

# On Approximating Restricted Cycle Covers\*

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## Abstract

A cycle cover of a graph is a set of cycles such that every vertex is part of exactly one cycle. An  $L$ -cycle cover is a cycle cover in which the length of every cycle is in the set  $L$ . The weight of a cycle cover of an edge-weighted graph is the sum of the weights of its edges.

We come close to settling the complexity and approximability of computing  $L$ -cycle covers. On the one hand, we show that for almost all  $L$ , computing  $L$ -cycle covers of maximum weight in directed and undirected graphs is APX-hard and NP-hard. Most of our hardness results hold even if the edge weights are restricted to zero and one.

On the other hand, we show that the problem of computing  $L$ -cycle covers of maximum weight can be approximated within a factor of 2 for undirected graphs and within a factor of  $8/3$  in the case of directed graphs. This holds for arbitrary sets  $L$ .

## 1 Introduction

A cycle cover of a graph is a spanning subgraph that consists solely of cycles such that every vertex is part of exactly one cycle. Cycle covers play an important role in the design of approximation algorithms for the travelling salesman problem [4, 6, 7, 10–13, 23], the shortest common superstring problem [9, 30], and vehicle routing problems [19].

In contrast to Hamiltonian cycles, which are special cases of cycle covers, cycle covers of maximum weight can be computed efficiently. This is exploited in the aforementioned approximation algorithms, which usually start by computing an initial cycle cover and then join cycles to obtain a Hamiltonian cycle. This technique is called *subtour patching* [16].

Short cycles in a cycle cover limit the approximation ratios achieved by such algorithms. In general, the longer the cycles in the initial cover, the better the

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approximation ratio. Thus, we are interested in computing cycle covers that do not contain short cycles. Moreover, there are approximation algorithms that perform particularly well if the cycle covers computed do not contain cycles of odd length [6]. Finally, some vehicle routing problems [19] require covering vertices with cycles of bounded length.

Therefore, we consider *restricted cycle covers*, where cycles of certain lengths are ruled out a priori: For  $L \subseteq \mathbb{N}$ , an  $L$ -*cycle cover* is a cycle cover in which the length of each cycle is in  $L$ . To fathom the possibility of designing approximation algorithms based on computing cycle covers, we aim to characterise the sets  $L$  for which  $L$ -cycle covers of maximum weight can be computed, or at least well approximated, efficiently.

Beyond being a basic tool for approximation algorithms, cycle covers are interesting in their own right. Matching theory and graph factorisation are important topics in graph theory. The classical matching problem is the problem of finding one-factors, i. e., spanning subgraphs, each vertex of which is incident to exactly one edge. Cycle covers of undirected graphs are also known as two-factors because every vertex is incident to exactly two edges. A considerable amount of research has been done on structural properties of graph factors and on the complexity of finding graph factors (cf. Lovász and Plummer [24] and Schrijver [29]). In particular, the complexity of finding restricted two-factors, i. e.,  $L$ -cycle covers in undirected graphs, has been investigated, and Hell et al. [22] showed that finding  $L$ -cycle covers in undirected graphs is **NP**-hard for almost all  $L$ . However, almost nothing is known so far about the complexity of finding directed  $L$ -cycle covers.

## 1.1 Preliminaries

Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . If  $G$  is undirected, then a **cycle cover** of  $G$  is a subset  $C \subseteq E$  of the edges of  $G$  such that all vertices in  $V$  are incident to exactly two edges in  $C$ . If  $G$  is a directed graph, then a cycle cover of  $G$  is a subset  $C \subseteq E$  such that all vertices are incident to exactly one incoming and one outgoing edge in  $C$ . Thus, the graph  $(V, C)$  consists solely of vertex-disjoint cycles. The length of a cycle is the number of edges it consists of. The shortest cycles of undirected and directed graphs are of length three and two, respectively.

We call a cycle of length  $\lambda$  a  $\lambda$ -cycle for short. Cycles of even or odd length will simply be called even or odd cycles, respectively.

An  **$L$ -cycle cover** of an undirected graph is a cycle cover in which the length of every cycle is in  $L \subseteq \mathcal{U} = \{3, 4, 5, \dots\}$ . An  $L$ -cycle cover of a directed graph is analogously defined except that  $L \subseteq \mathcal{D} = \{2, 3, 4, \dots\}$ . A  **$k$ -cycle cover** is a  $\{k, k + 1, \dots\}$ -cycle cover. In the following, let  $\bar{L} = \mathcal{U} \setminus L$  in the case of undirected graphs and  $\bar{L} = \mathcal{D} \setminus L$  in the case of directed graphs (this will be clear from the context).

Given edge weights  $w : E \rightarrow \mathbb{N}$ , the **weight**  $w(C)$  of a subset  $C \subseteq E$  of the edges of  $G$  is  $w(C) = \sum_{e \in C} w(e)$ . In particular, this defines the weight of a cycle cover since we view cycle covers as sets of edges. Let  $U \subseteq V$  be any subset of the vertices of  $G$ . The **internal edges of  $U$**  are all edges of  $G$  that have both vertices in  $U$ . We denote by  $w_U(C)$  the sum of the weights of all internal edges of  $U$  in  $C$ .

The **external edges at  $U$**  are all edges of  $G$  with exactly one vertex in  $U$ .

For  $L \subseteq \mathcal{U}$ ,  **$L$ -UCC** is the decision problem whether an undirected graph contains an  $L$ -cycle cover as spanning subgraph.

**Max- $L$ -UCC(0,1)** is the following optimisation problem: Given an undirected complete graph with edge weights zero and one, find an  $L$ -cycle cover of maximum weight. We can also consider the graph as being not complete and without edge weights. Then we try to find an  $L$ -cycle cover with a minimum number of “non-edges” (“non-edges” correspond to weight zero edges, edges to weight one edges), i. e., the  $L$ -cycle cover should contain as many edges as possible. Thus, Max- $L$ -UCC(0,1) generalises  $L$ -UCC.

**Max- $L$ -UCC** is the problem of finding  $L$ -cycle covers of maximum weight in graphs with arbitrary non-negative edge weights.

For  $k \in \mathcal{U}$ ,  **$k$ -UCC**, **Max- $k$ -UCC(0,1)**, and **Max- $k$ -UCC** are defined like  $L$ -UCC, Max- $L$ -UCC(0,1), and Max- $L$ -UCC except that  $k$ -cycle covers rather than  $L$ -cycle covers are sought.

The problems  **$L$ -DCC**, **Max- $L$ -DCC(0,1)**, and **Max- $L$ -DCC** as well as  **$k$ -DCC**, **Max- $k$ -DCC(0,1)**, and **Max- $k$ -DCC** are defined for directed graphs like their undirected counterparts except that  $L \subseteq \mathcal{D}$  and  $k \in \mathcal{D}$ .

An instance of **Min-Vertex-Cover( $\lambda$ )** is an undirected  $\lambda$ -regular graph  $H = (X, F)$ , i. e., every vertex in  $X$  is incident to exactly  $\lambda$  edges. A vertex cover of  $H$  is a subset  $\tilde{X} \subseteq X$  such that at least one vertex of every edge in  $F$  is in  $\tilde{X}$ . The aim is to find a subset  $\tilde{X} \subseteq X$  of minimum cardinality. Min-Vertex-Cover( $\lambda$ ) is APX-complete for  $\lambda \geq 3$  as follows easily from results by Alimonti and Kann [2] as well as Chlebík and Chlebíková [14].

An instance of the  $\lambda$ -dimensional matching problem  **$\lambda$ -DM** is a tuple  $(X, F)$  where  $X$  is a finite set and  $F$  is a collection  $F$  of subsets of  $X$ , each of cardinality  $\lambda$ . The question is whether there exists a subset  $\tilde{F} \subseteq F$  such that for every  $x \in X$  there is a unique  $a \in \tilde{F}$  with  $x \in a$ . For  $\lambda \geq 3$ ,  $\lambda$ -DM is NP-complete [15].

## 1.2 Previous Results

**Undirected Cycle Covers.** Max- $\mathcal{U}$ -UCC, and thus also  $\mathcal{U}$ -UCC and Max- $\mathcal{U}$ -UCC(0,1), can be solved in polynomial time via Tutte’s reduction to the classical perfect matching problem [24]. Hartvigsen presented a polynomial-time algorithm that can be used to decide 4-UCC in polynomial time [17]. Furthermore, it can be adapted to solve Max-4-UCC(0,1) as well.

Max- $k$ -UCC admits a simple factor  $3/2$  approximation for all  $k$ : Compute a maximum weight cycle cover, break the lightest edge of each cycle, and join the paths thus obtained to a Hamiltonian cycle. Unfortunately, this algorithm cannot be generalised to work for Max- $L$ -UCC for general  $L$ . For the problem of computing  $k$ -cycle covers of minimum weight in graphs with edge weights one and two, there exists a factor  $7/6$  approximation algorithm for all  $k$  [8]. Hassin and Rubinfeld [20, 21] devised a randomised approximation algorithm for Max- $\{3\}$ -UCC that achieves an approximation ratio of  $83/43 + \epsilon$ .

Hell et al. [22] proved that  $L$ -UCC is NP-hard for  $\overline{L} \not\subseteq \{3, 4\}$ . Vornberger showed that Max-5-UCC is NP-hard [31]. For  $k \geq 7$ , Max- $k$ -UCC(0,1) and Max- $k$ -UCC are

APX-complete [5].

The complexity of finding restricted cycle covers in undirected graphs seems to be well understood. However, hardly anything is known about the approximability of Max- $L$ -UCC(0,1) and Max- $L$ -UCC for arbitrary  $L$ .

**Directed Cycle Covers.**  $\mathcal{D}$ -DCC, Max- $\mathcal{D}$ -DCC(0,1), and Max- $\mathcal{D}$ -DCC (Max- $\mathcal{D}$ -DCC is also called the *assignment problem*) can be solved in polynomial time by reduction to the maximum weight perfect matching problem in bipartite graphs [1]. But already 3-DCC is NP-complete [15]. Max- $k$ -DCC(0,1) and Max- $k$ -DCC are APX-complete for all  $k \geq 3$  [5].

Similar to the factor  $3/2$  approximation algorithm for undirected cycle covers, Max- $k$ -DCC has a simple factor 2 approximation algorithm for all  $k$ : Compute a maximum weight cycle cover, break the lightest edge of every cycle, and join the cycles to obtain a Hamiltonian cycle. Again, this algorithm cannot be generalised to work for arbitrary  $L$ . There is a factor  $4/3$  approximation algorithm for Max-3-DCC [7] and a factor  $3/2$  approximation algorithm for Max- $k$ -DCC(0,1) for  $k \geq 3$  [5].

While the complexity of  $k$ -cycle covers in directed graphs is well understood, almost nothing is known about the complexity and approximability of  $L$ -cycle covers in general.

### 1.3 Our Results

We prove that Max- $L$ -UCC(0,1) is APX-hard for all  $L$  with  $\overline{L} \not\subseteq \{3, 4\}$  (Section 2.2) and that Max- $L$ -UCC is APX-hard if  $\overline{L} \not\subseteq \{3\}$  (Section 2.3). The hardness results for Max- $L$ -UCC hold even if we allow only the edge weights zero, one, and two.

We show a dichotomy for directed graphs: For all  $L$  with  $L \neq \{2\}$  and  $L \neq \mathcal{D}$ ,  $L$ -DCC is NP-hard and Max- $L$ -DCC(0,1) and Max- $L$ -DCC are APX-hard (Section 2.5), while all three problems are solvable in polynomial time if  $L = \{2\}$  or  $L = \mathcal{D}$ .

Of course, the hardness results for Max- $L$ -UCC(0,1) and Max- $L$ -DCC(0,1) carry over to the problem of computing  $L$ -cycle covers of minimum weight in graphs restricted to edge weights one and two.

To show the hardness of directed cycle covers, we show that certain kinds of graphs, called  *$L$ -clamps*, exist for non-empty  $L \subseteq \mathcal{D}$  if and only if  $L \neq \mathcal{D}$  (Theorem 2.10). This graph-theoretical result might be of independent interest.

Finally, we devise approximation algorithms for Max- $L$ -UCC and Max- $L$ -DCC that achieve ratios of 2 and  $8/3$ , respectively (Section 3). Both algorithms work for all sets  $L$ .

## 2 The Hardness of Approximating $L$ -Cycle Covers

### 2.1 Clamps and Gadgets

To begin the hardness proofs, we introduce *clamps*, which were defined by Hell et al. [22]. Clamps are crucial for our hardness proof.

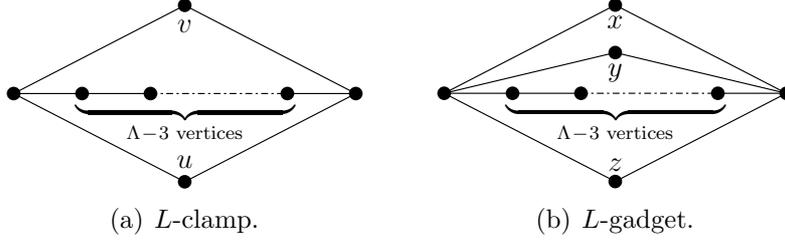


Figure 1: An  $L$ -clamp and an  $L$ -gadget for finite  $L$  with  $\max(L) = \Lambda$ .

Let  $K = (U, E)$  be an undirected graph, let  $u, v \in U$  be two vertices of  $K$ , which we call the **connectors** of  $K$ , and let  $L \subseteq \mathcal{U}$ . We denote by  $K_{-u}$  and  $K_{-v}$  the graphs obtained from  $K$  by deleting  $u$  and  $v$ , respectively, and their incident edges.  $K_{-u-v}$  is obtained from  $K$  by deleting both  $u$  and  $v$ . For  $k \in \mathbb{N}$ ,  $K^k$  is the following graph: Let  $y_1, \dots, y_k \notin U$  be new vertices, add edges  $\{u, y_1\}$ ,  $\{y_i, y_{i+1}\}$  for  $1 \leq i \leq k-1$ , and  $\{y_k, v\}$ . For  $k=0$ , we directly connect  $u$  to  $v$ . The graph  $K$  is called an  **$L$ -clamp** if the following properties hold:

1. Both  $K_{-u}$  and  $K_{-v}$  contain an  $L$ -cycle cover.
2. Neither  $K$  nor  $K_{-u-v}$  nor  $K^k$  for any  $k \in \mathbb{N}$  contains an  $L$ -cycle cover.

