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Structure Theorem and Strict Alternation Hierarchy for FO^2 on Words*

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It is well-known that every first-order property on words is expressible using at most three variables. The subclass of properties expressible with only two variables is also quite interesting and well-studied. We prove precise structure theorems that characterize the exact expressive power of first-order logic with two variables on words. Our results apply to $\mathrm{FO}^2[<]$ and $\mathrm{FO}^2[<, \mathrm{Suc}]$, the latter of which includes the binary successor relation in addition to the linear ordering on string positions.

For both languages, our structure theorems show exactly what is expressible using a given quantifier depth, n, and using m blocks of alternating quantifiers, for any $m \leq n$. Using these characterizations, we prove, among other results, that there is a strict hierarchy of alternating quantifiers for both languages. The question whether there was such a hierarchy had been completely open since it was asked in [EVW97].

1 Introduction

It is well-known that every first-order property on words is expressible using at most three variables [IK89, K68]. The subclass of properties expressible with only two variables is also quite interesting and well-studied (Fact 1.1).

In this paper we prove precise structure theorems that characterize the exact expressive power of first-order logic with two variables on words. Our results apply to $\mathrm{FO}^2[<]$ and

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$\mathrm{FO}^2[<, \mathrm{Suc}]$, the latter of which includes the binary successor relation in addition to the linear ordering on string positions.

For both languages, our structure theorems show exactly what is expressible using a given quantifier depth, n, and using m blocks of alternating quantifiers, for any $m \leq n$. Using these characterizations, we prove that there is a strict hierarchy of alternating quantifiers for both languages. The question whether there was such a hierarchy had been completely open since it was asked in [EVW97].

Our motivation for studying FO^2 on words comes from the desire to understand the trade-off between formula size and number of variables. This is of great interest because, as is well-known, this is equivalent to the trade-off between parallel time and number of processors [I99]. Adler and Immerman introduced a game that can be used to determine the minimum size of first-order formulas with a given number of variables needed to express a given property. These games, which are closely related to the communication complexity games of Karchmer and Widgerson [KW90], were used to prove two optimal size bounds for temporal logics [AI03]. Later Grohe and Schweikardt used similar methods to study the size versus variable trade-off for first-order logic on unary words [GS05]. They proved that all first-order expressible properties of unary words are already expressible with two variables and that the variable-size trade-off between two versus three variables is polynomial whereas the trade-off between three versus four variables is exponential. They left open the trade-off between k and $k+1$ variables for $k \geq 4$. While we do not directly address that question here, our classification of FO^2 on words is a step towards the general understanding of the expressive power of FO needed for progress on such trade-offs.

Our characterization of $\mathrm{FO}^2[<]$ and $\mathrm{FO}^2[<, \mathrm{Suc}]$ on words is based on the very natural notion of n-ranker (Definition 3.2). Informally, a ranker is the position of a certain combination of letters in a word. For example, $\triangleright_{\mathtt{a}}$ and $\triangleleft_{\mathtt{b}}$ are 1-rankers where $\triangleright_{\mathtt{a}}(w)$ is the position of the first $\mathtt{a}$ in w (from the left) and $\triangleleft_{\mathtt{b}}(w)$ is the position of the first $\mathtt{b}$ in w from the right. Similarly, the 2-ranker $r_2 = \triangleright_{\mathtt{a}}\triangleright_{\mathtt{c}}$ denotes the position of the first $\mathtt{c}$ to the right of the first $\mathtt{a}$, and the 3-ranker, $r_3 = \triangleright_{\mathtt{a}}\triangleright_{\mathtt{c}}\triangleleft_{\mathtt{b}}$ denotes the position of the first $\mathtt{b}$ to the left of r_2. If there is no such letter then the ranker is undefined. For example, $r_3(\mathtt{cababcba}) = 5$ and $r_3(\mathtt{acbbca})$ is undefined.

Our first structure theorem (Theorem 3.8) says that the properties expressible in $\mathrm{FO}^2_n[<]$, i.e., first-order logic with two variables and quantifier depth n, are exactly boolean combinations of statements of the form, "r is defined", and "r is to the left (right) of r'" for k-rankers, r, and k'-rankers, r', with $k \leq n$ and $k' < n$. A non-quantitative version of this theorem was previously known [STV01].[1]

Surprisingly, Theorem 3.8 can be generalized in almost exactly the same form to characterize $\mathrm{FO}^2_{m,n}[<]$ where there are at most m blocks of alternating quantifiers, $m \leq n$. This second structure theorem (Theorem 4.5) uses the notion of (m,n)-ranker where there are m blocks of $\triangleright$'s or $\triangleleft$'s, that is, changing direction in rankers corresponds exactly to alternation of quantifiers. Using Theorem 4.5 we prove that there is a strict alternation hierarchy for $\mathrm{FO}^2_n[<]$ (Theorem 4.10) but that exactly at most $|\Sigma| + 1$ alternations are useful, where $|\Sigma|$ is the size of the

[1] See item 7 in Fact 1.1: a "turtle language" is a language of the form "r is defined", for some ranker, r.

alphabet (Theorem 4.6).

The language $FO^2[<, Suc]$ is more expressive than $FO^2[<]$ because it allows us to talk about consecutive strings of symbols[2]. For $FO^2[<, Suc]$, a straightforward generalization of n-ranker to n-successor-ranker allows us to prove exact analogs of Theorems 3.8 and 4.5. We use the latter to prove that there is also a strict alternation hierarchy for $FO^2_n[<, Suc]$ (Theorem 5.6). Since in the presence of successor we can encode an arbitrary alphabet in binary, no analog of Theorem 4.6 holds for $FO^2[<, Suc]$.

The expressive power of first-order logic with three or more variables on words has been well-studied. The languages expressible are of course the star-free regular languages [MP71]. The dot-depth hierarchy is the natural hierarchy of these languages. This hierarchy is strict [BK78] and identical to the first-order quantifier alternation hierarchy [T82, T84].

Many beautiful results on FO^2 on words were also already known. The main significant outstanding question was whether there was an alternating hierarchy. The following is a summary of the main previously known characterizations of $FO^2[<]$ on words.

Fact 1.1. [EVW97, EVW02, PW97, S76, TW98, STV01] Let $R \subseteq \Sigma^\star$. The following conditions are equivalent:

1. $R \in FO^2[<]$
2. R is expressible in unary temporal logic
3. $R \in \Sigma_2 \cap \Pi_2[<]$
4. R is an unambiguous regular language
5. The syntactic semi-group of R is a member of **DA**
6. R is recognizable by a partially-ordered 2-way automaton
7. R is a boolean combination of "turtle languages"

The proofs of our structure theorems are self-contained applications of Ehrenfeucht-Fraïssé games. All of the above characterizations follow from these results. Furthermore, we have now exactly connected quantifier and alternation depth to the picture, thus adding tight bounds and further insight to the above results.

For example, one can best understand item 4 above – that $FO^2[<]$ on words corresponds to the unambiguous regular languages – via Theorem 3.12 which states that any $FO^2_n[<]$ formula with one free variable that is always true of at most one position in any string, necessarily denotes an n-ranker.

In the conclusion of [STV01], the authors define the subclasses of rankers with one and two blocks of alternation. They write that, "...turtle languages might turn out to be a helpful tool for futher studies in algebraic language theory." We feel that the present paper fully justifies that prediction. Turtle languages — aka rankers — do provide an exceptionally clear and precise understanding of the expressive power of FO^2 on words, with and without successor.

In summary, our structure theorems provide a complete classification of the expressive power of FO^2 on words in terms of both quantifier depth and alternation. They also tighten several previous characterizations and lead to the alternation hierarchy results.

[2]With three variables we can express $\mathrm{Suc}(x, y)$ using the ordering: $x < y \wedge \forall z(z \leq x \vee y \leq z)$.

We begin the remainder of this extended abstract with a brief review of logical background including Ehrenfeucht-Fraïssé games, our main tool. In section 3 we formally define rankers and prove our structure theorem for $\mathrm{FO}^2_n[<]$. The structure theorem for $\mathrm{FO}^2_{m,n}[<]$ is covered in section 4, including our alternation hierarchy result that follows from it. Section 5 extends our structure theorems and the alternation hierarchy result to $\mathrm{FO}^2[<,\mathrm{Suc}]$.

2 Background and Definitions

We briefly recall some notation concerning strings, first-order logic, and Ehrenfeucht-Fraïssé games. See [I99] for further details, including the proof of Facts 2.1 and 2.2.

Σ will always denote a finite alphabet and ε the empty string. For $w \in \Sigma^\ell$ and $i \in [1,\ell]$, let w_i be the i-th letter of w; and for $[a,b]$ a subinterval of $[1,\ell]$, let $w_{[a,b]}$ be the substring $w_a \dots w_b$. We identify a word, $w \in \Sigma^\ell$ with the logical structure, $w = (\{1,\dots,\ell\}, Q_\sigma, \sigma \in \Sigma)$, where $(w, i/x) \models Q_\sigma(x)$ iff $w_i = \sigma$.

