be a smooth scale function,*

$$dim_g(S) \leq \mathcal{L}^{\log}_g(S)$$

where,

$$\mathcal{L}^{\log}_g(S) = \inf\{\mathcal{L}^{\log}_{\Pi,g}(S) \mid \Pi \textit{ is a constructive predictor }\}$$

and

$$\mathcal{L}^{\log}_{\Pi,g}(S) = \liminf_n f^{n+1}(\mathcal{L}^{\log}_{\Pi}(S[0 \ldots n-1]))$$

and for the particular case of constructive dimension

$$dim(S) \leq \inf\{\mathcal{L}^{\log}_{\Pi}(S) \mid \Pi \textit{ is a constructive predictor }\}$$

where

$$\mathcal{L}^{\log}_{\Pi}(S) = \liminf_n \frac{\mathcal{L}^{\log}_{\Pi}(S[0 \ldots n-1])}{n}$$

The other inequality seems closely related to the open question of whether constructive prediction and constructive gales are equivalent.

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Appendix

Proof of Theorem 3.2. We will use the following lemma.

Lemma 4.9 *Let $g : H \times [0,\infty) \to [0,\infty)$ be a scale function and d_g a 0^g-supertermgale, then for all $u \in \{0,1\}^*$*

$$\sum_{w \in \{0,1\}^*} d_g(uw\Box) \le d_g(u)$$

Proof. *We use induction on m to show that*

$$\sum_{w \in \{0,1\}^{<m}} d_g(uw\Box) + \sum_{w \in \{0,1\}^m} d_g(uw) \le d_g(u)$$

for all $m \in [0,\infty)$. For $m = 0$, this is trivial. Assume that it holds for m. Then

$$\begin{aligned} & \sum_{w \in \{0,1\}^{<m+1}} d_g(uw\Box) + \sum_{w \in \{0,1\}^{m+1}} d_g(uw) \\ = & \sum_{w \in \{0,1\}^{<m}} d_g(uw\Box) + \sum_{w \in \{0,1\}^m} d_g(uw\Box) + \sum_{w \in \{0,1\}^m} [d_g(uw0) + d_g(uw1)] \\ \le & \sum_{w \in \{0,1\}^{<m}} d_g(uw\Box) + \sum_{w \in \{0,1\}^m} d_g(uw) \le d_g(u) \end{aligned}$$

□

It easy to see that $d_g[\tilde{p}\,]$ is a constructive g-supertermgale. We just have to prove that $d_g[\tilde{p}\,]$ is optimal. Let $d_g = \{d_g^s \mid s \in [0,\infty)\}$ be an arbitrary constructive g-supertermgale.
Define $p : \{0,1\}^* \to [0,\infty)$ by $p(w) = d_g^0(w\Box)$ for all $w \in \{0,1\}^*$. By the lemma (with $u = \lambda$), p is a subprobability measure on $\{0,1\}^*$ and p is constructive because d_g is constructive. For all $w \in \{0,1\}^*$ there exists $\alpha > 0$ such that $\tilde{p}(w) > \alpha\, p(w)$ because $\tilde{p}$ is optimal .
Then, for all $s \in [0,\infty)$ and $w \in \{0,1\}^*$ with $|w| \in H$,

$$\begin{aligned} d_g^s[\tilde{p}\,](w\Box) &= 2^{g(|w|+1,s)}\tilde{p}(w) \\ &> 2^{g(|w|+1,s)}\alpha\, p(w) \\ &= 2^{g(|w|+1,s)}\alpha\, d_g^0(w\Box) \\ &= \alpha\, C d_g^s(w\Box) \end{aligned}$$

and $d[\tilde{p}\,]$ is optimal.

Proof of Proposition 3.4. We will use the following lemma.

