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Approximations by Computationally-Efficient VCG-Based Mechanisms

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Abstract

We consider computationally-efficient incentive-compatible mechanisms that use the VCG payment scheme, and study how well they can approximate the social welfare in auction settings. We obtain a 2-approximation for multi-unit auctions, and show that this is best possible, even though from a purely computational perspective an FPTAS exists. For combinatorial auctions among submodular (or subadditive) bidders, we prove an $\Omega(m^{\frac{1}{6}})$ lower bound, which is close to the known upper bound of $O(m^{\frac{1}{2}})$, and qualitatively higher than the constant factor approximation possible from a purely computational point of view.

1 Introduction

1.1 Background

Algorithmic Mechanism design attempts to design protocols for distributed environments, such as the Internet, where the different participants each have their own selfish goals and are assumed to rationally attempt optimizing their own goals rather than just follow any prescribed protocol. The key target in this area is the design of *incentive-compatible* mechanisms – also called truthful or strategy proof mechanisms – whose payment schemes motivate the participants to correctly report their private information[1]. For a general introduction to the economic field of mechanism design see [19] and for an introduction to *algorithmic* mechanism design and further motivation see [23].

Typical problems in this setting involve allocation of various resources and a paradigmatic abstraction is that of *combinatorial auctions*. In this problem m heterogenous "items" need to be allocated between n "bidders". Each bidder i holds a *valuation* function v_i that specifies for each subset of the items $S \subseteq \{1...m\}$ the bidder's value $v_i(S)$ from winning the "bundle" S. The challenge is to find a partition $S_1...S_n$ of the items that maximizes the social welfare $\Sigma_i v_i(S_i)$. This problem presents a combination of algorithmic difficulty (it is NP-complete), representational difficulty (the valuation functions are exponential size objects) and strategic difficulty (ensuring incentive compatibility).

The key positive technique for achieving incentive compatibility is that of VCG mechanisms [26, 3, 10]: if player i's value from the chosen algorithmic outcome a is $v_i(a)$, then we charge player i the quantity $h_i(v_{-i}) - \Sigma_{j\neq i} v_j(a)$, where h_i is an arbitrary fixed function that does not depend on v_i. A powerful observation is that if the algorithmic outcome a always maximizes the *social welfare*, $\Sigma_i v_i(a)$, then the VCG payment rule results in an incentive compatible mechanism. However, in

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[1]More generally, one could design protocols where one gets the desired results as equilibria, but the revelation principle allows converting such general mechanisms to incentive compatible ones.

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most interesting computational scenarios, including combinatorial auctions, achieving exact optima is computationally intractable, and one must settle for heuristics or approximations. A key clash between the strategic and algorithmic considerations is that once only approximations or heuristics are chosen, the VCG payment rule no longer leads to incentive compatibility [18, 22].

In [22] a detailed examination was carried out of when are "VCG-based" mechanisms, i.e., those obtained using the "VCG" payment method – each bidder i pays $h_i(v_{-i}) - \Sigma_{j \neq i} v_j(a)$, where a is the algorithmic output) incentive compatible. It is easy to see that the following family of allocation algorithms do yield incentive-compatible VCG-based mechanisms:

Definition: An allocation algorithm (that produces an output $a \in \mathcal{A}$ for each input $v_1 ... v_n$, where $\mathcal{A}$ is the set of possible alternatives) is called "maximal-in-range" (henceforth MIR) if it completely optimizes the social welfare over some subrange $\mathcal{R} \subseteq \mathcal{A}$. I.e., for some $\mathcal{R} \subseteq \mathcal{A}$, we have that for all $v_1 ... v_n$, $a \in \arg\max_{a \in \mathcal{R}} \Sigma_i v_i(a)$.

The main result of [22] states that this is essentially it:

Theorem [22]: The allocation algorithm of any incentive-compatible VCG-based mechanism for combinatorial auctions is equivalent to a maximal-in-range algorithm.

"Equivalent" here means that the social utilities are identical for all inputs, i.e. if a and b are the outputs of the two allocation algorithms for input $v_1 ... v_n$ then $\Sigma_i v_i(a) = \Sigma_i v_i(b)$. In particular the outputs must coincide generically – except perhaps in case of ties. In [22] this is viewed as a negative result since "reasonable"[2] allocation algorithms will have a full range, and thus will never lead to truthful mechanisms (unless they are optimal and thus computationally intractable).

However, it turns out that one may construct "unreasonable" incentive-compatible VCG-based mechanisms that do have non-trivial approximation guarantees: In [11] it is shown that bundling the items into $O(\log m)$ equi-sized bundles and then allocating these bundles optimally (in time exponential in $\log m$ and thus polynomial in m) achieves a non-trivial $O(m/\sqrt{\log m})$-approximation ratio – yielding an truthful VCG-based mechanism[3]. In [5] the subcase of "complement-free" valuations was considered, and a truthful VCG-based mechanism with an $O(\sqrt{m})$ approximation ratio was obtained.

The basic question that we ask is how good an approximation ratio can be obtained this way – using VCG-based mechanisms or, equivalently, using maximum-in-range allocation algorithms? We should emphasize that our focus is not just on one technique among many. The VCG payment rule[4] is the *only* technique known for achieving incentive-compatibility except for very few exceptions: single dimensional domains (e.g., [18, 2, 20]) and a single additional example [1]. (This paper concerns deterministic mechanisms; slightly more is known for randomized ones [4, 16, 7][5].) In fact, it is known [25, 9] that indeed in sufficiently "rich" domains, the only incentive-compatible mechanisms are (weighted) VCG-based. Partial results along these lines for the case of combinatorial auctions were shown in [15], who left open the general question of whether non-VCG-based mechanisms (up to minor deviations) for combinatorial auctions exist.

1.2 Multi-Unit Auctions

It is perhaps best to illustrate the issues with a simple problem: multi-unit auctions. In this problem a set of M *identical* items are auctioned among n bidders. In the simple – "single minded" – case,

[2] In [22] a formal definition was used which stipulates that an item that has positive value for just a single bidder be allocated to him.