We use $\mathrm{FO}[<]$ to denote first-order logic with a binary linear order predicate $<$, and $\mathrm{FO} = \mathrm{FO}[<,\mathrm{Suc}]$ for first-order logic with an additional binary successor predicate. FO^k_n refers to the restriction of first-order logic to use at most k distinct variables, and quantifier depth n. $\mathrm{FO}^k_{m,n}$ is the further restriction to formulas such that any path in their parse tree has at most m blocks of alternating quantifiers, and $\mathrm{FO}^k\text{–ALT}\,[m] = \bigcup_{n\geq m} \mathrm{FO}^k_{m,n}$. We write $u \equiv^2_n v$ to mean that u and v agree on all formulas from FO^2_n, and $u \equiv^2_{m,n}$ to mean that they agree on $\mathrm{FO}^2_{m,n}$.

We assume that the reader is familiar with our main tool: the Ehrenfeucht-Fraïssé game. In each of the n moves of the game $\mathrm{FO}^2_n(u,v)$, Samson places one of the two pebbles pairs, x or y on a position in one of the two words and Delilah then answers by placing that pebble's mate on a position of the other word. Samson wins if after any move, the map from the chosen points in u to those in v, i.e., $x(u) \mapsto x(v)$, $y(u) \mapsto y(v)$ is not an isomorphism of the induced substructures; and Delilah wins otherwise. The fundamental theorem of Ehrenfeucht-Fraïssé games is the following:

Fact 2.1. Let $u, v \in \Sigma^\star$, $n \in \mathbb{N}$. Delilah has a winning strategy for the game $\mathrm{FO}^2_n(u,v)$ iff $u \equiv^2_n v$.

Thus, Ehrenfeucht-Fraïssé games are a perfect tool for determining what is expressible in first-order logic with a given quantifier-depth and number of variables. The game $\mathrm{FO}^2_{m,n}(u,v)$ is the restriction of the game $\mathrm{FO}^2_n(u,v)$ in which Samson may change which word he plays on at most $m-1$ times. For this game we have,

Fact 2.2. Let $u, v \in \Sigma^\star$ and let $m, n \in \mathbb{N}$ with $m \leq n$. Delilah has a winning strategy for the game $\mathrm{FO}^2_{m,n}(u,v)$ iff $u \equiv^2_{m,n} v$.

3 Structure Theorem for $\mathrm{FO}^2[<]$

We define boundary positions that point to the first or last occurrences of a letter in a word, and define an n-ranker as a sequence of n boundary positions. In terms of [STV01], boundary positions are turtle instructions and n-rankers are turtle programs of length n. The following three lemmas show that basic properties about the definedness and position of these rankers can be expressed in $\mathrm{FO}^2[<]$, and we use these results to prove our structure theorem.

Definition 3.1. A *boundary position* denotes the first or last occurrence of a letter in a given word. Boundary positions are of the form d_a where $d \in \{\triangleright, \triangleleft\}$ and $a \in \Sigma$. The interpretation of a boundary position d_a on a word $w = w_1 \dots w_{|w|} \in \Sigma^\star$ is defined as follows.

$$d_a(w) = \begin{cases} \min\{i \in [1, |w|] \mid w_i = a\} & \text{if } d = \triangleright \\ \max\{i \in [1, |w|] \mid w_i = a\} & \text{if } d = \triangleleft \end{cases}$$

Here we set $\min\{\}$ and $\max\{\}$ to be undefined, thus $d_a(w)$ is undefined if a does not occur in w. A boundary position can also be specified with respect to a position $q \in [1, |w|]$.

$$d_a(w, q) = \begin{cases} \min\{i \in [q+1, |w|] \mid w_i = a\} & \text{if } d = \triangleright \\ \max\{i \in [1, q-1] \mid w_i = a\} & \text{if } d = \triangleleft \end{cases}$$

Definition 3.2. Let n be a positive integer. An *n-ranker* r is a sequence of n boundary positions. The interpretation of an n-ranker $r = (p_1, \dots, p_n)$ on a word w is defined as follows.

$$r(w) := \begin{cases} p_1(w) & \text{if } r = (p_1) \\ \text{undefined} & \text{if } (p_1, \dots, p_{n-1})(w) \text{ is undefined} \\ p_n(w, (p_1, \dots, p_{n-1})(w)) & \text{otherwise} \end{cases}$$

Instead of writing n-rankers as a formal sequence $(p_1, \dots, p_n)$, we often use the simpler notation $p_1 \dots p_n$. We denote the set of all n-rankers by R_n, and the set of all n-rankers that are defined over a word w by $R_n(w)$. Furthermore, we set $R_n^\star := \bigcup_{i \in [1,n]} R_i$ and $R_n^\star(w) := \bigcup_{i \in [1,n]} R_i(w)$.

Definition 3.3. Let r be an n-ranker. As defined above, we have $r = (p_1, \dots, p_n)$ for boundary positions p_i. The *k-prefix ranker* of r for $k \in [1, n]$ is $r_k := (p_1, \dots, p_k)$.

Definition 3.4. Let $i, j \in \mathbb{N}$. The *order type* of i and j is defined as

$$\mathrm{ord}(i, j) = \begin{cases} < & \text{if } i < j \\ = & \text{if } i = j \\ > & \text{if } i > j \end{cases}$$

Lemma 3.5 (distinguishing points on opposite sides of a ranker)**.** Let n be a positive integer, let $u, v \in \Sigma^\star$ and let $r \in R_n(u) \cap R_n(v)$. Samson wins the game $\mathrm{FO}_n^2(u, v)$ where initially $\mathrm{ord}(x(u), r(u)) \neq \mathrm{ord}(x(v), r(v))$.

Proof. We only look at the case where $x(u) \geq r(u)$ and $x(v) < r(v)$ since all other cases are symmetric to this one. For $n = 1$ Samson has a winning strategy: If r is the first occurrence of a letter, then Samson places y on $r(u)$ and Delilah cannot reply. If r marks the last occurrence of a letter in the whole word, then Samson places y on $r(v)$. Again, Delilah cannot reply with any position and thus loses.

For $n > 1$, we look at the prefix ranker r_{n-1} of r. One of the following two cases applies.

(1) $r_{n-1}(u) < r(u)$, see Figure 1. Samson places y on $r(u)$, and Delilah has to reply with a position left of $x(v)$. She cannot choose any position in the interval $(r_{n-1}(v), r(v))$, because this section does not contain the letter $u_{r(u)}$. Thus she has to choose a position left of or equal to $r_{n-1}(v)$. By induction Samson wins the remaining game.

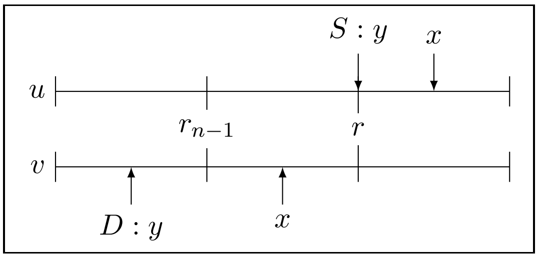

Figure 1: The case $r_{n-1}(u) < r(u)$

(2) $r(u) < r_{n-1}(u)$. This situation is illustrated in Figure 2. Samson places y on $r(v)$, and Delilah has to reply with a position right of $x(u)$ and thus right of $r(u)$. She cannot choose any position in the interval $(r(u), r_{n-1}(u))$ because this section does not contain the letter $v_{r(v)}$, thus Delilah has to choose a position right of or equal to $r_{n-1}(u)$. By induction Samson wins the remaining game. $\square$

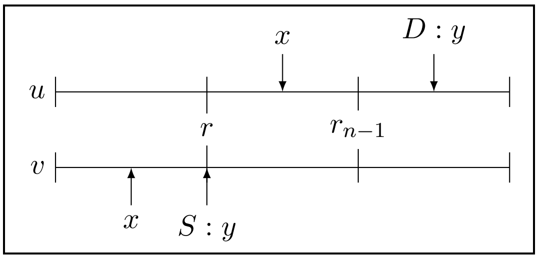

Figure 2: The case $r(u) < r_{n-1}(u)$

Lemma 3.6 (expressing the definedness of a ranker)**.** Let n be a positive integer, and let r be an n-ranker. There is a formula $\varphi_r \in \mathrm{FO}^2_n[<]$ such that for all $w \in \Sigma^\star$, $w \models \varphi_r \iff r \in R_n(w)$.

Proof. Let $u, v \in \Sigma^\star$ such that $r \in R_n(u)$ and $r \notin R_n(v)$. We show that Samson wins the game $\mathrm{FO}^2_n(u, v)$. If r_1, the shortest prefix ranker of r, is not defined over v, the letter referred to by r_1 occurs in u but does not occur in v. Thus Samson easily wins in one move.

Otherwise let $r_i = (p_1, \ldots, p_i)$ be the shortest prefix ranker of r that is undefined over v. Thus r_{i-1} is defined over both words. Without loss of generality we assume that $p_i = \lhd_{\mathtt{a}}$. This situation is illustrated in Figure 3. Notice that v does not contain any $\mathtt{a}$'s to the left of $r_{i-1}(v)$, otherwise r_i would be defined over v. Samson places x in u on $r_i(u)$, and Delilah has to reply with a position right of or equal to $r_{i-1}(v)$. Now Lemma 3.5 applies and Samson wins in $i - 1$ more moves. $\square$

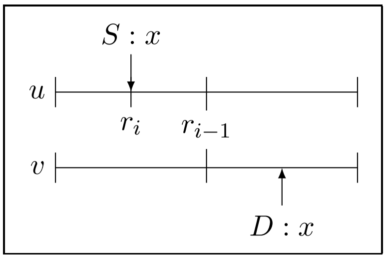

Figure 3: $r_i(v)$ is undefined

Lemma 3.7 (position of a ranker)**.** Let n be a positive integer and let $r \in R_n$. There is a formula $\varphi_r \in \mathrm{FO}^2_n[<]$ such that for all $w \in \Sigma^\star$ and for all $i \in [1, |w|]$, $(w, i/x) \models \varphi_r \iff i = r(w)$.

