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Polynomial time algorithms to approximate mixed volumes within a simply exponential factor

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Abstract

Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. The Minkowski polynomial $V_{\mathbf{K}}$ is defined as $V_{\mathbf{K}}(x_1, ..., x_n) = V(\lambda_1 K_1 + ... + \lambda_n K_n)$ and the mixed volume $V(K_1, ..., K_n)$ as

$$V(K_1...K_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n).$$

We study in this paper randomized algorithms to approximate the mixed volume of well-presented convex compact sets. Our main result is a poly-time algorithm which approximates $V(K_1, ..., K_n)$ with multiplicative error e^n and with better rates if the affine dimensions of most of the sets K_i are small.
Our approach is based on the particular convex relaxation of $\log(V(K_1, ..., K_n))$ via the geometric programming. We prove the mixed volume analogues of the Van der Waerden and the Schrijver/Valiant conjectures on the permanent. These results , interesting on their own, allow to "justify" the above mentioned convex relaxation, which is solved using the ellipsoid method and a randomized poly-time time algorithm for the approximation of the volume of a convex set.

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1 Introduction

Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. It is well known (see for instance [14]), that the value of $V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n)$ is a homogeneous polynomial of degree n in nonnegative variables $\lambda_1...\lambda_n$, where $''+''$ denotes Minkowski sum, and λK denotes the dilatation of K with coefficient λ. The coefficient $V(K_1...K_n)$ of $\lambda_1 \cdot \lambda_2 \ldots \cdot \lambda_n$ is called the *mixed volume* of $K_1...K_n$. Alternatively,

$$V(K_1...K_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n).$$

Mixed volume is known to be monotone [14] , namely $K_i \subseteq L_i$, for $i = 1, ..., n$, implies $V(K_1...K_n) \leq V(L_1...L_n)$. In particular, it is always nonnegative.

The problem of computing the mixed volume of convex bodies is important for combinatorics and algebraic geometry [13]. For instance, the number of toric solutions to a generic system of n polynomial equations on $\mathbf{C}^n$ is equal (and in a general case is upper bounded by) to the mixed volume of the Newton polytopes of the equations (see for instance [15] and [14]). This remarkable connection , called **BKH Theorem**, created an "industry" of computing the mixed volume of integer polytopes and its various generalizations, and most of algorithms in that area are of exponential runing time (see [13] for some references). Altough there was a substantial algorithmic activity on the mixed volume of polytopes prior to [13] , the paper [13] was first, to our knowledge, systematic complexity-theoretic study in the area. It followed (naturally) famous FPRAS algorithms [12] for volumes of convex bodies , solved several natural problems and posed many important hard questions.The existence of FPRAS for the mixed volume even for polytopes or ellipsoids is still very open problem.

Efficient polynomial-time probabilistic algorithms that approximate the mixed volume extremely tightly ($(1+\epsilon)$-factor) were developed for some classes of well-presented convex bodies [13]. The algorithms in [13] are based on the multivariate polynomial interpolation and work if and only if the number of distinct convex sets in the tuple $\mathbf{K}$ is "small".

How well can the mixed volume be approximated in polynomial time? The first efficient probabilistic algorithm that provides a $n^{O(n)}$-factor approximation for *arbitrary well-presented proper* [1] *convex bodies* was obtained by Barvinok in [9].

The question of existence of an efficient *deterministic* algorithm for approximating the mixed volume of arbitrary well-presented proper convex bodies with an error depending only on the dimension was posed in [13]. They quote a lower bound (**Barany-Furedi bound**) [10] of $\left(\Omega\left(\frac{n}{\log n}\right)\right)^{\frac{n}{2}}$ for the approximation factor of such an algorithm. (Notice that Barvinok's *randomized* algorithm [9] does not beat the **Barany-Furedi bound**.)

Deterministic polynomial-time algorithms that approximate the mixed volume with a factor of $n^{O(n)}$ were given, for a fixed number of distinct proper convex bodies in $\mathbf{K} = (K_1...K_n)$, in [9], [13]. Finally , a deterministic polynomial-time algorithm that approximate the mixed volume with a factor of $n^{O(n)}$ in the general case of well-presented compact convex sets was given in [6] ,[7]. Similarly to [9] , the algorithm in [6] ,[7] reduced the approximation of the mixed

[1] Recall that a convex body in $\mathbf{R}^n$ is *proper* if its interior is not empty.

volume to the approximation of the mixed discriminant. And the appoximation of the mixed discriminant was relaxed by some convex optimization problem (geometric programming). In order to prove the accuracy of the convex relaxation , the author proved in [16] the mixed discriminant analogue of the Van der Waerden conjecture on permanents of doubly stochastic matrices [1] , which was posed by R. V. Bapat in [2].