Figure 1(a) shows an example of an  $L$ -clamp for finite  $L$ . Hell et al. [22] proved the following result which we will exploit for our reduction.

**Lemma 2.1** (Hell et al. [22]). *Let  $L \subseteq \mathcal{U}$  be non-empty. Then there exists an  $L$ -clamp if and only if  $\bar{L} \not\subseteq \{3, 4\}$ .*

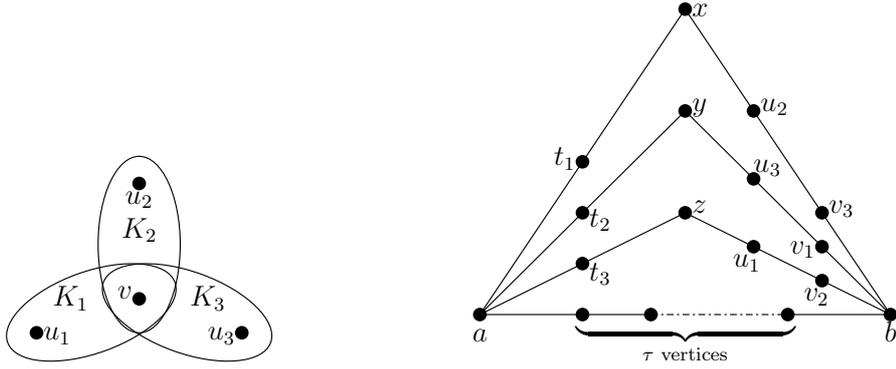
Let  $G$  be a graph with vertex set  $V$  and  $U \subseteq V$ . We say that  $U$  is an  $L$ -clamp with connectors  $u, v \in U$  in  $G$  if the subgraph of  $G$  induced by  $U$  is an  $L$ -clamp and the only external edges of  $U$  are incident to  $u$  or  $v$ .

Let us fix some technical terms. For this purpose, let  $C$  be a cycle cover of  $G$ . For any  $V' \subseteq V$ , we say that  $V'$  is **isolated in  $C$**  if there is no edge in  $C$  connecting  $V'$  to  $V \setminus V'$ . We say that the  $L$ -clamp  $U$  **absorbs  $u$  and expels  $v$**  if  $U \setminus \{v\}$  is isolated in  $C$ . Analogously,  $U$  **absorbs  $v$  expels  $u$**  if  $U \setminus \{u\}$  is isolated in  $C$ .

An  $L$ -clamp implements an exclusive-or of  $u$  and  $v$ : Exactly one of them is absorbed, the other one is expelled. For our purpose of reducing from Min-Vertex-Cover( $\lambda$ ), we need a one-out-of-three behaviour. A graph  $K$  with connectors  $x, y, z$  is called an  **$L$ -gadget** if the following property is fulfilled: Let  $K$  be a subgraph of a larger graph  $G$  as described above (now  $x, y$ , and  $z$  can be incident to external edges). Then, in all  $L$ -cycle covers  $C$  of  $G$ , exactly two of  $K$ 's connectors are expelled while the third one is absorbed. To put it another way: Either  $K_{-x-y}$  or  $K_{-x-z}$  or  $K_{-y-z}$  is isolated in  $C$ .

For finite sets  $L$ , we obtain an  $L$ -gadget, shown in Figure 1(b), immediately by equipping the  $L$ -clamp of Figure 1(a) with an additional connector.

For infinite sets  $L$ , we first build an intermediate subgraph. A **triple  $L$ -clamp** is built from three  $L$ -clamps and has three connectors  $u_1, u_2, u_3$ . Figure 2(a) shows the construction. Triple  $L$ -clamps show a two-out-of-three behaviour: Only one connector will be expelled, the other two will be absorbed. More precisely: One of



(a) A triple  $L$ -clamp with connectors  $u_1, u_2, u_3$ . The connectors of  $L$ -clamp  $K_i$  are  $u_i$  and  $v$ .

(b) An  $L$ -gadget with connectors  $x, y, z$ . The connectors of triple  $L$ -clamps  $T_i$  are  $t_i, u_i, v_i$ . For legibility, the triple  $L$ -clamps are not shown explicitly but only their connectors.

Figure 2: A triple  $L$ -clamp and an  $L$ -gadget.

the three clamps has to absorb  $v$ . The other two absorb their connectors  $u_i$ , which are also connectors of the triple clamp.

Now we are prepared to build  $L$ -gadgets. These graphs are built from three triple  $L$ -clamps, which have connectors  $u_i, v_i, t_i$  for  $i \in \{1, 2, 3\}$ . Figure 2(b) shows such an  $L$ -gadget. Since  $L$  is infinite, there exists a  $\tau \geq 1$  with  $\tau + 6 \in L$ . Let us argue why the  $L$ -gadget behaves as claimed. For this purpose, let  $C$  be an arbitrary  $L$ -cycle cover of  $G$ , where  $G$  contains the  $L$ -gadget as a subgraph. First, we observe that all  $\tau + 2$  vertices of the path connecting  $a$  to  $b$  must be on the same cycle  $c$  in  $C$ . The only other vertices to which  $a$  is incident are  $t_1, t_2$ , and  $t_3$ . By symmetry, we assume that  $t_1$  lies also in  $c$ . Therefore,  $T_1$  absorbs  $u_1$  and  $v_1$ . Thus,  $v_2$  and  $u_3$  are absorbed by  $T_2$  and  $T_3$ , respectively, and  $c$  runs through  $x, u_2, v_3$  back to  $b$  to form a  $(\tau + 6)$ -cycle. Thus,  $x$  is absorbed by the gadget.  $T_2$  expels  $u_2$  and absorbs  $u_3$ , while  $T_3$  expels  $v_3$  and absorbs  $v_2$ . Hence, the gadget expels  $y$  and  $z$  as claimed. The other two cases are symmetric.

To conclude this section about clamps, we transfer the notion of  $L$ -gadgets to edge weighted graphs and prove some properties. The transformation to graphs with edge weights zero and one is made in the obvious way: Let  $G$  be an undirected complete graph with vertex set  $V$  and edge weights zero and one. Let  $U \subseteq V$ . We say that  $U$  is an  $L$ -gadget with connectors  $x, y, z \in U$  if the subgraph of  $G$  induced by  $U$  restricted to the edges of weight one is an  $L$ -gadget.

Let  $\sigma$  be the number of vertices of an  $L$ -gadget. We call  $U$  **healthy in  $C$**  if  $U$  absorbs  $x, y$ , or  $z$ , expels its other two connectors, and  $w_U(C) = \sigma - 1$ . Since the edge weighted graph is complete,  $L$ -cycle may traverse  $L$ -gadgets arbitrarily. The following lemma shows that we lose weight by not traversing them healthily.

**Lemma 2.2.** *Let  $G$  be an undirected graph with vertex set  $V$  and edge weights zero and one, and let  $U \subseteq V$  be an  $L$ -gadget with connectors  $x, y, z$  in  $G$ . Let  $C$  be an arbitrary  $L$ -cycle cover of  $G$  and  $|U| = \sigma$ . Then the following properties hold:*

1.  $w_U(C) \leq \sigma - 1$ .
2. If there are  $2\alpha$  external edges at  $U$  in  $C$ , then  $w_U(C) \leq \sigma - \alpha$ .

3. Assume that  $U$  absorbs  $x$ . Then there exists an  $L$ -cycle cover  $\tilde{C}$  that differs from  $C$  only in the internal edges of  $U$  and has  $w_U(\tilde{C}) = \sigma - 2$ . The same holds if  $U$  absorbs  $y$  or  $z$ .
4. Assume that there are two external edges at  $U$  in  $C$  that are incident to two different connectors. Then  $w_U(C) \leq \sigma - 2$ .

*Proof.* If  $w_U(C) = \sigma$  was true, then  $U$  would contain an  $L$ -cycle cover consisting solely of weight one edges since  $|U| = \sigma$ . This would contradict  $U$  being an  $L$ -gadget.

The second claim follows immediately from  $|U| = \sigma$  and the fact that every vertex is incident to exactly two edges in a cycle cover.

Since  $U$  is an  $L$ -gadget,  $U \setminus \{y, z\}$  contains an  $L$ -cycle cover consisting of  $\sigma - 1$  weight one edges. This holds for  $U \setminus \{x, z\}$  and  $U \setminus \{x, y\}$  as well.

The fourth claim remains to be proved. If there are more than two external edges at  $U$  in  $C$ , we have at least four external edges and thus  $w_U(C) \leq \sigma - 2$ . So assume that there are exactly two external edges at  $U$  in  $C$  incident to, say,  $x$  and  $y$ . We have  $\sigma - 1$  internal edges of  $U$  in  $C$ . If all of them had weight one, this would contradict the property that in an unweighted  $L$ -gadget always  $U \setminus \{x, y\}$ ,  $U \setminus \{x, z\}$ , or  $U \setminus \{y, z\}$  is isolated.  $\square$

## 2.2 The Reduction for Undirected Graphs

Now we present an  $L$ -reduction from  $\text{Min-Vertex-Cover}(\lambda)$  to show the inapproximability of  $\text{Max-}L\text{-UCC}(0,1)$  for  $\bar{L} \not\subseteq \{3, 4\}$ . The inapproximability of  $\text{Max-}L\text{-UCC}$  for  $\bar{L} \not\subseteq \{3\}$  and  $\text{Max-}L\text{-DCC}(0,1)$  for  $L \neq \{2\}$  and  $L \neq \mathcal{D}$  will be shown in subsequent sections.  $L$ -reductions were introduced by Papadimitriou and Yannakakis [27] (cf. Ausiello et al. [3]).

Let  $L \subseteq \mathcal{U}$  be non-empty with  $\bar{L} \not\subseteq \{3, 4\}$ . Thus,  $L$ -gadgets exist and we fix one as in the previous section. Let  $\lambda = \min(L)$ . (This choice is arbitrary. We could choose any number in  $L$ .) We will reduce  $\text{Min-Vertex-Cover}(\lambda)$  to  $\text{Max-}L\text{-UCC}(0,1)$ .  $\text{Min-Vertex-Cover}(\lambda)$  is  $\text{APX}$ -complete since  $\lambda \geq 3$ .

Let  $H = (X, F)$  be an instance of  $\text{Min-Vertex-Cover}(\lambda)$  with  $|X| = n$  vertices and  $|F| = m = \lambda n/2$  edges. Our instance  $G$  for  $\text{Max-}L\text{-UCC}(0,1)$  consists of  $\lambda$  subgraphs  $G_1, \dots, G_\lambda$ , each containing  $\sigma m$  vertices. We start by describing  $G_1$ . Then we state the differences between  $G_1$  and  $G_2, \dots, G_\lambda$  and say to which edges between these graphs weight one is assigned.

Let  $a = \{x, y\} \in F$  be any edge of  $H$ . We construct an  $L$ -gadget  $F_a$  for  $a$  that has connectors  $x_a^1, y_a^1$  and  $z_a^1$ . We call  $F_a$  an **edge gadget**.

Now let  $x \in X$  be any vertex of  $H$  and let  $a_1, \dots, a_\lambda \in F$  be the  $\lambda$  edges that are incident to  $x$ . We connect the vertices  $x_{a_1}^1, \dots, x_{a_\lambda}^1$  to form a path by assigning weight one to the edges  $\{x_{a_\eta}^1, x_{a_{\eta+1}}^1\}$  for  $\eta \in \{1, \dots, \lambda - 1\}$ . Together with edge  $\{x_{a_\lambda}^1, x_{a_1}^1\}$ , these edges form a cycle of length  $\lambda \in L$ , but note that  $w(\{x_{a_\lambda}^1, x_{a_1}^1\}) = 0$ . These  $\lambda$  edges are called the **junctions of  $x$** . The **junctions at  $F_a$**  for some  $a = \{x, y\} \in F$  are the junctions of  $x$  and  $y$  that are incident to  $F_a$ . Overall, the graph  $G_1$  consists of  $\sigma m$  vertices since every edge gadget consists of  $\sigma$  vertices.

The graphs  $G_2, \dots, G_\lambda$  are almost exact copies of  $G_1$ . The graph  $G_\xi$  ( $\xi \in \{2, \dots, \lambda\}$ ) consists of  $L$ -gadgets with connectors  $x_a^\xi, y_a^\xi$ , and  $z_a^\xi$  for each edge

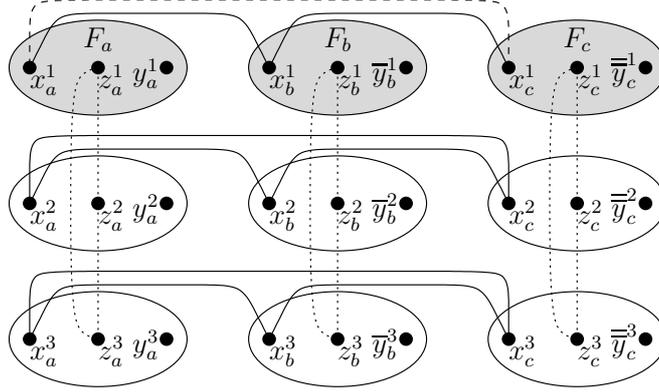


Figure 3: The construction for  $x \in X$  incident to  $a = \{x, y\}, b = \{x, \bar{y}\}, c = \{x, \bar{\bar{y}}\} \in F$  for  $\lambda = 3$ .  $F_a, F_b$ , and  $F_c$  are grey. The three ellipses in the second and third row build  $G_2$  and  $G_3$ , respectively. The cycles connecting the  $z$ -vertices are dotted. The junctions of  $x$  and their copies are solid, except for  $\{x_c^1, x_a^1\}$ , which has weight zero and is dashed.