Proof. Let $u, v \in \Sigma^\star$. We show that Samson wins the game $\mathrm{FO}^2_n(u, v)$ where initially $x(u) = r(u)$ and $x(v) \neq r(v)$. If $r(v)$ is defined over v, then we can apply Lemma 3.5 immediately to get the desired strategy for Samson. Otherwise we use the strategy from Lemma 3.6. $\square$

Theorem 3.8 (structure of $\mathrm{FO}^2_n[<]$)**.** Let u and v be finite words, and let $n \in \mathbb{N}$. The following two conditions are equivalent.

(i) (a) $R_n(u) = R_n(v)$, and,

(b) for all $r \in R^\star_n(u)$ and $r' \in R^\star_{n-1}(u)$, $\mathrm{ord}(r(u), r'(u)) = \mathrm{ord}(r(v), r'(v))$

(ii) $u \equiv^2_n v$

Notice that condition (i)(a) is equivalent to $R^\star_n(u) = R^\star_n(v)$. Instead of proving Theorem 3.8 directly, we prove the following more general version on words with two interpreted variables.

Theorem 3.9. Let u and v be finite words, let $i_1, i_2 \in [1, |u|]$, let $j_1, j_2 \in [1, |v|]$, and let $n \in \mathbb{N}$. The following two conditions are equivalent.

(i) (a) $R_n(u) = R_n(v)$, and,

(b) for all $r \in R^\star_n(u)$ and $r' \in R^\star_{n-1}(u)$, $\mathrm{ord}(r(u), r'(u)) = \mathrm{ord}(r(v), r'(v))$, and,

(c) $(u, i_1/x, i_2/y) \equiv^2_0 (v, j_1/x, j_2/y)$, and,

(d) for all $r \in R^\star_n(u)$, $\mathrm{ord}(i_1, r(u)) = \mathrm{ord}(j_1, r(v))$ and $\mathrm{ord}(i_2, r(u)) = \mathrm{ord}(j_2, r(v))$

(ii) $(u, i_1/x, i_2/y) \equiv^2_n (v, j_1/x, j_2/y)$

Proof. For $n = 0$, (i)(a), (i)(b) and (i)(d) are vacuous, and (i)(c) is equivalent to (ii). For $n \geq 1$, we prove the two implications individually using induction on n.

We first show "$\neg$(i) $\Rightarrow$ $\neg$(ii)". Assuming that (i) holds for $n \in \mathbb{N}$ but fails for $n + 1$, we show that $(u, i_1/x, i_2/y) \not\equiv^2_n (v, j_1/x, j_2/y)$ by giving a winning strategy for Samson in the FO^2_n game on the two structures. If (i)(c) does not hold, then Samson wins immediately. If (i)(d) does not hold for $n + 1$, then Samson wins by Lemma 3.5. If (i)(a) or (i)(b) do not hold for $n + 1$, then one of the following three cases applies.

(1) There are two n-rankers that don't agree on their ordering in u and v.

(2) There is an $(n + 1)$-ranker that is defined over one word but not over the other.

(3) There is an $(n+1)$-ranker that does not appear in the same order on both structures with respect to a k-ranker where $k \leq n$.

We first look at case (1) where there are $r, r' \in R^\star_n(u)$ such that $\mathrm{ord}(r(u), r'(u)) \neq \mathrm{ord}(r(v), r'(v))$. Without loss of generality we assume that $r(u) \leq r'(u)$ and $r(v) > r'(v)$, and present a winning strategy for Samson in the FO^2_{n+1} game. In the first move he places x on $r(u)$ in u. Delilah has to reply with $r(v)$ in v, otherwise she would lose the remaining n-move game as shown in Lemma 3.5. Let r'_{n-1} be the $(n - 1)$-prefix-ranker of r'. We look at two different cases depending on the ordering of r'_{n-1} and r'.

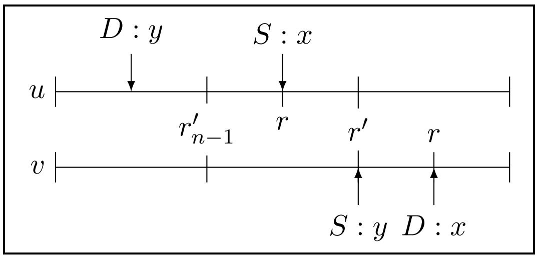

Figure 4: Two n-rankers appear in different order and r' ends with $\triangleright$

For $r'_{n-1}(u) < r'(u)$, the situation is illustrated in Figure 4. In his second move, Samson places y on $r'(v)$. Delilah has to reply with a position to the left of $x(u)$, but she cannot choose anything from the interval $(r'_{n-1}(u), r'(u))$ because this section does not contain the letter $v_{y(v)}$. Thus she has to reply with a position left of or equal to $r'_{n-1}(u)$, and Samson wins the remaining FO^2_{n-1} game as shown in Lemma 3.5.

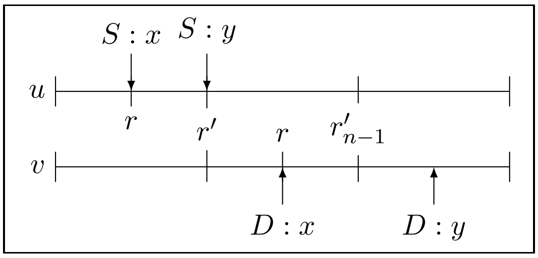

Figure 5: Two n-rankers appear in different order and r' ends with $\lhd$

For $r'_{n-1}(u) > r'(u)$, the situation is illustrated in Figure 5. In his second move, Samson places y on $r'(u)$. Delilah has to reply with a position to the right of $x(v)$, but she cannot choose anything from the interval $(r'(v), r'_{n-1}(v))$ because this section does not contain the letter $u_{y(u)}$. Thus she has to reply with a position right of or equal to $r'_{n-1}(v)$, and Samson wins the remaining FO^2_{n-1} game as shown in Lemma 3.5.

If (i) fails but all n-rankers agree on their ordering, then there are two consecutive n-rankers $r, r' \in R_n(u)$ with $r(u) < r'(u)$ and a letter $a \in \Sigma$ such that without loss of generality $\mathtt{a}$ occurs in the segment $u_{((r(u),r'(u))}$ but not in the segment $v_{(r(v),r'(v))}$. We describe a winning strategy for Samson in the game $\mathrm{FO}^2_{n+1}(u, v)$. He places x on an $\mathtt{a}$ in the segment $(r(u), r'(u))$ of u, as shown in Figure 6. Delilah cannot reply with anything in the interval $(r(v), r'(v))$. If she replies with a position left of or equal to $r(v)$, then x is on different sides of the n-ranker r in the two words. Thus Lemma 3.5 applies and Samson wins the remaining n-move game. If Delilah replies with a position right of or equal to $r'(v)$, then we can apply Lemma 3.5 to r' and get a winning strategy for the remaining game as well. This concludes the proof of "$\neg$(i) $\Rightarrow$ $\neg$(ii)".

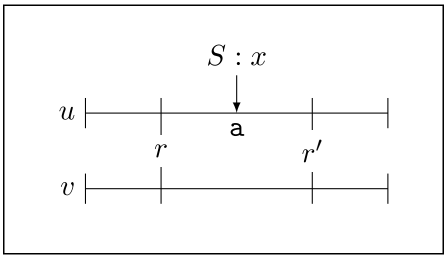

Figure 6: A letter $\mathtt{a}$ occurs between n-rankers r, r' in u but not in v

To show (i) $\Rightarrow$ (ii), we assume (i) for $n+1$, and present a winning strategy for Delilah in the FO^2_{n+1} game on the two structures. In his first move Samson picks up one of the two pebbles, and places it on a new position. Without loss of generality we assume that Samson picks up x and places it on u in his first move. If $x(u) = r(u)$ for any ranker $r \in R^\star_{n+1}(u)$, then Delilah replies with $x(v) = r(v)$. This establishes (i)(c) and (i)(d) for n, and thus Delilah has a winning strategy for the remaining FO^2_n game by induction.

If Samson does not place $x(u)$ on any ranker from $R^\star_{n+1}(u)$, then we look at the closest rankers from $R^\star_n(u)$ to the left and right of $x(u)$, denoted by r_ℓ and r_r, respectively. Let $\mathtt{a} := u_{x(u)}$ and define the $(n+1)$-ranker $s = (r_\ell, \rhd_{\mathtt{a}})$. On u we have $r_\ell(u) < s(u) < r_r(u)$. Because of (i)(a) s is defined on v as well, and because of (i)(b), we have $r_\ell(v) < s(v) < r_r(v)$. If $y(u)$ is not contained in the interval $(r_\ell(u), r_r(u))$, then Delilah places x on $s(v)$, which establishes (i)(c) and (i)(d) for n. Thus by induction Delilah has a winning strategy for the remaining FO^2_n game.