1.1 Our Approach

Assume , modulo deterministic poly-time preprocessing [13] , that the mixed volume $V(K_1...K_n) > 0$. We define the capacity of the volume polynomial $V_{\mathbf{K}}$ as $Cap(V_{\mathbf{K}}) = \inf_{x_i>0:1\leq i\leq n} \frac{V_{\mathbf{K}}(x_1,...,x_n)}{\prod_{1\leq i\leq n} x_i}$. Since the coeficients of the volume polynomial $V_{\mathbf{K}}$ are nonnegative real numbers we get the inequality $\frac{Cap(V_{\mathbf{K}})}{V(K_1...K_n)} \geq 1$. The trick is that $\log(Cap(V_{\mathbf{K}}))$ is a solution of the following convex minimization problem

$$\log(Cap(V_{\mathbf{K}})) = \inf_{y_1+...+y_n=0} \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})). \tag{1}$$

We view $Cap(V_{\mathbf{K}})$ as an approximation for the mixed volume $V(K_1...K_n)$, to justify this we prove the upper bound $\frac{Cap(V_{\mathbf{K}})}{V(K_1...K_n)} \leq \frac{n^n}{n!} \approx e^n$, which is the mixed volume analogue of the Van der Waerden conjecture. We also present a better upper bounds when "most" of the convex sets K_i have small affine dimension , which are analogues of Schrijver-Valiant conjecture [18] , [19].

After establishing this , we present a randomized poly-time algorithm to solve the problem (1) based on ellipsoid method and randomized poly-time algorithms for the volume approximation. Together with the proved Van Der Waerden conjecture for mixed volumes, this gives a randomized poly-time algorithm to approximate the mixed volume $V(K_1...K_n)$ within relative accuracy e^n. Notice that , in view of (**Barany-Furedi bound**) , this can not be achieved by a deterministic poly-time oracle algorithm. The idea of our approach is very similar to our treatment of POS-hyperbolic polynomials in [23]. Recall that a homogeneous polynomial $p(x_1, ..., x_n)$ with nonnegative coefficients is called POS-hyperbolic if all the roots of the univariate polynomials $b(t) = p(x_1 - ty_1, ..., x_n - ty_n)$ are real for all real vectors $(x_1, ..., x_n)$ and positive real vectors $(y_1, ..., y_n)$. Not all Minkowski polynomials $V_{\mathbf{K}}$ are POS-hyperbolic : any univariate polynomial with nonnegative coefficients $S(x) = \sum_{0\leq i\leq n} \binom{n}{i} a_i x^i; a_i^2 \geq a_{i-1}a_{i-1}, 1 \leq i \leq n-1$ can be presented as $S(x) = V(A + xB)$ for some convex compact subsets (simplexes) $A, B \subset R^n$ [17]. Fortunatelly, a modification of the induction in [23] works for Minkowski polynomials and presented in the next Section.

2 Van the Waerden conjecture for mixed volumes

Definition 2.1:

1. Let $n \geq k \geq 1$ be two integers. We define the univariate polynomial $sv_{n,k}(x) = 1 + \sum_{1\leq i\leq k} (\frac{x}{n})^i \binom{n}{i}$. Notice that $sv_{n,n}(x) = (1 + \frac{x}{n})^n$. We define the following, important for

what follows, functions :

$$\lambda(n,k) = (\min_{x>0}(\frac{sv_{n,k}(x)}{x})^{-1} \quad (2)$$

Remark 2.2: It was observed in [23] that $\lambda(k,k) = g(k) =: (\frac{k-1}{k})^{k-1}, k \geq 1$. The following inequalities are easy to prove :

$$\lambda(n,k) < \lambda(n,l) : n \geq k > l \geq 1; \lambda(m,k) > \lambda(n,k) : n > m \geq k. \quad (3)$$

The equality $\lambda(n,2) = (1+\sqrt{2}\sqrt{\frac{n-1}{n}})^{-1} \geq (1+\sqrt{2})^{-1}$ follows from basic calculus. ∎

2. An univariate polynomial with nonnegative coefficients $R(t) = \sum_{0 \leq i \leq m} a_i t^i$ is called $n-Newton$, where $n \geq m$ if it satifies the following inequalities :

$$NIs : (\frac{a_i}{\binom{n}{i}})^2 \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} : 1 \leq i \leq m-1 \quad (4)$$

The following weak Newton's inequalities $WNIs$ follow from NIs if the coefficients are nonnegative:

$$WNIs : a_i a_0^{i-1} \leq \frac{a_1{}^i}{n} \binom{n}{i} : 2 \leq i \leq k. \quad (5)$$

(Recall that the Newton's inequalities NIs with $n = k$ are satisfied if all the roots of p are real.)