[3] Actually, [11] studied a subclass or maximum-in-range mechanisms that arises as a characterization of communication-efficient ex-post Nash equilibria of an optimal allocation mechanism.

[4] More precisely, weighted versions of it that correspond to similarly weighted version of the social welfare. All our lower bounds apply also to arbitrary weighted-VCG-based mechanisms.

[5] But all our lower bounds apply also to randomized VCG-based mechanisms.

each bidder i has a value of p_i for obtaining at least q_i items. The problem is to find the set of "winning bidders" W such that $\Sigma_{i \in W} q_i \leq M$ with maximum value of $\Sigma_{i \in W} p_i$. In the more general case, each bidder i may hold a function $v_i : \{1..m\} \to \mathbb{R}$, that specifies i's value $v_i(q)$ for every possible number of elements obtained. In this case the auction must decide on a quantity q_i to allocate to each bidder such that $\Sigma_i q_i \leq M$ and $\Sigma_i v_i(q_i)$ is maximized.

The reader may have already noticed that, computationally speaking, the single-minded problem is exactly the well known knapsack problem. (The general problem turns out to be *computationally* very similar.) While the knapsack problem is known to be NP-complete, it is also well known that it has a fully polynomial approximation scheme, based on the simple observation that the unary version of the problem has a polynomial time algorithm. More specifically, the knapsack problem can be solved in polynomial time using dynamic programming either if all p_i's are given in unary (and the q_i's are arbitrary) or if all q_i's are given in unary (and the p_i's are arbitrary). Both of these algorithms generalize to the general multi-unit auction problem.

However, none of these algorithms suffices for providing incentive-compatible approximation mechanisms. Versions that round the p_i's will provide a fully polynomial approximation scheme but are not maximal in range and thus turn out to lose all incentive properties. Version that round the q_i's to unary will be maximal in range but loses the approximation properties. Can one get a variant that maintains both properties? A negative answer is suggested by [15] where it is shown that in the special case of two player mechanisms that *always allocate all items*, no approximation factor better than 2 may be obtained by any non-optimal incentive-compatible mechanism. For the strategically simpler single-minded case, a fully polynomial approximation mechanism was recently obtained in [2] improving upon a 2-approximation mechanism of [20]. To date, no truthful approximation algorithm, with any non-trivial quality of approximation, is known for general multi-unit auctions[6].

We first show that a variant of the method of rounding the q_i's, which is maximal in its range and thus incentive-compatible, yields a 2-approximation mechanism:

Theorem: There exists a polynomial-time computable incentive-compatible VCG-based mechanism for multi-unit auctions that gives a 2-approximation.

We also show that this is best possible. Specifically, that every non-optimal VCG-based mechanism for multi-unit auctions among two bidders must lose a factor of 2 for some inputs. This implies that no computationally efficient VCG-based mechanisms can obtain a better than 2-approximation.

1.3 Our Main Result

Our main result provides a lower bound for the approximation factor that can be achieved by incentive-compatible VCG-based mechanisms for combinatorial auctions. Our lower bound applies to the subclass of submodular valuations ($v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + V_i(T)$ for all S, T) and thus also to its superset class of complement-free valuations ($v_i(S \cup T) \leq v_i(S) + v_i(T)$ for all S, T) – two classes of valuations which have been extensively studied [17, 5, 7, 6, 8, 12]. Table 1 summarizes the known lower and upper bounds on the approximation factors that may be achieved in polynomial time for these subclasses of valuations (a) algorithmically, (b) using incentive-compatible mechanisms and (c) using incentive-compatible VCG-based mechanisms. A word about the computational model is in place here: the "inputs" to the mechanism, the v_i's, are exponential sized objects (in the number of items m), but the mechanisms should run in time polynomial in n and m. Thus it is assumed that the mechanism repeatedly queries the bidders. The upper bounds in the table always assume some

[6]A *randomized* 2-approximation truthful in expectation mechanism was presented in [16]. An "almost truthful" mechanism for this problem, with further restrictions on the valuations, was obtained in [13].

	General Algorithms	Incentive Compatible	Maximum in Range
General	$\Theta(\sqrt{m})$ [21]	$\Theta(\sqrt{m})$ [4] (randomized) $O(\frac{m}{\sqrt{\log m}})$ [11]	$O(\frac{m}{\sqrt{\log m}})$ [11]
Complement Free	≤ 2 [7] ≥ 2 [5]	$O(\sqrt{m})$ [5]	$O(\sqrt{m})$ [5] $\Omega(m^{\frac{1}{6}})$ (Section 3)
Submodular	$\leq \frac{e}{e-1} - 10^{-4}$ [8] $\geq \frac{20}{19}$ [8]	$O(\log^2 m)$ [4] (randomized) $O(\sqrt{m})$ [5]	$O(\sqrt{m})$ [5] $\Omega(m^{\frac{1}{6}})$ (Section 3)
Multi Unit	FPTAS	≤ 2 (Section 2)	2 (Section 2)

Table 1: All upper bounds require demand queries. All lower bounds are for any family of queries.

specific natural type of query (usually a "demand query"), while all lower bounds apply for every type of query and are in fact communication lower bounds.

As is evident from the table, there is very little known specifically regarding truthful mechanisms: all deterministic upper bounds are by VCG-based mechanisms and all lower bounds apply to general algorithms. We show that the known non-trivial but weak $O(\sqrt{m})$ approximation factor obtained by truthful VCG-based mechanism [5] for complement-free valuations is close to optimal.

Theorem: Every VCG-based mechanism for approximating the welfare in combinatorial auctions with submodular bidders that uses a sub-exponential number of queries to the bidders achieves an approximation factor of $\min(\Omega(n), \Omega(m^{1/6}))$.

The proof proceeds by combinatorially analyzing maximum in range allocation algorithms[7]. The analysis shows that if the range is "large" then optimizing over it requires exponential communication, while if it is "small" then it can not achieve a good approximation ratio. It turns out that "large" and "small" in this sense cannot just be interpreted in terms of the size of the range. Instead we define two "complexity measures" of a set of partitions (which is what the range is). One of them, termed the intersection number, is shown to bound from below the communication complexity of optimization over the range. The other, termed the cover number, is shown to bound from above the approximation ratio achieved by allocations in the range. Our main combinatorial lemma, which may be of independent interest, shows that these two complexity measures are related to each other.