If both pebbles $x(u)$ and $y(u)$ are placed in the interval $(r_\ell(u), r_r(u))$, then we have to be more careful. Without loss of generality we assume $y(u) < x(u)$ as illustrated in Figure 7. Thus Delilah has to place x somewhere in the segment $(y(v), r_r(v))$ and at a position with the letter $\mathtt{a} := u_{x(u)}$. We define the $n+1$-ranker $s = (r_r, \triangleleft_\mathtt{a})$. From (i)(d) we know that s appears on the same side of y in both structures, thus we have $y(v) < s(v) < r_r(v)$. Delilah places her pebble x on $s(v)$, and thus establishes (i)(c) and (i)(d) for n. By induction, Delilah has a winning strategy for the remaining FO^2_n game. $\square$

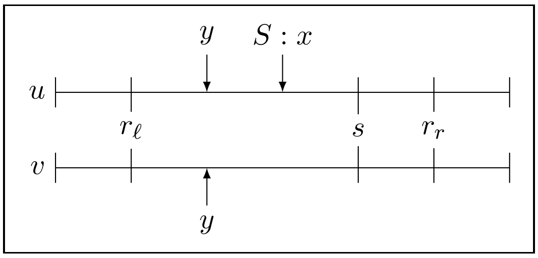

Figure 7: x and y are in the same section

A fundamental property of an n-ranker is that it uniquely describes a position in a given word. Now we show that the converse of this holds as well: any unique position in a word can be described by a ranker. Thus rankers completely characterize unique positions.

Definition 3.10 (unique position formula)**.** A formula $\varphi \in \mathrm{FO}^2[<]$ with x as a free variable is a *unique position formula* if for all $w \in \Sigma^\star$ there is at most one $i \in [1, |w|]$ such that $(w, i/x) \models \varphi$.

Lemma 3.11. Let n be a positive integer and let $\varphi \in \mathrm{FO}^2_n[<]$ be a unique position formula. Let $u \in \Sigma^\star$ and let $i \in [1, |u|]$ such that $(u, i/x) \models \varphi$. Then $i = r(u)$ for some ranker $r \in R^\star_n$.

Proof. Suppose for the sake of a contradiction that there is no ranker $r \in R^\star_n$ such that $(u, i/x) \models \varphi_r$. Because the first and last positions in u are described by 1-rankers, we know that $i \notin \{1, |u|\}$. Let $r_\ell, r_r \in R^\star_n(u)$ be the closest rankers to the left and right of i, respectively. We construct a new word v by doubling the symbol at position i in u, $v = u_1 \ldots u_{i-1} u_i u_i u_{i+1} \ldots u_{|u|}$. Because no ranker points to u_i, the two words u and v agree on the definedness of all n-rankers and on their ordering. Furthermore, position i in u and positions i and $i+1$ in v all appear in the same order with respect to all n-rankers. By Theorem 3.9, we thus have $(u, i/x) \equiv^2_n (v, i/x) \equiv^2_n (v, i+1/x)$, which contradicts the fact that φ is a unique position formula. $\square$

Theorem 3.12. Let n be a positive integer and let $\varphi \in \mathrm{FO}^2_n[<]$ be a unique position formula. There is a $k \in \mathbb{N}$, and there are mutually exclusive formulas $\alpha_i \in \mathrm{FO}^2_n[<]$ and rankers $r_i \in R^\star_n$ such that

$$\varphi \equiv \bigvee_{i \in [1,k]} (\alpha_i \wedge \varphi_{r_i})$$

where $\varphi_{r_i} \in \mathrm{FO}^2_n[<]$ is the formula from Lemma 3.7 that uniquely describes the ranker r_i.

Proof. Let $\mathcal{T}$ be the set of all $\mathrm{FO}^2_n[<]$ types of words over Σ with one interpreted variable. Because there are only finitely many inequivalent formulas in $\mathrm{FO}^2_n[<]$, $\mathcal{T}$ is finite. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the set of all types that satisfy φ. We set $\mathcal{T}' = \{T_1, \ldots, T_k\}$ and let $\alpha_i \in \mathrm{FO}^2_n[<]$ be a description of type T_i. Thus $\varphi \equiv \bigvee_{i \in [1,k]} \alpha_i$.

Now suppose that $(u, j/x) \models \varphi$. Thus $(u, j/x) \models \alpha_i$ for some i. By Lemma 3.11 $(u, j/x) \models \varphi_{r_i}$ for some $r_i \in R_n^\star$. Thus $\alpha_i \to \varphi_{r_i}$ since $\varphi_{r_i} \in \mathrm{FO}_n^2$ and α_i is a complete FO_n^2 formula. Thus $\alpha_i \equiv \alpha_i \wedge \varphi_{r_i}$ so φ is in the desired form. $\square$

4 Alternation hierarchy for $\mathrm{FO}^2[<]$

We define alternation rankers and prove our structure theorem (Theorem 4.5) for $\mathrm{FO}^2_{m,n}[<]$. Surprisingly the number of alternating blocks of $\triangleleft$ and $\triangleright$ in the rankers corresponds exactly to the number of alternating quantifier blocks. The main ideas from our proof of Theorem 3.8 still apply here, but keeping track of the number of alternations does add complications.

Definition 4.1 (m-alternation n-ranker)**.** Let $m, n \in \mathbb{N}$ with $m \leq n$. An *m-alternation n-ranker*, or (m, n)-ranker, is an n-ranker with exactly m blocks of boundary positions that alternate between $\triangleright$ and $\triangleleft$.

We use the following notation for alternation rankers.

$$
\begin{aligned}
R_{m,n}(w) &:= \{r \mid r \text{ is an } m\text{-alternation } n\text{-rankers that is defined over the word } w\} \\
R_{m\triangleright,n}(w) &:= \{r \in R_{m,n}(w) \mid r \text{ ends with } \triangleright\} \\
R^\star_{m,n}(w) &:= \bigcup_{i\in[1,m],j\in[1,n]} R_{i,j}(w) \\
R^\star_{m\triangleright,n}(w) &:= R^\star_{m-1,n}(w) \cup \bigcup_{i\in[1,n]} R_{m\triangleright,i}(w)
\end{aligned}
$$

Lemma 4.2. Let m and n be positive integers with $m \leq n$, let $u, v \in \Sigma^\star$, and let $r \in R_{m,n}(u) \cap R_{m,n}(v)$. Samson wins the game $\mathrm{FO}^2_{m,n}(u, v)$ where initially $\mathrm{ord}(r(u), x(u)) \neq \mathrm{ord}(r(v), x(v))$.

Furthermore, Samson can start the game with a move on u if r ends with $\triangleright$, $r(u) \leq x(u)$ and $r(v) \geq x(v)$, or if r ends with $\triangleleft$, $r(u) \geq x(u)$ and $r(v) \leq x(v)$. He can start the game with a move on v if r ends with $\triangleright$, $r(u) \geq x(u)$ and $r(v) \leq x(v)$, or if r ends with $\triangleleft$, $r(u) \leq x(u)$ and $r(v) \geq x(v)$.

Proof. If $m = n = 1$, then we can immediately apply the base case from the proof of Lemma 3.5. Samson wins in one move, placing his pebble on u or v as specified.

For the remaining cases, we assume without loss of generality that r ends with $\triangleright$ and that $x(u) \geq r(u)$ and $x(v) \leq r(v)$. Let r_{n-1} be the $(n-1)$-prefix ranker of r. This situation is illustrated in Figure 1 of Lemma 3.5. Samson places y on $r(u)$, and creates a situation where $y(u) > r_{n-1}(u)$ and $y(v) \leq r_{n-1}(v)$. If r_{n-1} ends with $\triangleleft$, then by induction Samson wins the remaining $\mathrm{FO}^2_{m-1,n-1}$ game and thus he has a winning strategy for the $\mathrm{FO}^2_{m,n}$ game. If r_{n-1} ends with $\triangleright$, then by induction Samson wins the remaining $\mathrm{FO}^2_{m,n-1}$ game starting with a move on u, and thus he has a winning strategy for the $\mathrm{FO}^2_{m,n}$ game. $\square$

Lemma 4.3. Let m and n be positive integers with $m \leq n$ and let $r \in R_{m,n}$. There is a formula $\varphi_r \in \mathrm{FO}^2_{m,n}[<]$ such that for all $w \in \Sigma^\star$, $w \models \varphi_r \iff r \in R_{m,n}(w)$.

Proof. Let $u, v \in \Sigma^\star$ such that $r \in R_{m,n}(u)$ and $r \notin R_{m,n}(v)$. Let $r_i = (p_1, \dots, p_i)$ be the shortest prefix ranker of r that is undefined over v, and we assume without loss of generality that this ranker ends with the boundary position $p_i = \lhd_a$ for some $a \in \Sigma$. This situation is illustrated in Figure 3 for Lemma 3.7. In his first move Samson places x on $r_i(u)$ and thus forces a situation where $x(u) < r_{i-1}(u)$ and $x(v) \geq r_{i-1}(v)$. If r_{i-1} ends with $\lhd$, then according to Lemma 4.2, Samson wins the remaining $\mathrm{FO}^2_{m,n-1}$ game starting with a move on u. Otherwise r_{i-1} ends with $\rhd$, and thus by Lemma 4.2 Samson wins the remaining $\mathrm{FO}^2_{m-1,n-1}$ game starting with a move on v. ◻

Lemma 4.4. Let m and n be positive integers with $m \leq n$ and let $r \in R_{m,n}$. There is a formula $\varphi_r \in \mathrm{FO}^2_{m,n}[<]$ such that for all $w \in \Sigma^\star$ and for all $i \in [1, |w|]$, $(w, i/x) \models \varphi_r \iff i = r(w)$.