$a = \{x, y\} \in F$ , just as above. The edge weights are also identical with the single exception that the edge  $\{x_{a_\lambda}^\xi, x_{a_1}^\xi\}$  also has weight one. Note that we only use the term “edge gadget” for the subgraphs of  $G_1$  defined above although almost the same subgraphs occur in  $G_2, \dots, G_\lambda$  as well. Similarly, the term “junction” refers only to edges in  $G_1$  as defined above.

Finally, we describe how to connect  $G_1, \dots, G_\lambda$  with each other. For every edge  $a \in F$ , there are  $\lambda$  vertices  $z_a^1, \dots, z_a^\lambda$ . These are connected to form a cycle consisting solely of weight one edges, i.e., we assign weight one to all edges  $\{z_a^\xi, z_a^{\xi+1}\}$  for  $\xi \in \{1, \dots, \lambda - 1\}$  and to  $\{z_a^\lambda, z_a^1\}$ . Figure 3 shows an example of the whole construction from the viewpoint of a single vertex.

We call edges that are not junctions but connect two different gadgets **illegal**. Edges with both vertices in the same gadget are called internal edges. The fourth kind of edges are the  **$z$ -edges** of  $F_a$  for  $a \in F$ , which are the two edges  $\{z_a^1, z_a^2\}$  and  $\{z_a^1, z_a^\lambda\}$ . The  $z$ -edges are not illegal.

Let  $C$  be any subset of the edges of the graph  $G$  thus constructed, and let  $a = \{x, y\} \in F$  be an arbitrary edge of  $H$ . We say that  $C$  **legally connects  $F_a$**  if the following properties are fulfilled:

- $C$  contains no illegal edges incident to  $F_a$  and exactly two or four junctions at  $F_a$ .
- If  $C$  contains exactly two junctions at  $F_a$ , then these belong to the same vertex and there are two  $z$ -edges at  $F_a$  in  $C$ .
- If  $C$  contains four junctions at  $F_a$ , then  $C$  does not contain the  $z$ -edges at  $F_a$ .

We call  $C$  **legal** if  $C$  legally connects all gadgets. If  $\tilde{C}$  is a legal  $L$ -cycle cover, then for all  $x \in X$  either all junctions of  $x$  or no junction of  $x$  is in  $\tilde{C}$ . From a legal  $L$ -cycle cover  $\tilde{C}$ , we obtain the subset  $\tilde{X} = \{x \mid \text{the junctions of } x \text{ are in } \tilde{C}\} \subseteq X$ . Since at least two junctions at  $F_a$  are in  $\tilde{C}$  for every  $a \in F$ , the set  $\tilde{X}$  is a vertex cover of  $H$ .

The idea behind the reduction is as follows: Consider an edge  $a = \{x, y\} \in F$ . We interpret  $x_a^1$  being expelled to mean that  $x$  is in the vertex cover. (In this case, the junctions of  $x$  are in the cycle cover.) Analogously,  $y$  is in the vertex cover if  $y_a^1$  is expelled. The vertex  $z_a^1$  is only absorbed if both  $x$  and  $y$  are in the vertex cover. If only one of  $x$  and  $y$  is in the vertex cover,  $z_a^1$  forms a  $\lambda$ -cycle with  $z_a^2, \dots, z_a^\lambda$ .

We only considered  $G_1$  when defining the terms “legally connected” and “legal”. This is because in  $G_1$ , we lose weight one for putting  $x$  into the vertex cover since the junction  $\{x_{a_\lambda}^1, x_{a_1}^1\}$  weighs zero. The other  $\lambda - 1$  copies of the construction are only needed because  $z_a^1$  must be part of some cycle if  $z_a^1$  is not absorbed.

**Lemma 2.3.** *Let  $\tilde{X}$  be a vertex cover of size  $\tilde{n}$  of  $H$ . Then  $G$  contains an  $L$ -cycle cover  $\tilde{C}$  with  $w(\tilde{C}) = \sigma\lambda m - \tilde{n}$ .*

*Proof.* We start by describing  $\tilde{C}$  in  $G_1$ . For every vertex  $x \in \tilde{X}$ , the cycle consisting of all  $\lambda$  junctions is in  $\tilde{C}$ . Let  $a = \{x, y\} \in F$  be any edge. Then either  $x$  or  $y$  or both are in  $\tilde{X}$ . If only  $x$  is in  $\tilde{X}$ , we let  $F_a$  absorb  $y_a^1$ , while  $z_a^1$  is expelled. If only  $y$  is in  $\tilde{X}$ , we let  $F_a$  absorb  $x_a^1$ , while  $z_a^1$  is again expelled. If both  $x$  and  $y$  are in  $\tilde{X}$ , then we let  $x_a^1$  and  $y_a^1$  be expelled while  $z_a^1$  is absorbed.

We perform the same construction as for  $G_1$  for all copies  $G_2, \dots, G_\lambda$ . If  $z_a^1$  is expelled, then  $t_a^2, \dots, t_a^\lambda$  are expelled as well. We let them form a  $\lambda$ -cycle in  $\tilde{C}$ .

Clearly,  $\tilde{C}$  is legal. Furthermore,  $\tilde{C}$  is an  $L$ -cycle cover: Every cycle either has length  $\lambda \in L$  or lies totally inside a single  $L$ -gadget. Since all  $L$ -gadget are healthy in  $\tilde{C}$ ,  $\tilde{C}$  is an  $L$ -cycle cover.

All edges of  $\tilde{C}$  within  $G_2, \dots, G_\lambda$  have weight one. The only edges that connect different copies  $G_\xi$  and  $G_{\xi'}$  are edges  $\{z_a^\xi, z_a^{\xi+1}\}$  and  $\{z_a^\lambda, x_a^1\}$ , which have weight one as well. Almost all edges used in  $G_1$  also have weight one; the only exception is one junction of weight zero for each  $x \in \tilde{X}$ . Since  $|\tilde{X}| = \tilde{n}$ , there are  $\tilde{n}$  edges of weight zero in  $\tilde{C}$ . The graph  $G$  contains  $\sigma\lambda m$  vertices, thus  $\tilde{C}$  contains  $\sigma\lambda m$  edges,  $\sigma\lambda m - \tilde{n}$  of which have weight one.  $\square$

Let  $C$  be an  $L$ -cycle cover of  $G$  and let  $a \in F$ . We define  $W_{F_a}(C)$  as the sum of the weights of all internal edges of  $F_a$  plus half the number of  $z$ -edges in  $C$  at  $F_a$ . Analogously,  $W_{G_\xi}(C)$  is the number of weight one edges with both vertices in  $G_\xi$  plus half the number of weight one edges with exactly one vertex in  $G_\xi$ .

**Lemma 2.4.** *Let  $C$  be an  $L$ -cycle cover and let  $j$  be the number of weight one junctions in  $C$ . Then  $w(C) = j + \sum_{a \in F} W_{F_a}(C) + \sum_{\xi=2}^\lambda W_{G_\xi}(C)$ .*

*Proof.* Every edge with both vertices in the same  $G_\xi$  is counted once. The only edges of weight one between different  $G_\xi$  are the edges  $\{z_a^\xi, z_a^{\xi+1}\}$  and  $\{z_a^\lambda, z_a^1\}$ . These are counted with one half in both  $W_{G_\xi}(C)$  and  $W_{G_{\xi+1}}(C)$  for  $2 \leq \xi \leq \lambda - 1$  or one half in both  $W_{G_\xi}(C)$  and  $W_{F_a}(C)$  for  $\xi \in \{2, \lambda\}$ .  $\square$

In a legal  $L$ -cycle cover  $\tilde{C}$  as described in Lemma 2.3, we have  $W_{G_\xi}(\tilde{C}) = \sigma m$  for all  $\xi \in \{2, \dots, \lambda\}$  since every vertex in  $G_\xi$  is only incident to edges of weight one of the cycle cover by construction. Now we show that it is always best to traverse the gadgets legally and to keep the gadgets healthy.

**Lemma 2.5.** *Given an arbitrary  $L$ -cycle cover  $C$ , we can compute a legal  $L$ -cycle cover  $\tilde{C}$  with  $w(\tilde{C}) \geq w(C)$  in polynomial time.*

*Proof.* We proceed as follows to obtain  $\tilde{C}$ :

1. Let  $C'$  be  $C$  with all illegal edges with at least one endpoint in  $G_1$  removed.
2. For all  $x \in X$  in arbitrary order: If at least one junction of  $x$  is in  $C$ , then put all junctions of  $x$  into  $C'$ .
3. For all  $a = \{x, y\} \in F$  in arbitrary order: If neither the junctions of  $x$  nor the junctions of  $y$  are in  $C'$ , choose arbitrarily one vertex of  $a$ , say  $x$ , and add all junctions of  $x$  to  $C'$ .
4. Rearrange  $C'$  within  $G_1$  such that all clamps are healthy in  $C'$ .
5. Rearrange  $C'$  such that all  $G_2, \dots, G_\lambda$  are traversed exactly like  $G_1$ .
6. For all  $a \in F$ : If  $z_a^1, \dots, z_a^\xi$  are not absorbed, let them form a  $\lambda$ -cycle. Call the result  $\tilde{C}$ .

The running-time of the algorithm is polynomial. Moreover,  $\tilde{C}$  is a legal  $L$ -cycle cover by construction. What remains is to prove  $w(\tilde{C}) \geq w(C)$ .

Let  $w(C) = j + \sum_{a \in F} W_{F_a}(C) + \sum_{\xi=2}^\lambda W_{G_\xi}(C)$  be the weight of  $C$  according to Lemma 2.4, i. e.,  $C$  contains  $j$  junctions of weight one. Analogously, let  $w(\tilde{C}) = \tilde{j} + \sum_{a \in F} W_{F_a}(\tilde{C}) + \sum_{\xi=2}^\lambda W_{G_\xi}(\tilde{C})$ , i. e.,  $\tilde{j}$  is the number of junctions of weight one in  $\tilde{C}$ .

All illegal edges have weight zero, and we do not remove any junctions. We have  $W_{G_\xi}(\tilde{C}) = \sigma m$  for all  $\xi$ , which is maximal. Thus, no weight is lost in this way. What remains is to consider the internal edges of the gadgets and the  $z$ -edges.

Let  $a = \{x, y\}$  be an arbitrary edge of  $H$ . If  $W_{F_a}(C) \leq W_{F_a}(\tilde{C})$ , then nothing has to be shown. Those gadgets  $F_a$  with  $W_{F_a}(C) > W_{F_a}(\tilde{C})$  remain to be considered. We have  $W_{F_a}(\tilde{C}) \geq \sigma - 2$  and  $W_{F_a}(C) \leq \sigma - 1$  according to Lemma 2.2. Thus,  $W_{F_a}(C) = \sigma - 1$  and  $W_{F_a}(\tilde{C}) = \sigma - 2 = W_{F_a}(C) - 1$  for all  $a \in F$  with  $W_{F_a}(C) > W_{F_a}(\tilde{C})$ . What remains to be proved is that for all such gadgets, there is a junction of weight one in  $\tilde{C}$  that is not in  $C$  and can thus compensate for the loss of weight one in  $F_a$ . This means that we have to show that  $\tilde{j}$  is at least  $j$  plus the number of edges  $a$  with  $W_{F_a}(C) > W_{F_a}(\tilde{C})$ .

If  $W_{F_a}(C) = \sigma - 1$ , then according to Lemma 2.2(4), the junctions at  $F_a$  in  $C$  (if there are any) belong to the same vertex. Since  $W_{F_a}(\tilde{C}) = \sigma - 2$ , all four junctions at  $F_a$  are in  $\tilde{C}$ . Thus, while executing the above algorithm, there is a moment at which at least one of, say,  $y$ 's junctions at  $F_a$  is in  $C'$ , and the junctions of  $x$  are added in the next step. We say that a vertex  $x$  **compensates**  $F_a$  if

1.  $\tilde{C}$  contains  $x$ 's junctions,
2. no junction of  $x$  at  $F_a$  is in  $C$ , and
3. at the moment at which  $x$ 's junctions are added,  $C'$  already contains at least one junction of  $y$  at  $F_a$ .

Thus, every gadget  $F_a$  with  $W_{F_a}(\tilde{C}) < W_{F_a}(C)$  is compensated by some vertex  $x \in a$ .

It remains to be shown that the number of gadgets that are compensated by some vertex is at most equal to the number of weight one junctions added to  $C'$ . Let  $\eta \in \{0, \dots, \lambda\}$  be the number of junctions of  $x$  in  $C$ . If  $\eta = \lambda$ , then  $x$  does not compensate any gadget. If  $\eta = 0$ , i. e.,  $C$  does not contain any of  $x$ 's junctions, then the junctions of  $x$  are added during Step 3 of the algorithm because there is some edge  $a \in F$  with  $x \in a$  such that there is no junction at all in  $C'$  at  $F_a$  before adding  $x$ 's junctions. Thus,  $x$  does not compensate  $F_a$ . At most  $\lambda - 1$  gadgets are compensated by  $x$ , and  $\lambda - 1$  junctions of  $x$  have weight one. The case that remains is  $\eta \in \{1, \dots, \lambda - 1\}$ . Then  $\lambda - \eta$  junctions of  $x$  are added and at least  $\lambda - \eta - 1$  of them have weight one. On the other hand, there are at least  $\eta + 1$  gadgets  $F_a$  such that at least one junction of  $x$  at  $F_a$  is already in  $C$ : Every junction is at two gadgets, and thus  $\eta$  junctions are at  $\eta + 1$  or more gadgets. Thus, at most  $\lambda - \eta - 1$  gadgets are compensated by  $x$ .  $\square$

Finally, we prove the following counterpart to Lemma 2.3.