∎

The next definition is adapted from [23].

Definition 2.3:

1. Let $p \in Hom_+(n,m), p(x_1,...,x_n) = \sum_{(r_1,...,r_n) \in I_{n,m}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be a homogeneous polynomial with nonnegative real coefficients of degree m in n variables. Here $I_{n,m}$ stands for the set of vectors $r = (r_1,...,r_n)$ with nonnegative integer components and $\sum_{1 \leq i \leq n} r_i = m$.

   The support of the polynomial p as above is defined as $supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p.
   For a subset $A \subset \{1,...,n\}$ we define $S_p(A) = \max_{(r_1,...,r_n) \in supp(p)} \sum_{i \in A} r_i$. *(If $p = V_{\mathbf{K}}$ then $S_p(A)$ is equal to the affine dimension of the Minkowski sum $\sum_{i \in A} K_i$.)*

   The following linear differential operator maps $Hom(n,n)$ onto $Hom(n-1,n-1)$:

$$p_{x_1}(x_2,...,x_n) = \frac{\partial}{\partial x_1} p(0,x_2,...,x_n).$$

We define $p_{x_i}, 2 \leq i \leq n$ in the same way . We will use the notation $p_{i_1,...,i_k}$ for the composition $((p_{x_{i_1}})_{x_{i_2}}...)_{x_{i_k}}$. Notice that the operator $p_{i_1,...,i_k}$ maps $H(n,n)$ onto $Hom(n-k, n-k)$.

The following inequality follows straight from the definition :

$$S_{p_{x_1}}(A) \leq \min(n-1, S_p(A)) : A \subset \{2,...,n\}, p \in Hom_+(n,n). \tag{6}$$

2. A homogeneous polynomial $p \in Hom_+(n,m)$ is called $Newton$ if for all vectors $X, Y \in R^n_+$ the polynomial $R(t) = p(tX+Y)$ is $(m - Newton)$. A notion of the $Weak - Newton$ is defined analogously.
   Finally, a homogeneous polynomial $p \in Hom_+(n,n)$ is called AF-polynomial if the polynomials $p_{i_1,...,i_k}$ are $Newton$ for all $1 \leq i_1 < ... < i_k \leq n$.

3. We define **Capacity** of a homogeneous polynomial $p \in Hom_+(n,n)$ as

$$Cap(p) = \inf x_i > 0 : 1 \leq i \leq n \frac{p(x_1,...,x_n)}{\prod_{1 \leq i \leq n} x_i} \tag{7}$$

∎

The main result in this section is the following theorem :

Theorem 2.4: *Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple convex compact subsets in the Euclidean space $\mathbf{R}^n$ and $af(i)$ be the affine dimension of K_i , $1 \leq i \leq n$. Then the following inequality holds :*

$$Cap(V_{\mathbf{K}}) \geq V(K_1,...,K_n) \geq \prod_{1 \leq i \leq n} \lambda(i, \min(i, af(i)))Cap(V_{\mathbf{K}}) \tag{8}$$

Corollary 2.5:

1. $$V(K_1,...,K_n) \geq \frac{n!}{n^n} Cap(V_{\mathbf{K}}) \tag{9}$$

   The equality in (9) is attained if and only if either the mixed volume $V(K_1,...,K_n) = 0$ or $K_i = a_i K_1 + b_i : a_i > 0, b_i \in R^n; 2 \leq i \leq n$.

2. *Suppose that $af(i) \leq k : k+1 \leq i \leq n$ then*

$$V(K_1,...,K_n) \geq \frac{k!}{k^k} \lambda(n,k)^{n-k} Cap(V_{\mathbf{K}}). \tag{10}$$

   If $k = 2$ we get the inequality $V(K_1,...,K_n) \geq \frac{1}{2}(1+\sqrt{2})^{2-n} Cap(V_{\mathbf{K}})$.

Remark 2.6: The inequality (9) is an analoque of the famous Van der Waerden conjecture on the permanent of doubly-stochastic matrices. Indeed , let $K_i = \{(x_1,...,x_n) : 0 \leq x_j \leq A(i,j), 1 \leq j \leq n\}$. Then the mixed volume $V(\mathbf{K}) = (V(K_1,...,K_n) = Perm(A)$; and if the $n \times n$ matrix A is doubly-stochastic then $Cap(V_{\mathbf{K}}) = 1$.

The inequality (10) is an analoque of the Schrijver lower on the number of perfect matchings in k-regular bipartite graphs.

