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# Randomness-Efficient Sampling within $NC^1$

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**Abstract**

We construct a randomness-efficient *averaging sampler* that is computable by uniform constant-depth circuits with parity gates (i.e., in uniform $AC^0[\oplus]$). Our sampler matches the parameters achieved by random walks on constant-degree expander graphs, allowing us to apply a variety expander-based techniques within $NC^1$. For example, we obtain the following results:

- Randomness-efficient error-reduction for uniform probabilistic $NC^1, TC^0, AC^0[\oplus]$ and $AC^0$: Any function computable by uniform probabilistic circuits with error $1/3$ using $r$ random bits is computable by uniform probabilistic circuits with error $\delta$ using $O(r + \log(1/\delta))$ random bits.
- An optimal explicit $\epsilon$-biased generator in $AC^0[\oplus]$: There exists a $1/2^{\Omega(n)}$-biased generator $G : \{0,1\}^{O(n)} \to \{0,1\}^{2^n}$ for which poly($n$)-size uniform $AC^0[\oplus]$ circuits can compute $G(s)_i$ given $(s,i) \in \{0,1\}^{O(n)} \times \{0,1\}^n$. This resolves a question raised by Gutfreund and Viola (*Random 2004*).
- uniform $BP \cdot AC^0 \subseteq$ uniform $AC^0/O(n)$.

Our sampler is based on the *zig-zag graph product* of Reingold, Vadhan and Wigderson (*Annals of Math 2002*) and as part of our analysis we give an elementary proof of a generalization of Gillman's *Chernoff Bound for Expander Walks* (*FOCS 1998*).

## 1 Introduction

Over the last three decades, *expander graphs* have found a wide variety of applications in Theoretical Computer Science. They have been used in designing novel algorithms (e.g., [AKS83], [JS89], [Rei05]), in the study of circuit complexity (e.g., [Val77], [IW97]) and to derandomize probabilistic computation (e.g., [CW89], [IZ89]), just to name a few notable examples from this vast literature.

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Many of these applications involve a *random walk* on an expander. That is, we choose a random starting node $v$ in an expander graph $G$, take a $k$-step random walk and use the $k$ nodes visited by this walk in some way – often as a substitute for $k$ independently-chosen nodes. Despite its simplicity, this processes has some remarkable sampling properties which we discuss shortly. For the moment, we address the computational efficiency of expanders walks.

In applications, one often requires an expander graph that is exponentially large, say on $2^n$ nodes. In this case, a random walk on the graph is performed using an efficient *explicit* representation – that is, a representation in which each node is identified with an $n$-bit string and it is possible to efficiently (e.g., in time $\text{poly}(n)$) find all the neighbors of a given node $v \in G$. Several beautiful constructions [Mar73, GG81, LPS88, RVW02] are known of such explicit constant-degree expander graphs of exponential size.

At first glance, the act of taking a random walk on an expander graph seems like an inherently *sequential* process – indeed, each step of the walk seems to rely on the previous step in an essential way. A natural question, therefore, is whether the wealth of expander-based techniques from the literature can be applied within highly-*parallel* models of computation, such as log-depth circuits (i.e., $NC^1$) or even constant depth circuits.

The main technical contribution of this work is a *sampler* which is just as good as a random walk on an expander graphs (in a sense that is made precise in the next section), but which is computable in parallel time $O(\log n)$, i.e. computable by uniform $NC^1$ circuits. In fact, our sampler is computable by uniform constant-depth circuits with parity gates (i.e. $AC^0[\oplus]$), a class which is strictly weaker than $NC^1$ as it cannot even compute the majority of $n$ bits [Raz87].

We now discuss the important sampling properties of random walks on expander graphs in order to better understand what properties we require of our sampler. A more formal definition of expander graphs will be given in Section 3, but for the moment the reader may simply think of an expander graph as a constant-degree undirected graph, $G$, that is "highly-connected".

A fundamental sampling property of expander walks is the *hitting* property, first shown by Ajtai, Komlós and Szemerédi [AKS87]:

**The Hitting Property:** For any subset $S$ of half the nodes of $G$, the probability that a $k$-step random walk never visits a node in $S$ is at most $2^{-\Omega(k)}$.

This hitting property is quite useful (e.g. to reduce the error of $RP$ algorithms), but some applications require an even stronger property, which we call the *strong hitting* property:

**The Strong Hitting Property:** For any sequence of subsets $S_1, \ldots, S_k$, each consisting of half the nodes of $G$, the probability that a $k$-step random walk does not pass through $S_i$ on the $i$-th step for any $i \in \{1, \ldots, k\}$ is at most $2^{-\Omega(k)}$.

It turns out that this strong hitting property is what is necessary for the randomness-efficient error reduction techniques of [CW89] and [IZ89] and for the derandomized XOR Lemma of [IW97], as well as a variety of other applications.

Clearly, the strong hitting property is a generalization of the (non-strong) hitting property. Another natural generalization of the hitting property is the following, first proved by Gillman [Gil94]:

**The Chernoff Bound for Expander Walks:** For any subset $S$ of half the nodes of $G$, the fraction of time that a $k$-step random walk spends in $S$ is $1/2 \pm \epsilon$ with probability $1 - 2^{-\Omega(\epsilon^2 k)}$.

This Chernoff Bound is quite powerful and has applications Markov-Chain Monte Carlo algorithms (see [Gil94]). However, it is not clear that it subsumes the *strong* hitting property. The following property, however, generalizes both the strong hitting property *and* the Chernoff bound:

**The Strong Chernoff Bound for Expander Walks:** Fix a sequence of subsets $S_1, \dots, S_k$, each consisting of half the nodes of $G$. Then for a $k$-step random walk on $G$, the fraction of indices $i$ such that the $i$-th step of the walk lands in $S_i$ is $1/2 \pm \epsilon$ with probability $1 - 2^{-\Omega(\epsilon^2 k)}$.

Thus, the Strong Chernoff Bound for Expander Walks subsumes all the aforementioned sampling properties, and it seems to represent the essential abstract property of random walks on expanders that is necessary for most natural applications. This bound has only been proved recently – it follows from a more general result of Wigderson and Xiao [WX05].

In this paper, we give a direct and elementary proof of the Strong Chernoff Bound for Expander Walks (Theorem 1). In contrast to most of the proofs in this area, our proof uses only basic linear algebra and, in particular, does not require any perturbation theory or complex analysis in order to obtain a bound that matches the parameters of Gillman's (non-strong) Chernoff bound.[1] Since this bound is important to our analysis, we give a more formal statement of the bound before describing our results in more detail. (In the following, a $\lambda$-*expander* is a regular graph whose normalized second-largest eigenvalue (in absolute value) is at most $\lambda$ – see Section 3 for a precise definition.)