**Lemma 2.6.** *Let  $\tilde{C}$  be the  $L$ -cycle cover constructed as described in the proof of Lemma 2.5 and let  $\tilde{X} = \{x \mid x\text{'s junctions are in } \tilde{C}\}$  be the subset of  $X$  obtained from  $\tilde{C}$ . Choose  $\tilde{n}$  such that  $w(\tilde{C}) = \sigma\lambda m - \tilde{n}$ . Then  $|\tilde{X}| = \tilde{n}$ .*

*Proof.* The proof is similar to the proof of Lemma 2.3. We set the weight of all junctions to one. With respect to the modified edge weights, the weight of  $\tilde{C}$  is  $\sigma\lambda m$ . Thus,  $\tilde{n}$  is the number of weight zero junctions in  $\tilde{C}$ , which is just  $|\tilde{X}|$ .  $\square$

Now we are prepared to prove the main theorem of this section.

**Theorem 2.7.** *For all  $L \subseteq \mathcal{U}$  with  $\bar{L} \not\subseteq \{3, 4\}$ , Max- $L$ -UCC(0,1) is APX-hard.*

*Proof.* We show that the reduction presented is an L-reduction. Then the result follows from the APX-hardness of Min-Vertex-Cover( $\lambda$ ). Let  $\text{opt}(H)$  be the size of a minimum vertex cover of  $H$  and  $\text{opt}(G)$  be the weight of a maximum weight  $L$ -cycle cover of  $G$ . From Lemmas 2.3, 2.5, and 2.6, we obtain that  $\text{opt}(G) = \sigma\lambda m - \text{opt}(H) \leq \sigma\lambda m$ . Since  $H$  is  $\lambda$ -regular, we have  $\text{opt}(H) \geq n/2$ . Thus,

$$\text{opt}(G) \leq \sigma\lambda m = \sigma\lambda(\lambda/2) \cdot n \leq (\sigma\lambda^2/2) \cdot \text{opt}(H).$$

Let  $C$  be an arbitrary  $L$ -cycle cover of  $G$ ,  $\tilde{C}$  be a legal  $L$ -cycle cover obtained from  $C$  as in Lemma 2.5, and  $\tilde{X} \subseteq X$  obtained from  $\tilde{C}$ . Then

$$||\tilde{X}| - \text{opt}(H)| = |w(\tilde{C}) - \text{opt}(G)| \leq |w(C) - \text{opt}(G)|,$$

which completes the proof.  $\square$

### 2.3 Adaption of the Reduction to Max- $L$ -UCC

To prove the APX-hardness of Max- $L$ -UCC for  $\bar{L} \not\subseteq \{3\}$ , all we have to do is to deal with  $\bar{L} = \{4\}$  and  $\bar{L} = \{3, 4\}$ . For all other sets  $L$ , the inapproximability follows from Theorem 2.7. We will adapt the reduction presented in the previous section.

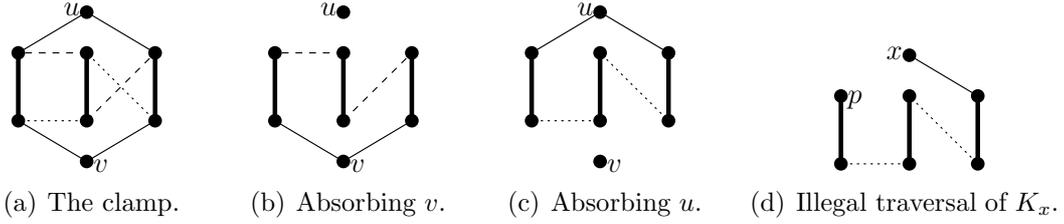


Figure 4: A weighted  $L$ -clamp for  $\{4\} \subseteq \bar{L} \subseteq \{3, 4\}$  and how to traverse it. Bold edges have weight two; solid, dashed, and dotted edges have weight one.

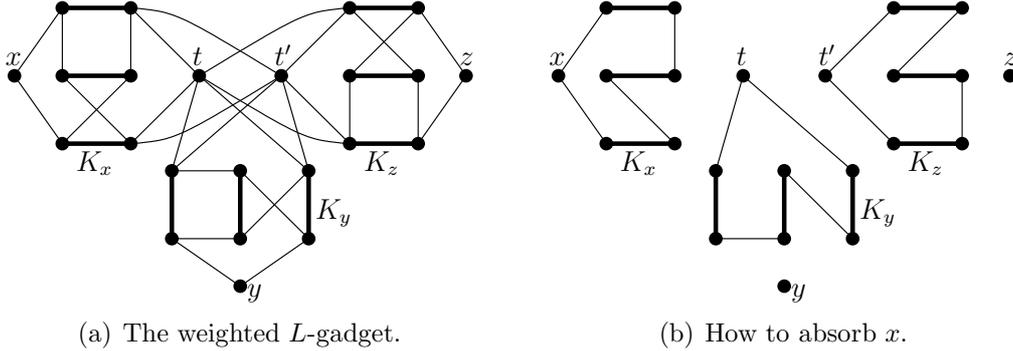


Figure 5: A weighted  $L$ -gadget and how to use it.

To do this, we have to find an edge weighted analogue of an  $L$ -clamp. We do not explicitly define the properties a weighted  $L$ -clamp has to fulfil. Instead, we just call the graph shown in Figure 4(a) a **weighted  $L$ -clamp** for  $\bar{L} = \{3, 4\}$  and  $L = \{4\}$ .

The basic idea is that all three edges of weight two of the weighted clamp have to be traversed in a cycle cover. Since 4-cycles are forbidden, we have to take either the two dotted edges or the two dashed edges. Otherwise, we would have to take an edge of weight zero. Furthermore, if we take the dashed edges, we have to absorb  $v$  and to expel  $u$ , and if we take the dotted edges, we have to absorb  $u$  and to expel  $v$  (Figures 4(b) and 4(c)). Again, we would have to take edges of weight zero otherwise.

Using three weighted  $L$ -clamps  $K_x, K_y, K_z$ , we build an  $L$ -gadget as shown in Figure 5(a). Note that both  $t$  and  $t'$  can serve as a connector for each of the clamps. This weighted  $L$ -gadget has essentially the same properties as the  $L$ -gadgets of Section 2.1, which were stated as Lemma 2.2. The difference is that  $\sigma = 32$  is no longer the number of vertices, but the number of vertices plus the number of edges of weight two.

**Lemma 2.8.** *Let  $G$  be an undirected graph with vertex set  $V$  and edge weights zero and one, and let  $U \subseteq V$  be a weighted  $L$ -gadget with connectors  $x, y, z$  in  $G$ . Let  $C$  be an arbitrary  $L$ -cycle cover of  $G$ . Then the following properties hold:*

1.  $w_U(C) \leq 31$ .
2. If there are  $2\alpha$  external edges at  $U$  in  $C$ , then  $w_U(C) \leq 32 - \alpha$ .

3. If  $U$  absorbs  $x$ , then there exists an  $L$ -cycle cover  $\tilde{C}$  that differs from  $C$  only in the internal edges of  $U$  and has  $w_U(\tilde{C}) = 30$ . The same holds if  $U$  absorbs  $y$  or  $z$ .
4. Assume that there are two external edges at  $U$  in  $C$  that are incident to two different connectors. Then  $w_U(C) \leq 30$ .

*Proof.* The only way to achieve  $w_U(C) > 31$  is  $w_U(C) = 32$ , which requires that we have 23 internal edges including all nine edges of weight two. Since 4-cycles are forbidden, such an  $L$ -cycle cover does not exist.

If we have  $2\alpha$  external edges, then we have  $23 - \alpha$  internal edges. At most nine of them are of weight two.

If  $U$  absorbs  $x$ , then we can achieve a weight of 30 by letting  $K_y$  and  $K_z$  absorb  $t_1$  and  $t_2$ , respectively (Figure 5(b)). (We can also connect  $K_y$  and  $K_z$  via  $t$  and  $t'$  to obtain a 14-cycle. The weight would be the same.) In the same way, we can achieve weight 30 if  $U$  absorbs  $y$  or  $z$ .

What remains is to show claim 4. We have  $w_U(C) \leq 31$  and 22 internal edges. If  $w_U(C) > 30$ , then  $w_U(C) = 31$ , and  $C$  contains all nine edges of weight two and no internal edge of weight zero of  $U$ . By symmetry, it suffices to consider the case that  $x$  is incident to one external edge. Figure 4(d) shows which edges are mandatory in order to keep all three edges of weight two. Since the cycle that contains  $x$  must be continued at  $p$ , vertex  $p$  is incident to an edge of weight zero in  $C$ , which proves the claim.  $\square$

Given these properties, we can plug the  $L$ -gadget into the reduction described in the previous section to obtain the APX-hardness of Max- $L$ -UCC for  $\bar{L} = \{4\}$  and  $\bar{L} = \{3, 4\}$ . Together with Theorem 2.7, we obtain the following result.

**Theorem 2.9.** *Max- $L$ -UCC is APX-hard for all  $L$  with  $\bar{L} \not\subseteq \{3\}$  even if the edge weights are restricted to be zero, one, or two.*

## 2.4 Clamps in Directed Graphs

The aim of this section is to prove a counterpart to Lemma 2.1 (for the existence of  $L$ -clamps) for directed graphs. Let  $K = (V, E)$  be a directed graph and  $u, v \in V$ . Again,  $K_{-u}$ ,  $K_{-v}$ , and  $K_{-u-v}$  denote the graphs obtained by deleting  $u$ ,  $v$ , and both  $u$  and  $v$ , respectively. For  $k \in \mathbb{N}$ ,  $K_u^k$  denotes the following graph: Let  $y_1, \dots, y_k \notin V$  be new vertices and add edges  $(u, y_1), (y_1, y_2), \dots, (y_k, v)$ . For  $k = 0$ , we add the edge  $(u, v)$ . The graph  $K_v^k$  is similarly defined, except that we now start at  $v$ , i. e., we add the edges  $(v, y_1), (y_1, y_2), \dots, (y_k, u)$ .  $K_v^0$  is  $K$  with the additional edge  $(v, u)$ .

Now we can define clamps for directed graphs: Let  $L \subseteq \mathcal{D}$ . A directed graph  $K = (V, E)$  with  $u, v \in V$  is a **directed  $L$ -clamp** with connectors  $u$  and  $v$  if the following properties hold:

- Both  $K_{-u}$  and  $K_{-v}$  contain an  $L$ -cycle cover.
- Neither  $K$  nor  $K_{-u-v}$  nor  $K_u^k$  nor  $K_v^k$  for any  $k \in \mathbb{N}$  contains an  $L$ -cycle cover.

Let us now prove that directed  $L$ -clamps exist for almost all  $L$ .

**Theorem 2.10.** *Let  $L \subseteq \mathcal{D}$  be non-empty. Then there exists a directed  $L$ -clamp if and only if  $L \neq \mathcal{D}$ .*

*Proof.* We first prove that directed  $L$ -clamps exist for all non-empty sets  $L \subseteq \mathcal{D}$  with  $L \neq \mathcal{D}$ . We start by considering finite  $L$ . If  $L$  is finite,  $\max(L) = \Lambda$  exists. For  $L = \{2\}$ , the graph shown in Figure 6(a) is a directed  $L$ -clamp: Either  $u$  or  $v$  forms a 2-cycle with  $x_1$ , and there are no other possibilities. Otherwise, we have  $\Lambda \geq 3$ . Figure 6(b) shows a directed  $L$ -clamp for this case, which is a directed variant of the undirected clamp shown in Figure 1(a).

Now we consider finite  $\bar{L}$ . Figure 6(c) shows an  $L$ -clamp for  $\bar{L} = \{2\}$ :  $x_1, x_2$ , and  $x_3$  must be on the same path since length two is forbidden. This cycle must include  $u$  or  $v$  but cannot include both of them

Otherwise,  $\max(\bar{L}) = \Lambda \geq 3$  and  $\Lambda + 2 \in L$  and the graph shown in Figure 6(d) is an  $L$ -clamp: The vertices  $x_1, \dots, x_{\Lambda-1}$  must all be on the same cycle. Thus, either  $(y, x_1)$  or  $(z, x_1)$  is in the cycle cover. By symmetry, it suffices to consider the first case. Since  $\Lambda \notin L$ , the edge  $(x_{\Lambda-1}, y)$  cannot be in the cycle cover. Thus,  $(v, y)$  and  $(x_{\Lambda-1}, z)$  and hence  $(z, v)$  are in the cycle cover.

The case that remains to be considered is that both  $L$  and  $\bar{L}$  are infinite. We distinguish two subcases. Either there exists a  $\Lambda \geq 4$  with  $\Lambda, \Lambda + 2 \notin L$  and  $\Lambda + 1 \in L$ . In this case, the graph shown in Figure 6(e) is an  $L$ -clamp:  $x_1, \dots, x_\Lambda$  must be on the same cycle. Since the lengths  $\Lambda$  and  $\Lambda + 2$  are not allowed, either  $v$  or  $u$  is expelled and the other vertex is absorbed.

If no  $\Lambda$  exists with  $\Lambda, \Lambda + 2 \notin L$  and  $\Lambda + 1 \in L$  (but  $L$  and  $\bar{L}$  are infinite), then there exists a  $\Lambda \geq 3$  with  $\Lambda \notin L$  and  $\Lambda + 2 \in L$  and we can use the graph already used for finite  $\bar{L}$  (Figure 6(d)) as a directed  $L$ -clamp.

From Lemma 2.11 below we obtain the fact that  $\mathcal{D}$ -clamps do not exist, which completes the proof.  $\square$

**Lemma 2.11.** *Let  $G = (V, E)$  be a directed graph and let  $u, v \in V$ . If  $G_{-u}$  and  $G_{-v}$  both contain a cycle cover, then*

- both  $G$  and  $G_{-u-v}$  contain cycle covers or
- all  $G_u^k$  and  $G_v^k$  for  $k \in \mathbb{N}$  contain cycle covers.