Our main open problem is to determine how good an approximation can be achieved by efficient VCG-based mechanisms for *general valuations*. The only upper bound known achieves the the nearly trivial approximation ratio of $O(m/\sqrt{\log m})$, while the known lower bound is $\Omega(m^{1/2-\epsilon})$ which holds for general algorithms. We believe that the truth is close to the upper bound (and that, in fact, this holds for general incentive-compatible deterministic mechanisms, even non VCG-based ones). Closing the gap between our lower bound of $\Omega(m^{1/6})$ for submodular or complement-free bidders (which we can push to $m^{1/5-\epsilon}$) and the $O(\sqrt{m})$ upper bound is another open problem.

Paper Structure

As a warm-up in Section 2 we present the multi-unit case. In Section 3 we prove the main theorem. Appendix A brings the characterization of VCG-based mechanisms of [22], modified for our scenarios.

2 Multi-Unit Auctions

We consider the multi-unit auction problem with general valuations: m *identical* items are auctioned between n bidders. Each bidder has a valuation function $v_i : \{0..m\} \rightarrow \mathbb{R}$, where $v_i(q)$ denotes the

[7] While formally the characterization of [22] applies to mechanisms over general valuations, it needs only minor modifications to apply also to submodular valuations. For completeness we bring the modified proof in the appendix.

value that i gets from receiving q elements. We assume that $v_i(0) = 0$ (normalization) and that v_i is weakly monotone (free disposal). We consider the case where m is "large" – given in binary – and desire algorithms that are polynomial in n and $\log m$. As a full description of the valuation function v_i is exponential in $\log m$, we will assume in our algorithm an "oracle access" to it that may be queried for $v_i(q)$, where q is the given bundle size[8].

We will design an MIR algorithm for this problem, which directly yields an incentive compatible VCG-based mechanism. Our MIR approximation algorithm will first split the items into n^2 equi-sized bundles of size $b = \lfloor \frac{m}{n^2} \rfloor$ as well as a single extra bundle of size r that holds the remaining elements (thus $n^2 b + r = m$). The maximum in range algorithm will optimally allocate these whole bundles among the n bidders. What we need to show is the following two simple facts:

Lemma 2.1 *An optimal allocation of the bundles can be found in time polynomial in n and $\log m$.*

Lemma 2.2 *Let $a_1 ... a_n$ be an optimal allocation of the bundles that was found by the algorithm, and $o_1 ... o_n$ an optimal unrestricted allocation, then $\Sigma_i v_i(o_i) \leq 2\Sigma_i v_i(a_i)$.*

The proofs are simple:

Proof: (of Lemma 2.1): The algorithm is by dynamic programming. We calculate the following information for every $1 \leq i \leq n$ and $1 \leq q \leq n^2$: $M(i,q)$ is the maximum value that can be obtained by allocating at most q regular bundles among bidders $1...i$, and $M^+(i,q)$ is the maximum value that can be obtained by allocating at most q regular bundles and the "remainder" bundle among bidders $1...i$. Each entry can be filled in polynomial time using the realtions: $M(i,q) = \max_{q' \leq q} v_i(q'b) + M(i-1, q-q')$ and $M^+(i,q) = \max(\max_{q' \leq q} v_i(q'b) + M^+(i-1, q-q'), \max_{q' \leq q} v_i(q'b+r) + M(i-1, q-q'))$. □

Proof: (of Lemma 2.2): Let us start with an optimal unrestricted allocation $o_1 ... o_n$ where all items are allocated (without loss of generality since the valuations are monotone), and look at the bidder j that got the largest number of items $o_j \geq m/n$. There are now two possibilities: if $v_j(o_j) \geq \Sigma_{i \neq j} v_i(o_i)$ then by allocating all items to j (i.e. all regular-sized bundles as well as the remainder bundle) we get the required 2-approximation. Otherwise, round *up* each o_i to the nearest multiple of b (i.e. to full bundles), except for bidder j that gets nothing. This is a valid allocation since we added at most $nb \leq m/n$ items by rounding up, but deleted at least m/n items by removing o_j, and its value is certainly at least $\Sigma_{i \neq j} v_i(o_i)$ which gives the required approximation. □

We have thus proved:

Theorem 2.3 *There exists an incentive-compatible computationally-efficient VCG-based mechanism that gives a 2-approximation for multi-unit auctions with general valuations.*

We now move on to show that this is the best possible. Consider an MIR algorithm for two bidders that does not have full range. I.e., for some $0 \leq q^* \leq m$ it never outputs the allocation $(a_1 = q^*, a_2 = m - q^*)$. Now consider the pair of valuations where $v_1(q) = 1$ iff $q \geq q^*$ (and 0 otherwise), and $v_2(q) = 1$ iff $q \geq m - q^*$ (and 0 otherwise). The only allocation with value 2 is $(a_1 = q^*, a_2 = m - q^*)$ which is not in the range, while all other allocations have a value of at most 1.

From this we can easily get a lower bound any any computationally efficient MIR algorithm. The lower bound is on the number of queries that the bidders must be queried, and holds for any type of query – i.e., in a general communication setting[9].

[8] This is analogous to the weakest "value query" in combinatorial auction setting. Our lower bounds presented later will apply to all other query types as well.

[9] Note that we can not get an NP-completeness result when the valuations are simply encoded by lists of $(q_i^j, p_i^j)_j$ simply since when there are only 2 (or generally $O(1)$) bidders this problem is trivially optimally solvable in polynomial time by exhaustive search. The NP-completeness of approximation does follow for any encoding of the valuations for which finding the optimal allocation among 2 bidders is NP-complete.