The reader familiar with [23] can recognize the similarity of the inequalities 8, 9, 10 with inequalities in [23] , proved for POS-hyperbolic polynomials. The method of proof in this paper is similar to [23], inspite the fact that there exist non POS-hyperbolic Minkowski polynomials $V_{\mathbf{K}}$. But we get the worse constants: for instance ,if $k = 2$,in notations of (10) , then in *pos*-hyperbolic case one gets the factor 2^{-n+1} instead of $\frac{1}{2}(1+\sqrt{2})^{n-2}$ in this paper. Whether the latter factor is assymptoticaly sharp is an open problem. ∎

2.1 Proofs

We recall here the fundamental Alexandrov-Fenchel inequality for the mixed volume of n convex sets in R^n :

$$V(K_1, K_2, K_3, ..., K_n)^2 \geq V(K_1, K_1, K_3, ..., K_n)V(K_2, K_2, K_3, ..., K_n) \tag{11}$$

Using our definition, the inequality (11) can be stated in the following equivalent form (implicit in [14]) :

Proposition 2.7: *Let* $\mathbf{K} = (K_1...K_n)$ *be a* n*-tuple of convex compact subsets* R^n *and* $V_{\mathbf{K}}$ *is the corresponding Minkowski polynomial. Then for all* $1 \leq i_1, ..., i_k \leq n, k \leq n-1$ *the polynomials* $(V_{\mathbf{K}})_{i_1,...,i_k}$ *are* $Newton$*. Or , in other words , the Minkowski polynomials* $V_{\mathbf{K}}$ *are* AF*-polynomials.*

We need the next (elementary) result:

Lemma 2.8:

1. *Let* $R(t) = \sum_{0 \leq i \leq k} a_i t^i$ *be* $n - Newton$ *polynomial,* $n \geq k$*. Then*

$$a_1 = R^{'}(0) \geq \lambda(n, k) \inf_{t>0} \frac{R(t)}{t} \tag{12}$$

   The inequality (12) is attained if and only if $R(t) = R(0)(1 + \sum_{1 \leq i \leq k} (\frac{t}{n})^i \binom{n}{i})$*. If* $n = k$ *, it attained iff* $R(t) = R(0)(1 + \frac{t}{n})^n$*.*

2. *Let* $p \in Hom_+(n, n)$ *be* $Newton$ *polynomial of degree* n *in* $n \geq 2$ *variables. Then* $Cap(p_{x_i}) \geq \lambda(n, S_p(\{i\}))Cap(p)$*.*

Proof: The first part of Lemma 2.8 is a minor modification of Lemma 2.7 in [23].
To prove the second part, let us consider WLOG the case $i = 1$ and fix $n-1$ positive numbers $x_2, .., x_{n-1}$ such that $\prod_{2 \leq i \leq n} x_i = 1$. Then $p(t, x_2, ..., x_n) = R(t) = \sum_{0 \leq i \leq k} a_i t^i$; the degree k of R is equal to $S_p(\{1\}$ and $R^{'}(0) = a_1 = p_{x_1}(x_2, ..., x_n)$. Since the polynomial p is $Newton$ hence the univariate polynomial r is $n - Newton$. It follows from the definition (7) of the **Capacity** that $R(t) \geq Cap(p)tx_2, .., x_{n-1} = Cap(p)$. Therefore, using inequality (12) ,we get that $p_{x_1}(x_2, ..., x_n) = R^{'}(0) \geq \lambda(n, k)Cap(p)$ or eqivalently that $Cap(p_{x_1}) \geq \lambda(n, S_p(\{1\}))Cap(p)$. ∎

The next theorem follows from Lemma 2.8 by a direct induction , using the inequalities (6) , (3).

Theorem 2.9: *Let $p \in Hom_+(n,n)$ be AF polynomial. Then*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \geq \prod_{1 \leq i \leq n} \lambda(i, \min(i, S_p(\{i\}))Cap(p) \geq \prod_{1 \leq i \leq n} \lambda(i,i)Cap(p) = \frac{n!}{n^n} Cap(p). \tag{13}$$

Let $V_{\mathbf{K}}(x_1,...,x_n)$ be Minkowski polynomial. Then $S_{V_{\mathbf{K}}}(A) = af(\sum_{i \in A} K_i)$. Also $V_{\mathbf{K}}$ is AF-polynomial. Therefore , Theorem (2.4) and Corollary (2.5) follow from Theorem (2.9).

We sketch below the uniqueness part of the first part of Corollary (2.5).