Proof. Let $u, v \in \Sigma^\star$. We show that Samson wins the game $\mathrm{FO}^2_{m,n}(u, v)$ where initially $x(u) = r(u)$ and $x(v) \neq r(v)$. Depending on whether r is defined over v, we use the strategies from Lemma 4.2 or Lemma 4.3. ◻

Theorem 4.5 (structure of $\mathrm{FO}^2_{m,n}[<]$)**.** Let u and v be finite words, and let $m, n \in \mathbb{N}$ with $m \leq n$. The following two conditions are equivalent.

- (i) (a) $R_{m,n}(u) = R_{m,n}(v)$, and,
  - (b) for all $r \in R^\star_{m,n}(u)$ and $r' \in R^\star_{m-1,n-1}(u)$, $\mathrm{ord}(r(u), r'(u)) = \mathrm{ord}(r(v), r'(v))$, and,
  - (c) for all $r \in R^\star_{m,n}(u)$ and $r' \in R^\star_{m,n-1}(u)$ such that r and r' end with different directions, $\mathrm{ord}(r(u), r'(u)) = \mathrm{ord}(r(v), r'(v))$
- (ii) $u \equiv^2_{m,n} v$

Proof. As in the proof of Theorem 3.8, we first show "¬ (i) ⇒ ¬ (ii)". For $m = n$ the statement of this theorem is equivalent to Theorem 3.8. For $n > m$ we use induction on n.

Suppose that (i) holds for (m, n), but fails for $(m, n + 1)$. Thus one of the following cases applies.

(1) There are rankers $r \in R_{m,n}(u)$ and $r' \in R_{m-1,n}(u)$ such that $\mathrm{ord}(r(u), r'(u)) \neq \mathrm{ord}(r(v), r'(v))$.

(2) There are rankers $r, r' \in R_{m,n}(u)$ that end on different directions and $\mathrm{ord}(r(u), r'(u)) \neq \mathrm{ord}(r(v), r'(v))$.

(3) There is a ranker $r \in R_{m,n+1}$ that is defined over one structure but not over the other.

(4) There is a ranker $r \in R_{m,n+1}(u)$ that does not appear in the same order on both structures with respect to a ranker $r' \in R_{m-1,n}(u)$ or with respect to a ranker $r' \in R_{m,n}(u)$ that ends on a different direction than r.

For all of the above cases we present a winning strategy for Samson in the game $\mathrm{FO}^2_{m,n+1}(u,v)$, and thereby show that $u \not\equiv^2_{m,n+1} v$. We look at case (1) first, and we assume that $r(u) \le r'(u)$, as illustrated in Figure 8. The situation for $r(u) \ge r'(u)$ is completely symmetric. Depending on the last boundary position of r, one of the following two subcases applies.

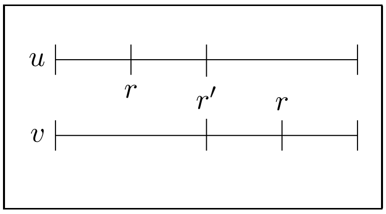

Figure 8: r and r' appear in different order

- r ends with $\triangleright$. Samson places x on $r(u)$ in his first move.
  If Delilah replies with a position to the left of $r(v)$, then we can apply Lemma 4.2 to get a winning strategy for Samson in the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on u. If Delilah replies with a position to the right of r', Samson has a winning strategy for the remaining $\mathrm{FO}^2_{m-1,n}$ game. Thus we have a winning strategy for Samson in the $\mathrm{FO}^2_{m,n+1}$ game.

- r ends with $\triangleleft$. This is similar to the previous case, but now Samson places x on $r(v)$ in his first move. If Delilah replies with a position to the right of $r(u)$, then as above we get a winning strategy for Samson in the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on v. Otherwise we get a winning strategy for Samson with only $m-1$ alternations for the remaining game. Thus again he has a winning strategy for the $\mathrm{FO}^2_{m,n+1}$ game.

For case (2), Samson's winning strategy is very similar to the previous case. If $r(u) \le r'(u)$ and r ends with $\triangleright$, then Samson places x on $r(u)$ in his first move. If Delilah replies with a position to the right of $r(u)$, then Samson's winning strategy is as above. Otherwise x is on different sides of r' and Samson has a winning strategy for the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on u. All together, he has a winning strategy for the $\mathrm{FO}^2_{m,n+1}$ game. The remaining three cases work in the same way.

Similar to what we did in the proof of Theorem 3.8, we can reduce cases (3) and (4) to an easier situation where a certain segment contains a certain letter in one structure, but not in the other structure.

In case (3), we assume without loss of generality that the $(m, n+1)$-ranker r is defined over u but not over v. Let $\mathtt{a} := u_{r(u)}$ be the letter in u at position $r(u)$. We define the following sets of rankers.

$$R_\ell := \{s \in R^\star_{m\triangleright,n}(u) \mid s(u) < r(u)\}$$
$$R_r := \{s \in R^\star_{m\triangleleft,n}(u) \mid s(u) > r(u)\}$$

Notice that all rankers from R_ℓ appear to the left of all rankers from R_r in u. By our inductive hypothesis, we know that this is also true in v. However, the rankers from R_ℓ and R_r by themselves do not necessarily appear in the same order in both structures. We look at the ordering of these rankers in v, and let r_ℓ be the rightmost ranker from R_ℓ and r_r be the leftmost ranker from R_r according to this ordering. By construction, we have $r_\ell(u) < r(u) < r_r(u)$, so the segment (r_ℓ, r_r) in u contains the letter $\mathtt{a}$. Let r_n be the n-prefix-ranker of r, and observe that r_n is defined on both structures and that r_n is contained in either R_ℓ or R_r. Because r is not defined on v, the letter a does not occur in v either to the right of $r_n \in R_\ell$ or to the left of $r_n \in R_r$. Thus the segment (r_ℓ, r_r) does not contain the letter a in v.

In case (4), we look at the same sets of rankers, R_ℓ and R_r, and at r_n, the n-prefix-ranker of r. We assume that $r(u) \leq r'(u)$ and that r ends with $\triangleright$, all other three cases are completely symmetric. Notice that r_n is either an $(m-1,n)$-ranker or an (m,n)-ranker that ends with $\triangleright$. Thus both structures agree on the ordering of r_n and r'. The relative positions of all these rankers are illustrated in Figure 9. As above, let r_ℓ be the rightmost ranker from R_ℓ and let r_r be the leftmost ranker from R_r, with respect to the ordering of these rankers on v. Again we know that $r_\ell(u) < r(u) < r_r(u)$ and therefore the segment (r_ℓ, r_r) of u contains an a. Notice that $r_n \in R_\ell$ and $r' \in R_r$, thus $r_n(v) \leq r_\ell(v) < r_r(v) \leq r'(v)$. Thus the segment (r_ℓ, r_r) does not contain the letter a in v.

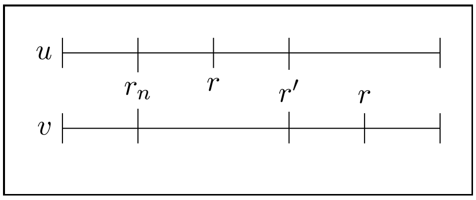

Figure 9: Ranker positions in case (4)

Now we know that a occurs in the segment (r_ℓ, r_r) in u but not in v, and thus we have established the situation illustrated in Figure 10. Samson places his first pebble on an a within this section of u, and Delilah has to reply with a position outside of this section. No matter at what side of the segment she chooses, with Lemma 4.2 Samson has a winning strategy for the remaining game and thus wins the $\mathrm{FO}^2_{m,n+1}$ game.

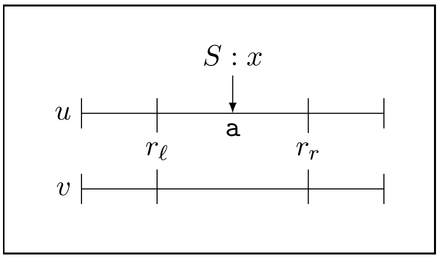

Figure 10: A letter occurs between rankers r, r' in u but not in v

To prove "(i) $\Rightarrow$ (ii)", we present a winning strategy for Delilah in the game $\mathrm{FO}^2_{m,n}(u,v)$, very similar to the one presented in the proof of Theorem 3.8. Delilah maintains the following invariant after each move $k \in [1,n]$, where j is the number of alternations between the two structures in Samson's moves so far. Thus we have $k = 1$ and $j = 0$ after the first move.