*Proof.* Let  $E_{-u}$  and  $E_{-v}$  be the sets of edges of the cycle covers of  $G_{-u}$  and  $G_{-v}$ , respectively. We construct two sequences of edges  $P = (e_1, e_2, \dots)$  and  $P' = (e'_1, e'_2, \dots)$ . These sequences can be viewed as augmenting paths and we use them to construct cycle covers of  $G_{-u-v}$  and  $G$  or  $G_u^k$  and  $G_v^k$ . The sequence  $P$  is given uniquely by traversing edges of  $E_{-v}$  forwards and edges of  $E_{-u}$  backwards:

- $e_1 = (u, x_1)$  is the unique outgoing edge of  $u = x_0$  in  $E_{-v}$ .
- If  $e_i = (x_{i-1}, x_i) \in E_{-v}$ , i. e., if  $i$  is odd, then  $e_{i+1} = (x_{i+1}, x_i) \in E_{-u}$  is the unique incoming edge of  $x_i$  in  $E_{-u}$ .
- If  $e_i = (x_i, x_{i-1}) \in E_{-u}$ , i. e., if  $i$  is even, then  $e_{i+1} = (x_i, x_{i+1}) \in E_{-v}$  is the unique outgoing edge of  $x_i$  in  $E_{-v}$ .

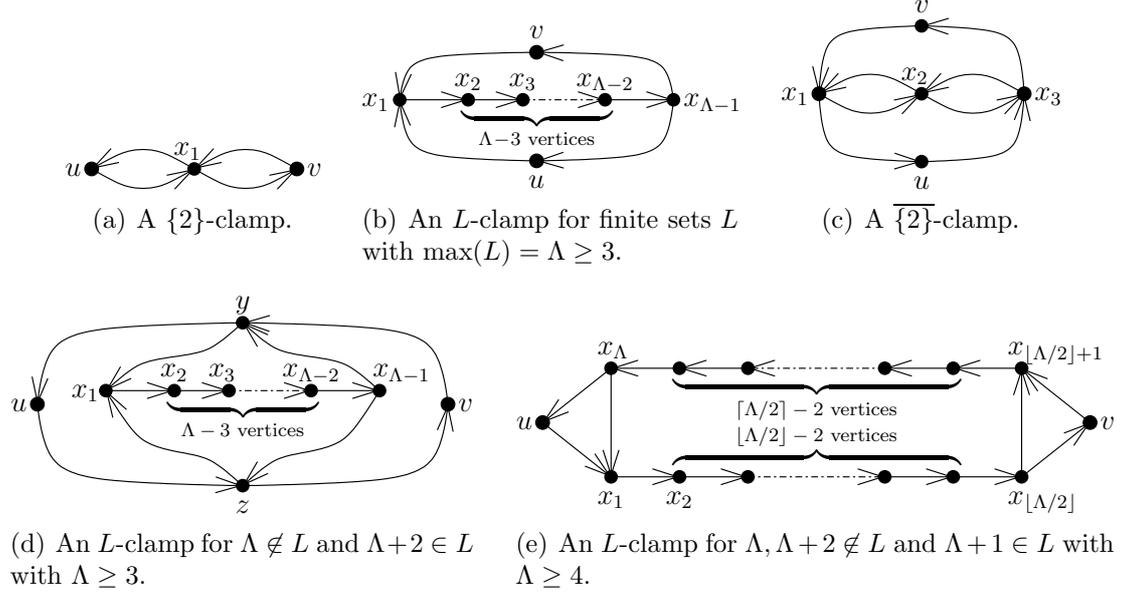


Figure 6: Directed  $L$ -clamps. The connectors are  $u$  and  $v$ , the internal vertices are  $x_1, x_2, \dots$  and  $y, z$ .

- If in any of the above steps no extension of  $P$  is possible, then stop.

Let  $P = (e_1, \dots, e_\ell)$ . We observe two properties of the sequence  $P$ :

**Lemma 2.12.** 1. No edge appears more than once in  $P$ .

2. If  $\ell$  is odd, i. e.,  $e_\ell \in E_{-v}$ , then  $e_\ell = (x_{\ell-1}, u)$ . If  $\ell$  is even, i. e.,  $e_\ell \in E_{-u}$ , then  $e_\ell = (v, x_{\ell-1})$ .

*Proof.* Assume the contrary of the first claim and let  $e_i = e_j$  ( $i \neq j$ ) be an edge that appears at least twice in  $P$  such that  $i$  is minimal. If  $i = 1$ , then  $e_j = (u, x_1) \in E_{-v}$ . This would imply  $e_{j-1} = (u, x_{j-2}) \in E_{-u}$ , a contradiction. If  $i > 1$ , then assume  $e_i = (x_{i-1}, x_i) \in E_{-v}$  without loss of generality. Since exactly one edge leaves  $x_{i-1}$  in  $E_{-u}$ , the edge  $e_{i-1} = e_{j-1}$  is uniquely determined, contradicting the minimality of  $i$ .

Let us now prove the second claim. Without loss of generality, we assume that the last edge  $e_\ell$  belongs to  $E_{-v}$ . Let  $e_\ell = (x_{\ell-1}, x_\ell)$ . The path  $P$  cannot be extended, which implies that there does not exist an edge  $(x_{\ell+1}, x_\ell) \in E_{-u}$ . Since  $E_{-u}$  is a cycle cover of  $G_{-u}$ , this implies  $x_\ell = u$  and completes the proof.  $\square$

Now we build the sequence  $P'$  analogously, except that we start with the edge  $e'_1 = (x'_1, v) \in E_{-u}$ . Again, we traverse edges of  $E_{-v}$  forwards and edges of  $E_{-u}$  backwards. Let  $P' = (e'_1, \dots, e'_{\ell'})$ .

No edge appears in both  $P$  and  $P'$  as can be proved similarly to the first observation above. Moreover, either  $P$  ends at  $u$  and  $P'$  ends at  $v$  or vice versa:  $e_\ell = (x_{\ell-1}, u)$  if and only if  $e'_{\ell'} = (v, x_{\ell'-1})$  and  $e_\ell = (v, x_{\ell-1})$  if and only if  $e'_{\ell'} = (x_{\ell'-1}, u)$ . Let  $P_{-u} \subseteq E_{-u}$  denote the set of edges of  $E_{-u}$  that are part of  $P$ . The sets  $P_{-v}$ ,  $P'_{-u}$ ,  $P'_{-v}$  are defined similarly.

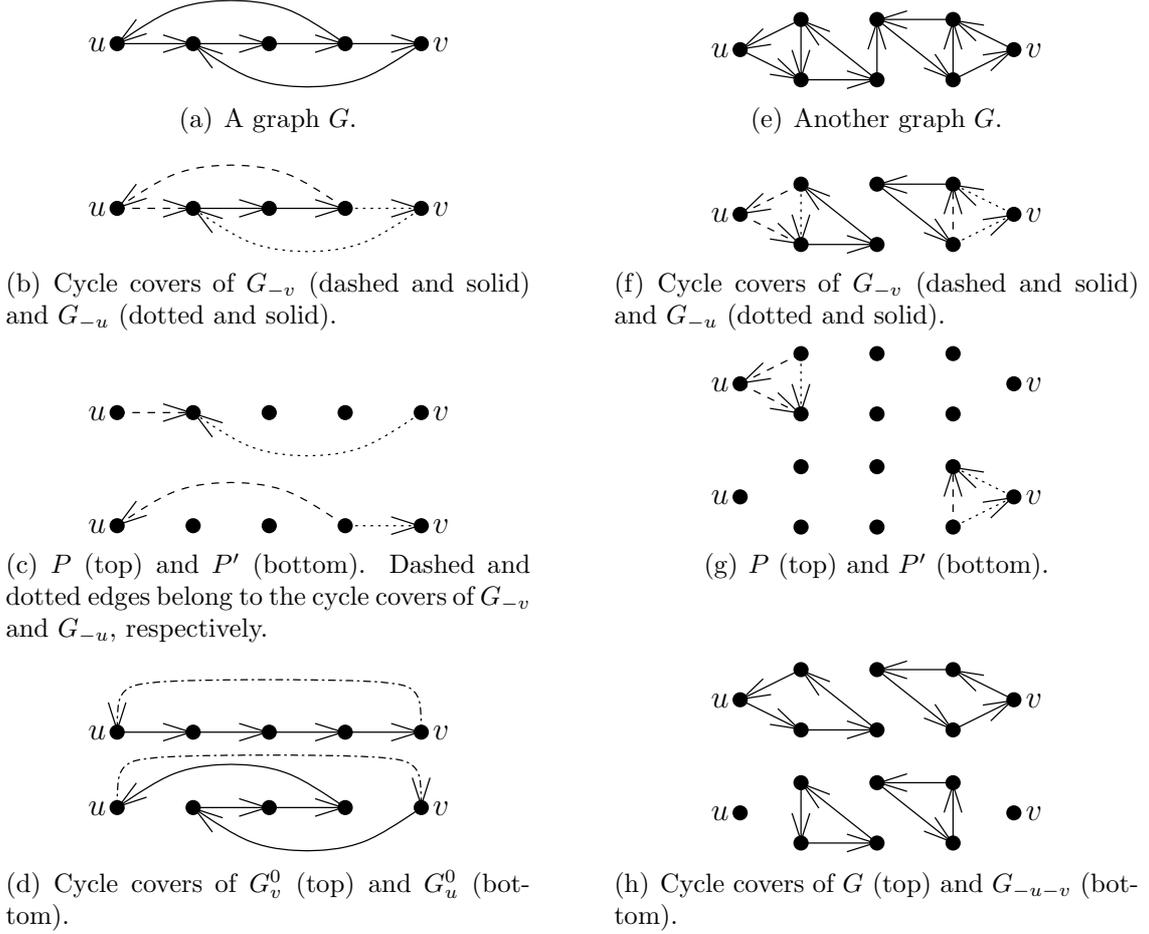


Figure 7: Constructing new cycle covers from the sequences  $P$  and  $P'$ .

Two examples are shown in Figure 7: Figures 7(a) and 7(b) show a graph with its cycle covers, while Figure 7(c) depicts  $P$  and  $P'$ , the former starting at  $u$  and ending at  $v$  and the latter starting at  $v$  and ending at  $u$ . Figures 7(e), 7(f), and 7(g) show another example graph, this time  $P$  starts and ends at  $u$  and  $P'$  starts and ends at  $v$ .

We distinguish two cases. Let us start with the case that  $P$  starts at  $u$  and ends at  $v$  and, consequently,  $P'$  starts at  $v$  and ends at  $u$ . Then

$$E_u^0 = (E_{-v} \setminus P_{-v}) \cup P_{-u} \cup \{(u, v)\}$$

is a cycle cover of  $G_u^0$ . To prove this, we have to show  $\text{indeg}_{E_u^0}(x) = \text{outdeg}_{E_u^0}(x) = 1$  for all  $x \in V$ :

- We removed the outgoing edge of  $u$  in  $E_{-v}$ , which is in  $P_{-v}$ . The incoming edge of  $u$  in  $E_{-v}$  is left.  $P_{-u}$  does not contain any edge incident to  $u$  and  $(u, v)$  is an outgoing edge of  $u$ . Thus,  $\text{indeg}_{E_u^0}(u) = \text{outdeg}_{E_u^0}(u) = 1$ .
- There is no edge incident to  $v$  in  $E_{-v}$ .  $P_{-u}$  contains an outgoing edge of  $v$  and  $(u, v)$  is an incoming edge of  $v$ . Thus,  $\text{indeg}_{E_u^0}(v) = \text{outdeg}_{E_u^0}(v) = 1$ .

- For all  $x \in V \setminus \{u, v\}$ , either both  $P_{-v}$  and  $P_{-u}$  contain an incoming edge of  $x$  or none of them does. Analogously, either both  $P_{-v}$  and  $P_{-u}$  contain an outgoing edge of  $x$  or none of them does. Thus, replacing  $P_{-v}$  by  $P_{-u}$  changes neither  $\text{indeg}(x)$  nor  $\text{outdeg}(x)$ .

By replacing the edge  $(u, v)$  by a path  $(u, y_1), \dots, (y_k, v)$ , we obtain a cycle cover of  $G_u^k$  for all  $k \in \mathbb{N}$ . A cycle cover of  $G_v^0$  is obtained similarly:

$$E_v^0 = (E_{-u} \setminus P_{-u}) \cup P_{-v} \cup \{(v, u)\}.$$

As above, we get cycle covers of  $G_v^k$  by replacing  $(v, u)$  by a path  $(v, y_1), \dots, (y_k, u)$ . Figure 7(d) shows an example how the new cycle covers are obtained.

The case that remains to be considered is that  $P$  starts and ends at  $u$  and  $P'$  starts and ends at  $v$ . In this case,

$$(E_{-v} \setminus P_{-u}) \cup P_{-v} \quad \text{and} \quad (E_{-u} \setminus P'_{-v}) \cup P'_{-u}$$

are cycle covers of  $G$  and

$$(E_{-v} \setminus P_{-v}) \cup P_{-u} \quad \text{and} \quad (E_{-u} \setminus P'_{-u}) \cup P'_{-v}$$

are cycle covers of  $G_{-u-v}$ . The proof is similar to the previous case above. Figure 7(h) shows an example.  $\square$

## 2.5 Intractability for Directed Graphs

From the hardness results in the previous sections and the work by Hell et al. [22], we obtain the NP-hardness and APX-hardness of  $L$ -DCC and Max- $L$ -DCC(0,1), respectively, for all  $L$  with  $2 \notin L$  and  $\bar{L} \not\subseteq \{2, 3, 4\}$ : We use the same reduction as for undirected cycle covers and replace every undirected edge  $\{u, v\}$  by a pair of directed edges  $(u, v)$  and  $(v, u)$ . However, this does not work if  $2 \in L$  and also leaves open the cases when  $\bar{L} \subsetneq \{2, 3, 4\}$ .  $\mathcal{D}$ -DCC, Max- $\mathcal{D}$ -DCC(0,1), and Max- $\mathcal{D}$ -DCC can be solved in polynomial time, but the case  $L = \{2\}$  is also easy: Replace two opposite edges  $(u, v)$  and  $(v, u)$  by an edge  $\{u, v\}$  of weight  $w(u, v) + w(v, u)$  and compute a matching of maximum weight on the undirected graph thus obtained.

We will settle the complexity of the directed cycle cover problems by showing that  $L = \{2\}$  and  $L = \mathcal{D}$  are the only tractable cases. For all other  $L$ ,  $L$ -DCC is NP-hard and Max- $L$ -DCC(0,1) and Max- $L$ -DCC are APX-hard. Let us start by proving the APX-hardness.