**Theorem 1** (Implicit in [WX05]). *Let $G$ be a regular $\lambda$-expander on $V$. Fix a sequence of subsets $S_i \subseteq V$ each of density $\rho_i = |S_i|/|V|$, and for a random walk $v_1, \dots, v_k$ on $G$, let $T$ be the random variable that counts the number of steps $i$ such that $v_i \in S_i$. Then for all $\epsilon > 0$,*

$$\Pr\left[\left|T - \sum_{i=1}^{k} \rho_i\right| \geq \epsilon k\right] \leq 2e^{-\epsilon^2(1-\lambda)k/36}.$$

[1] [WX05] also gives a proof of a (strong) Chernoff bound using no perturbation theory, but their bound does not match Gillman's. It should be noted, however, that [WX05] considers the more general setting of matrix-valued functions.

## 2 Our Results

Our main result is the construction of a *sampler* that is computable by $AC^0[\oplus]$ circuits and possesses all the "sampling properties" of a random walk on a constant-degree expander graphs of size $2^n$. To make this notion precise, we recall the following definition (essentially from [Zuc97]):

**Definition 2.** *A function* $\Gamma : \{0,1\}^m \to (\{0,1\}^n)^k$ *is said to be a* strong $(\gamma, \epsilon)$-averaging[2] boolean sampler *if: for any sequence of functions* $f_i : \{0,1\}^n \to \{0,1\}$ *each with mean* $\mu_i = \Pr_x[f_i(x) = 1]$,

$$\Pr_s\left[\left|\frac{1}{k}\sum_{i=1}^{k}(f_i(\Gamma(s)_i) - \mu_i)\right| \leq \epsilon\right] \geq 1 - \gamma.$$

*We call* $m$ *the* seed-length *of the sampler, and we call* $k$ *the* sample complexity *of the sampler.*

It is not hard to check that Theorem 1 implies that a random walk on a constant-degree expander (with $\lambda = 1 - \Omega(1)$) of size $2^n$ is a strong averaging boolean sampler with seed-length $m = n + O(\log(1/\gamma)/\epsilon^2)$ and sample complexity $k = O(\log(1/\gamma)/\epsilon^2)$. Our main theorem is that uniform $AC^0[\oplus]$ can compute a sampler that is just as good (up to constant factors):

**Theorem 3.** *There exists a strong* $(\gamma, \epsilon)$*-averaging boolean sampler* $\Gamma : \{0,1\}^m \to (\{0,1\}^n)^k$*, with* $m = O(n + \log(1/\gamma)/\epsilon^2)$ *and* $k = O(\log(1/\gamma)/\epsilon^2)$*, that is computable by* uniform $AC^0[\oplus]$ *circuits of size* $\mathrm{poly}(n, 1/\epsilon, \log(1/\gamma))$.

At this point, the reader may wish to disregard the exact parameters of our construction, and instead think of our construction as computing (intuitively) a walk of length $k$ on a constant-degree expander graph of size $2^n$. Indeed, in most natural applications that employ random walks on expander graphs, one can safely substitute a sampler with the above parameters in place of the expander walk.

Gutfreund and Viola have shown [GV04] that walks on the Margulis/Gabber-Galil expander graph [Mar73, GG81] with $2^n$ nodes are computable in space $O(\log n)$ (and therefore that logspace has strong samplers that match the above parameters). To the best of our knowledge, ours is the first work that implies the existence of such strong samplers within the class $NC^1$ of log-depth circuits; in fact, our construction is in the strictly-weaker class $AC^0[\oplus] \subsetneq TC^0 \subseteq NC^1 \subseteq L$.

Since expander walks are a powerful and widely-applicable tool it is not surprising that our sampler construction should have a variety of applications. Indeed, we apply our construction to obtain the following new results:

**Randomness-Efficient Error Reduction within $\mathbf{NC^1}$** One important application of random walks on expander graphs is in reducing the error of probabilistic algorithms. Such error reduction

[2] [Zuc97] uses the term "oblivious sampler". We follow [Gol97] and use the more-accurate "averaging sampler".

was achieved for $BPP$ by Cohen and Wigderson [CW89] and Impagliazzo and Zuckerman [IZ89]. Bar-Yosef, Goldreich and Wigderson [BYGW99] show how to achieve modest-but-optimal error reduction for randomized logspace, and the expander walks of Gutfreund and Viola [GV04] imply randomness-efficient error reduction for the class $BP{\cdot}L$.[3] By applying our sampler construction, we obtain analogous error-reduction for a variety of classes below logspace:

**Lemma 4.** *Let $f : \{0,1\}^n \to \{0,1\}$ be a function computable by polynomial-size* uniform $BP \cdot AC^0[\oplus]$ *(respectively, $BP{\cdot}TC^0$ or $BP{\cdot}NC^1$) circuits with error at most $1/3$ using $r = r(n)$ random bits. Then for any $\delta = \delta(n) > 1/2^{O(\mathrm{poly}(n))}$, $f$ has polynomial-size* uniform $BP \cdot AC^0[\oplus]$ *(respectively $BP \cdot TC^0$ or $BP \cdot NC^1$) circuits with error at most $\delta$ using $O(r + \log(1/\delta))$ random bits.*

Combining our sampler with Nisan's unconditional pseudorandom generator for constant depth circuits [Nis91], we obtain an even stronger result for uniform $AC^0$:

**Lemma 5.** *Let $f : \{0,1\}^n \to \{0,1\}$ be a function computable by polynomial-size* uniform $BP \cdot AC^0$ *circuits with error at most $1/3$ using $r = r(n)$ random bits. Then for any $\delta = \delta(n) > 1/2^{O(\mathrm{poly}(n))}$, $f$ has polynomial-size* uniform $BP \cdot AC^0$ *circuits with error at most $\delta$ using $O(\min\{r, \mathrm{polylog}(n)\} + \log(1/\delta))$ random bits.*

**Derandomization with Linear Advice** Recently, Fortnow and Klivans [FK06] have proved that $RL \subseteq L/O(n)$ – that is, one can derandomize randomized logspace computation at the cost of only a linear amount of non-uniform advice. Their approach is based on a clever combination of Nisan's pseudorandom generator for space-bounded computation [Nis92] and the logspace expander walks of Gutfreund and Viola [GV04]. Our techniques yield an analogous result for uniform probabilistic constant-depth circuits:

**Corollary 6.** uniform $BP \cdot AC^0 \subseteq$ uniform $AC^0/O(n)$.

Ajtai and Ben-Or [ABO84] have shown that nonuniform $BP \cdot AC^0 =$ nonuniform $AC^0$; even for derandomizing uniform $BP \cdot AC^0$ [Ajt93], however, their technique seems to require an arbitrary polynomial amount of non-uniform advice. Theorem 6 quantifies the amount of nonuniformity that is necessary to derandomize a probabilistic $AC^0$ circuit, and therefore can be viewed as a refinement of their result.

A similar approach, together with a new pseudorandom generator of Viola [Vio05], yields the following:

**Corollary 7.** *Let $AC^0[\oplus_{\log}]$ be the class of boolean functions computable by* $\mathrm{poly}(n)$*-size $AC^0$ circuits having $O(\log n)$ parity gates, and similarly let $AC^0[\mathrm{SYM}_{\log}]$ be the class of boolean functions computable by* $\mathrm{poly}(n)$*-size $AC^0$ circuits having $O(\log n)$ arbitrary symmetric gates (e.g., parity and majority gates). Then the following inclusions hold:*

[3] $BP \cdot L$ refers to randomized logspace computations that are allowed *two-way* access to the random bits, whereas the result of Bar-Yosef et al. refers to algorithms that have only *one-way* access to the random bits. See the survey of Saks [Sak96] for a discussion of the subtleties surrounding different notions of randomized space-bounded computation.