Proof:

1. Assume that $Cap(V_{\mathbf{K}}) > 0$. If there is equality in (9) then necessary the affine dimensions $af(K_i) = n, 1 \leq i \leq n$. This fact implies that all coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive.

2. **Scaling**.
As all coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive, hence there exist unique positive numbers $a_1,...,a_n$ such that the scaled polynomial $p = V_{\{a_1K_1,...,a_nK_n\}}$ is doubly stochastic (see [7]) : $\frac{\partial}{\partial x_i} p(1,1,...,1) = 1, 1 \leq i \leq n$. We will deal , without loss of generality, only with this doubly stochastic case.

3. **Brunn-Minkowski**.
Let $(z_2,...,z_n))$ be the unique minimizer of the problem
$\min_{x_i > 0, 2 \leq i \leq n; \prod_{2 \leq i \leq n} x_i = 1} p_{x_1}(x_2,...,x_n)$. Such unique minimizer exists for all the coefficients of p_{x_1} are positive. It follows from the proof of second part of Lemma 2.8 , that $V_{\mathbf{K}}(t, z_2,...,z_n) = (at+b)^n$ for some positive numbers a, b. It follows from the equality case of the Brunn-Minkowski inequality [14] that $K_1 = \sum_{2 \leq j \leq n} A(1,j)K_j + \{T_1\}$, where $A_{1,j} > 0$ and $T_1 \in R^n$. In the same way , we get that there exist a $n \times n$ matrix A , with the zero diagonal and positive off-diagonal part , and vectors $T_1,...,T_n \in R^n$ such that $K_i = \sum_{j \neq i} A(1,j)K_j + \{T_i\}$. It follows from the doubly-stochasticity that all row sums of the matrix A are equal to one.

4. Associate with the convex compact set $K_i \subset R^n$ its support function
$\gamma_i(X) = \max_{Y \in K_i} < X, Y >, X \in R^n$. We get that
$\gamma_i(X) = \sum_{j \neq i} A(1,j)\gamma_j(X) + < X, T_i >, X \in R^n$.
As the kernel $\{Y \in R^n : Y = AY\} = \{c(1,1,...,1), c \in R\}$, we get finally that
$\gamma_i(X) = \alpha(X) + < X, L_j >, X \in R^n$ for some functional $\alpha(X)$ and vectors $L_1,...,L_n \in R^n$.
Which means , in the doubly-stochastic case , that $K_i = K_1 + \{L_i - L_i\}, 2 \leq i \leq n$.

∎

3 Convex Optimization

3.1 Presentations of convex compact sets

. Following [13] we consider the following well-presentation of convex compact set $K_i \subset R^n, 1 \leq i \leq n$: A weak membership oracle for K and a rational $n \times n$ matrix A_i , a rational vector

$y_i \in R^n$ such that

$$y_i + A_i(Ball_n) \subset K_i \subset y + n\sqrt{n+1}A_i(Ball_n) \quad (14)$$

Here $Ball_n = \{x \in R^n : ||x||_2 \leq 1\}$ is a standard unit ball in R^n. We define the size $< \mathbf{K} >$ as the maximum of bit sizes of entries of matrices $A_i, 1 \leq i \leq n$. Since the mixed volume $V(K_1, ..., K_n) = V(K_1 + \{-y_1\}, ..., K_n + \{-y_n\}$, we will assume WLOG that $y_i = 0, 1 \leq i \leq n$. This assumption implies that the following identity for affine dimensions

$$af(\sum i \in SK_i) = Rank(\sum i \in SA_iA_I^T), S \subset \{1, ..., n\}. \quad (15)$$

Definition 3.1:

1. We recall a notion of the mixed discriminant $D(Q_1, .., Q_n)$, where $Q_i; 1 \leq i \leq n$ are $n \times n$ complex matrices :

$$D(Q_1, .., Q_n) = \frac{\partial^n}{\partial x_1 ... \partial x_n} \det(x_1Q_1 + \cdots x_nQ_n) \quad (16)$$

2. A n-tuple $\mathbf{K} = (K_1...K_n)$ of convex compact subsets in R^n is called indecomposable if $aff(\sum i \in SK_i) > Card(S) : S \subset \{1, ..., n\}, 1 \leq Card(S) < n$.
   We consider , similarly to [7] , $n(n-1)$ auxiliary n-tuples $\mathbf{K}^{ij}$, where $\mathbf{K}^{ij}$ is obtained from $\mathbf{K}$ by substituting K_i instead of K_j. Notice that

$$V(x_1K_1 + ... + x_nK_n) = x_1x_2...x_n(V(\mathbf{K}) + \frac{1}{2}\sum_{1 \leq i,j \leq n} \frac{x_i}{x_j} V(\mathbf{K}^{ij})) + ... \quad (17)$$

