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Note on MAX 2SAT

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Abstract

In this note we present an approximation algorithm for MAX 2SAT that given a $(1-\varepsilon)$ satisfiable instance finds an assignment of variables satisfying a

$$1 - O(\sqrt{\varepsilon})$$

fraction of all constraints. This result is optimal assuming the Unique Games Conjecture.

The best previously known result, due to Zwick, was $1 - O(\varepsilon^{1/3})$. We believe that the analysis of our algorithm is much simpler than the analysis of Zwick's algorithm.

1 Introduction

In the seminal paper [4], Goemans and Williamson constructed an approximation algorithm for MAX CUT, that given a $1-\varepsilon$ satisfiable instance finds an assignment satisfying a $1-O(\sqrt{\varepsilon})$ fraction of all constraints. In 1998, Zwick developed an approximation algorithm for a more general problem, MAX 2SAT. Given a $1-\varepsilon$ satisfiable instance his algorithm satisfies a $1 - O(\varepsilon^{\frac{1}{3}})$ fraction of all constraints. In this note we close the gap between the approximation guarantees for these two problems. Namely, we present an algorithm that satisfies a $1 - O(\sqrt{\varepsilon})$ fraction of all constraint.

Khot, Kindler, Mossel, and O'Donnell [6] showed that the approximation guarantee of Goemans and Williamson is optimal assuming the Unique Games Conjecture of Khot [5]. Thus our result is also tight assuming the Unique Games Conjecture (since MAX 2SAT is a generalization of MAX CUT).

Let us now formally define the problem.

Definition 1.1 (MAX 2SAT). *We are given a set of boolean variables $x_1, \dots, x_n$ and a set of clauses of the form $x_i \to x_j$, $\bar{x}_i \to x_j$, $\bar{x}_i \to x_j$, and $\bar{x}_i \to \bar{x}_j$. Our goal is to assign a value "0" or "1" to each variable x_i so as to maximize the number of satisfied clauses.*

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Remark 1.1. *In MAX CUT the clauses are of the form* $x_i \leftrightarrow \bar{x}_j$.

The following table shows the best previously known results for these problems. It is interesting to note that the approximation guarantees for MAX 2SAT and MAX CUT were the same for $\varepsilon < 1/\log n$.

Range	MAX CUT	MAX 2SAT
$\varepsilon > 1/\log n$	$1 - O(\sqrt{\varepsilon})$ [4]	$1 - \varepsilon^{1/3}$ [7]
$\varepsilon < 1/\log n$	$1 - O(\sqrt{\varepsilon \log n})$ [1]	$1 - O(\sqrt{\varepsilon \log n})$ [1]

2 Approximation Algorithm

2.1 SDP relaxation

In this section we describe the vector program (SDP) for MAX 2SAT. For convenience we replace each negation $\bar{x}_i$ with a new variable x_{-i} that is equal by the definition to $\bar{x}_i$. We now rewrite all clauses in the form $x_i \to x_j$, where $i, j \in \{\pm 1, \pm 2, \dots, \pm n\}$.

For each x_i, we introduce a vector variable v_i in the SDP. We also define a special unit vector v_0 that "corresponds" to the value 1: in the intended (integral) solution $v_i = v_0$, if $x_i = 1$; and $v_i = -v_0$, if $x_i = 0$. The SDP contains the constraints that all vectors are unit vectors; v_i and v_{-i} are opposite; and some ℓ_2^2-triangle inequalities.

For each clause $x_i \to x_j$ we add the term

$$\frac{1}{8}\left(\|v_j - v_i\|^2 - 2\langle v_j - v_i, v_0\rangle\right)$$

to the objective function. In the intended solution this expression equals to 1, if the clause is not satisfied; and 0, if it is satisfied. Therefore, our SDP is a relaxation of MAX 2SAT (the objective function measures how many clauses are not satisfied). Note that each term in the SDP is positive due to the triangle inequality constraints.

We get the following SDP:

$$\text{minimize} \quad \frac{1}{8} \sum_{\text{clauses } x_i \to x_j} \|v_j - v_i\|^2 - 2\langle v_j - v_i, v_0\rangle$$

subject to

$$\begin{aligned} \|v_j - v_i\|^2 - 2\langle v_j - v_i, v_0\rangle &\geq 0 && \text{for all clauses } v_i \to v_j \\ \|v_i\|^2 &= 1 && \text{for all } i \in \{0, \pm 1, \dots, \pm n\} \\ v_i &= -v_{-i} && \text{for all } i \in \{\pm 1, \dots, \pm n\} \end{aligned}$$

In a slightly different form, this semidefinite program was introduced by Feige and Goemans [3]. Later, Zwick [7] used this SDP in his algorithm.