Lemma 4.10 *Let $g : H \times [0,\infty] \to [0,\infty)$ be a scale function. Suppose that for $m > m_0$ the functions $g(m,\cdot)$ are strictly increasing.*
Let $\tilde{d_g}$ be an optimal constructive g-supertermgale and d_g an arbitrary constructive g-supertermgale with $\tilde{d}_g^s(w\Box) > \alpha\, d_g^s(w\Box)$ for all $s \in [0,\infty)$ and $w \in \{0,1\}^*$ with $|w| \in H$.
Define $h : \{0,1\}^{>m_0} \to [0,\infty)$ by $h(w) = s'$, where s' is such that

$$g(|w|+1, \dim_{d_g}(w) + s') - g(|w|+1, \dim_{d_g}(w)) = \log \tfrac{1}{\alpha}.$$

Then

$$\dim_{\tilde{d_g}}(w) \le \dim_{d_g}(w) + h(w) \qquad \forall\, |w| > m_0$$

Proof. *Observe h is well defined because $g(m,\cdot)$ are continuous and strictly increasing for all $m > m_0$.*
Denote $s = \dim_{d_g}(w) + h(w)$. In order to see that $\dim_{\tilde{d_g}}(w) \le \dim_{d_g}(w) + h(w) \quad \forall\, |w| > m_0$ it suffices to show that $\tilde{d_g}^{\,s}(w\Box) > 1$

$$\begin{aligned} \tilde{d_g}^{\,s}(w\Box) > \alpha\, d_g^s(w\Box) &= \\ \alpha\, 2^{g(|w|+1,s) - g(|w|+1,\dim_{d_g}(w))} &= 1 \end{aligned}$$

□

By this lemma, $\dim_{\tilde{d_g}}(w) \le \dim_{d_g}(w) + h(w)\ \forall\, |w| > m_0$, where $h(w)$ is defined by

$$g(|w|+1, \dim_{d_g}(w) + h(w)) - g(|w|+1, \dim_{d_g}(w)) = \log \tfrac{1}{\alpha}.$$

By the mean value theorem, there exists $s' \in (\dim_{d_g}(w), \dim_{d_g}(w) + h(w))$ such that

$$g(|w|+1, \dim_{d_g}(w) + h(w)) - g(|w|+1, \dim_{d_g}(w)) = \frac{\partial g}{\partial s}(|w|+1, s')h(w)$$

so

$$h(w) = \frac{\log \frac{1}{\alpha}}{\frac{\partial g}{\partial s}(|w|+1, s')} \le \frac{\log \frac{1}{\alpha}}{\frac{\partial g}{\partial s}(|w|+1, \dim_{d_g}(w))}$$

Proof of Theorem 3.6.

To see that $\dim_g(S) \le \liminf_n \dim_g(S[0,\ldots,n-1])$, let s and s' be rational numbers such that $s' > s > \liminf_n \dim_g(S[0,\ldots n-1])$. It suffices to show that $\dim_g(S) \leq s'$. By our choice of s, there is an infinite set $J \subseteq N$ such that for all $n \in J$, $\dim_g(S[0,\dots n-1]) < s$, whence $\tilde{d}_{g\square}^{\,s}(S[0,\dots n-1]) > 1$. Define $d'_g : \{0,1\}^* \to [0,\infty)$ by $d'_g(w) = \tilde{d}_{g\square}^{\,s'}(w) + \frac{1}{2}\,\tilde{d}_{g\square}^{\,s'}(u\square)$, $\quad u = w[0,\dots|w|-1]$. Then d'_g is a constructive g-supergale and for all $n \in J$,

$$\begin{aligned}
d'_g(S[0,\dots,n]) &= \frac{1}{2}\tilde{d}_{g\square}^{\,s'}(S[0,\dots n-1]\square) + \tilde{d}_{g\square}^{\,s'}(S[0,\dots,n]) \\
&\geq \frac{1}{2}\,\tilde{d}_{g\square}^{\,s'}(S[0,\dots n-1]\square) \\
&= \frac{1}{2}\,2^{g(n,s)-g(n,s')}\tilde{d}_{g\square}^{\,s}(S[0,\dots n-1]\square) \\
&> \frac{1}{2}\,2^{g(n,s)-g(n,s')}
\end{aligned}$$

Since J is infinite, this implies that $S \in S^\infty[d'_g]$, whence $\dim_g(S) \leq s'$.