∎

It follows from 15 that the n-tuple $\mathbf{K} = (K_1...K_n)$ of well-presented convex sets is indecomposable iff the n-tuple of positive semidefinite matrices $\mathbf{Q} = (Q_1...Q_n) : Q_i = A_iA_I^T$ is fully indecomposable as defined in [7] , which implies that indecomposability of $\mathbf{K}$ is equivalent to the inequalities $V(\mathbf{K}^{ij})) > 0 : 1 \leq i, j \leq n$. Again , using Theorem 1.9 , Lemma 2.3 , Lemma 2.4 from [7] and "decomposition lemma" from [14] , we can , by deterministic poly-time pre-proprocessing , to check either the tuple $\mathbf{K}$ is indecomposable or to factor the mixed volume $V(\mathbf{K}) = \prod_{1 \leq j \leq m \leq n} V(\mathbf{K_j})$. Here the $n(j)$-tuple $\mathbf{K_j} = (K_{j,1}, ..., K_{j,n(j)} \subset R^{n(j)}$ is well presented and indecomposable ; $\sum_{1 \leq j \leq m} n(j) = n$ and the sizes $< \mathbf{K_j} > \leq < \mathbf{K} > +poly(n)$.
Based on the above remarks , we will deal from now on only with indecomposable well-presented tuples of convex compact sets. Moreover , to simplify the exposition , we assume WLOG that the matrices A_i in (14) are integer.

Let $\mathcal{E}_A$ be the ellipsoid $A(Ball_n)$ in R^n. The next inequality , proved in [9], connects the mixed volume of ellipsoids and the corresponding mixed discriminant

$$3^{-\frac{n+1}{2}} v_n D^{\frac{1}{2}}(A_1(A_1)^T, ..., A_n(A_n)^T) \leq V(\mathcal{E}_{A_1}...\mathcal{E}_{A_n}) \leq v_n D^{\frac{1}{2}}(A_1(A_1)^T, ..., A_n(A_n)^T). \quad (18)$$

Here v_n is the volume of the unit ball in $\mathbf{R}^n$.

3.2 Properties of Volume polynomials : Lipshitz , bound on the second derivative , a priori ball

Proposition 3.2:

1. **Lipshitz Porperty**.
Let $p(x_1, ..., x_n)$ be a nonzero homogeneous polynomial of degree n with nonnegative coefficients , $x_i = e^{y_i}$. Then

$$\frac{\partial}{\partial y_i} \log(p(e^{y_1}, ..., e^{y_n})) = \frac{\frac{\partial}{\partial x_i} p(x_1, ..., x_n) e^{y_i}}{p(x_1, ..., x_n)} \tag{19}$$

It follows from the Eyler's identity that $\sum_{1 \leq i \leq n} \frac{\partial}{\partial y_i} \log(p(e^{y_1}, ..., e^{y_n})) = n$, therefore the functional $f(y_1, ..., y_n) = \log(p(e^{y_1}, ..., e^{y_n}))$ is Lipshitz on R^n :

$$|f(y_1 + \delta_1, ..., y_n + \delta_n) - f(y_1, ..., y_n)| \leq n||\Delta||_\infty \leq n||\Delta||_2 \tag{20}$$

2. **Upper bound on second derivatives**.
Let us fix real numbers $y_1, ..., y_{i-1}, y_{i+1}, ..., y_n$ and define univariate function $q(y_i) = \log(p(e^{y_1}, ..., e^{y_i}, ..., e^{y_n}))$. We also define $deg_p(i)$ as a maximum degree of the variable x_i in polynomial $p(x_1, ..., x_n)$. Notive that $e^{q(y_i)} = \sum_{0 \leq j \leq deg_p(i)} a_j e^{jy}, a_j \geq 0$.
Then the second derivative

$$0 \leq q^{''}(y) \leq \frac{1}{4}(deg_p(i))^2 \leq \frac{1}{4}n^2 \tag{21}$$

. The inequality 21 follows from the following probabilistic representation :
$q^{''}(y) = E(D^2) - (E(D))^2$, where the random variable $D = j, 0 \leq j \leq deg_p(i)$ with the probability $\frac{a_j e^{jy}}{\sum_{0 \leq j \leq deg_p(i)} a_j e^{jy}}$.