1. $BP \cdot AC^0[\oplus_{\log}] \subseteq AC^0[\oplus]/O(n)$
2. $BP \cdot AC^0[\mathrm{SYM}_{\log}] \subseteq TC^0/O(n)$

**An Optimal Explicit $\epsilon$-Biased Generator in $\mathbf{AC^0}[\oplus]$** Gutfreund and Viola [GV04] study the complexity of constructing *explicit* $\epsilon$-biased generators (see Definition 9). They give a construction in $AC^0[\oplus]$ whose seed-length is optimal for $\epsilon = \Omega(1/\mathrm{poly}\log\log(m))$ (where $m$ is the number of output bits) and sub-optimal for smaller $\epsilon$. Healy and Viola [HV06] give an optimal construction in $TC^0$ and a sub-optimal construction in $AC^0[\oplus]$ whose parameters are incomparable to those of [GV04]. In this work, we resolve this question entirely: using our sampler construction, we construct an *optimal* explicit $\epsilon$-biased generator in $AC^0[\oplus]$:

**Corollary 8** ([NN90] + [GV04] + Theorem 3)**.** *For every $\epsilon > 0$ and $m$, there is an $\epsilon$-biased generator $G : \{0,1\}^n \to \{0,1\}^m$ with $n = O(\log m + \log(1/\epsilon))$ for which uniform $AC^0[\oplus]$ circuits of size* $\mathrm{poly}(n, \log m) = \mathrm{poly}(n)$ *can compute $G(s)_i$ given $(s,i) \in \{0,1\}^n \times [m]$.*

It can be shown that an explicit $\epsilon$-biased generators acheiving the parameters of Corollary 8 requires $AC^0$ circuits of exponential size (see [GV04] and [MNT90]). Therefore, the construction of Corollary 8 is tight both in terms of seed-length and computational complexity.

## 3 Preliminaries

For a positive integer $n$, we denote the set $\{1, \ldots, n\}$ by $[n]$.

**$\epsilon$-Biased Sets and Generators** Small-biased spaces appear in two ways in this work. First, poly-size $\epsilon$-biased sets are used to construct expander graphs on which our sampler construction is based (Lemma 11). Second, one of the applications of our sampler is to build exponential-size $\epsilon$-biased sets which are computable *explicitly* (see the definition below and Corollary 8).

**Definition 9.** *For $a, b \in \mathbb{F}_2^m$, let $\langle a, b\rangle_2$ denote the inner product of $a$ and $b$ modulo 2.*

*A multi-set $S \subseteq \mathbb{F}_2^m$ is* $\epsilon$-biased *if for all non-zero $y \in \mathbb{F}_2^m$, $\Pr_{x\in S}[\langle x, y\rangle_2 = 1] \in [1/2 - \epsilon, 1/2 + \epsilon]$.*

*An* $\epsilon$-biased generator *is a function $\Gamma : \{0,1\}^\ell \to \{0,1\}^m$ whose range is an $\epsilon$-biased multi-set.*

*An* explicit *$\epsilon$-biased generator is a function $\Gamma : \{0,1\}^\ell \times [m] \to \{0,1\}$ such that the function $\Gamma'(s) = (\Gamma(s,0), \Gamma(s,1), \ldots, \Gamma(s, m-1))$ is an $\epsilon$-biased generator.*

**Expander Graphs** Informally, expander graphs are sparse-yet-highly-connected graphs. While there are a variety of equivalent notions of graph expansion (see, e.g., [Gol99] and the references therein), it will be most convenient for us to work with the spectral definition.

**Definition 10.** *A regular graph $G$ of degree $d$ is a* $\lambda$-expander *if the second-largest eigenvalue (in absolute value) of its probability transition matrix (i.e., $1/d$ times its adjacency matrix) is at most $\lambda$.*

We will often abuse language and refer to an "$\lambda$-expander", when we really mean a "family of $\lambda(n)$-expanders of size $s(n)$" for some function $s(n)$. Also, when we simply refer to an "expander graph", without mention of $\lambda$, it is understood that we mean a $(1 - \Omega(1))$-expander.

By a *random walk* $v_1, \ldots, v_k$ on an $d$-regular graph $G$, we mean the following process: Choose a random starting vertex $v_0 \in G$, and for $i = 1, \ldots, k$, let $v_i$ be a uniformly random neighbor of $v_{i-1}$ in $G$. Note that we are implicitly discarding the start vertex $v_0$ – while it is easy to see that the distribution is unchanged even if we keep $v_0$, we prefer this convention as it will simplify our notation and presentation. We also note that such a walk is described by a tuple $(v_0, s_1, \ldots, s_k) \in [|G|] \times [d] \times \cdots \times [d]$, and hence by a string of $O(\log |G| + k \log d)$ bits.

**Constant-Depth Circuits** We consider three classes of unbounded fan-in constant-depth circuits: circuits over the bases $\{\wedge, \vee\}$ (i.e., $AC^0$), $\{\wedge, \vee, \mathit{Parity}\}$ (i.e., $AC^0[\oplus]$), and $\{\wedge, \vee, \mathit{Majority}\}$ (i.e., $TC^0$). Unless explicitly stated otherwise, all circuits are of polynomial size and *uniform* – specifically, we adopt the standard of *Dlogtime*-uniformity, a notion of uniformity which is even more restrictive than logspace-uniformity and which has become the generally-accepted convention for uniformity in constant-depth circuits [BIS90]. Informally, a circuit is *Dlogtime*-uniform if, given indices of two gates in the circuit, one can determine the types of the gates and whether they are connected in linear time in the length of the indices (which is logarithmic in the size of the circuit).

When referring to non-uniform circuits, we always indicate this explicitly using *slash* notation: for example, $AC^0/O(n)$ is the class of boolean functions $f$ that are computable by a *Dlogtime*-uniform $AC^0$ circuit family $C_n : \{0,1\}^n \times \{0,1\}^{O(n)} \to \{0,1\}$ for which there is a single advice string $a_n$ of length $O(n)$ such that $C_n(x, a_n) = f(x)$ for all $x \in \{0,1\}^n$.

The probabilistic classes $BP \cdot AC^0, BP \cdot AC^0[\oplus], BP \cdot TC^0$ and $BP \cdot NC^1$ are all defined in the natural way: the circuit takes two inputs, one of $n$ bits and one of $r(n)$ random bits for some polynomially-bounded function $r(n)$, and for any fixed input $x \in \{0,1\}^n$, the circuit should correctly compute the function with probability at least $2/3$ over the $r(n)$ random bits.