**Theorem 2.13.** *Let  $L \subseteq \mathcal{D}$  be a non-empty set. If  $L \neq \{2\}$  and  $L \neq \mathcal{D}$ , then Max- $L$ -DCC(0,1) and Max- $L$ -DCC are APX-hard.*

*Proof.* We adapt the proof presented in Section 2.2. Since  $L \neq \{2\}$ , there exists a  $\lambda \in L$  with  $\lambda \geq 3$ . Thus, Min-Vertex-Cover( $\lambda$ ) is APX-complete. All we need is such a  $\lambda$  and a directed  $L$ -clamp. Then we can reduce Min-Vertex-Cover( $\lambda$ ) to Max- $L$ -DCC(0,1).

We use the  $L$ -clamps to build  $L$ -gadgets, which again should have the property that they absorb one of their connectors and expel the other two. In case of  $L$  being

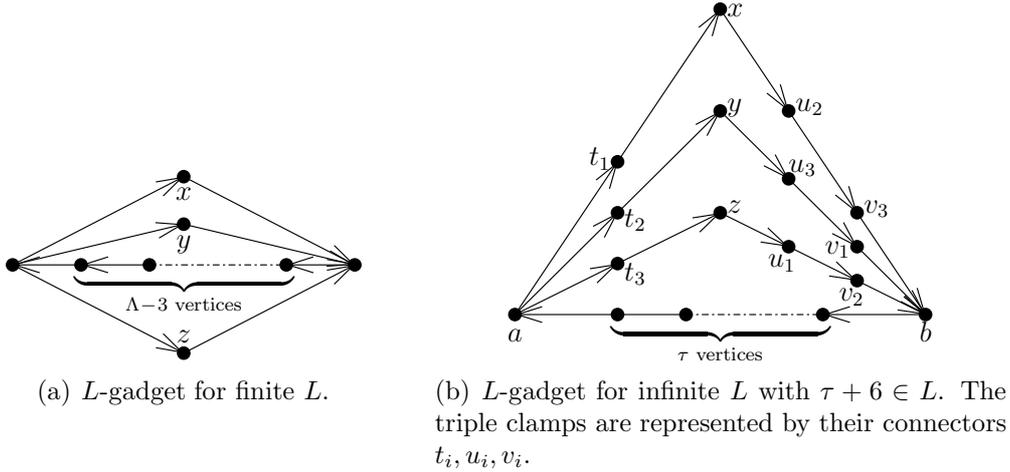


Figure 8: Directed  $L$ -gadgets with connectors  $x, y, z$ .

finite, the graph shown in Figure 8(a) is a directed  $L$ -gadget. In case of infinite  $L$ , we can build directed triple  $L$ -clamps exactly as for undirected graphs. Using these, we can build directed  $L$ -gadgets, which are simply directed variants of their undirected counterparts (Figure 8(b)).

The edge gadgets build the graph  $G_1$ : Let  $x \in X$  be a vertex of  $H$  and  $a_1, \dots, a_\lambda \in F$  be the edges incident to  $x$  in  $H$  (in arbitrary order). Then we assign weight one to the edges  $(x_{a_\xi}^1, x_{a_{\xi+1}}^1)$  for all  $\xi \in \{1, \dots, \lambda - 1\}$ . The edge  $(x_{a_\lambda}^1, x_{a_1}^1)$  has weight zero. These  $\lambda$  edges are called the junctions of  $x$ .

Again,  $G_2, \dots, G_\lambda$  are exact copies of  $G_1$  except that weight one is assigned also to  $(x_{a_\lambda}^\xi, x_{a_1}^\xi)$  for all  $\xi \in \{2, 3, \dots, \lambda\}$ .

Again, we let the  $z$ -vertices form  $\lambda$ -cycles: For all edges  $a \in F$ , we assign weight one to  $(z_a^\xi, z_a^{\xi+1})$  for  $\xi \in \{1, 2, \dots, \lambda - 1\}$  and to  $(z_a^\lambda, z_a^1)$ .

Weight zero is assigned to all edges that are not mentioned.

The remainder of the proof goes along the same lines as the APX-hardness proof for undirected  $L$ -cycle covers.  $\square$

Note that the NP-hardness of  $L$ -DCC for  $L \notin \{\{2\}, \mathcal{D}\}$  does not follow directly from the APX-hardness of Max- $L$ -DCC(0,1): A famous counterexample is 2SAT, for which it is APX-hard to maximise the number of simultaneously satisfied clauses [27], although testing whether a 2CNF formula is satisfiable takes only linear time.

**Theorem 2.14.** *Let  $L \subseteq \mathcal{D}$  be a non-empty set. If  $L \neq \{2\}$  and  $L \neq \mathcal{D}$ , then  $L$ -DCC is NP-hard.*

*Proof.* All we need is an  $L$ -clamp and some  $\lambda \in L$  with  $\lambda \geq 3$ . We present a reduction from  $\lambda$ -DM (which is NP-complete since  $\lambda \geq 3$ ) that is similar to the reduction of Hell et al. [22] used to prove the NP-hardness of  $L$ -UCC for  $\bar{L} \not\subseteq \{3, 4\}$ .

Let  $(X, F)$  be an instance of  $\lambda$ -DM. Note that we will construct a directed graph  $G$  as an instance of  $L$ -DCC, i. e.,  $G$  is neither complete nor edge-weighted. For each  $x \in X$ , we have a vertex in  $G$  that we again call  $x$ . For  $a = \{x_1, \dots, x_\lambda\} \in F$ , we construct a  $\lambda$ -cycle consisting of the vertices  $a_1, \dots, a_\lambda$ . Then we add  $\lambda$   $L$ -clamps

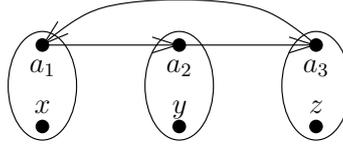


Figure 9: The construction for the NP-hardness of  $L$ -DCC from the viewpoint of  $a = \{x, y, z\} \in F$ . Each ellipse represents an  $L$ -clamp.

$K_a^{x_\eta}$  with  $a_\eta$  and  $x_\eta$  as connectors for all  $\eta \in \{1, \dots, \lambda\}$ . See Figure 9 for an example.

What remains to be shown is that  $G \in L$ -DCC if and only if  $(X, F) \in \lambda$ -DM. Assume first that  $(X, F) \in \lambda$ -DM. Thus, there exists a subset  $\tilde{F} \subseteq F$  such that  $\bigcup_{a \in \tilde{F}} a = X$  and every element  $x \in X$  is contained in exactly one set of  $\tilde{F}$ . We construct an  $L$ -cycle cover of  $G$  in which all clamps are healthy: Let  $a = \{x_1, \dots, x_\lambda\} \in F$ . If  $a \in \tilde{F}$ , then let  $K_a^{x_\eta}$  expel  $a_\eta$  and absorb  $x_\eta$  for all  $\eta \in \{1, \dots, \lambda\}$ , and let  $a_1, a_2, \dots, a_\lambda$  form a  $\lambda$ -cycle. If  $a \notin \tilde{F}$ , let  $K_a^{x_\eta}$  expel  $x_\eta$  and absorb  $a_\eta$  for all  $\eta \in \{1, \dots, \lambda\}$ . All connectors are absorbed by exactly one clamp or are covered by a  $\lambda$ -cycle since  $\tilde{F}$  is a matching.

Now we prove the reverse direction. Assume that  $G \in L$ -DCC, and let  $C$  be an  $L$ -cycle cover of  $G$ . Then every clamp of  $G$  is healthy in  $C$ , i. e., it absorbs one of its connectors and expels the other one. Let  $a = \{x_1, \dots, x_\lambda\} \in F$  and assume that  $K_a^{x_\eta}$  expels  $a_\eta$ . Since  $a_\eta$  must be part of a cycle in  $C$ ,  $(a_{\eta-1}, a_\eta)$  and  $(a_\eta, a_{\eta+1})$  must be in  $C$ . We obtain that either all  $a_1, \dots, a_\lambda$  are absorbed by  $K_a^{x_1}, \dots, K_a^{x_\lambda}$  or that all are expelled by  $K_a^{x_1}, \dots, K_a^{x_\lambda}$ . Now consider any  $x \in X$  and let  $a_1, a_2, \dots, a_\ell \in F$  be all the sets that contain  $x$ . All clamps  $K_{a_1}^x, \dots, K_{a_\ell}^x$  are healthy,  $C$  is an  $L$ -cycle cover of  $G$ , and  $x$  is not incident to any further edges. Hence, there must be a unique  $a_i$  such that  $K_{a_i}^x$  absorbs  $x$ . Thus,

$$\tilde{F} = \{a = \{x_1, \dots, x_\lambda\} \in F \mid K_a^{x_\eta} \text{ absorbs } x_\eta \text{ for all } \eta \in \{1, \dots, \lambda\}\}$$

is a  $\lambda$ -dimensional matching, proving  $(X, F) \in \lambda$ -DM.  $\square$

If the language  $\{1^\lambda \mid \lambda \in L\}$  is in NP, then  $L$ -DCC is also in NP and therefore NP-complete if  $L \notin \{\{2\}, \mathcal{D}\}$ : We can nondeterministically guess a cycle cover and then check if  $\lambda \in L$  for every cycle length  $\lambda$  occurring in that cover. Conversely, if  $\{1^\lambda \mid \lambda \in L\}$  is not in NP, then  $L$ -DCC is not in NP either since there is a reduction of  $\{1^\lambda \mid \lambda \in L\}$  to  $L$ -DCC: On input  $x = 1^\lambda$ , construct a graph  $G$  on  $\lambda$  vertices that consists solely of a Hamiltonian cycle. Then  $x \in L$  if and only if  $G \in L$ -DCC.

### 3 Approximation Algorithms

The goal of this section is to devise approximation algorithms for Max- $L$ -UCC and Max- $L$ -DCC that work for arbitrary  $L$ . The catch is that for most  $L$  it is impossible to decide whether some cycle length is in  $L$  or not. One possibility would be to restrict ourselves to sets  $L$  such that  $\{1^\lambda \mid \lambda \in L\}$  is in P. Another possibility to cope with this problem is to include the permitted cycle lengths in the input. However, it turns out that such restrictions are not necessary.

A necessary and sufficient condition for a complete graph with  $n$  vertices to have an  $L$ -cycle cover is that there exist (not necessarily distinct) lengths  $\lambda_1, \dots, \lambda_k \in L$  for some  $k \in \mathbb{N}$  with  $\sum_{i=1}^k \lambda_i = n$ . We call such an  $n$   **$L$ -admissible** and define  $\langle L \rangle = \{n \mid n \text{ is } L\text{-admissible}\}$ . Although  $L$  can be arbitrarily complicated,  $\langle L \rangle$  always allows efficient membership testing.

**Lemma 3.1.** *For all  $L \subseteq \mathbb{N}$ , there exists a finite set  $L' \subseteq L$  with  $\langle L' \rangle = \langle L \rangle$ .*

*Proof.* Let  $L_{\leq \ell} = \{n \in L \mid n \leq \ell\} \subseteq L$ . Let  $g_L \in \mathbb{N}$  be the greatest common divisor of all numbers in  $L$ . (This exists even if  $L$  is infinite.) There exists an  $\ell_0 \in L$  such that  $g_L$  is also the greatest common divisor of  $L_{\leq \ell_0}$ .

If  $g_L \in L$ , then  $\langle \{g_L\} \rangle = \langle L \rangle$  and we are done. Thus, we assume  $g_L \notin L$ . There exist  $\xi_1, \dots, \xi_k \in \mathbb{Z}$  and  $\lambda_1, \dots, \lambda_k \in L_{\leq \ell_0}$  for some  $k \in \mathbb{N}$  with  $\sum_{i=1}^k \xi_i \lambda_i = g_L$ . Let  $\xi = \min_{1 \leq i \leq k} \xi_i$ . We have  $\xi < 0$  since  $g_L \notin L$ . Choose any  $\lambda \in L_{\leq \ell_0}$  and let  $\ell = -\xi \lambda \cdot \sum_{i=1}^k \lambda_i$ . Let  $n \in \langle L \rangle$  with  $n \geq \ell$ , let  $m = \text{mod}(n - \ell, \lambda)$ , and let  $s = \lfloor \frac{n - \ell}{\lambda} \rfloor$ . We can write  $n$  as

$$n = \lambda s + m + \ell = \lambda s + \frac{m}{g_L} \cdot \sum_{i=1}^k \xi_i \lambda_i - \lambda \xi \cdot \sum_{i=1}^k \lambda_i = \lambda s + \sum_{i=1}^k (m \xi_i - \lambda \xi) \cdot \lambda_i.$$

Since  $m < \lambda$  and  $\xi_i \geq \xi < 0$ , we have  $(m \xi_i - \lambda \xi) \geq 0$  for all  $i$ . Hence,  $\langle L_{\leq \ell_0} \rangle$  contains all elements  $n \in \langle L \rangle$  with  $n \geq \ell$ . Elements of  $\langle L \rangle$  smaller than  $\ell$  are contained in  $\langle L_{\leq \ell} \rangle \supseteq \langle L_{\leq \ell_0} \rangle$ . Hence,  $\langle L_{\leq \ell} \rangle = \langle L \rangle$  and  $L' = L_{\leq \ell}$  is the finite set we are looking for.  $\square$

For every fixed  $L$ , we can not only test in time polynomial in  $n$  whether  $n$  is  $L$ -admissible, but we can, provided that  $n \in \langle L \rangle$ , also find numbers  $\lambda_1, \dots, \lambda_k \in L'$  that add up to  $n$ , where  $L' \subseteq L$  denotes a finite set with  $\langle L' \rangle = \langle L \rangle$ . This can be done via dynamic programming in time  $O(n \cdot |L'|)$ , which is  $O(n)$  for fixed  $L$ .