Invariant: For each of the two pebbles $\hat{x} \in \{x, y\}$,

(a) for all $r \in R^\star_{m-j-1,n-k}(u)$, $\mathrm{ord}(r(u), \hat{x}(u)) = \mathrm{ord}(r(v), \hat{x}(v))$

(b) for all $r \in R^\star_{m-j,n-k}(u)$,

- (b$_1$) if Samson played on u, r ends on $\triangleright$, and $r(u) = \hat{x}(u)$, then $r(v) \leq \hat{x}(v)$
- (b$_2$) if Samson played on u, r ends on $\triangleright$, and $r(u) < \hat{x}(u)$, then $r(v) < \hat{x}(v)$
- (b$_3$) if Samson played on u, r ends on $\triangleleft$, and $r(u) = \hat{x}(u)$, then $r(v) \geq \hat{x}(v)$
- (b$_4$) if Samson played on u, r ends on $\triangleleft$, and $r(u) > \hat{x}(u)$, then $r(v) > \hat{x}(v)$
- (b$_5$) if Samson played on v, r ends on $\triangleright$, and $r(v) = \hat{x}(v)$, then $r(u) \leq \hat{x}(u)$
- (b$_6$) if Samson played on v, r ends on $\triangleright$, and $r(v) < \hat{x}(v)$, then $r(u) < \hat{x}(u)$
- (b$_7$) if Samson played on v, r ends on $\triangleleft$, and $r(v) = \hat{x}(v)$, then $r(u) \geq \hat{x}(u)$
- (b$_8$) if Samson played on v, r ends on $\triangleleft$, and $r(v) > \hat{x}(v)$, then $r(u) > \hat{x}(u)$

First we argue that Delilah can establish this invariant in the first move. We assume without loss of generality that Samson places pebble x on u. Delilah's move depends on where exactly Samson places his pebble. If $x(u) = r(u)$ for any ranker $r \in R^\star_{m-1,n-1}(u)$, then Delilah replies with $x(v) = r(v)$ and thus establishes the invariant immediately. Otherwise we look at the following two sets of rankers.

$$R_\ell := \{r \in R^\star_{m\triangleright,n-1}(u) \mid r(u) < x(u)\}$$
$$R_r := \{r \in R^\star_{m\triangleleft,n-1}(u) \mid r(u) > x(u)\}$$

Let $\mathtt{a} := u_{x(u)}$ be the letter Samson places his pebble on. Delilah needs to find a position in v that is labeled with $\mathtt{a}$, and that is to the right of all rankers from R_ℓ and to the left of all rankers from R_r. Additionally, if Samson placed his pebble on a ranker from $R^\star_{m,n-1}(u)$, then we need to make sure that Delilah satisfied the relevant equality conditions from the invariant. We define

$$R'_\ell := \{r\triangleright_{\mathtt{a}} \mid r \in R_\ell\} \cup \{r \in R^\star_{m\triangleright,n-1}(u) \mid r(u) = x(u)\}$$
$$R'_r := \{r\triangleleft_{\mathtt{a}} \mid r \in R_r\} \cup \{r \in R^\star_{m\triangleleft,n-1}(u) \mid r(u) = x(u)\}$$

Delilah places her pebble on the rightmost ranker from R'_ℓ in v, and thus establishes (b_1) and (b_2). Because of (i)(c) this position is to the left of or equal to any ranker from R'_r, and thus (b_3) and (b_4) hold as well. Similarly (a) follows directly from (i)(b). Therefore Delilah establishes the invariant in her first move.

Now suppose that the invariant holds after move k, and suppose that Samson has used j alternations between the two structures so far. We also assume that Samson places y in move $k+1$ on u, and that $y(u) \leq x(u)$, as the other cases are symmetric.

If $y(u) = x(u)$, then of course Delilah replies with $y(v) = x(v)$ and establishes the invariant immediately. Otherwise we need to look at where Samson placed a pebble in the previous move. We first look at the case where Samson played on u in the previous move. If he places y on a ranker $r \in R^\star_{m-j-1,n-k-1}$, then Delilah replies by placing y on the same ranker on v and establishes the invariant immediately. Otherwise we look at the following two sets of rankers, very similar to what we did for Delilah's first move.

$$R_\ell := \{r \in R^\star_{m-j\triangleright,n-k-1}(u) \mid r(u) < y(u)\}$$
$$R_r := \{r \in R^\star_{m-j\triangleleft,n-k-1}(u) \mid r(u) > y(u)\}$$

Let $\mathtt{a} := u_{y(u)}$ be the letter Samson places his pebble on. Delilah needs to find a position in v that is labeled with $\mathtt{a}$, that is to the left of $x(u)$, and that is to the right of all rankers from R_ℓ and to the left of all rankers from R_r. Additionally, if Samson placed his pebble on a ranker from $R^\star_{m,n-1}(u)$, then we need to make sure that Delilah satisfied the relevant equality conditions from the invariant. We define

$$R'_\ell := \{r\triangleright_{\mathtt{a}} \mid r \in R_\ell\} \cup \{r \in R^\star_{m\triangleright,n-1}(u) \mid r(u) = y(u)\}$$
$$R'_r := \{r\triangleleft_{\mathtt{a}} \mid r \in R_r\} \cup \{r \in R^\star_{m\triangleleft,n-1}(u) \mid r(u) = y(u)\}$$

Delilah places her pebble on the rightmost ranker from R'_ℓ in v. All rankers from R'_ℓ appear left of or at $y(u)$ in u and thus also to the left of $x(u)$ in u. From (b_2) we know that all these rankers also appear to the left of $x(v)$ in v, so we have in fact $y(v) < x(v)$. It is also clear that (b_1) and (b_2) hold for y. Because of (i)(c), $y(v)$ appears to the left of or equal to any ranker from R'_r, and thus (b_3) and (b_4) hold as well. Similarly (a) follows directly from (i)(b). Therefore Delilah establishes the invariant again.

If Samson played on v in the previous move, we proceed in a similar way, but now the number of alternations increases as well. If Samson places y on a ranker $r \in R^\star_{m-j-2,n-k-1}$, then Delilah replies by placing y on the same ranker on v and establishes the invariant immediately. Otherwise we look at R_ℓ and R_r again, defined almost as above.

$$R_\ell := \{r \in R^\star_{m-j-1\triangleright,n-k-1}(u) \mid r(u) < y(u)\}$$
$$R_r := \{r \in R^\star_{m-j-1\triangleleft,n-k-1}(u) \mid r(u) > y(u)\}$$

R'_ℓ and R'_r are defined exactly as above, using our new definitions of R_ℓ and R_r. Delilah places her pebble on the rightmost ranker from R'_ℓ in v. Notice that $R'_\ell \subseteq R^\star_{m-j-1,n-k}$. Thus part (a) of the old invariant applies to all rankers from R'_ℓ and thus all these rankers appear to the left of or at the position of y on v. Therefore parts (b_1) and (b_2) of the invariant now hold. And because of (i)(b), $y(v)$ appears to the left of or at the position of any ranker from R'_r, so (b_3) and (b_4) hold as well. Part (a) of the invariant follows directly from (i)(b). Thus Delilah establishes the invariant again.

At the end of the game Delilah has managed to maintain the invariant without losing at any move, thus she wins the game. ◻

Using Theorem 4.5, we show that for any fixed alphabet Σ, at most $|\Sigma| + 1$ alternations are useful. Intuitively, each boundary position in a ranker says that a certain letter does not occur in some part of a word. Alternations are only useful if they visit one of these previous parts again. Once we visited one part of a word $|\Sigma|$ times, this part cannot contain any letters anymore and thus has to be empty.

Theorem 4.6. Let Σ be a finite alphabet, let $u, v \in \Sigma^\star$ and let $n \in \mathbb{N}$. If $u \equiv^2_{|\Sigma|+1,n} v$, then $u \equiv^2_n v$.

Proof. Suppose for the sake of a contradiction that $u \equiv^2_{|\Sigma|+1,n} v$ and $u \not\equiv^2_n v$. By Theorem 4.5, there is at least one m-alternation ranker such that $m > |\Sigma|$ and u and v disagree on the definedness of this ranker, or they disagree on the ordering of this ranker with respect to some other ranker. Let r be the shortest such ranker.

We write the ranker r in blocks of alternating directions,

$$r = D^1_{a_1} \dots D^1_{a_{k_1}} \;\; D^2_{a_{k_1+1}} \dots D^2_{a_{k_2}} \;\; \dots \;\; D^m_{a_{k_{m-1}+1}} \dots D^m_{a_{k_m}}$$

where $0 < k_1$, $k_{i-1} < k_i$, $D^i \in \{\triangleleft, \triangleright\}$, $D^i \neq D^{i-1}$, and $D^i = D^{i-2}$. We look at the prefix rankers of r at the end of each alternating block, $r_{k_1}, \dots, r_{k_m}$, and the intervals defined by these rankers. We set $I_0(u) := [1, |u|]$, $r_0(u) = 0$ if $D^1 = \triangleright$ and $r_0(u) = |u| + 1$ if $D^1 = \triangleleft$. For all $i \in [1, m]$ let,

$$I_i(u) := \begin{cases} [r_{k_i-1}(u) + 1, r_{k_i}(u) - 1] & \text{if } D^i = \triangleright \\ [r_{k_i}(u) + 1, r_{k_i-1}(u) - 1] & \text{if } D^i = \triangleleft \end{cases}$$

Notice that by definition the letter a_{k_i} does not occur in the interval I_i.

Suppose that for all $i \in [1, m]$ we have $r_{k_i}(u) \in I_{i-1}(u)$. Then the letter a_{k_i} has to occur in the interval $I_{i-1}(u)$ of u, but the interval $I_{|\Sigma|}(u)$ of u cannot contain any of the $|\Sigma|$ distinct letters. Therefore $r_{k_{|\Sigma|+1}} \notin I_{|\Sigma|}$ and we have a contradiction.

Otherwise there is an $i \in [1, m]$ such that $r_{k_i}(u) \notin I_{i-1}(u)$. We will construct a ranker r' that is shorter than r, does not have more alternations than r and occurs at exactly the same position as r in both u and v. By our assumption, u and v disagree on some property of the ranker r, and thus on some property of the shorter ranker r'. This contradicts our assumption that r was the shortest such ranker.