Recall that $AC^0 \subsetneq AC^0[\oplus] \subsetneq TC^0 \subseteq NC^1 \subseteq$ logspace, where the last inclusion holds under logspace uniformity and the separations follow from works by Furst et al. [FSS84] and Razborov [Raz87], respectively (and hold even for non-uniform circuits). Despite these lower-bounds, $AC^0$ can compute the *approximate* majority of $n$ bits [Ajt93] – in particular, for any constant $\epsilon > 0$, there exists a family of $AC^0$ circuits that correctly computes the majority function for all inputs with at most a $n/2 - \epsilon n$ ones and for all inputs with at least $n/2 + \epsilon n$ ones. See, e.g., [Hås87, Vol99] for additional background on constant-depth circuits.

## 4 The Sampler Construction

In this section, we describe our sampler construction and prove Theorem 3. Recall that our goal is to construct a sampler $\Gamma : \{0,1\}^m \to (\{0,1\}^n)^k$ that matches the parameters of random walks on expander graphs. Naturally, one way to achieve this would be to exhibit a family of constant-degree expander graphs on $2^n$ nodes and show that walks of length $k$ on these expanders can be computed in $AC^0[\oplus]$ of size $\mathrm{poly}(n, k)$. Unfortunately, we do not know of any such family of expanders. Instead, we begin with a family of expander graphs of degree $\mathrm{poly}(n)$ where walks are computable in $AC^0[\oplus]$ – note that a walk of length $k$ on such a graph is described by a seed of length $n + O(k \cdot \log n)$ – and then we *derandomize* the walk on this graph to achieve the optimal seed length $O(n+k)$. This derandomization uses random walks on a smaller expander graph, and its analysis is based on the *zig-zag graph product* of [RVW02]. We now describe the construction in more detail.

Our first graph, $G$, is a Cayley graph on the group $\mathbb{F}_2^n$. Specifically, we construct a $1/n$-biased set $S \subset \mathbb{F}_2^n$ of size $\mathrm{poly}(n)$ (see Definition 9) and let $\{v, w\}$ be an edge if and only if $v - w \in S$. The following well-know fact guarantees that $G$ has second-largest eigenvalue at most $2/n$ (e.g., see [AR94]).

**Lemma 11.** *A Cayley graph on $\mathbb{F}_2^n$ with generators $S \subset \mathbb{F}_2^n$ is a $2\epsilon$-expander if and only if $S$ is $\epsilon$-biased.*

Before continuing, let us see how walks on $G$ can be computed in $AC^0[\oplus]$. First, we note that a $1/n$-biased set $S$ of size $\mathrm{poly}(n)$ can be constructed in $AC^0$. For instance, we may use the "Powering Construction" of an $\epsilon$-biased generator from [AGHP92] together with the results on field arithmetic of [HV06].[4] (Note that if we only wished to give a non-uniform construction, we could simply hard-wire such an $\epsilon$-biased set into the circuit.)

Thus, given the description a walk $(v, s_1, \ldots, s_k) \in \{0,1\}^n \times \{0,1\}^{O(\log n)} \times \cdots \times \{0,1\}^{O(\log n)}$, to determine the $i$-th vertex visited by the walk, the circuit need only compute from each index $s_j$ (in parallel) the appropriate vector $v_{s_j} \in S$ and then compute the sum

$$v + \sum_{j=1}^{i} v_{s_j}.$$

[4]Specifically, let $m = \log n$ (assuming that $\log n$ is an integer for simplicity) and consider the finite field $\mathbb{F}_{2^{2m}}$ with $2^{2m}$ elements (viewed as the ring of polynomials over $\mathbb{F}_2$ modulo an irreducible polynomial of degree $2m$). The generator outputs $2^{4m} = n^4$ vectors $v_{\alpha,\beta}$ of dimension $2^m = n$, indexed by pairs of elements $\alpha, \beta \in \mathbb{F}_{2^{2m}}$, where the $i$-th bit of $v_{\alpha,\beta}$ is given by $\langle \alpha^i, \beta \rangle \pmod 2$. It is shown in [AGHP92] that such a generator has bias less than $2^m/2^{2m} = 1/n$, and it is shown in [HV06] that all the necessary field arithmetic can be carried out in uniform $AC^0$ of size $\mathrm{poly}(n)$ for this range of parameters.

Since the summation is modulo 2, this is easily seen to be computable in $AC^0[\oplus]$ of size $\mathrm{poly}(n,k)$.

Now we turn to the problem of producing a *pseudorandom* sequence of steps $s_i$, with the goal of reducing the seed length of a walk on $G$, while at the same time preserving the sampling properties of such walks. Our approach is motivated by the zig-zag product of Reingold, Vadhan and Wigderson [RVW02]. Roughly speaking, one may interpret their results as saying the following: to derandomize a walk on a graph $G$ of degree d, it suffices to choose the steps in $G$ according to a random walk on a constant-degree expander graph $H$ of size $d$. (For technical reasons, their result requires the graph $H$ to be the *square* of an expander graph, but we will ignore this for the moment.) Specifically, to take a pseudorandom $k$-step walk in $G$:

1. Choose a random starting vertex $v_0 \in G$
2. Choose a random $w_0 \in H$ and take a random walk of length $k$, visiting nodes $w_1, \ldots, w_k$
3. View $w_1, \ldots, w_k$ as indices in $[d] = [|H|]$
4. Use $w_1, \ldots, w_k$ as the steps of a walk (starting at $v_0$) in $G$
5. Output the nodes $v_1, \ldots, v_k \in G$ visited by this walk

Note that the seed-length of such a sampler is of size $|v_0| + (|w_0| + O(k)) = n + \log|H| + O(k) = O(n+k)$, as desired. Moreover, one can show (using the results of [RVW02]) that the above construction is a strong averaging sampler. What is not clear, however, is how to compute this sampler in $AC^0[\oplus]$. The reason is that it requires a long walk on the graph $H$, and while $H$ is small (only $\mathrm{poly}(n)$ nodes) compared to $G$ (which has $2^n$ nodes), we do not know how to take such a long walk on any constant-degree expander family in $AC^0[\oplus]$ (or even in $NC^1$ for that matter).

In order to circumvent this obstacle, we derandomize the walk on $G$ by using many short walks on $H$, rather than a single long walk.

**Construction 12.**

1. *Choose a random starting vertex* $v_0 \in G$
2. *Take* $k/\log n$ *random walks of length* $\log n$ *in* $H$*, where the* $i$*-th walk visits* $w_1^{(i)}, \ldots, w_{\log n}^{(i)} \in H$
3. *View* $w_1^{(1)}, \ldots, w_{\log n}^{(1)}, w_1^{(2)}, \ldots, w_{\log n}^{(2)}, \ldots, w_1^{(k/\log n)}, \ldots, w_{\log n}^{(k/\log n)}$ *as indices in* $[d] = [|H|]$
4. *Use* $w_1^{(1)}, \ldots, w_{\log n}^{(1)}, \ldots, w_1^{(k/\log n)}, \ldots, w_{\log n}^{(k/\log n)}$ *as the steps of a walk (starting at* $v_0$*) in* $G$
5. *Output the nodes* $v_1, \ldots, v_k \in G$ *visited by this walk*

This sampler has seed-length

$$|v_0| + \sum_{i=1}^{k/\log n} (|w_0^{(i)}| + O(\log n)) = n + \frac{k}{\log n} \cdot O(\log n) = O(n+k).$$

Furthermore, we show below that this construction is a strong averaging sampler, achieving essentially the same parameters as a random walk on an expander graph. Before proving this, however, we observe that it is computable in $AC^0[\oplus]$. Indeed, it is known how to compute walks of length $O(\log n)$ on poly-sized explicit expanders of constant degree in $AC^0$ [Ajt93, GV04],[5] and thus each of the five steps above is computable in constant depth.