Lemma 2.4 *An MIR algorithm for multi-unit auctions that achieves an approximation ratio better than 2 requires exponential communication. The result also applies for randomized settings.*

Proof: In the case of two bidders, an optimal algorithm is known to require exponential communication:

Lemma 2.5 ([24]) *Finding the optimal allocation in multi-unit auctions requires exponential communication, even if there are only two bidders and even for just finding the value of the allocation. This lower bound also applies for both randomized and nondeterministic settings.*

Thus, any MIR algorithm for 2 bidders that uses sub-exponential communication will be non-optimal and thus, as argued above, gives no better than a 2-approximation. The case of more than 2 bidders follows by setting all valuations but the first two to 0. □

This concludes the proof that no VCG-based mechanism can obtain a better than 2-approximation, except for two technical details that should be explicitly mentioned:

- The fact that incentive compatible VCG-based mechanisms are equivalent to MIR algorithms was shown in [22] for combinatorial auctions, while we need it for the somewhat different case of multi-unit auctions. However, the proof carries over naturally, and, for completeness, in Appendix A we give the complete proof for our case.
- Our lower bound was for MIR algorithms, while VCG-based mechanisms only proved to give algorithms that are *equivalent* to MIR algorithms. However, since the lower bound holds even for finding the *value* of the optimal allocation and it directly applies also to algorithms that are equivalent to MIR algorithms.

3 Combinatorial Auctions with Submodular Bidders

3.1 Combinatorial Auctions: Preliminaries

In a combinatorial auction we have a set M, $|M| = m$, of heterogeneous items and a set of N bidders, $|N| = n$. Each bidder i has a valuation function $v_i : 2^M \to \mathbb{R}$. We assume that each valuation v_i is normalized (i.e., $v_i(\emptyset) = 0$) and monotone (for each $S \subseteq T$, $v_i(S) \leq v_i(T)$). An *allocation* is an n-tuple $S = (S_1, ..., S_n)$, where for each i, $S_i \subseteq M$, and for each $i \neq i'$, $S_i \cap S_{i'} = \emptyset$. Our goal is to find an allocation S that maximizes the welfare $\Sigma_i v_i(S_i)$.

A valuation v is said to be *submodular* if it exhibits decreasing marginal utilities. I.e., for each $S \subseteq T \subseteq M$ and $j \notin S$, we have that $v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S)$. We will also use a very simple subset of submodular valuations called additive valuations. A valuation v is said to be *additive* if for each $S \subseteq M$, we have that $v(S) = \Sigma_{j \in S} v(\{j\})$.

3.2 The Main Result

In this section we analyze the power of MIR algorithms in the context of combinatorial auctions with submodular bidders. For this setting, an $O(\sqrt{m})$-approximation MIR algorithm is known [5]. We will show that this is (almost) the best approximation one can get using MIR algorithms. The theorem is stated only for MIR algorithms but we will point out how it can be extended to algorithms that are *equivalent* to MIR algorithms, and thus to all VCG-based mechanisms.

Theorem 3.1 *Every MIR mechanism for approximating the welfare in combinatorial auctions with submodular bidders that uses $O(e^{m^{\frac{1}{15}}})$ bits of communication achieves an approximation factor of $\min(\Omega(n), \Omega(m^{1/6}))$. This result also holds for the randomized and non-deterministic settings.*

We define two complexity measures for the range $\mathcal{R}$ of an MIR algorithm A: the cover number, and the intersection number. The cover number roughly corresponds to the size of the range $\mathcal{R}$. We will show, using the probabilistic method, that if the cover number is "small" then there exists an instance such that A fails to provide a good approximation. Therefore, the range $\mathcal{R}$ must be "large". In this case we will show that the intersection number of A must be exponential. We will see that the intersection number serves as a lower bound to the communication complexity of A, and so we get that any MIR-approximation algorithm that provides a good approximation ratio must have exponential communication complexity.

The proof of the theorem starts with Subsection 3.3, where the cover number is formally defined and its relation to the approximation ratio is shown. In Subsection 3.4 we define and discuss the second measure: the intersection number. The proof concludes in Subsection 3.5 by showing the relationship between the measures.

3.3 Complexity Measure I: The Cover Number

Intuitively we wish to rely on the size of the range. Yet, naive counting will fail to provide good results, since a single allocation in the range may contain many "degenerate allocations". For example, if the range contains an allocation that assigns all items to some bidder i, it also contains all allocations such that i is assigned any subset of the items, and the rest of the bidders get nothing. These exponentially many allocations are degenerate in the sense that we can assume that they are not in the range of the algorithm without changing the guaranteed approximation ratio of the A. We therefore use an alternative measure for describing the "size" of the range.

Definition 3.2 *A set $\mathcal{C}$ of allocations is said to be a* cover set *of another set of allocations $\mathcal{R}$ if for each $S \in \mathcal{R}$ there exists some $C \in \mathcal{C}$ such that for all i, $S_i \subseteq C_i$.*

The cover number *of a set of allocations $\mathcal{R}$ is defined to be the size of the minimum cardinality cover set of $\mathcal{R}$. The cover number is denoted by $cover(\mathcal{R})$.*

In the next lemma we prove that if $cover(\mathcal{R})$ is small, then there exists some instance in which A provides only $\Omega(n)$-approximation.

Lemma 3.3 *Let A be an MIR-algorithm with range $\mathcal{R}$. If $cover(\mathcal{R}) < e^{\frac{m}{300n}}$ then there is an instance in which A provides no more than $\frac{1.01}{n}$-fraction of the welfare.*

Proof: We randomly construct an instance of a combinatorial auction with additive valuations. Since the valuations are additive, we only need to specify the value of $v_i(\{j\})$ for each bidder i and item j. This is done in the following way: for each item $j \in M$ choose exactly one bidder, where each bidder is selected with probability of exactly $\frac{1}{n}$. Let i be the selected bidder. We set the value of $v_i(\{j\})$ to be 1. For each $i' \neq i$ we set the value of $v_{i'}(\{j\})$ to be 0.

First, observe that the value of the optimal solution in the random instance is exactly m. Nevertheless we will see that with non-negative probability the welfare provided by the MIR-algorithm A is only $\frac{1.01}{n}m$. Hence, the approximation ratio provided by A is no better than $\frac{n}{1.01}$. The following version of the Chernoff bounds will be useful.