Now we show how to construct a shorter ranker r' that occurs at the same position at r. Recall that the prefix ranker

$$r_{k_i} = D^1_{a_1} \dots D^1_{a_{k_1}} \;\; D^2_{a_{k_1+1}} \dots D^2_{a_{k_2}} \;\; \dots \;\; D^{k_i-1}_{a_{k_{i-2}+1}} \dots D^{k_i-1}_{a_{k_{i-1}}} \;\; D^{k_i}_{a_{k_{i-1}+1}} \dots D^{k_i}_{a_{k_i}}$$

does not occur in the interval $I_{i-1}(u)$ in the word u. We assume without loss of generality that $D^{k_i} = \triangleleft$, and look at the relative positions of the rankers $r_{k_{i-1}+1}, \dots, r_{k_i}$ with respect to the ranker $r_{k_{i-1}-1}$. We know that $r_{k_i}(u) \leq r_{k_{i-1}-1}(u)$. Let $j \in [k_{i-1} + 1, k_i]$ be the index of the right-most of these rankers that is still to the left of $r_{k_{i-1}-1}$. Thus we have

$$r_{k_i}(u) < \dots < r_j(u) \leq r_{k_{i-1}-1}(u) < r_{j-1}(u) < \dots < r_{k_{i-1}+1}(u) < r_{k_{i-1}}(u)$$

We know that $u \equiv^2_{|\Sigma|+1,n} v$, thus by Theorem 4.5, these rankers occur in exactly the same order in v. Now we set

$$s := r_{k_{i-1}-1} \;\; D^{k_i}_{a_{k_j}} \dots D^{k_i}_{a_{k_i}}$$

Because u and v agree on the ordering of the relevant rankers, we have $s(u) = r_{k_i}(u)$ and $s(v) = r_{k_i}(v)$. Therefore we have reduced the size of a prefix of r without increasing the number of alternations, and thus have a shorter ranker r' that occurs at the same position as r in both structures. ☐

In order to prove that the alternation hierarchy for FO^2 is strict, we define example languages that can be separated by a formula of a given alternation depth m, but that cannot be separated by any formula of lower alternation depth. As Theorem 4.6 shows, we need to increase the size of the alphabet with increasing alternation depth. We inductively define the example words $u_{m,n}$ and $v_{m,n}$ and the example languages K_m and L_m over finite alphabets $\Sigma_m = \{a_0, \dots, a_{m-1}\}$. Here i, m and n are positive integers.

$$\begin{aligned} u_{1,n} &:= a_0 & v_{1,n} &:= \varepsilon \\ u_{2,n} &:= a_0(a_1a_0)^{2n} & v_{2,n} &:= (a_1a_0)^{2n} \\ u_{2i+1,n} &:= (a_0 \dots a_{2i})^n \; u_{2i,n} & v_{2i+1,n} &:= (a_0 \dots a_{2i})^n \; v_{2i,n} \\ u_{2i+2,n} &:= u_{2i+1,n} \; (a_{2i+1} \dots a_0)^n & v_{2i+2,n} &:= v_{2i+1,n} \; (a_{2i+1} \dots a_0)^n \end{aligned}$$

Notice that $u_{m,n}$ and $v_{m,n}$ are almost identical — if we delete the a_0 in the center of $u_{m,n}$, we get $v_{m,n}$. Finally, we set $K_m := \bigcup_{n\geq 1}\{u_{m,n}\}$ and $L_m := \bigcup_{n\geq 1}\{v_{m,n}\}$.

Definition 4.7. A formula φ *separates* two languages $K, L \subseteq \Sigma^\star$ if for all $w \in K$ we have $w \models \varphi$ and for all $w \in L$ we have $w \not\models \varphi$ or vice versa.

Lemma 4.8. For all $m \in \mathbb{N}$, there is a formula $\varphi_m \in \mathrm{FO}^2[<]$–ALT $[m]$ that separates K_m and L_m.

Proof. For $m = 1$, we can easily separate $K_1 = \{a_0\}$ and $L_1 = \{\varepsilon\}$ with the formula $\exists x(x = x)$.

For $m = 2$, we have $K_2 = \{a_0(a_1a_0)^{2n} \mid n \geq 1\}$ and $L_2 = \{(a_1a_0)^{2n} \mid n \geq 1\}$, and we define the ranker $r_2 := \triangleright_{a_1}\triangleleft_{a_0}$. On any word from K_2, r_2 evaluates to the first position in this word, but r_2 is not defined over any word from L_2, since all these words start with a_1. Thus we can separate K_2 and L_2 with an $\mathrm{FO}^2_{2,2}[<]$ formula by Lemma 4.4.

For $m \geq 3$, we show that the two languages K_m and L_m differ on the ordering of two $(m-1)$-alternation rankers. Then by Theorem 4.5 there is an $\mathrm{FO}^2_{m,m}[<]$ formula that separates K_m and L_m. We inductively define the rankers

$$
\begin{aligned}
r_3 &:= \triangleleft_{a_2}\triangleright_{a_0} & s_3 &:= \triangleleft_{a_2}\triangleright_{a_1}\\
r_{2i} &:= \triangleright_{a_{2i-1}} r_{2i-1} & s_{2i} &:= \triangleright_{a_{2i-1}} r'_{2i-1}\\
r_{2i+1} &:= \triangleleft_{a_{2i}} r_{2i} & s_{2i+1} &:= \triangleleft_{a_{2i}} r'_{2i}
\end{aligned}
$$

For $m \geq 3$, all words from K_m contain the substring $a_0a_1a_2\ a_0\ a_1a_0$ in the middle, whereas all words from L_m have the substring $a_0a_1a_2\ a_1a_0$ in the middle. For both the words from K_m and those from L_m, s_m evaluates to the position of a_1 at the end of this section. For the words from K_m, r evaluates to the position of the a_0 in the middle, whereas for the words from L_m r evaluates to the position of the next a_0. Thus we have $r_m(u) < s_m(u)$ for all $u \in K_m$ and $r_m(v) > s_m(v)$ for all $v \in L_m$. Therefore condition (i)(b) of Theorem 4.5 fails for any pair of words, and there is a formula in $\mathrm{FO}^2_{m,m}[<]$ that separates K_m and L_m. $\square$

Lemma 4.9. For all positive integers m and for all $n \in \mathbb{N}$, we have $u_{m,n} \equiv^2_{m-1,n} v_{m,n}$.

Proof. Because we do not have constants, there are no quantifier-free sentences. Thus $\mathrm{FO}^2_{0,n}[<]$ does not contain any formulas and the statement holds trivially for $m = 1$.

For $m \geq 2$ and any $n \geq m$, we claim that exactly the same $(m-1,n)$-rankers occur in $u_{m,n}$ and $v_{m,n}$, and that all $(m-1,n)$-rankers appear in the same order with respect to all $(m-2,n-1)$-rankers and all $(m-1,n-1)$-rankers that end on a different direction. Once we established this claim, the lemma follows immediately with Theorem 4.5. We already observed that $u_{m,n}$ and $v_{m,n}$ are almost identical. The only difference between the two words is that $u_{m,n}$ contains the letter a_0 in the middle whereas $v_{m,n}$ does not. Thus we only have to consider rankers that are affected by this middle a_0.

We claim that any ranker that points to the middle a_0 of $u_{m,n}$ requires at least $m-1$ alternations. Furthermore, we claim that any such ranker needs to start with $\triangleright$ for even m and with $\triangleleft$ for odd m. We prove this by induction on m.

For $m = 2$ we have $u_{2,n} = a_0(a_1a_0)^n$. Any n-ranker that starts with $\triangleleft$ cannot reach the first a_0, thus we need a ranker that starts with $\triangleright$.

For odd $m > 2$ we have $u_{m,n} = (a_0 \ldots a_{m-1})^n u_{m-1,n}$. Any n-ranker that starts with $\triangleright$ cannot leave the first block of $n \cdot m$ symbols of this word and thus not reach the middle a_0. Therefore we need to start with $\triangleleft$, and in fact use $\triangleleft_{a_{m-1}}$ at some point, because we would not be able to leave the last section of $u_{m-1,n}$ otherwise. But with $\triangleleft_{a_{m-1}}$ we move past all of $u_{m-1,n}$, and we need one alternation to turn around again. By induction, we need at least $m-2$ alternations within $u_{m-1,n}$, and thus $m-1$ alternations total.

The argument for even m is completely symmetric. Thus we showed that we need at least $m-1$ alternation blocks to point to the middle a_0. Furthermore, we showed that if we have exactly $m-1$ alternation blocks, then the last of these blocks uses $\triangleright$. Therefore we only need to consider $(m-1)$-alternation rankers that end on $\triangleright$ and pass through the middle a_0. It is easy to see that all of these rankers agree on their ordering with respect to all other $(m-2)$-alternation rankers, and with respect to all $(m-1)$-alternation rankers that end on $\triangleleft$.

To summarize, we showed that $u_{m,n}$ and $v_{m,n}$ satisfy condition (i) from Theorem 4.5 for $m-1$ alternations. Thus the two words agree on all formulas from $\mathrm{FO}^2_{m-1,n}[<]$. □

Theorem 4.10 (alternation hierarchy for $\mathrm{FO}^2[<]$)**.** For any positive integer m, there is a $\varphi_m \in \mathrm{FO}^2[<]$–ALT $[m]$ and there are two languages $K_m, L_m \subseteq \Sigma^\star$ such that φ_m separates K_m and L_m, but no $\psi \in \mathrm{FO}^2[<]$–ALT $[m-1]$ separates K_m and L_m.