To see that $\dim(S) \geq \liminf_n \dim_g(S[0,\dots n-1])$, let s' and s rational numbers such that $s' > s > \dim_g(S)$. It suffices to show that there exist infinitely many $n \in N$ for which $\dim_g(S[0,\dots,n-1]) \leq s'$.
Since $s > \dim_g(S)$, there is a constructive s^g-supergale d_g such that $S \in S^\infty[d_g]$. Define

$$\begin{aligned}
d'_g :\quad \mathcal{T} &\to [0,\infty) \\
w &\mapsto d_g(w) \\
w\square &\mapsto [2^{\triangle g(|w|,s')} - 2^{\triangle g(|w|,s)}]d_g(w)
\end{aligned}$$

Then d'_g is a constructive s'^g-supertermgale, so if for each $s \in [0,\infty)$ we define
$\tilde{d}_g^{\,t} : \mathcal{T} \to [0,\infty)$ by $\tilde{d}_g^{\,t} = 2^{g(|w|,t)+g(|w|,s')}d'_g(w)$, then the family
$\tilde{d}_g = \{\tilde{d}_g^{\,s} | s \in [0,\infty)\}$ is a constructive g-termgale. It follows by the optimality of $\tilde{d}_{g\square}$ that there is a constant $\alpha > 0$ such that for all $t \in [0,\infty)$ and $w \in \{0,1\}^*, |w| \in H$, $\tilde{d}_{g\square}^{\,t}(w\square) > \alpha\,\tilde{d}_g^{\,t}(w\square)$. Since $S \in S^\infty[d_g]$, there are infinitely many $n \in N$ such that $\alpha\,[2^{\triangle g(n,s')} - 2^{\triangle g(n,s)}]d_g((S[0,\dots,n-1]) > 1$. For all such n with $n \in H$ we have

$$\begin{aligned}
\tilde{d}_{g\square}^{\,s'}(S[0,\dots,n-1]\square) &> \alpha\,\tilde{d}_g^{\,s'}(S[0,\dots,n-1]\square) \\
&= \alpha\,[2^{\triangle g(n,s')} - 2^{\triangle g(n,s)}]d_g(S[0,\dots,n-1]) > 1
\end{aligned}$$

whence $\dim_g(S[0,\dots,n-1]) \leq s'$.

Proof of lemma 3.7. We will use the following lemma.

Lemma 4.11 *Let g be a smooth scale function, let $s \in [0,\infty)$ and let d_g^s be a constructive s^g-supertermgale, then, for all $\epsilon < 0$ computable, there exists a contructive $(s+\epsilon)^g$-termgale $\widetilde{d}_g^{(s+\epsilon)}$ such that $\widetilde{d}_g^{(s+\epsilon)}(w\Box) > 1$ when $d_g^s(w\Box) > 1$.*

Proof. *Let d_g^s be an s^g-supertermgale and define for each $n \in \mathbb{N}$ $d_n : \mathcal{T} \to [0,\infty)$*

$$d_n(w) = \begin{cases} 2^{-|w|+\sum_{n=0}^{|w|-1}\Delta g(n,s+\epsilon)} \sharp\{v | w \sqsubseteq v, d_g^s(v\Box) > 1\} & \textit{if } |w|-1 < n \\ 0, & \textit{otherwise} \end{cases}$$

and

$$d_n(w\Box) = \begin{cases} 2^{\Delta g(n,s+\epsilon)} d_n(w) & \textit{if } |w| = n \\ 0, & \textit{otherwise} \end{cases}$$

then d_n is a constructive $(s+\epsilon)$-termgale for each n. let t such that $s < t < s+\epsilon$ and let $d = \sum_{n=1}^{\infty} 2^{g(n,t)-g(n,s)} d_n$

For $w \in \mathcal{T}$, $d(w) = \sum_{n=1}^{|w|} 2^{g(n,t)-g(n,s)} d_n(w)$ so, d is a constructive $(s+\epsilon)^g$-termgale and if $d_g^s(w) > 1$ then

$$\begin{aligned} d(w\Box) &= 2^{\Delta g(|w|,s+\epsilon)} d_{|w|}(w) 2^{g(|w|,t)-g(|w|,s)} \\ &= 2^{|w|-\sum_{n=1}^{\infty}\Delta g(n,s+\epsilon)} 2^{g(|w|,t)-g(|w|,s)} \geq 1 \end{aligned}$$

□

Suppose that $\dim_g(w) = s_0$ it's suffices to show that for all $\epsilon > 0$ there exists a constructive g-termgale (depending on ϵ) such that $\dim_d(w) \leq s_0 + \epsilon$.