Claim 3.4 (Chernoff bound) *Let $X_1, ...X_m$ be independent random variables that take values in $\{0,1\}$, such that for all i, $\Pr[X_i = 1] = p$ for some p. Then for every $0 \leq \delta \leq 2e - 1$ it holds that:*

$$\Pr[\Sigma_i X_i > (1+\delta)pm] \leq e^{-\frac{pm\delta^2}{3}}$$

Let $\mathcal{C}$ be the minimum cardinality cover set of $\mathcal{R}$ with $|\mathcal{C}| = cover(\mathcal{R})$. Fix some $C \in \mathcal{C}$. The probability that $v_i(\{j\}) = 1$, and that $j \in C_i$ is exactly $\frac{1}{n}$, for any bidder i and item j. By the Chernoff bound, $\Pr[\Sigma_i v_i(C_i) > \frac{1+\delta}{n}m] \leq e^{-\frac{\delta^2 m}{3n}}$. We now claim, by using the union bound, that if $cover(\mathcal{R}) < e^{\frac{\delta^2 m}{3n}}$ then there exists some instance such that no allocation in $\mathcal{C}$ provides a welfare of more than $\frac{1+\delta}{n}m$. Therefore it is obvious that no allocation in $\mathcal{R}$ can provide a welfare of more than $(\frac{1+\delta}{n})m$ for this instance. The lemma follows by choosing $\delta = .01$. $\square$

3.4 Complexity Measure II: The Intersection Number

The second complexity measure to be defined is the intersection number. We will show that the intersection number of the range of an MIR algorithm A is a lower bound to the communication complexity of A. Before defining the intersection number, we need a structural definition of a set of allocations.

Definition 3.5 *We say that a set of allocations $\mathcal{R}$ is* regular *if there exist constants $s_1, ..., s_n$ such that for all $S \in \mathcal{R}$ and for all $1 \leq i \leq n$ it holds that $|S_i| = s_i$.*

We are now ready to define the complexity measure itself.

Definition 3.6 *A set of allocations $\mathcal{D}$ is called an* (i,j)-intersection set *if for each $D, D' \in \mathcal{D}$, $D \neq D'$, it holds that $D_i \cap D'_j \neq \emptyset$.*

Define the intersection number *of a set of allocations $\mathcal{R}$, denoted by $intersect(\mathcal{R})$, to be the maximum cardinality regular (i,j)-intersecting set which is a subset of $\mathcal{R}$, where the maximum is taken over all pairs of bidders i and j.*

Notice that in the definition of the intersection number we require that the intersection set will be regular.

The next lemma shows that we can use the intersection number as a lower bound to the communication complexity of the algorithm.

Lemma 3.7 *Let A be an MIR-algorithm for combinatorial auctions with submodular bidders with range $\mathcal{R}$. Let $intersect(\mathcal{R}) = d$. Then, the communication complexity of A is $\Omega(d)$. This result holds even for randomized protocols and for non-deterministic protocols.*

Proof: We reduce from the disjointness problem (see [14]). In this problem Alice holds a d-bit string $a_1, ..., a_d$, and Bob holds a d-bit string $b_1, ..., b_d$. The goal is to decide whether there exists some index k such that $a_k = b_k = 1$. It is known that solving the disjointness problem requires $\Omega(d)$ communication bits, even for nondeterministic and randomized protocols.

Let $\mathcal{D} = \{D^1, ..., D^d\}$ be an (i,j)-intersection set of $\mathcal{R}$. $\mathcal{D}$ is regular, so for each bidder t there exists a constant s_t such that $|D_t| = s_t$, for all $D \in \mathcal{D}$. Construct a combinatorial auction with m items in the following way: Alice will play the role of bidder i, and Bob will play the role of the rest of the bidders, in particular bidder j. We now define the valuations of the bidders. Let the valuation of bidder i played by Alice be:

$$v_i(S) = \begin{cases} |S|, & |S| \leq s_i - 1; \\ s_i, & \exists k \text{ s.t. } D_i^k \subseteq S \text{ and } a_k = 1; \\ s_i - 2^{-(|S| - s_i + 1)}, & \text{otherwise.} \end{cases}$$

The valuation v_j is defined in an analogous way. Let the valuations of the rest of the bidders be zero on any bundle. The reader is encouraged to verify that all valuations are indeed submodular.

Observe that if there exists some index k such that the k'th input bit of both players is 1, then the optimal welfare is $s_i + s_j$. Otherwise, the optimal welfare is strictly less than $s_i + s_j$. To see this notice that if bidder i gains a value of s_i from the bundle S_1 he was assigned by A, then there must be an index k such that $D_i^k \subseteq S_1$ and $a_k = 1$. In order of bidder j to gain a value of s_j he must have an index k' such that $D_j^{k'} \subseteq S_2$. However, $\mathcal{D}$ is an (i,j)-intersection set and so it must hold that $D_i^k \cap D_j^{k'} \neq \emptyset$, and thus $S_1 \cap S_2 \neq \emptyset$. Clearly, the optimal welfare in this case is less than $s_i + s_j$.

By construction, if the optimal welfare is $s_i + s_j$ then it can be achieved by an allocation in $\mathcal{R}$. A is a maximal-in-range algorithm, and so the value of the allocation returned by A in this case must be $s_i + s_j$. Thus, we will be able to decide if there is a some index k such that $a_k = b_k = 1$. Hence, the communication complexity of A is at least as that of the disjointness problem: $\Omega(d)$. $\square$

Notice that our lower bound applies even for computing the *value* of the optimal allocation in $\mathcal{R}$, and thus applies not only to MIR algorithms but also to algorithms that are equivalent to MIR.

3.5 The Relationship between the Measures

It is easy to see that $cover(\mathcal{R}) \geq intersect(\mathcal{R})$. This subsection shows that the gap between the two is not too large. Specifically, if $intersect(\mathcal{R})$ is small, then $cover(\mathcal{R})$ is small too.

Lemma 3.8 *Let $\mathcal{R}$ be a set of allocations with $intersect(\mathcal{R}) \leq d$. Then*

$$cover(\mathcal{R}) < (8d)^{m^{\frac{3}{5}}n} \cdot m^{4m^{\frac{2}{5}}n^2}$$

As a corollary[10], let $n = m^{\frac{1}{6}}$. If $cover(\mathcal{R}) > e^{\frac{m}{300m}}$ then $intersect(\mathcal{R}) \geq e^{m^{\frac{1}{15}}}$. Thus, proving the lemma, together with Lemmas 3.3 and 3.7, derives Theorem 3.1.