3. **A Priori Ball result from [7]**.
Let $p \in Hom_+(n, n)$, $p(x_1, ..., x_n) = x_1 x_2 ... x_n(a + \frac{1}{2} \sum_{1 \leq i \neq j \leq n} b^{i,j} \frac{x_i}{x_j}) + ...$
Assume that $min_{1 \leq i \neq j \leq n} b^{i,j} = Stf(p) > 0$. Then there exists an unique minimizer $(z_1, ..., z_n) = Argmin(\log(p(e^{y_1}, ..., y_1)) : \sum_{1 \leq i \leq n} y_i = 0$. Moreover ,

$$|z_i - z_j| \leq \log(\frac{2Cap(p)}{Stf(p)}) \tag{22}$$

The next proposition adapts Lemma 4.1 from [7] to the "mixed volume situation" , using the Barvinok's inequality (18).

Proposition 3.3: *Consider an indecomposable n-tuple of convex compact sets $\mathbf{K} = (K_1, .., K_n)$ with the well-presentation $A_i(Ball_n) \subset K_i \subset y + n\sqrt{n+1}A_i(Ball_n), 1 \leq i \leq n$ with integer $n \times n$ matrices A_i. Then the minimum in the convex optimization problem 1 is attained and unique. The unique minimizer vector*
$(z_1, ..., z_n), \sum_{1 \leq i \leq n} z_i = 0$ satisfies the following inequalities

$$|z_i - z_j| \leq O(n^{\frac{3}{2}}(\log(n) + <\mathbf{K}>)); ||z_1, ..., z_n)||_2 \leq O(n^2(\log(n) + <\mathbf{K}>)) \qquad (23)$$

In other words the convex optimization problem (1) can be solved on the following ball in R^{n-1}:

$$Apr(\mathbf{K}) = \{(z_1, ..., z_n) : ||z_1, ..., z_n)||_2 \leq O(n^2(\log(n) + <\mathbf{K}>)), \sum_{1 \leq i \leq n} z_i = 0\}$$

The following inequality follows from the Lipshitz property (20):

$$|\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}) - \log(V_{\mathbf{K}}(e^{l_1}, ..., e^{l_n})| \leq O(n^3(\log(n) + <\mathbf{K}>)) : Y, L \in Apr(\mathbf{K}). \qquad (24)$$

3.3 Ellipsoid method with noisy first order oracles

. We recall the following fundamental result [21]:
Let $f(Y)$ be differentiable convex functional defined on the ball $Ball_n(r) = \{Y \in R^n :< Y, Y > \leq r^2$ of radius r. Let $Var(f) = \max_{Y \in Ball_n(r)} f(Y) - \min_{Y \in Ball_n(r)} f(Y)$. Assume that at each vector $Y \in Ball_n(r)$ we have an oracle evaluating a value $g(Y)$ such that $|g(Y) - f(Y)|0.2\delta Var(f)$ and the vector $gr(Y) \in R^n$ such that $||gr(Y) - (\bigtriangledown f)(Y)||_2 \leq 0.2\delta r^{-1}$ (here $(\bigtriangledown f)(Y)$ is a gradient of f evaluated at Y).
Then the **Ellipsoid method** *finds a vector $Z \in Ball_n(r)$ such that $f(Z) \leq \min_{Y \in Ball_n(r)} f(Y) + \epsilon Var(f), \epsilon > \delta$. The method requires $O(n^2 \log(\frac{1}{\epsilon - \delta}))$ oracle calls plus $O(n^2)$ elementary operations to run the algorithm itself.*

3.4 Putting things together

We take advantage here of randomized algorithms which can evaluate,for a well-presented convex set, K $\log(Vol(K))$ with additive error ϵ and failure probability δ in $O(\epsilon^{-k} n^l \log(\frac{1}{\delta}))$ oracle calls. For instance , the best current algorithm [22] gives $k = 2, l = 4$. We will need below to evaluate volumes $V(\sum_{1 \leq i \leq n} x_i K_i)$. It will require to get well-presentation of the sum $\sum_{1 \leq i \leq n} x_i K_i$ from the well-presentation of individual K_i. And it is possible , provided that the bit size of the weigths x_i is bounded by the size $<\mathbf{K}>$ [13]. In our context, it will require a rounding procedure based on the Lipshitz property from Propostion (3.2) which will be described in the full version of the paper.

In our case the fuctional $f = \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})$ defined on ball $Apr(\mathbf{K})$ of radius $O(n^2(\log(n) + < \mathbf{K} >))$ with the variance $Var(f) \leq O(n^3(\log(n) + <\mathbf{K}>))$. Theorem 2.4 gives the bound :
$\log(V(\mathbf{K}) \leq (\min_{Y \in Apr(\mathbf{K})} f(Y) \leq \log(V(\mathbf{K}) + \log(\frac{n^n}{n!}) \approx \log(V(\mathbf{K}) + n$.
Therefore , to get e^n approximation of the mixed volume $V(\mathbf{K}$ it is sufficient to find out $Z \in Apr(\mathbf{K})$ such that $f(Z) \leq \min_{Y \in Apr(\mathbf{K})} f(Y) + O(1)$. In order to do it via the Ellipsoid method we need to approximate $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})$ with the additive error $O(Var(f)^{-1}) = O(n^{-3}(\log(n) + <\mathbf{K}>)^{-1})$ and its gradient with the additive l_2 error $O(n^{-2}(\log(n) + <\mathbf{K}>)^{-1})$.