By definition, $\dim_d(w) = \inf\{s | d^s(w\Box) > 1\}$ and $\dim_d(w) \leq s_0 + \epsilon$ if and only if $d^{s_0+\epsilon}(w\Box) > 1$.

Since $\dim_g(w) = s_0$ then $d_\Box^{(s_0+\epsilon/2)}(w\Box) > 1$ and by the last lemma, there exist a constructive $[(s_0+\epsilon/2)+\epsilon/2]^g$-termgale such that $d^{s_0+\epsilon} > 1$. Let d the constructive g-termgale constructed using this $(s_0+\epsilon)^g$-termgale.

Proof of Theorem 4.2.

Let $\mathbf{m}$ be the optimal subprobability measure on $\{0,1\}^*$ such that for all $w \in \{0,1\}^*$,

$$\left| K(w) - \log \frac{1}{\mathbf{m}(w)} \right| \leq c$$

For all $w \in \{0,1\}^*, |w| \in H$ and $s \in [0,\infty)$,

$$\begin{aligned} d_g[\mathbf{m}](w\square) > 1 &\Leftrightarrow 2^{g(|w|+1,s)}\mathbf{m}(w) > 1 \\ &\Leftrightarrow g(|w|+1,s) > \log \frac{1}{\mathbf{m}(w)} \end{aligned}$$

If $|w|$ is sufficiently large, $g(m,0) > \log \frac{1}{\mathbf{m}(w)}$ and exists $s_w \in [0,\infty)$ such that
$g(|w|+1,s_w) = \log \frac{1}{\mathbf{m}(w)}$. Then $\dim_{d_g[\mathbf{m}]}(w) = s_w$. So,

$$|K(w) - g(|w|+1,s_w)| = \left|K(w) - \log \frac{1}{\mathbf{m}(w)}\right| < c.$$

On the other hand,

$$\begin{aligned} |K(w) - g(|w|+1,s_w)| &= |g(|w|+1, f^{|w|+1}(K(w)) - g(|w|+1,s_w)| \\ &= \frac{\partial g}{\partial s}(|w|+1,s'_w)|f^{|w|+1}(K(w)) - s_w| \end{aligned}$$

$$\min\{f^{|w|+1}(K(w)), s_w\} \le s'_w \le \max\{f^{|w|+1}(K(w)), s_w\}.$$

Then

$$|f^{|w|+1}(K(w)) - \dim_{d_g[\mathbf{m}]}(w)| < \frac{c}{\frac{\partial g}{\partial s}(|w|+1,s'_w)} < \frac{c}{\frac{\partial g}{\partial s}(|w|+1,0)}.$$

By corollary 3.5,

$$|\dim_{d_g[\mathbf{m}]}(w) - \dim_g(w)| \le \frac{c'}{\frac{\partial g}{\partial s}(|w|+1,0)}$$

$$\begin{aligned} |f^{|w|+1}(K(w)) - \dim_g(w)| &\le |f^{|w|+1}(K(w)) - \dim_{d_g[\mathbf{m}]}(w)| \\ &+ |\dim_{d_g[\mathbf{m}]}(w) - \dim_g(w)| \\ &< \frac{c+c'}{\frac{\partial g}{\partial s}(|w|+1,0)} \end{aligned}$$

Proof of Theorem 4.6.