We now show that Construction 12 is a strong averaging sampler. In particular, Theorem 3 is a consequence of the following lemma:

**Lemma 13.** *Let $H = \tilde{H}^2$ where $\tilde{H}$ is a constant-degree expander graph on* $\mathrm{poly}(n)$ *nodes. Then Construction 12 is a strong averaging boolean sampler with seed length $O(n + \log(1/\gamma)/\epsilon^2)$ and sample complexity $O(\log(1/\gamma)/\epsilon^2)$.*

*Proof.* Our proof relies on the zig-zag product of [RVW02], so we briefly recall that construction.

**Zig-Zag Product** Let $G$ be a regular graph of degree $d$ on vertices $V_G$ whose edges are labeled with the names $1, \ldots, d$ in such a way that no two incident edges share the same label.[6] (Note that under such a labeling, if $w$ is the "$i$-th neighbor of $v$", then $v$ is the "$i$-th neighbor of $w$" – the graph $G$, defined above, clearly has this property, as it is a Cayley graph on a group of characteristic 2.) Then if $g$ is a regular graph on vertices $V_g$ where $|V_g| = d$, we may form the *zig-zag product* graph $G Ⓩ g$ where:

- $G Ⓩ g$ has vertices $V_G \times V_g$
- $\{(v, w), (v', w')\}$ is an edge if there is an $x \in g$ such that $v'$ is the $x$-th neighbor of $v$ in $G$ and $(w, x, w')$ is a path in $g$. (Note that the labeling condition on $G$ ensures this is symmetric.)

Thus, if we start at $(v, w) \in G Ⓩ g$, a step to a random neighbor $(v', w')$ has following form:

- Choose a random neighbor $x$ of $w$ in $g$.
- Set $v'$ to be the $x$-th neighbor of $v$ in $G$.
- Choose a random neighbor $w'$ of $x$ in $g$.

In particular, if we only consider the $V_G$-coordinate of a random walk of length $\ell$ in $G Ⓩ g$ (starting at a random vertex), it has the same distribution as the following process:

- Choose a random start vertex $v_0 \in V_G$.

[5] As with the $1/n$-biased set $S$ above, the delicate issue here is the uniformity of the circuits; if we only wish to give a nonuniform construction we could simply hard-wire all the possible walks of length $\log n$ into the circuit.

[6] The zig-zag product of [RVW02] actually holds in much greater generality; however, this simplification suffices for our application.

- Take a random walk $w_1, w_2 \ldots, w_\ell$ in $g^2$.
- For $i > 0$, let $v_i$ to be the $w_i$-th neighbor of $v_{i-1}$ in $G$.
- Output $v_1, v_2, \ldots, v_\ell$.

Thus, each of of the segments of length $k/\log n$ in our sampler construction corresponds to a random walk on $G \text{ⓩ} \tilde{H}$, projected onto the $V_G$-coordinate. But what about the boundaries between these segments? In this case, Construction 12 says we choose a new, entirely-random node of $\tilde{H}$ and then continue the walk on $G$. This is equivalent to taking a step on $G \text{ⓩ} K_d$, i.e. the zig-zag product of $G$ with a complete graph (with self-loops) on $d$ nodes. Therefore, the output of our sampler is the projection onto the $V_G$-coordinate of a random walk on a time-varying graph that is $G \text{ⓩ} \tilde{H}$ most of the time, and $G \text{ⓩ} K_d$ once every $\log n$ steps. We now show that this output satisfies Definition 2 for the desired parameters.

First we note for any function $f : V_G \to \{0, 1\}$ there is a natural *lift* of $f$ to $\hat{f} : V_G \times V_{\tilde{H}} \to \{0, 1\}$, defined by $\hat{f}(v, w) = f(v)$. It is clear that the lift $\hat{f}$ has the same average as $f$. Therefore, to conclude that the projection of a random walk yields a strong averaging sampler, it suffices to show that a random walk on the time-varying graph is a strong averaging sampler. By the remark after the proof of Theorem 1, it does not matter if the graph is varying over time: as long as it is a regular $\lambda$-expander at every point in time, Theorem 1 holds (and so the random walk is a good sampler). Thus, we are left with the task of showing that $G \text{ⓩ} \tilde{H}$ and $G \text{ⓩ} K_d$ are expanders. For this, we apply the following consequence of the main theorem of [RVW02]:

**Lemma 14** ([RVW02], Corollary to Theorem 4.3)**.** *Let $G$ be a regular graph of degree $d$ whose edges are labeled with $1, \ldots, d$ in such a way that no two incident edges share the same label, and let $g$ be a regular graph on $d$ nodes. If $G$ is a $\lambda_G$-expander and $g$ is a $\lambda_g$-expander, then $G \text{ⓩ} g$ is a $(\lambda_G + \lambda_g)$-expander.*

By Lemma 11, $G$ is a $2/n$-expander, and by assumption $\tilde{H}$ is a $(1 - \Omega(1))$-expander, and so by Lemma 14, $G \text{ⓩ} \tilde{H}$ is a $(1 - \Omega(1))$-expander.

It is not hard to see that $K_d$, the complete graph (with self-loops) on $d$ nodes, is a 0-expander, and therefore by Lemma 14, $G \text{ⓩ} K_d$ is a $2/n$-expander.

Thus our sampler stretches a seed of length $O(n+k)$ into $k$ samples of $n$ bits each that satisfy the bound from Theorem 1 with $\lambda = 1-\Omega(1)$. Specifically, the sampler approximates the mean of the $f_i$'s with error $\epsilon$ and confidence $1 - \gamma = 1 - e^{-\Omega(\epsilon^2 k)}$; in other words, the seed length is $O(n + k) = O(n + \log(1/\gamma)/\epsilon^2)$ and the sample complexity is $k = O(\log(1/\gamma)/\epsilon^2)$. $\square$

# 5 Proofs of Other Results

**Lemma 4.** *Let $f : \{0,1\}^n \to \{0,1\}$ be a function computable by polynomial-size* uniform $BP \cdot AC^0[\oplus]$ *(respectively, $BP \cdot TC^0$ or $BP \cdot NC^1$) circuits with error at most $1/3$ using $r = r(n)$ random bits. Then for any $\delta = \delta(n) > 1/2^{O(\mathrm{poly}(n))}$, $f$ has polynomial-size* uniform $BP \cdot AC^0[\oplus]$ *(respectively $BP \cdot TC^0$ or $BP \cdot NC^1$) circuits with error at most $\delta$ using $O(r + \log(1/\delta))$ random bits.*

*Proof sketch.* Let $C_f$ be a circuit computing $f$. Construct the circuit that, on input $x \in \{0,1\}^n$, runs $k = \Theta(\log(1/\delta))$ copies of $C_f$ in parallel, using independent random $r$-bit blocks of randomness, and then computes the $(5/12, 7/12)$-approximate majority of the outputs [Ajt93]. (For $BP \cdot TC^0$ and $BP \cdot NC$ we can just compute the majority exactly.) Now, instead of using independent random bits for each block, we apply the construction of $\Gamma : \{0,1\}^{O(r+k)} \to (\{0,1\}^r)^k$ from Theorem 3 (with $\epsilon = 1/12$ and $\gamma = \delta$) to generate the necessary random bits from a seed of length $O(r + k)$.