2.2 Algorithm and Analysis

We now present the approximation algorithm.

Approximation Algorithm

1. Solve the SDP. Denote by SDP the objective value of the solution and by ε the fraction of the constraints "unsatisfied" by the vector solution, that is,

$$\varepsilon = \frac{SDP}{\#\text{constraints}}.$$

2. Pick a random Gaussian vector g with independent components distributed as $\mathcal{N}(0,1)$.

3. For every i,

   (a) Project the vector g to v_i:
   $$\xi_i = \langle g, v_i \rangle.$$
   Note, that ξ_i is a standard normal random variable, since v_i is a unit vector.

   (b) Pick a threshold t_i as follows:
   $$t_i = -\langle v_i, v_0 \rangle \big/ \sqrt{\varepsilon}\ .$$

   (c) If $\xi_i \geq t_i$, set $x_i = 1$, otherwise set $x_i = 0$.

It is easy to see that the algorithm always obtains a valid assignment to variables: if $x_i = 1$, then $x_{-i} = 0$ and vice versa. We will need several facts about normal random variables. Denote the probability that a standard normal random variable is greater than $t \in \mathbb{R}$ by $\tilde{\Phi}(t)$, in other words

$$\tilde{\Phi}(t) \equiv 1 - \Phi_{0,1}(t) = \Phi_{0,1}(-t),$$

where $\Phi_{0,1}$ is the normal distribution function. The following lemma gives a lower and upper bounds on $\tilde{\Phi}(t)$ (for the proof see, *e.g.* [2]).

Lemma 2.1. *For every positive t,*

$$\frac{t}{\sqrt{2\pi}(t^2+1)} e^{-\frac{t^2}{2}} < \tilde{\Phi}(t) < \frac{1}{\sqrt{2\pi}t} e^{-\frac{t^2}{2}}.$$

Corollary 2.2. *There exists a constant C such that for every positive t, the following inequality holds $\tilde{\Phi}(t) \leq C \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$.*

A clause $x_i \to x_j$ is not satisfied by the algorithm if $\xi_i \geq t_i$ and $\xi_j \leq t_j$ (*i.e.* x_i is set to 1; and x_j is set to 0). The following lemma bounds the probability of this event.

Lemma 2.3. *Let ξ_i and ξ_j be two standard normal random variables with covariance $1-2\Delta^2$ (where $\Delta \geq 0$). For all real numbers t_i, t_j and $\delta = (t_j - t_i)/2$ we have (for some absolute constant C)*

1. *If $t_j \leq t_i$,*
$$\Pr\left(\xi_i \geq t_i \text{ and } \xi_j \leq t_j\right) \leq C \min(\Delta^2/|\delta|, \Delta).$$

2. *If $t_j \geq t_i$,*
$$\Pr\left(\xi_i \geq t_i \text{ and } \xi_j \leq t_j\right) \leq C(\Delta + 2\delta).$$

Proof. First note that if $\Delta = 0$, then the above inequalities hold (since $\xi_j = \xi_i$ almost surely). If $\Delta \geq 1/2$, then the right hand sides of the inequalities are greater than 1 (for sufficiently large C) and thus the inequalities hold. So we assume $0 < \Delta < 1/2$.

1. Let $\xi = (\xi_j + \xi_i)/2$ and $\eta = (\xi_i - \xi_j)/2$. Notice that $\mathrm{Var}[\xi] = 1-\Delta^2$, $\mathrm{Var}[\eta] = \Delta^2$; and random variables ξ and η are independent. We estimate the desired probability as follows:

$$\begin{aligned}
\Pr\left(\xi_j \leq t_j \text{ and } \xi_i \geq t_i\right) &= \Pr\left(\eta \geq \left|\xi - \frac{t_j + t_i}{2}\right| + \frac{t_i - t_j}{2}\right) \\
&= \int_{-\infty}^{+\infty} \Pr\left(\eta \geq \left|\xi - \frac{t_j + t_i}{2}\right| + \frac{t_i - t_j}{2} \quad | \quad \xi = t\right) dF_\xi(t).
\end{aligned}$$