Although  $\langle L \rangle = \langle L' \rangle$ , there are clearly graphs for which the weights of an optimal  $L$ -cycle cover and an optimal  $L'$ -cycle cover differ: Let  $\lambda \in L \setminus L'$  and consider a graph on  $\lambda$  vertices. We assign weight one to  $\lambda$  edges that form a Hamiltonian cycle, all other edges are assigned weight zero. However, this does not matter for our approximation algorithms.

The two approximation algorithms presented in Sections 3.2 and 3.3 are based on a decomposition technique for cycle covers presented in Section 3.1.

### 3.1 Decomposing Cycle Covers

In this section, we present a decomposition technique for cycle covers. The technique can be applied to cycle covers of undirected graphs but also to directed cycle covers that do not contain 2-cycles.

A **single** is a single edge (or a path of length one) in a graph, while a **double** is a path of length two. Our aim is to decompose a cycle cover  $C$  on  $n$  vertices into roughly  $n/6$  singles,  $n/6$  doubles, and  $n/6$  isolated vertices. If  $n$  is not divisible by six, we replace  $n/6$  by  $\lfloor n/6 \rfloor$  or  $\lceil n/6 \rceil$ : If  $n = 6k + \ell$  for  $k, \ell \in \mathbb{N}$  and  $\ell \leq 5$ , then we take  $k + \alpha_\ell$  singles and  $k + \beta_\ell$  doubles, where  $\alpha_\ell$  and  $\beta_\ell$  are given in Table 1. Thus,

$\ell$	0	1	2	3	4	5
$\alpha_\ell$	0	1	1	0	0	1
$\beta_\ell$	0	0	0	1	1	1

Table 1: A cycle cover on  $n = 6k + \ell$  vertices will be decomposed into  $k + \alpha_\ell$  singles and  $k + \beta_\ell$  doubles.

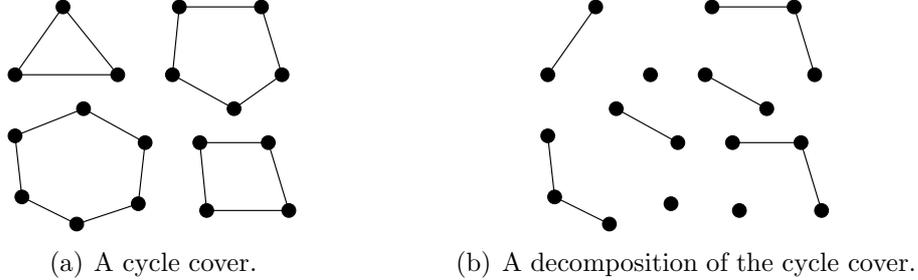


Figure 10: An example of a decomposition according to Lemma 3.2.

we retain half of the edges of  $C$ . We aim to decompose the cycle covers such that at least half of the weight of the cycle cover is preserved.

The reason why we decompose cycle covers into singles and doubles is the following: We cannot decompose them into longer paths since this does not work for  $\{3\}$ -cycle covers. If we restricted ourselves to decomposing the cycle covers into singles only, then 3-cycles would limit the weight preserved: We would retain only one third of the edges of the 3-cycles, thus at most one third of their weight in general. Finally, if we restricted ourselves to doubles, then 5-cycles would limit the weight we could obtain since we would retain only two of their five edges.

In our approximation algorithms, we exploit the following observation: If every cycle cover on  $n$  vertices can be decomposed into  $\alpha$  singles and  $\beta$  doubles, then, for every  $L$ , every  $L$ -cycle cover on  $n$  vertices can be decomposed in the same way. This implies that we can build cycle covers from such a decomposition: Given  $\alpha$  singles and  $\beta$  doubles, and  $n - 2\alpha - 3\beta$  isolated vertices, we can join them to form an  $L$ -cycle cover. (The only restriction is that  $n$  must be  $L$ -admissible.)

Let us now state the decomposition lemma.

**Lemma 3.2.** *Let  $C = (V, E)$  be a cycle cover on  $n = 6k + \ell$  vertices such that the length of each cycle is at least three. Let  $w : E \rightarrow \mathbb{N}$  be an edge weight function.*

*Then there exists a decomposition  $D \subseteq E$  of  $C$  such that  $(V, D)$  consists of vertex-disjoint  $k + \alpha_\ell$  singles,  $k + \beta_\ell$  doubles, and  $n - 5k - 3\beta_\ell - 2\alpha_\ell$  isolated vertices and  $w(D) \geq w(E)/2$ .*

Figure 10 illustrates how a cycle cover is decomposed into singles and doubles. Let us first prove some helpful lemmas.

**Lemma 3.3.** *Let  $\lambda, \alpha, \beta \in \mathbb{N}$  with  $\alpha + 2\beta \geq \lambda/2$  and  $2\alpha + 3\beta \leq \lambda$ . Then every cycle  $c$  of length  $\lambda$  can be decomposed into  $\alpha$  singles and  $\beta$  doubles such that the weight of the decomposition is at least  $w(c)/2$ .*

*Proof.* Every single involves two vertices of  $c$  while every double involves three vertices. Thus,  $2\alpha + 3\beta \leq \lambda$  is a necessary condition for  $c$  being decomposable into  $\alpha$  singles and  $\beta$  doubles. It is also a sufficient condition.

We assign an arbitrary orientation to  $c$ . Let  $e_0, \dots, e_{\lambda-1}$  be the consecutive edges of  $c$ , where  $e_0$  is chosen uniformly at random among the edges of  $c$ . We take  $\alpha$  singles  $e_0, e_2, \dots, e_{2\alpha-2}$  and  $\beta$  doubles  $(e_{2\alpha}, e_{2\alpha+1}), (e_{2\alpha+3}, e_{2\alpha+4}), \dots, (e_{2\alpha+3\beta-3}, e_{2\alpha+3\beta-2})$ . Since  $2\alpha + 3\beta \leq \lambda$ , this is a feasible decomposition. The probability that any fixed edge of  $c$  is included in the decomposition is  $\frac{\alpha+2\beta}{\lambda}$ . Thus, the expected weight of the decomposition is  $\frac{\alpha+2\beta}{\lambda} \cdot w(c) \geq w(c)/2$ .  $\square$

**Lemma 3.4.** *Let  $\lambda \in \mathbb{N}$ . Suppose that every cycle  $c$  of length  $\lambda$  can be decomposed into  $\alpha$  singles and  $\beta$  doubles of weight at least  $w(c)/2$ . Then every cycle  $c'$  of length  $\lambda + 6$  can be decomposed into  $\alpha + 1$  singles and  $\beta + 1$  doubles of weight at least  $w(c')/2$ .*

*Proof.* We have  $\alpha + 2\beta \geq \lambda/2$  and  $2\alpha + 3\beta \leq \lambda$ . Thus,  $\alpha + 1 + 2(\beta + 1) \geq (\lambda + 6)/2$  and  $2(\alpha + 1) + 3(\beta + 1) \leq \lambda + 6$ . The lemma follows from Lemma 3.3.  $\square$

Lemma 3.4 also holds if we consider more than one cycle: Assume that every collection of  $k$  cycles of lengths  $\lambda_1, \dots, \lambda_k$  can be decomposed into  $\alpha$  singles and  $\beta$  doubles such that the weight of the decomposition is at least half the weight of the cycles. Then  $k$  cycles of lengths  $\lambda_1 + 6, \lambda_2, \dots, \lambda_k$  can be decomposed into  $\alpha + 1$  singles and  $\beta + 1$  doubles such that also at least half of the weight of the cycles is preserved. Due to Lemma 3.4, we can restrict ourselves to cycles of length at most eight in the following. The reason for this is the following: If we know how to decompose cycles of length  $\lambda$ , then we also know how to decompose cycles of length  $\lambda + 6, \lambda + 12, \dots$  from Lemma 3.4.

We now come to the proof of the decomposition lemma.

*Proof of Lemma 3.2.* We prove the lemma by induction on the number of cycles. As the induction basis, we consider a cycle cover consisting of either a single cycle or of two odd cycles. Due to Lemma 3.4, we can restrict ourselves to considering cycles of length at most eight. Tables 2(a) and 2(b) show how to decompose a single cycle and two odd cycles, respectively. We always perform the decomposition such that the weight preserved is maximised. In particular, if there are two odd cycles of different length, we have two options in how to decompose these cycles, and we choose the one that yields larger weight. Overall, we obtain a decomposition with an appropriate number of singles and doubles such that at least one half of the weight is preserved.

As the induction hypothesis, we assume that the lemma holds if the number of cycles is less than  $r$ . Assume that we have a cycle cover  $C$  consisting of  $r$  cycles. Let  $n = 6k + \ell$  for the number of its vertices for  $k, \ell \in \mathbb{N}$  and  $\ell \leq 5$ . We remove either an even cycle or two odd cycles. In the following, let  $C'$  be the new cycle cover obtained by removing one or two cycles from  $C$ . A little more care is needed than in the induction basis: Consider for instance the case of removing a 4-cycle. If  $\ell = 4$ , then  $C$  has to be decomposed into  $k$  singles and  $k + 1$  doubles, while we have to take  $k$  singles and  $k$  doubles from  $C'$ . Thus, the 4-cycle has to be decomposed into a double. But if  $\ell = 1$ , then we need  $k + 1$  singles and  $k$  doubles from  $C$  and  $k - 1$

length	$\ell$	$\alpha$	$\beta$
3	3	0	1
4	4	0	1
5	5	1	1
6	0	1	1
7	1	2	1
8	2	2	1

(a) One cycle.

lengths	$\ell$	$\alpha$	$\beta$	decomposition
3 + 3	0	1	1	$3 \rightsquigarrow (1,0) + 3 \rightsquigarrow (0,1)$
3 + 5	2	2	1	$3 \rightsquigarrow (1,0) + 5 \rightsquigarrow (1,1)$ or $3 \rightsquigarrow (0,1) + 5 \rightsquigarrow (2,0)$
3 + 7	4	1	2	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (0,2)$ or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (1,1)$
5 + 5	4	1	2	$5 \rightsquigarrow (0,1) + 5 \rightsquigarrow (1,1)$
5 + 7	0	2	2	$5 \rightsquigarrow (2,0) + 7 \rightsquigarrow (0,2)$ or $5 \rightsquigarrow (1,1) + 7 \rightsquigarrow (1,1)$
7 + 7	2	3	2	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (2,1)$

(b) Two odd cycles.

Table 2: The induction basis. The columns  $\alpha$  and  $\beta$  show the number of singles and doubles needed, respectively. We denote by  $\lambda \rightsquigarrow (\alpha, \beta)$  that a  $\lambda$ -cycle is decomposed into  $\alpha$  singles and  $\beta$  doubles. If there are two lines for a case, then the option that yields more weight is chosen.

length	$\ell$	$\alpha$	$\beta$
4	0,3,4,5	0	1
4	1,2	2	0
6	all	1	1
8	0,1,2,5	2	1
8	3,4	0	2

(a) Removing an even cycle.

lengths	$\ell$	$\alpha$	$\beta$	decomposition
3 + 3	all	1	1	$3 \rightsquigarrow (1,0) + 3 \rightsquigarrow (0,1)$
3 + 7	0,3,4,5	1	2	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (0,2)$ or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (1,1)$
3 + 7	1,2	3	1	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (2,1)$ or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (3,0)$
5 + 5	0,3,4,5	1	2	$5 \rightsquigarrow (0,1) + 5 \rightsquigarrow (1,1)$
5 + 5	1,2	3	1	$5 \rightsquigarrow (2,0) + 5 \rightsquigarrow (1,1)$
5 + 7	all	2	2	$5 \rightsquigarrow (2,0) + 7 \rightsquigarrow (0,2)$ or $5 \rightsquigarrow (1,1) + 7 \rightsquigarrow (1,1)$
7 + 7	0,1,2,5	3	2	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (2,1)$
7 + 7	3,4	1	3	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (0,2)$

(b) Removing two odd cycles.

Table 3: Induction step.

singles and  $k$  doubles from  $C'$ . Thus, the 4-cycle has to be decomposed into two singles. Overall, the 4-cycle has to be decomposed into a double if  $\ell \in \{0, 3, 4, 5\}$  and into two singles if  $\ell \in \{1, 2\}$ . Similar case distinctions hold for all other cases. How to remove one even or two odd cycles is shown in Tables 3(a) and 3(b), respectively.

To complete the proof, we have to deal with the case of a 3- and a 5-cycle, which is slightly more complicated and not covered by Table 3(b). We run into trouble if, for instance,  $\ell = 3$ . In this case, we have to take two doubles. If the 5-cycle is much heavier than the 3-cycle, then it is impossible to preserve half of the weight of the two cycles. But we can avoid this problem: As long as there is an even cycle, we decompose this one. After that, as long as there are at least three odd cycles, we can choose two of them such that we do not have a pair of one  $(3 + 6\xi)$ -cycle and one  $(5 + 6\xi')$ -cycle for some  $\xi, \xi' \in \mathbb{N}$ . The only situation in which it can happen that we cannot avoid decomposing a  $(3 + 6\xi)$ -cycle and a  $(5 + 6\xi')$ -cycle is when there are only two cycles left. In this case, we have  $\ell = 2$ , and we have treated this

case already in the induction basis.  $\square$

The decomposition can clearly be done in polynomial time.

If we consider directed graphs where 2-cycles can also occur, only one third of the weight can be preserved. This can be done by decomposing the cycle cover into a matching of cardinality  $\lceil n/3 \rceil$ . (Every  $\lambda$ -cycle can be decomposed into a matching of up to size  $\lfloor \lambda/2 \rfloor$ . The bottleneck are 3-cycles, which yield only one edge.)

An obvious question is whether the decomposition lemma can be improved in order to preserve more than half of the weight or more than one third of the weight if we additionally allow 2-cycles. Unfortunately, this is not the case.