For any fixed input $x$, the probability that a randomly chosen $O(r + k)$-bit random string causes the algorithm to fail (i.e. that more than 5/12 of the outputs of $\Gamma$ fall in the $\leq 1/3$ fraction of random strings that cause $C_f$ to fail) is at most $2^{-\Omega(k)} = 2^{-\Omega(\Theta(\log 1/\delta))}$ since $\Gamma$ is an averaging sampler. By choosing the constants appropriately, this is at most $\delta$ and the result follows. ☐

**Lemma 5.** *Let $f : \{0,1\}^n \to \{0,1\}$ be a function computable by polynomial-size* uniform $BP \cdot AC^0$ *circuits with error at most $1/3$ using $r = r(n)$ random bits. Then for any $\delta = \delta(n) > 1/2^{O(\mathrm{poly}(n))}$, $f$ has polynomial-size* uniform $BP \cdot AC^0$ *circuits with error at most $\delta$ using $O(\min\{r, \mathrm{polylog}(n)\} + \log(1/\delta))$ random bits.*

*Proof sketch.* Let $C_f$ be a circuit computing $f$. By applying Nisan's pseudorandom generator for $BP \cdot AC^0$ [Nis91] (which has been shown to be computable in $AC^0$ in [Vio04]), we may assume, with no loss of generality, that $C_f$ uses only $r' = r'(n) = \min\{r(n), \log^c(n)\}$ random bits for some constant $c$ that may depend on $f$.

By applying the construction of Lemma 4 with $\delta(n) = 2^{-r'}$, we obtain a $BP \cdot AC^0$ circuit that has error at most $\delta(n) = 2^{-r'}$ and uses $O(r')$ bits of randomness. (The circuit is in $BP \cdot AC^0$, and not just $BP \cdot AC^0[\oplus]$ because one can readily check that all the necessary parities are on $r' = O(\log^c n)$ bits, and can therefore be computed in $AC^0$.) By applying $\Theta(n/r')$ such circuits in parallel (on the same input, but independent random strings), and taking the approximate majority of their $\Theta(n/r')$ outputs, we have a circuit taking $O(r') \cdot \Theta(n/r') = O(n)$ random bits and having error less than $2^{-n}$ (by a standard Chernoff bound and an appropriate setting of constants). ☐

### Derandomization with Linear Advice

**Corollary 6.** uniform $BP \cdot AC^0 \subseteq$ uniform $AC^0/O(n)$.

*Proof.* Apply Lemma 5 to obtain a $BP \cdot AC^0$ circuit with error less than $2^{-n}$ using $r = O(n)$ random bits. By a union bound, at least one $r$-bit string causes the circuit to correctly decide all inputs. Fix one such string as the non-uniform advice and the result follows. $\square$

**Corollary 7.** *Let $AC^0[\oplus_{\log}]$ be the class of boolean functions computable by $\mathrm{poly}(n)$-size $AC^0$ circuits having $O(\log n)$ parity gates, and similarly let $AC^0[\mathrm{SYM}_{\log}]$ be the class of boolean functions computable by $\mathrm{poly}(n)$-size $AC^0$ circuits having $O(\log n)$ arbitrary symmetric gates (e.g., parity and majority gates). Then the following inclusions hold:*

1. $BP \cdot AC^0[\oplus_{\log}] \subseteq AC^0[\oplus]/O(n)$
2. $BP \cdot AC^0[\mathrm{SYM}_{\log}] \subseteq TC^0/O(n)$

*Proof sketch.* The proof is similar to the proof of Lemma 5 and Theorem 6, except that we use the generator of Viola [Vio04] instead of Nisan's. Specifically, the generator from [Vio04] allows us to assume, without loss of generality, that any function $f \in BP \cdot AC^0[\oplus_{\log}]$ (respectively, $BP \cdot AC^0[\mathrm{SYM}_{\log}]$) can be computed by a $BP \cdot AC^0[\oplus]$ (respectively, $BP \cdot TC^0$) circuit using only $n^{o(1)}$ random bits. By applying Lemma 4, we may reduce the error to less than $2^{-n}$ using only $O(n^{o(1)} + n) = O(n)$ random bits. Finally, a union bound yields a single advice string of $O(n)$ bits that works for all inputs. $\square$

### Optimal explicit $\epsilon$-biased generator in $AC^0[\oplus]$

**Corollary 8** ([NN90] + [GV04] + Theorem 3)**.** *For every $\epsilon > 0$ and $m$, there is an $\epsilon$-biased generator $G : \{0,1\}^n \to \{0,1\}^m$ with $n = O(\log m + \log(1/\epsilon))$ for which uniform $AC^0[\oplus]$ circuits of size $\mathrm{poly}(n, \log m) = \mathrm{poly}(n)$ can compute $G(s)_i$ given $(s, i) \in \{0,1\}^n \times [m]$.*

*Proof idea.* We follow the approach of [GV04] and implement the $\epsilon$-biased generator of Naor and Naor [NN90]. This generator requires a 7-wise independent generator and a long walk on an expander graph. Constructions of 7-wise independent generators in $AC^0[\oplus]$ are known [GV04, HV06]. Since the use of an expander walk in [NN90] is simply as a hitting generator, our construction from Section 4 is more than adequate for this purpose. $\square$

# 6 The Proof of Theorem 1

In this section we give an elementary proof of the following generalization of Gillman's *Chernoff Bound for Expander Walks* [Gil94].

**Theorem 1.** *Let $G$ be a regular $\lambda$-expander on $V$. Fix a sequence of subsets $S_i \subseteq V$ each of density $\rho_i = |S_i|/|V|$, and for a random walk $v_1, \ldots, v_k$ on $G$, let $T$ be the random variable that counts the number of steps $i$ such that $v_i \in S_i$. Then for all $\epsilon > 0$,*

$$\Pr\left[\left|T - \sum_{i=1}^{k} \rho_i\right| \geq \epsilon k\right] \leq 2e^{-\epsilon^2(1-\lambda)k/36}.$$

Wigderson and Xiao [WX05] have recently established the same bound (up to constants) using techniques from perturbation theory – Gillman's proof (which treats the case $S_1 = \cdots = S_k$) also employs results from perturbation and complex analysis to obtain a simlar bound. In contrast, the proof presented here has only very modest prerequisites, which are summarized in the following paragraph.