Proof. The theorem immediately follows from lemmas 4.8 and 4.9. □

Theorem 4.10 resolves an open question from [EVW97, EVW02].

5 Structure Theorem and Alternation Hierarchy for $\mathrm{FO}^2[<, \mathrm{Suc}]$

We extend our definitions of boundary positions and rankers from section 3 to include the substrings of a given length that occur immediately before and after the position of the ranker.

Definition 5.1. A (k, ℓ)-*neighborhood boundary position* denotes the first or last occurrence of a substring in a given word. More precisely, a (k, ℓ)-neighborhood boundary position is of the form $d_{(s,a,t)}$ with $d \in \{\triangleright, \triangleleft\}$, $s \in \Sigma^k$, $a \in \Sigma$ and $t \in \Sigma^\ell$. The interpretation of a (k, ℓ)-neighborhood boundary position $p = d_{(s,a,t)}$ on a word $w = w_1 \ldots w_{|w|}$ is defined as follows.

$$p(w) = \begin{cases} \min\{i \in [k+1, |w| - \ell] \mid w_{i-k} \ldots w_{i+\ell} = sat\} & \text{if } d = \triangleright \\ \max\{i \in [k+1, |w| - \ell] \mid w_{i-k} \ldots w_{i+\ell} = sat\} & \text{if } d = \triangleleft \end{cases}$$

Notice that $p(w)$ is undefined if the sequence sat does not occur in w. A (k, ℓ)-neighborhood boundary position can also be specified with respect to a position $q \in [1, |w|]$.

$$p(w, q) = \begin{cases} \min\{i \in [\max\{q+1, k+1\}, |w| - \ell] \mid w_{i-k} \ldots w_{i+\ell} = sat\} & \text{if } d = \triangleright \\ \max\{i \in [k+1, \min\{q-1, |w| - \ell\}] \mid w_{i-k} \ldots w_{i+\ell} = sat\} & \text{if } d = \triangleleft \end{cases}$$

Observe that $(0,0)$-neighborhood boundary positions coincide with the boundary positions from Definition 3.1. As before in the case without successor, we build rankers out of these boundary positions.

Definition 5.2. An *n-successor-ranker* r is an n-sequence of neighborhood boundary positions, $r = (p_1, \dots, p_n)$, where p_i is a (k_i, ℓ_i)-neighborhood boundary position and $k_i, \ell_i \in [0, (i-1)]$. The interpretation of an n-successor-ranker r on a word w is defined as follows.

$$r(w) := \begin{cases} p_1(w) & \text{if } r = (p_1) \\ \text{undefined} & \text{if } (p_1, \dots, p_{n-1})(w) \text{ is undefined} \\ p_n(w, (p_1, \dots, p_{n-1})(w)) & \text{otherwise} \end{cases}$$

We denote the set of all n-successor-rankers that are defined over a word w by $SR_n(w)$, and set $SR^\star_n(w) := \bigcup_{i \in [1,n]} SR_i(w)$.

Because we now have the additional atomic relation Suc, we need to extend our definition of order type as well.

Definition 5.3. Let $i, j \in \mathbb{N}$. The *successor order type* of i and j is defined as

$$\mathrm{ord}_{\mathrm{S}}(i,j) = \begin{cases} \ll & \text{if } i < j-1 \\ -1 & \text{if } i = j-1 \\ = & \text{if } i = j \\ +1 & \text{if } i = j+1 \\ \gg & \text{if } i > j+1 \end{cases}$$

With this new definition of n-successor-rankers, our proofs for Lemmas 3.5, 3.6, 3.7 and Theorem 3.8 go through with only minor modifications. Instead of working through all the details again, we simply point out the differences.

First we notice that 1-successor-rankers are simply 1-rankers, so the base case of all inductions remains unchanged. In the proofs of Lemmas 3.5, 3.6 and 3.7, and in the proof of (ii) $\Rightarrow$ (i) from Theorem 3.8, we argued that Delilah cannot reply with a position in a given section because it does not contain a certain ranker and therefore it does not contain the symbol used to define this ranker. Now we need to know more – we need to show that Delilah cannot reply with a certain letter in a given section that is surrounded by a specified neighborhood, given that this section does not contain the corresponding successor-ranker. Whenever Samson's winning strategy depends on the fact that an n-successor-ranker does not occur in a given section, he has $n-1$ additional moves left. So if Delilah does not reply with a position with the same letter and the same neighborhood, Samson can point out a difference in the neighborhood with at most $(n-1)$ additional moves.

For the other direction of Theorem 3.8, we need to make sure that Delilah can reply with a position that is contained in the correct interval, has the same symbol and is surrounded by the same neighborhood. Where we previously defined the n-ranker $s := (r_\ell, \triangleright_{\mathtt{a}})$ or $s := (r_r, \triangleleft_{\mathtt{a}})$, we now include the $(n-1)$-neighborhood of the respective positions chosen by Samson. Thus we make sure that Samson cannot point out a difference in the two words, and Delilah still has a winning strategy. Thus we have the following three theorems for $\mathrm{FO}^2[<,\mathrm{Suc}]$.

Theorem 5.4 (structure of $\mathrm{FO}^2_n[<,\mathrm{Suc}]$)**.** Let u and v be finite words, and let $n \in \mathbb{N}$. The following two conditions are equivalent.

(i) (a) $SR_n(u) = SR_n(v)$, and,

  (b) for all $r \in SR^\star_n(u)$ and $r' \in SR^\star_{n-1}(u)$, $\mathrm{ord_S}(r(u), r'(u)) = \mathrm{ord_S}(r(v), r'(v))$

(ii) $u \equiv^2_n v$

Theorem 5.5 (structure of $\mathrm{FO}^2_{m,n}[<,\mathrm{Suc}]$)**.** Let u and v be finite words, and let $m, n \in \mathbb{N}$ with $m \leq n$. The following two conditions are equivalent.

(i) (a) $SR_{m,n}(u) = SR_{m,n}(v)$, and,

  (b) for all $r \in SR^\star_{m,n}(u)$ and $r' \in SR^\star_{m-1,n-1}(u)$, $\mathrm{ord_S}(r(u), r'(u)) = \mathrm{ord_S}(r(v), r'(v))$, and,

  (c) for all $r \in SR^\star_{m,n}(u)$ and $r' \in SR^\star_{m,n-1}(u)$ such that r and r' end with different directions, $\mathrm{ord_S}(r(u), r'(u)) = \mathrm{ord_S}(r(v), r'(v))$

(ii) $u \equiv^2_{m,n} v$

Theorem 5.6 (alternation hierarchy for $\mathrm{FO}^2[<,\mathrm{Suc}]$)**.** Let m be a positive integer. There is a $\varphi_m \in \mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}\,[m]$ and there are two languages $K_m, L_m \subseteq \Sigma^\star$ such that φ_m separates K_m and L_m, but no $\psi \in \mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}\,[m-1]$ separates K_m and L_m.

Proof. We use the same ideas as before in Theorem 4.10. We define example languages that now include an extra letter b to ensure that the successor predicate is of no use. As before, we inductively construct the words $u_{m,n}$ and $v_{m,n}$ and use them to define the languages K_m and L_m.

$$
\begin{aligned}
u_{1,n} &:= b^{2n} a_0 b^{2n} & v_{1,n} &:= b^{2n} \\
u_{2,n} &:= u_{1,n}\ (a_1 b^{2n} a_0 b^{2n})^{2n} & v_{2,n} &:= v_{1,n}\ (a_1 b^{2n} a_0 b^{2n})^{2n} \\
u_{2i+1,n} &:= (b^{2n} a_0 b^{2n} \dots b^{2n} a_{2i})^n\ u_{2i,n} & v_{2i+1,n} &:= (b^{2n} a_0 b^{2n} \dots b^{2n} a_{2i})^n\ v_{2i,n} \\
u_{2i+2,n} &:= u_{2i+1,n}\ (a_{2i+1} b^{2n} \dots b^{2n} a_0 b^{2n})^n & v_{2i+2,n} &:= v_{2i+1,n}\ (a_{2i+1} b^{2n} \dots b^{2n} a_0 b^{2n})^n
\end{aligned}
$$

Finally we set $K_m := \bigcup_{n \geq 1} \{u_{m,n}\}$ and $L_m := \bigcup_{n \geq 1} \{v_{m,n}\}$. Notice that the b's are not necessary to distinguish between the two languages K_m and L_m, and thus the proof of Lemma 4.8 goes through unchanged and we have a formula $\varphi_m \in \mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}\,[m]$ that separates K_m and L_m. To see that no $\mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}\,[m-1]$ formula can separate K_m and L_m, we observe that any $(n-1)$-neighborhood in the words $u_{m,n}$ and $v_{m,n}$ contains all b's except for at most one letter a_i for some $i \in [0, m-1]$. Thus the proof of Lemma 4.9 goes through here as well. ☐

6 Conclusion

We proved precise structure theorems for FO^2, with and without the successor predicate, that completely characterize the expressive power of the respective logics, including exact bounds on the quantifier depth and on the alternation depth. Using our structure theorems, we show that the quantifier alternation hierarchy for FO^2 is strict, settling an open question from [EVW97, EVW02]. Both our structure theorems and the alternation hierarchy results add further insight to and simplify previous characterizations of FO^2. We also hope that the insights gained in our study of FO^2 on words will be useful in future investigations of the trade-off between formula size and number of variables.

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