The following lemma exploits the fact that the fraction of edges that are preserved in a cycle cover decomposition is a lower bound for the fraction of the weight that can be preserved. A generic decomposition lemma states the following: For every  $n \in \mathbb{N}$ , every  $k$ -cycle cover on  $n$  vertices can be decomposed into  $\alpha$  singles and  $\beta$  doubles such that at least a fraction  $r$  of the weight of the cycle cover is preserved. (As already mentioned, longer paths are impossible due to 3-cycles.) Lemma 3.2 instantiates this generic lemma with  $\alpha \approx n/6$ ,  $\beta \approx n/6$ , and  $r = 1/2$ . In case of the presence of 2-cycles, we have sketched a decomposition with  $\alpha \approx n/3$ ,  $\beta = 0$ , and  $r = 1/3$ .

**Lemma 3.5.** *No decomposition technique for 3-cycle covers can in general preserve more than one half of the weight of the 3-cycle covers.*

*Furthermore, no decomposition technique for 2-cycle covers can in general preserve more than one third of the weight of the 2-cycle covers.*

*Proof.* Since, in particular,  $\{3\}$ -cycle covers have to be decomposed, we cannot decompose the cycle cover into paths of length more than two. Now consider decomposing a  $\{4\}$ -cycle cover. Since paths of length 3 are not allowed, we have to discard two edges of every 4-cycle. Thus, at most 2 edges of every 4-cycle are preserved, which proves the first part of the lemma.

The second part follows analogously by considering 3-cycles and observing that paths of length two or more are not allowed.  $\square$

Overall, Lemma 3.5 shows that every approximation algorithm for Max- $L$ -UCC or Max- $L$ -DCC that works for arbitrary sets  $L$  and is purely decomposition-based achieves approximation ratios of at best 2 or 3, respectively. We achieve an approximation ratio of  $8/3 < 3$  for Max- $L$ -DCC by paying special attention to 2-cycles (Section 3.3).

## 3.2 Undirected Cycle Covers

Our approximation algorithm for Max- $L$ -UCC (Algorithm 1) directly exploits Lemma 3.2.

**Theorem 3.6.** *Algorithm 1 is a factor 2 approximation algorithm for Max- $L$ -UCC for all  $L \subseteq \mathcal{U}$ . Its running-time is  $O(n^3)$ .*

**Input:** undirected complete graph  $G = (V, E)$ ,  $|V| = n$ ; edge weights  $w : E \rightarrow \mathbb{N}$   
**Output:** an  $L$ -cycle cover  $C^{\text{apx}}$  of  $G$  if  $n$  is  $L$ -admissible,  $\perp$  otherwise

- 1: **if**  $n \notin \langle L \rangle$  **then**
- 2:     **return**  $\perp$
- 3: compute a cycle cover  $C^{\text{init}}$  in  $G$  of maximum weight
- 4: decompose  $C^{\text{init}}$  into a set  $D \subseteq C^{\text{init}}$  of edges according to Lemma 3.2
- 5: join the singles and doubles in  $D$  to obtain an  $L$ -cycle cover  $C^{\text{apx}}$
- 6: **return**  $C^{\text{apx}}$

Algorithm 1: A 2-approximation algorithm for Max- $L$ -UCC.

*Proof.* If  $L$  is infinite, we replace  $L$  by a finite set  $L' \subseteq L$  with  $\langle L' \rangle = \langle L \rangle$  according to Lemma 3.1. Algorithm 1 returns  $\perp$  if and only if  $n \notin \langle L \rangle$ . Otherwise, an  $L$ -cycle cover  $C^{\text{apx}}$  is returned. Let  $C^*$  denote an  $L$ -cycle cover of maximum weight of  $G$ . We have  $w(C^*) \leq w(C^{\text{init}}) \leq 2 \cdot w(D) \leq 2 \cdot w(C^{\text{apx}})$ . The first inequality holds because  $L$ -cycle covers are special cases of cycle covers. The second inequality holds due to the decomposition lemma (Lemma 3.2). The last inequality holds since no weight is lost during the joining. Overall, the algorithm achieves an approximation ratio of 2.

The running-time of the algorithm is dominated by the time needed to compute the initial cycle cover, which is  $O(n^3)$  [1].  $\square$

### 3.3 Directed Cycle Covers

In the following, let  $C^{\text{opt}}$  be an  $L$ -cycle cover of maximum weight. Let  $w_\lambda$  denote the weight of the  $\lambda$ -cycles in  $C^{\text{opt}}$ , i. e.,  $w(C^{\text{opt}}) = \sum_{\lambda \geq 2} w_\lambda$ .

We use the decomposition lemma only if  $2 \notin L$ . In this case, the weight of an optimal  $L$ -cycle cover is at most the weight of an optimal 3-cycle cover  $C_3^{\text{opt}}$ . Thus, we proceed as follows: First, we compute a  $4/3$  approximation  $C_3^{\text{init}}$  for Max-3-DCC, which can be done by using the algorithm of Bläser et al. [7]. We have  $w(C_3^{\text{init}}) \geq \frac{3}{4} \cdot w(C_3^{\text{opt}}) \geq \frac{3}{4} \cdot w(C^{\text{opt}})$ . Now we decompose  $C_3^{\text{init}}$  into a collection  $D$  of singles and doubles according to Lemma 3.2. Finally, we join the singles, doubles, and isolated vertices of  $D$  to form an  $L$ -cycle cover  $C^{\text{apx}}$ . We obtain a factor  $8/3$  approximation for the case that  $2 \notin L$ :

$$w(C^{\text{apx}}) \geq w(D) \geq \frac{1}{2} \cdot w(C_3^{\text{init}}) \geq \frac{3}{8} \cdot w(C^{\text{opt}}).$$

We divide the case that  $2 \in L$  into two subcases. First, consider the case that  $3 \notin L$ . In this case, a matching-based algorithm achieves an approximation ratio of  $5/2$ : We compute a matching of a certain cardinality, which we will specify in a moment, and then we join the edges of the matching to obtain an  $L$ -cycle cover. The cardinality of the matching is chosen such that an  $L$ -cycle cover can be built from such a matching. A  $\lambda$ -cycle yields a matching of cardinality  $\lfloor \lambda/2 \rfloor$ . Thus, a matching of cardinality  $d$  in a graph of  $n$  vertices can be extended to form an  $L$ -cycle

**Input:** directed complete graph  $G = (V, E)$ ,  $|V| = n$ ; edge weights  $w : E \rightarrow \mathbb{N}$

**Output:** an  $L$ -cycle cover  $C^{\text{apx}}$  of  $G$  if  $n$  is  $L$ -admissible,  $\perp$  otherwise

```

1: if  $n \notin \langle L \rangle$  then
2:   return  $\perp$ 
3: if  $2 \in L$  and  $3 \in L$  then
4:   compute a cycle cover  $C^{\text{init}}$  (without restrictions)
5:   for all even cycles  $c$  of  $C^{\text{init}}$  do
6:     take every other edge of  $c$  such that at least one half of  $c$ 's weight is
       preserved
7:     add the converse edges to obtain 2-cycles; add these cycles to  $C^{\text{apx}}$ 
8:   for all odd cycles  $c$  of  $C^{\text{init}}$  do
9:     take every other edge and one path of length two of  $c$  such that at least
       one half of  $c$ 's weight is preserved
10:    add edges to obtain 2-cycles plus one 3-cycle; add these cycles to  $C^{\text{apx}}$ 
11: else if  $2 \in L$ ,  $3 \notin L$  then
12:   compute a matching  $M$  of maximum weight of cardinality at most  $D(n, L)$ 
13:   join the edges of  $M$  to form an  $L$ -cycle cover  $C^{\text{apx}}$ 
14: else ( $2 \notin L$ )
15:   compute a 4/3-approximation  $C_3^{\text{init}}$  to an optimal 3-cycle cover
16:   decompose  $C_3^{\text{init}}$  into a set  $D \subseteq C_3^{\text{init}}$  of edges according to Lemma 3.2
17:   join the singles and doubles in  $D$  to obtain an  $L$ -cycle  $C^{\text{apx}}$ 
18: return  $C^{\text{apx}}$ 

```

Algorithm 2: A factor 8/3 approximation algorithm for Max- $L$ -DCC.

cover if and only if  $d \leq D(n, L)$ , where

$$D(n, L) = \max \left\{ \sum_{i=1}^k \lfloor \lambda_i / 2 \rfloor \mid k \in \mathbb{N}, \sum_{i=1}^k \lambda_i = n, \text{ and } \lambda_i \in L \text{ for } 1 \leq i \leq k \right\}.$$

Let us now estimate the weight of a matching of cardinality at most  $D(n, L)$  that has maximum weight among all such matchings. From  $C^{\text{opt}}$ , we obtain a matching with a weight of at least

$$\sum_{\lambda \geq 2} \left\lfloor \frac{\lambda}{2} \right\rfloor \cdot w_\lambda \geq \sum_{\lambda \geq 2} \frac{2}{5} \cdot w_\lambda = \frac{2}{5} \cdot w(C^{\text{opt}}).$$

The reason is that  $w_3 = 0$  because  $3 \notin L$  and that  $\min_{\lambda \in \{2, 4, 5, 6, 7, \dots\}} \lfloor \lambda / 2 \rfloor \geq 2/5$ . Thus, by computing a maximum-weight matching  $M$  of cardinality at most  $D(n, L)$  and joining the edges to form an  $L$ -cycle cover  $C^{\text{apx}}$ , we obtain a factor 5/2 approximation.

What remains to be considered is the case that  $L$  contains both 2 and 3. In this case, we start by computing an initial cycle cover  $C^{\text{init}}$  (without any restrictions). Then we do the following: For every even cycle, we take every other edge such that at least one half of its weight is preserved. For every edge thus obtained, we add the converse edge to obtain a collection of 2-cycles. For every odd cycle, we take every other edge and one path of length two such that at least half of the

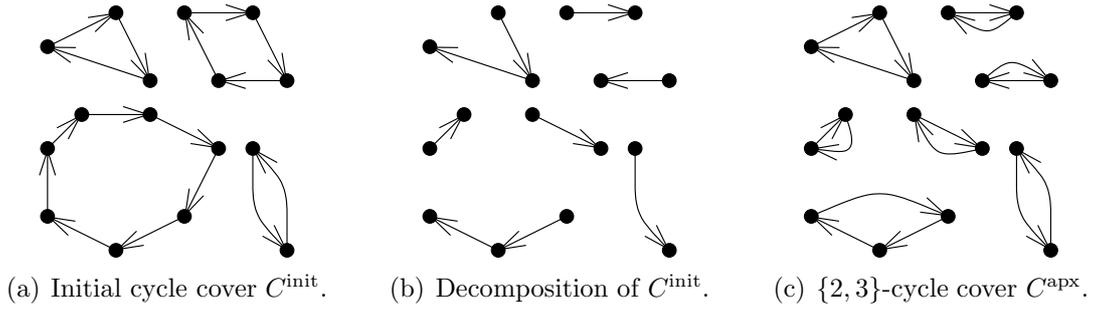


Figure 11: Sketch of the algorithm for  $\{2, 3\} \subseteq L$ .

	$L$ -UCC	Max- $L$ -UCC(0,1)	Max- $L$ -UCC
$\overline{L} = \emptyset$	in P	in PO	in PO
$\overline{L} = \{3\}$	in P	in PO	
$\overline{L} = \{4\}, \{3, 4\}$			APX-complete
$\overline{L} \not\subseteq \{3, 4\}$	NP-hard	APX-hard	APX-hard

(a) Undirected cycle covers.

	$L$ -DCC	Max- $L$ -DCC(0,1)	Max- $L$ -DCC
$L = \{2\}, \mathcal{D}$	in P	in PO	in PO
$L \notin \{\{2\}, \mathcal{D}\}$	NP-hard	APX-hard	APX-hard

(b) Directed cycle covers.

Table 4: The complexity of computing  $L$ -cycle covers.

weight is preserved. Then we add edges to obtain 2-cycles and one 3-cycle. In this way, we obtain a  $\{2, 3\}$ -cycle cover  $C^{\text{apx}}$ , which is also an  $L$ -cycle cover. We have  $w(C^{\text{apx}}) \geq \frac{1}{2} \cdot w(C^{\text{init}}) \geq \frac{1}{2} \cdot w(C^{\text{opt}})$ . Figure 11 shows an example.

Our approximation algorithm is summarised as Algorithm 2. The running-time of the algorithm of Bläser et al. is polynomial [7] and all other steps can be executed in polynomial time as well. Thus, the running-time of Algorithm 2 is also polynomial.

**Theorem 3.7.** *Algorithm 2 is a factor  $8/3$  approximation algorithm for Max- $L$ -UCC for all non-empty sets  $L \subseteq \mathcal{D}$ . Its running-time is polynomial.*

## 4 Conclusions

For almost all  $L$ , finding  $L$ -cycle covers is NP-hard and finding  $L$ -cycle covers of maximum weight is APX-hard. Table 4 shows an overview. Although this shows that computing restricted cycle covers is generally very hard, we have proved that  $L$ -cycle covers of maximum weight can be approximated within a constant factor in polynomial time for all  $L$ .

For directed graphs, we have settled the complexity: If  $L = \{2\}$  or  $L = \mathcal{D}$ , then  $L$ -DCC, Max- $L$ -DCC(0,1), and Max- $L$ -DCC are solvable in polynomial time, otherwise they are intractable. For undirected graphs, the status of only five cycle

cover problems remains open:  $L$ -UCC and Max- $L$ -UCC(0,1) for  $\overline{L} = \{4\}, \{3, 4\}$  and Max-4-UCC.

There are some reasons for optimism that  $L$ -UCC and Max- $L$ -UCC(0,1) for  $\overline{L} = \{4\}, \{3, 4\}$  are solvable in polynomial time: Hartvigsen [18] devised a polynomial-time algorithm for finding  $\overline{\{4\}}$ -cycle covers in bipartite graphs (forbidding 3-cycles does not change the problem for bipartite graphs). Moreover, there are augmenting path theorems for  $L$ -cycle covers for all  $L$  with  $\overline{L} \subseteq \{3, 4\}$  [28], which includes the two cases that are known to be polynomial-time solvable. Augmenting path theorems are often a building block for matching algorithms. But there are also augmenting path theorems for  $L \subseteq \{3, 4\}$  [28], even though these  $L$ -cycle cover problems are intractable.

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