**Background** Throughout, we work with a regular undirected graph $G$ on $N$ nodes. For any such graph, its probability transition matrix $P$ is clearly real and symmetric, and therefore we may form an orthogonal basis of $\mathbb{R}^N$ consisting of eigenvectors of $P$. Since $G$ is regular, the vector $\mathbf{1} = (1, \ldots, 1)$ is an eigenvector with eigenvalue $\lambda_1 = 1$. By the Perron-Frobenius Theorem, all other eigenvalues $\lambda_2 \geq \ldots \geq \lambda_n$ are between 1 and $-1$. We denote by $\lambda$, the quantity $\max\{|\lambda_2|, |\lambda_n|\}$. For any vector $\mathbf{v} \in \mathbb{R}^N$, we let $\mathbf{v}^{\|}$ denote the component of $\mathbf{v}$ in the direction of $\mathbf{1}$ and we let $\mathbf{v}^{\perp}$ denote the component of $\mathbf{v}$ that lies in the orthogonal complement of $\mathbf{1}$; that is, $\mathbf{v}^{\|} = \langle \mathbf{1}, \mathbf{v} \rangle \mathbf{u}$ and $\mathbf{v}^{\perp} = \mathbf{v} - \mathbf{v}^{\|} = \mathbf{v} - \langle \mathbf{1}, \mathbf{v} \rangle \mathbf{u}$, where $\mathbf{u} = (1/N, \ldots, 1/N)$. Since $\mathbb{R}^N$ has an orthogonal basis of eigenvectors of $P$, it is not hard to see that $\|P\mathbf{v}^{\perp}\| \leq \lambda \|\mathbf{v}^{\perp}\|$ for any vector $\mathbf{v} \in \mathbb{R}^N$.

*Proof of Theorem 1.* We shall bound the quantity $\Pr\left[T - \sum_i \rho_i \geq \epsilon k\right]$ and the same bound will follow for $\Pr\left[T - \sum_i \rho_i \leq -\epsilon k\right]$ by replacing the sets $S_i$ with their complements. Let $r \leq \min\{1, \log(1/\lambda)/2\}$ be a positive parameter to be specified later.

$$\Pr\left[T - \sum_i \rho_i \geq \epsilon k\right] = \Pr\left[T \geq \epsilon k + \sum_i \rho_i\right] = \Pr\left[e^{rT} \geq e^{r\left(\epsilon k + \sum_i \rho_i\right)}\right] \leq \frac{\mathrm{E}\left[e^{rT}\right]}{e^{r\left(\epsilon k + \sum_i \rho_i\right)}} \tag{1}$$

where the last step follows by applying Markov's inequality.

We now bound $\mathrm{E}\left[e^{rT}\right]$. Let $P$ be the probability transition matrix for $G$, and for each set $S_i$ let $E_i$ be a diagonal matrix with $e_{j,j} = e^r$ if $j \in S_i$ and $e_{j,j} = 1$ otherwise. It is not hard to see that

$$\mathrm{E}\left[e^{rT}\right] = \mathbf{1}^T E_k P E_{k-1} P \cdots E_1 P \mathbf{u}. \tag{2}$$

To bound this quantity, we require the following lemma.

**Lemma 15.** *Let $P$ be as above, and assume that $r \leq \log(1/\lambda)/2$. Let $S \subseteq V$ be of density $\rho = |S|/|V|$, and let $E$ be the diagonal matrix with $e_{j,j} = e^r$ for $j \in S$ and $e_{j,j} = 1$ otherwise. Then for any $\mathbf{v} \in \mathbb{R}^N$:*

- $\|(EP\mathbf{v})^{\|}\| \leq (1 + \rho(e^r - 1)) \cdot \|\mathbf{v}^{\|}\| + (e^r - 1) \cdot \|\mathbf{v}^{\perp}\|$
- $\|(EP\mathbf{v})^{\perp}\| \leq (e^r - 1) \cdot \|\mathbf{v}^{\|}\| + \sqrt{\lambda} \cdot \|\mathbf{v}^{\perp}\|.$

*Proof.* By the triangle inequality $\|(EP\mathbf{v})^{\|}\| = \|(EP\mathbf{v}^{\|})^{\|} + (EP\mathbf{v}^{\perp})^{\|}\| \leq \|(EP\mathbf{v}^{\|})^{\|}\| + \|(EP\mathbf{v}^{\perp})^{\|}\|$.

$$(EP\mathbf{v}^{\|})^{\|} = (E\mathbf{v}^{\|})^{\|} = \langle \mathbf{1}, E\mathbf{v}^{\|} \rangle \mathbf{u} = \langle \mathbf{1}, E\mathbf{u} \rangle \mathbf{v}^{\|} = (1 - \rho + \rho e^r)\mathbf{v}^{\|}. \tag{3}$$

$$(EP\mathbf{v}^{\perp})^{\|} = \langle \mathbf{1}, EP\mathbf{v}^{\perp} \rangle \mathbf{u} = \langle \mathbf{1}, (E - I)P\mathbf{v}^{\perp} \rangle \mathbf{u} + \langle \mathbf{1}, P\mathbf{v}^{\perp} \rangle \mathbf{u} = \langle \mathbf{1}, (E - I)P\mathbf{v}^{\perp} \rangle \mathbf{u},$$

and by the Cauchy-Schwartz inequality,

$$\|\langle \mathbf{1}, (E - I)P\mathbf{v}^{\perp} \rangle \mathbf{u}\| \leq \|\mathbf{1}\| \cdot \|(E - I)P\mathbf{v}^{\perp}\| \cdot \|\mathbf{u}\| \leq (e^r - 1) \cdot \|P\mathbf{v}^{\perp}\| \leq (e^r - 1) \cdot \|\mathbf{v}^{\perp}\|.$$

Similarly, for the second item, $\|(EP\mathbf{v})^{\perp}\| = \|(EP\mathbf{v}^{\|})^{\perp} + (EP\mathbf{v}^{\perp})^{\perp}\| \leq \|(EP\mathbf{v}^{\|})^{\perp}\| + \|(EP\mathbf{v}^{\perp})^{\perp}\|$.

$(EP\mathbf{v}^{\|})^{\perp} = (E\mathbf{v}^{\|})^{\perp}$, and by equation (3),

$$(E\mathbf{v}^{\|})^{\perp} = E\mathbf{v}^{\|} - (E\mathbf{v}^{\|})^{\|} = E\mathbf{v}^{\|} - (1 + \rho(e^r - 1))\mathbf{v}^{\|} = (E - (1 + \rho(e^r - 1))I)\mathbf{v}^{\|}.$$

The matrix $E - (1 + \rho(e^r - 1))I$ has diagonal entries $(1 - \rho)(e^r - 1)$ and $-\rho(e^r - 1)$ each of which is at most $(e^r - 1)$ in absolute value. Therefore, $\|(EP\mathbf{v}^{\|})^{\perp}\| \leq (e^r - 1) \cdot \|\mathbf{v}^{\|}\|$.