Proof: (of Lemma 3.8) The lemma will follow from the following claim.

Claim 3.9 *Fix some w, $1 \leq w \leq m$. Let $\widetilde{\mathcal{R}}$ be a regular set of allocations. If $intersect(\widetilde{\mathcal{R}}) \leq d$ then there is a subset $\mathcal{E}$ of $\widetilde{\mathcal{R}}$ where $\frac{|\mathcal{E}|}{|\widetilde{\mathcal{R}}|} \geq (8d)^{-mn/w} 4^{-n^2}$, and $cover(\mathcal{E}) \leq w^n m^{wn^2}$.*

The lemma is proved by partitioning $\mathcal{R}$ to up to m^n classes of regular allocations, $\mathcal{R}_1, ..., \mathcal{R}_{m^n}$, one for each possible choice of constants $s_1, ..., s_n$ from Definition 3.5. Since each s_i is between 1 and m, there are at most m^n classes. The cover number of each class $\mathcal{R}_s$ will be upper bounded separately, in the following way:

Let $\mathcal{E}_1^s$ be the set obtained from the claim. Look at $\mathcal{R}_s \setminus \mathcal{E}_1^s$, and obtain from the claim another set $\mathcal{E}_2^s \subseteq \mathcal{R}_s \setminus \mathcal{E}_1^s$ with small cover, and so on. After $(8d)^{\frac{mn}{w}} \cdot 4^{n^2} \cdot \log |\mathcal{R}_s|$ steps $\mathcal{R}_s$ is completely covered. Now $cover(\mathcal{R}_s)$ can be bounded from above by $\Sigma_k cover(\mathcal{E}_k^s)$. By bounding from above the size of each class $|\mathcal{R}_s|$ by $|\mathcal{R}| \leq n^m$, we have that (by choosing $w = m^{\frac{2}{5}}$):

$$cover(\mathcal{R}) \leq \Sigma_{a=1}^{m^n} cover(\mathcal{R}_s) \leq \Sigma_{a=1}^{m^n} \Sigma_k |\mathcal{E}_k^s| \leq m^n \cdot (8d)^{\frac{mn}{w}} \cdot 4^{n^2} \cdot m \log n \cdot w^n m^{wn^2} \leq (8d)^{m^{\frac{3}{5}}n} \cdot m^{4m^{\frac{2}{5}}n^2}$$

Before proving Claim 3.9 itself, and thus Lemma 3.8, we will need some notation.

Definition 3.10 *Let $T_1, ..., T_n \subseteq M$. We say that a set of allocations $\mathcal{R}$ is $(T_1, ..., T_n)$-*structured *if for all $S \in \mathcal{R}$ it holds that $S_i \subseteq T_i$.*

Definition 3.11 *We will say that an allocation S is w-(i,j)-*aligned *in structure $(T_1, ..., T_n)$, if $|S_i \cap T_j| \leq w$. We will omit w when it will be clear from the context.*

[10]The result is actually stronger: fix a constant $\epsilon > 0$, and let $n < m^{\frac{1}{5}-\epsilon}$. If $cover(\mathcal{R}) > e^{\frac{m}{300m}}$ then $intersect(\mathcal{R}) \geq e^{m^{\epsilon}}$. The statement of the theorem improves accordingly.

The idea in proving Claim 3.9 will be to find a large subset $\mathcal{E} \subseteq \widetilde{\mathcal{R}}$, which is "sufficiently" aligned. Next we show that such subset has a small cover number.

Claim 3.12 *Let $\mathcal{E}$ be a $T = (T_1, ..., T_n)$-structured set of allocations. If for each pair of bidders i and j either all allocations in $\mathcal{E}$ are w-(i,j)-aligned in T or all allocations in $\mathcal{E}$ are w-(j,i)-aligned in T, then $cover(\mathcal{E}) \leq w^n m^{n^2 w}$.*

Proof: (of Claim 3.12) For each bidder i define a set of bidders I_i, where bidder j is in I_i if all allocations in $\mathcal{E}$ are (i,j)-aligned in T. Clearly, for each i and j, either $j \in I_i$ or $i \in I_j$. Let $B_i = T_i \setminus (\cup_{j \in I_i} T_j)$. The construction guarantees that $(B_1, ..., B_n)$ "almost" covers $\mathcal{E}$ in the sense that for bidder i and $S \in \mathcal{E}$, $|S_i \setminus B_i| \leq nw$. Also notice that by construction for each two different bidders i and j, $B_i \cap B_j = \emptyset$. Define the cover $\mathcal{C}$ as follows:

$$\mathcal{C} = \{P | P \text{ is an allocation in the form } ((B_1 \cup Q_1) \setminus \cup_{j \neq 1} Q_j, ..., (B_n \cup Q_n) \setminus \cup_{j \neq n} Q_j), \\ \text{and for each } i,\ |Q_i| \leq nw\}$$

Observe that since each $|Q_i| \leq nw$ we have that $|\mathcal{C}| \leq (\Sigma_{r=1}^{w} r \binom{m}{nr})^n \leq (w(m)^{nw})^n = w^n m^{n^2 w}$. Also notice that $\mathcal{C}$ is a cover set of $\mathcal{E}$. To see this, fix an allocation $S \in \mathcal{E}$. For each i, let $Q_i = S_i \setminus B_i$. Observe that each $|Q_i| \leq nw$, and that the Q_i's define an allocation that is in $\mathcal{C}$ and covers S. $\square$

Now we are ready to prove the main claim, and thus finish the proof of Lemma 3.8.

Proof: (of Claim 3.9) The construction of $\mathcal{E}$ will be divided into several steps. During the construction we maintain a sequence of subsets of $\widetilde{\mathcal{R}} : \mathcal{R}_0, \mathcal{R}_1, ...$ and structures T^0, $T^1, ...,$ such that each $\mathcal{R}_t$ is T^t-structured. We start by setting $\mathcal{R}_0 = \widetilde{\mathcal{R}}$ and $T^0 = (M, ..., M)$.