For each $s \in [0,\infty)$ we have that,

$$\begin{aligned}\log d_g^s(w\Box) &= g(|w|+1,s) + \sum_{i=0}^{|w|-1} \log \Pi_d(w[0\ldots i-1], w[i]) + \log \Pi_d(w,\Box) \\ &= g(|w|+1,s) - \mathcal{L}_{\Pi_d}^{\log}(w\Box)\end{aligned}$$

Then,

$$\begin{aligned}\dim_{d_g}(w) &= \inf\{s | d_\Pi(w\Box) > 1\} \\ &= \inf\{s \mid g(|w|+1,s) - \mathcal{L}_{\Pi_d}^{\log}(w\Box) > 0\} \\ &= f^{|w|+1}(\mathcal{L}_{\Pi_d}^{\log}(w\Box))\end{aligned}$$

Proof of Theorem 4.7.

By proposition 3.4 and theorem 3.6, for all Π constructive termpredictor, if $d_{\Pi,g}$ is the constructive termgale defined as in lemma 4.5 then

$$\dim_g(S) = \liminf_n \dim_g(S[0\ldots n-1]) \leq \liminf_n \dim_{d_{\Pi,g}}(S[0\ldots n-1])$$

and by theorem 4.6 we have the result.

Proof of Theorem 4.8.

We will use the following lemma.

Lemma 4.12 *Let g be a smooth scale function,*

1. *For all $m \in \mathbb{N}$, the function f^m is increasing.*
2. *f^m is differentiable for all $m \in \mathbb{N}$ and $\frac{\partial f^m}{\partial x}(\cdot)$ are strictly decreasing for all $m \in H$.*
3. $\lim_m \frac{\partial f^m}{\partial x}(g(m,0)) = 0$

Proof. *The result is a direct consequence of the properties of smooth scale functions.* □

It's suffices to show that for a fixed $S \in \mathbf{C}$, for all constructive predictor Π with $\mathcal{L}^{\log}_{\Pi,g}(S) < \infty$, there exists a constructive termpredictor $\widetilde{\Pi}$ such that

$$\mathcal{L}^{\log}_{\Pi,g}(S) = \mathcal{L}^{\log}_{\widetilde{\Pi},g}(S)$$

That is,

$$\liminf_n \left[f^{n+1}(\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1])\Box) - f^{n+1}(\mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1])) \right] = 0$$

By the last lemma and the Mean Value Theorem,

$$\liminf_n \left[f^{n+1}(\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box)) - f^{n+1}(\mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1])) \right] =$$
$$\liminf_n \frac{\partial f^{n+1}}{\partial x}(\xi)[\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box)) - \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]))] \leq$$
$$\liminf_n \frac{\partial f^{n+1}}{\partial x}(g(n+1,0))(\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box)) - \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]))]$$

where $\xi \in [\mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]), \mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box)]$.

Then, if for all constructive predictor Π with $\mathcal{L}^{\log}_{\Pi,g}(S) < \infty$, there exists a constructive termpredictor $\widetilde{\Pi}$ such that, for n enough large,

$$\liminf \left(\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box) - \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]) \right) \leq C$$

then we have the result because $\liminf_n \frac{\partial f^{n+1}}{\partial x}(g(n+1,0)) = 0$.

Let for all $w \in \{0,1\}^*$, $b \in \{0,1\}$

$$\begin{aligned} \widetilde{\Pi}(w,b) &= \frac{\Pi(w,b)}{2} \\ \widetilde{\Pi}(w,b) &= 1/2 \end{aligned}$$

It's clear that $\widetilde{\Pi}$ is a constructive termpredictor and

$$\liminf_n \left(\mathcal{L}^{\log}_{\widetilde{\Pi}}(S[0\ldots n-1]\Box) - \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]) \right) =$$
$$\sum_{k=0}^{n-1} \left(\frac{1}{\widetilde{\Pi}(S[0\ldots k-1], S[k])} - \frac{1}{\Pi(S[0\ldots k-1], S[k])} \right) + \frac{1}{\widetilde{\Pi}(S[0\ldots n-1], \Box)} =$$
$$\liminf_n \sum_{k=0}^{n-1} \frac{1}{\Pi(S[0\ldots k-1], S[k])} - 2 = \liminf_n \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1]) - 2 < C$$

since $\mathcal{L}^{\log}_{\Pi,g}(S) < \infty$ implies that $\liminf_n \mathcal{L}^{\log}_{\Pi}(S[0\ldots n-1] < \infty$.

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