Note that the density of the normal distribution with variance $1 - \Delta^2$ is always less than $1/\sqrt{2\pi(1-\Delta^2)} < 1$, thus we can replace $dF_\xi(t)$ with dt.

$$\begin{aligned}
\Pr\left(\xi_j \leq t_j \text{ and } \xi_i \geq t_i\right) &\leq \int_{-\infty}^{+\infty} \tilde{\Phi}\left(\frac{\left|t - \frac{t_j+t_i}{2}\right| + \frac{t_i - t_j}{2}}{\Delta}\right) dt \\
&= \int_{-\infty}^{+\infty} \tilde{\Phi}\left(\frac{|t| + |\delta|}{\Delta}\right) dt = \Delta \int_{-\infty}^{+\infty} \tilde{\Phi}\left(|s| + |\delta|/\Delta\right) ds \\
\text{(by Corollary 2.2)} &\leq C'\Delta \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(|s|+|\delta|/\Delta)^2/2} ds = 2C'\Delta \cdot \tilde{\Phi}(|\delta|/\Delta) \\
\text{(by Lemma 2.1)} &\leq 2C' \min(\Delta^2/|\delta|, \Delta).
\end{aligned}$$

2. We have

$$\begin{aligned}
\Pr\left(\xi_j \leq t_j \text{ and } \xi_i \geq t_i\right) &\leq \Pr\left(\xi_j \leq t_j \text{ and } \xi_i \geq t_j\right) + \Pr\left(t_i \leq \xi_i \leq t_j\right) \\
&\leq C(\Delta + 2\delta).
\end{aligned}$$

For estimating the probability $\Pr\left(\xi_j \leq t_j \text{ and } \xi_i \geq t_j\right)$ we used part 1 with $t_i = t_j$. ☐

Theorem 2.4. *The approximation algorithm finds an assignment satisfying a $1 - O(\sqrt{\varepsilon})$ fraction of all constraints, if a $1 - \varepsilon$ fraction of all constraints is satisfied in the optimal solution.*

Proof. We shall estimate the probability of satisfying a clause $x_i \to x_j$. Set $\Delta_{ij} = \|v_j - v_i\|/2$ (so that $\mathrm{cov}(\xi_i, \xi_j) = 1 - 2\Delta_{ij}^2$) and $\delta_{ij} = (t_j - t_i)/2 \equiv \langle v_j - v_i, v_0 \rangle / (2\sqrt{\varepsilon})$. The contribution of the term to the SDP is equal to $c_{ij} = (\Delta_{ij}^2 + \delta_{ij}\sqrt{\varepsilon})/2$.

Consider the following cases (we use Lemma 2.3 in all of them):

1. If $\delta_{ij} \geq 0$, then the probability that the constraint is not satisfied is at most
$$C(\Delta_{ij} + 2\delta_{ij}) \leq C(\sqrt{2c_{ij}} + 4c_{ij}/\sqrt{\varepsilon}).$$

2. If $\delta_{ij} < 0$ and $\Delta_{ij}^2 \leq 4c_{ij}$, then the probability that the constraint is not satisfied is at most
$$C\Delta_{ij} \leq 2C\sqrt{c_{ij}}.$$

3. If $\delta_{ij} < 0$ and $\Delta_{ij}^2 > 4c_{ij}$, then the probability that the constraint is not satisfied is at most
$$\frac{C\Delta_{ij}^2}{|\delta_{ij}|} = \frac{C\Delta_{ij}^2}{(\Delta_{ij}^2 - 2c_{ij})/\sqrt{\varepsilon}} \leq \frac{C\sqrt{\varepsilon}\,\Delta_{ij}^2}{\Delta_{ij}^2 - \Delta_{ij}^2/2} = 2C\sqrt{\varepsilon}.$$

Combining these cases we get that the probability that the clause is not satisfied is at most
$$4C(\sqrt{c_{ij}} + c_{ij}/\sqrt{\varepsilon} + \sqrt{\varepsilon}).$$

Summing over all clauses and using convexity of the function $\sqrt{\cdot}$ we get that the expected fraction of unsatisfied constraints is $O(\sqrt{\varepsilon})$. □

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