In each step we look at a pair of bidders i and j such that either all allocations in $\mathcal{R}_t$ are (i,j)-aligned in T^t or all allocations in $\mathcal{E}$ are (j,i)-aligned in T^t. If there is no such pair then let $\mathcal{E} = \mathcal{R}_t$ and the construction is over. Otherwise, look at all allocations in $\mathcal{R}_t$ that are either (i,j)-aligned or (j,i)-aligned in T^t. If there are at least $|\mathcal{R}_t|/2$ such allocations then we set $\mathcal{R}_{t+1}$ to be the largest set of the two: all allocations in $\mathcal{R}_t$ that are (i,j)-aligned, or all allocations in $\mathcal{R}_t$ that are (j,i)-aligned. Set the structure T^{t+1} to be T^t. Notice that $|\mathcal{R}_{t+1}| \geq |\mathcal{R}_t|/4$, and that $\mathcal{R}_{t+1}$ is T^{t+1}-structured. We call this step an *alignment* step, and proceed to the next step.

Otherwise, let $\mathcal{R}'_t$ be the set of allocations in $\mathcal{R}_t$ that are neither (i,j)-aligned nor (j,i)-aligned. Notice that $|\mathcal{R}'_t \geq \frac{\mathcal{R}_t}{2}|$. Take a maximal (i,j)-intersection set $\mathcal{D} \subseteq \mathcal{R}'_t$ – of size at most d. Now for every allocation $S \in \mathcal{R}'_t \setminus \mathcal{D}$ there exists some $D \in \mathcal{D}$ such that $D_i \cap S_j = \emptyset$ or $D_j \cap S_i = \emptyset$. Otherwise we have that $S \in \mathcal{D}$, contradicting the fact that $\mathcal{D}$ is a maximal intersection set. Thus, for some $D \in \mathcal{D}$ we have that for at least $(|\mathcal{R}'_t| - d)/(2d)$ allocations in $\mathcal{R}'_t$ either $D_i \cap S_j = \emptyset$ or $D_j \cap S_i = \emptyset$. Let us assume that for at least $(|\mathcal{R}'_t| - d)/(2d)$ allocations in $\mathcal{R}'_t$ the first option occurs. Define $\mathcal{R}_{t+1}$ to be this set of $(|\mathcal{R}'_t| - d)/(2d) \geq |\mathcal{R}_t|/(8d)$ allocations. Let $T_j^{t+1} = T_j^t \setminus D_i$. Also let $T_k^{t+1} = T_k^t$, for each $k \neq i$. Now notice that since $\mathcal{D}$ is a set of allocations that are not (i,j)-aligned in T_t, we have that $D_i \cap T_j^t > w$. We therefore have that $|T_j^{t+1}| < |T_j^t| - w$. (The other case is handled similarly, but this time by shrinking T_i^{t+1} rather than T_j^{t+1}.) By construction we have that $\mathcal{R}_{t+1}$ is T^{t+1}-structured. Term this step a *shrinkage* step, and continue to the next step.

Denote by l the number of steps the process went on. At most $\frac{nm}{w}$ steps are shrinkage steps, since in each shrinkage step $\Sigma_i |T_i^t|$ loses an additive of at least w. In addition, there are at most $\binom{n}{2}$ alignment steps, one for each pair of bidders. Therefore $|\mathcal{E}| = |\mathcal{R}_l| \geq \frac{|\widetilde{\mathcal{R}}|}{(8d)^{mn/w} 4^{(n^2)}}$. Also note that in the end of the process for each pair of bidders i and j either all allocations in $\mathcal{E}$ are (i,j)-aligned in T^l or all allocations in $\mathcal{E}$ are (j,i)-aligned in T^l (observe that an allocation that became properly aligned after an alignment step will remain so during the rest of the process.) By Claim 3.12 we have that $cover(E) \leq w^n m^{n^2 w}$, and thus Claim 3.9 is proved. $\square$

This concludes the proof of Lemma 3.8. $\square$

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A Characterization of VCG-Based Algorithms

In [22] it was proved that any VCG-based mechanism for general combinatorial auctions is equivalent to MIR algorithm. We slightly generalize this proof to hold for more settings, including the ones considered in this paper.

Let $\mathcal{A}$ be a set of alternatives (in our application, $\mathcal{A}$ will be the set of allocations). For all i let $V_i \subseteq R^{\mathcal{A}}$ be a set of valuations on $\mathcal{A}$ and denote $V = V_1 \times ... \times V_n$. A mechanism is composed of an allocation rule $f : V \to \mathcal{A}$ and payment rules $p = (p_1, ..., p_n)$, where $p_i : V \to R$.

Definition A.1 *A mechanism (f, p) is called VCG-based (VCGB) if for every i and some $h_i : V_{-i} \to R$ we have that for all v, $p_i(v) = h_i(v_{-i}) - \Sigma_{j \neq i} v_j(f(v))$.*

Definition A.2 *A mechanism (f, p) is called incentive compatible (IC) if for every v_i, v'_i, v_{-i} we have that $v_i(f(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$.*

Definition A.3 *An allocation rule f is called maximal in its range (MIR) if for every v, $f(v) \in \arg\max_{r \in \mathcal{R}} \Sigma v_i(r)$, where $\mathcal{R} = \{f(v) | v \in V\}$ is the range of f.*

Definition A.4 *An allocation rule f is equivalent to an allocation rule g if for all v, $\Sigma_i v_i(f(v)) = \Sigma_i v_i(g(v))$.*

Theorem A.5 (slight extension of [22]) *: Assume that V satisfies Condition 1 and Condition 2 below. If a mechanism (f, p) is VCGB and IC then f is equivalent to a MIR allocation rule.*

For Condition 1 and Condition 2 we will need notations:

Definition A.6 $V' = \{v \in V | \forall a \neq b \in A,\ \Sigma_i v_i(a) \neq \Sigma_i v_i(b)\}$.