Finally, $\|(EP\mathbf{v}^{\perp})^{\perp}\| \leq \|(EP\mathbf{v}^{\perp})\| \leq e^r \cdot \|P\mathbf{v}^{\perp}\|$, and $\|P\mathbf{v}^{\perp}\| \leq \lambda \cdot \|\mathbf{v}^{\perp}\|$. Thus, $\|(EP\mathbf{v}^{\perp})^{\perp}\| \leq e^r \lambda \cdot \|\mathbf{v}^{\perp}\|$, and since we assume that $r \leq \log(1/\lambda)/2$, this is at most $\sqrt{\lambda} \cdot \|\mathbf{v}^{\perp}\|$. □

We now define a sequence of vectors $\mathbf{v}_0 = \mathbf{u}$ and $\mathbf{v}_i = E_i P \mathbf{v}_{i-1}$ for $i > 0$, noting that

$$\mathrm{E}\left[e^{rT}\right] = \mathbf{1}^T E_k P E_{k-1} P \cdots E_1 P \mathbf{u} = \langle \mathbf{1}, \mathbf{v}_k \rangle = \langle \mathbf{1}, \mathbf{v}_k^{\|} \rangle \leq \|\mathbf{1}\| \cdot \|\mathbf{v}_k^{\|}\| = \sqrt{N} \cdot \|\mathbf{v}_k^{\|}\|. \tag{4}$$

By Lemma 15,

- $\|\mathbf{v}_i^{\|}\| \leq (1 + \rho_i(e^r - 1)) \cdot \|\mathbf{v}_{i-1}^{\|}\| + (e^r - 1) \cdot \|\mathbf{v}_{i-1}^{\perp}\|$
- $\|\mathbf{v}_i^{\perp}\| \leq (e^r - 1) \cdot \|\mathbf{v}_{i-1}^{\|}\| + \sqrt{\lambda} \cdot \|\mathbf{v}_{i-1}^{\perp}\|.$

Recursively applying the bound on $\|\mathbf{v}_i^{\perp}\|$ and noting that $\|\mathbf{v}_0^{\perp}\| = 0$,

$$\|\mathbf{v}_i^{\perp}\| \leq (e^r - 1) \cdot \sum_{j=0}^{i-1} (\sqrt{\lambda})^j \|\mathbf{v}_{i-j-1}^{\|}\|,$$

and substituting into the bound for $\|\mathbf{v}_i^{\|}\|$,

$$\|\mathbf{v}_i^{\|}\| \leq (1 + \rho_i(e^r - 1)) \cdot \|\mathbf{v}_{i-1}^{\|}\| + (e^r - 1)^2 \cdot \sum_{j=0}^{i-2} (\sqrt{\lambda})^j \|\mathbf{v}_{i-j-2}^{\|}\|$$

which is at most

$$\left(1+\rho_i(e^r-1)+(e^r-1)^2\cdot\sum_{j=0}^{i-2}(\sqrt{\lambda})^j\right)\max_{j<i}\{\|\mathbf{v}_j^{\|}\|\}\leq\left(1+\rho_i(e^r-1)+\frac{2(e^r-1)^2}{1-\lambda}\right)\max_{j<i}\{\|\mathbf{v}_j^{\|}\|\}.$$

Recalling that $\|v_0^{\|}\|=1/\sqrt{N}$, by induction on $j$ it is clear that for all $j\geq 0$

$$\|\mathbf{v}_j^{\|}\|\leq\frac{1}{\sqrt{N}}\cdot\prod_{i=1}^{j}\left(1+\rho_i(e^r-1)+\frac{2(e^r-1)^2}{1-\lambda}\right)$$

and in particular, by equation (4)

$$\mathrm{E}\left[e^{rT}\right]\leq\sqrt{N}\cdot\|\mathbf{v}_k^{\|}\|\leq\prod_{i=1}^{k}\left(1+\rho_i(e^r-1)+\frac{2(e^r-1)^2}{1-\lambda}\right).$$

Using the fact that $e^r-1\leq r+r^2\leq 2r$ for $0\leq r\leq 1$,

$$\mathrm{E}\left[e^{rT}\right]\leq\prod_{i=1}^{k}\left(1+\rho_i(r+r^2)+\frac{8r^2}{1-\lambda}\right)\leq\prod_{i=1}^{k}\left(1+r\rho_i+\frac{9r^2}{1-\lambda}\right).$$

Taking logarithms and using the fact that $\log(1+x)\leq x$ for all $x\geq 0$, we have

$$\log\mathrm{E}\left[e^{rT}\right]\leq k\cdot\frac{9r^2}{1-\lambda}+r\cdot\sum_{i=1}^{k}\rho_i.$$

Thus, by equation (1),

$$\log\Pr\left[T-\sum_i\rho_i\geq\epsilon k\right]\leq\log\left(\mathrm{E}\left[e^{rT}\right]\right)-r\left(\epsilon k+\sum_i\rho_i\right)\leq k\left(\frac{9r^2}{1-\lambda}-\epsilon r\right).$$

Finally, we minimize the right-hand side by setting $r=\epsilon(1-\lambda)/18$, noting that $r$ is indeed at most $\min\{1,\log(1/\lambda)/2\}$ simply because $1-\lambda\leq\log(1/\lambda)$ for all $\lambda\in[0,1]$.

$$\log\Pr\left[T-\sum_i\rho_i\geq\epsilon k\right]\leq k\left(\frac{9}{1-\lambda}\cdot\frac{\epsilon^2(1-\lambda)^2}{18^2}-\frac{\epsilon^2(1-\lambda)}{18}\right)=-\frac{\epsilon^2(1-\lambda)k}{36}.$$

□

**Remark 16.** *One can readily see that the same proof works even if the graph is different for each of the $k$ steps, as long as it is a $\lambda$-expander at each step. This is observation is important for the proof of correctness of our sampler (Theorem 3), as that construction concerns a walk on an expander graph that is varying from one step to the next step. This observation is not unique to our proof of the Chernoff bound, and this same property has been exploited before, most notably in the hardness amplification result of Goldreich et al. [GIL+90] (although there, they only require the hitting property of expander walks, and not the stronger sampling properties guaranteed here).*

# 7 Open Questions

It is well-known that expander walks yield averaging-samplers that are optimal (up to constants) for $\epsilon = \Omega(1)$, but sub-optimal for smaller $\epsilon$. Since pairwise-hashing is in $AC^0[\oplus]$ [GV04, HV06], one can implement the *median-of-averages sampler* of [BGG93] in $TC^0$ by using our sampler in lieu of the expander walks. (Majority gates are only necessary to compute the medians and averages – the actual samples can be computed in $AC^0[\oplus]$.) Can $AC^0[\oplus]$ compute an optimal *averaging* sampler?

There is also the question of lower-bounds. We suspect that $AC^0$ cannot compute samplers that match the parameters of our $AC^0[\oplus]$ construction. One approach to showing this is to use the equivalence of samplers and extractors from [Zuc97] and show that $AC^0$ cannot compute a (strong) extractor for sources of high constant min-entropy. Viola [Vio04] has shown that $AC^0$ cannot compute an extractor for sources of low min-entropy; however, his techniques do not seem to apply directly in this setting.

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