Condition 1 *V' is dense in V (in the usual metric in $R^{\mathcal{A}}$).*

Definition A.7 *For $a \in \mathcal{A}$ and $v_i, z_i \in V_i$, We say that z_i is a-above v_i if for every $b \in \mathcal{A}$, $z_i(a) - v_i(a) \geq z_i(b) - v_i(b)$.*

Condition 2 *For every $v_i, w_i \in V_i$ there exists $z_i \in V_i$ that is a-above v_i and a-above w_i.*

Before we prove the theorem, let us just look at the two applications needed for this paper:

1. **Multi-unit auctions**: $A = \{(a_1 ... a_n) | \Sigma_i a_i \leq M\}$, V_i is all weakly monotone functions from $1...M$ to R. Condition 1 is met since V' has measure 0. Condition 2 is met by giving a sufficiently high value q to getting at least a_i items.
2. **Combinatorial auctions with submodular bidders:** $\mathcal{A}$ is the set of all allocations $S_1 ... S_n$, and v_i is the set of submodular valuations. Condition 1 is met since again V' has zero measure while V is fully dimensional. Condition 2 is met by defining an additive valuation (which in particular is submodular) that gives a sufficiently high value for each element in S_i.

Proof: Let us denote $\mathcal{R}' = \{f(v) | v \in V'\}$. Notice that by definition $\Sigma_i v_i(a) \neq \Sigma_i v_i(b)$ for every $v \in V'$ and $a \neq b \in \mathcal{R}'$ and in particular the argmax is unique. We will follow [22] and first show that over V', f is exactly maximal in the range $\mathcal{R}'$. I.e. that for all $v \in V'$, $f(v) = \arg\max_{r \in \mathcal{R}'} \Sigma v_i(r)$. Let us also assume wlog that all $h_i = 0$.

Before proceeding with the proof let us note two simple claims:

Claim A.8 *If $f(w) = a$ and z_i is a-above w_i then $f(z_i, w_{-i}) = a$.*

Proof: Assume to the contrary $f(z_i, w_{-i}) = b \neq a$. Since the VCG mechanism based on f is IC, we get by looking at a player with valuation w_i that $w_i(a) + \Sigma_{j \neq i} w_j(a) \geq w_i(b) + \Sigma_{j \neq i} w_j(b)$ while by looking at a player with valuation z_i we get $z_i(a) + \Sigma_{j \neq i} w_j(a) \leq z_i(b) + \Sigma_{j \neq i} w_j(b)$. Subtracting the two inequalities we get $w_i(a) - z_i(a) \geq w_i(b) - z_i(b)$ but notice that the fact that z_i is a-above w_i gives the opposite inequality which means that in fact $w_i(a) - z_i(a) = w_i(b) - z_i(b)$. Adding this equality to the second inequality above gives $w_i(a) + \Sigma_{j \neq i} w_j(a) \leq w_i(b) + \Sigma_{j \neq i} w_j(b)$, and thus equality holds in contradiction to w being in V'. □

Claim A.9 *If $f(v_i, v_{-i}) \neq a = \arg\max_{c \in \mathcal{R}'} \Sigma_i v_i(c)$ and z_i is a-above v_i then $f(z_i, v_{-i}) \neq a = \arg\max_{c \in \mathcal{R}'} z_i(c) + \Sigma_{j \neq i} v_j(c)$.*

Proof: The fact that $a = \arg\max_{c \in \mathcal{R}'} z_i(c) + \Sigma_{j \neq i} v_j(c)$ is simply since in moving from v_i to z_i, the value of the argument to the argmax increased more for a than for any other $c \in A$. The fact that $f(z_i, v_{-i}) \neq a$ is since otherwise a bidder with valuation v_i will benifit from reporting z_i. □

We are now ready to prove that f is exactly maximal in the range $\mathcal{R}'$. Assume towards contradiction that for $v, w \in V'$, $f(v) = b \neq a = \arg\max_{c \in \mathcal{R}'} \Sigma_i v_i(c)$, and $f(w) = a$. For every i fix z_i that is a-above both v_i and w_i (using Condition 2). Using Claim A.8 repeatedly, for all i, we get that $f(z) = a$ (at every stage i we look at $z_1 ... z_{i-1}, w_i, w_{i+1} ... w_n$ vs $z_1 ... z_{i-1}, z_i, w_{i+1} ... w_n$.)

On the other hand, using Claim A.9 repeatedly we get that $f(y) \neq a$ (at every stage i we look at $z_1...z_{i-1}, v_i, v_{i+1}...v_n$ vs $z_1...z_{i-1}, z_i, v_{i+1}...v_n$). Contradiction.

We now need to handle $V - V'$. Due to Condition 2, for every $v \in V - V'$ we can find an infinite sequence of points $v^j \in V'$ such that $v^j \to v$ and for all j, $f(v^j) = a$ for some fixed $a \in \mathcal{R}'$. Our equivalent MIR allocation rule f' will define $f'(v) = a$ (using e.g. the lexicographic first possible a). It remains to see that $\Sigma_i v_i(a) = \Sigma_i v_i(f(v))$. This follows since (1) $\Sigma_i v_i(a) = \lim_{j \to \inf} \Sigma_i v_i^j(a)$ (simply since $v^j \to v$) and (2) $\Sigma_i v_i(f(v)) = \lim_{j \to \inf} \Sigma_i v_i^j(f(v^j))$. This last equality just means the continuity of the function $\Sigma_i v_i(f(v))$ in v and can be established by looking at $|\Sigma_i v_i(f(v)) - \Sigma_i w_i(f(w))|$ which can be bounded by a telescopic sum of n elements of a similar form but with only a single index i with $v_i \neq w_i$, i.e. $|(v_i(a) + \Sigma_{j \neq i} v_j(a)) - (w_i(b) + \Sigma_{j \neq i} v_j(b))|$, where $a = f(v_i, v_{-i})$ and $b = f(w_i, v_{-i}$. This last difference can be bounded by $\max(v_i(a) - w_i(a), v_i(b) - w_i(b))$ since otherwise a player with valuation v_i would rather declare w_i (in case the LHS of the difference is smaller), or vice versa (otherwise).

$\square$

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