1. **Approximation of** $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}))$ **with failure probability** δ. The complexity is $O(n^10(\log(n) + <\mathbf{K}>)^2 \log(\delta^{-1}))$

2. **Approximation of the partial derivatives**.

   Let $x_i = e^{y_i}$ and recall that the partial derivatives

$$\beta_i = \frac{\partial}{\partial y_i} \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})) = \frac{\frac{\partial}{\partial x_i} V_{\mathbf{K}}(x_1, ..., x_n) e^{y_i}}{V_{\mathbf{K}}(x_1, ..., x_n)}$$

   Suppose that $0 \leq 1-a \leq \frac{\gamma_i}{\beta_i} \leq 1+a$. It follows from the Eyler's identity that $\sum_{1\leq i\leq n} |\gamma_i - \beta_i| \leq a$. If $a = O(n^{-2}(\log(n) + < \mathbf{K} >)^{-1})$, then such vector $(\gamma_1, ..., \gamma_n)$ is the needed approximation of the gradient.

   Now $\Gamma_i = \frac{\partial}{\partial x_i} V_{\mathbf{K}}(x_1, ..., x_n) = \frac{1}{(n-1)!} V(A, B, ..., B)$, where the convex sets $A = K_i$ and $B = \sum_{1\leq i\leq n} e^{y_j} K_j$. The randomized algorithm from [13] approximates $V(A, B, ..., B)$ with the complexity $O(n^{4+o(1)} \epsilon^{-(2+o(1)} \log(\delta)$. This gives the needed approximation of the gradient with the complexity $nO(n^{8+o(1)}(\log(n) + < \mathbf{K} >)^{2+o(1)} \log(\delta^{-1}))$.

3. **Controlling the failure probability** δ . We need to approximate $O(n^2 \log(Var(f)))$ values and gradients. To achieve a probability of success $\frac{3}{4}$ we need $\delta \approx \frac{1}{4}(n^2(n^{\frac{5}{2}}(\log(n) + < \mathbf{K} >)))^{-1}$. Which gives $-\log(\delta) \approx O(\log(n) + \log(\log(n) + < \mathbf{K} >))$.

Theorem 3.4: *Given n-tuple* $\mathbf{K}$ *of well-presented convex compact sets in* R^n *there is a poly-time algorithm which computes the number* $AV(\mathbf{K})$ *such that*

$$Prob\{1 \leq \frac{AV(\mathbf{K})}{V(\mathbf{K})} \leq 2 \prod_{1\leq i\leq n} \lambda(i, \min(i, af(i))) \leq 2\frac{n^n}{n!}\} \geq .25$$

The complexity of the algorithm , neglecting the log *terms, is bounded by* $O(n^{10}(\log(n) + < \mathbf{K} >)^2)$.

Next, we focus on the case of the Newton polytopes, in other words ,polytopes with integer vertices. I.e. we will consider the mixed volumes $V_{\mathbf{P}} = V(P_1, ..., P_n)$, where $P_i = CO(v_{i,j} : 1 \leq j \; m(i), v_{i,j} \in Z_+^n$. We define $d(i) = \min\{k : P_i \subset CO(0, e_1, ..., e_n)\}$, i.e. $d(i)$ is the maximum coordinate sum attained on P_i. It follows that $V(P_1, ..., P_n) \leq \prod_{1\leq i\leq k} d(i)$. Such polytopes are well-presented if, for instance, they are given as a least of $poly(n)$ vertices. This case corresponds to the system of sparse polynomial equations. The next theorem is proved in the same way as Theorem (3.4).

Theorem 3.5: *Given n-tuple of* $\mathbf{P} = (P_1, ..., P_n)$ *of well-presented integer polytopes in* R^n *there is a poly-time algorithm which computes the number* $AV(\mathbf{P})$ *such that*

$$Prob\{1 \leq \frac{AV(\mathbf{P})}{V(\mathbf{P})} \leq 2 \prod_{1\leq i\leq n} \lambda(i, \min(i, af(P_i))) \leq 2\frac{n^n}{n!}\} \geq .25$$

The complexity of the algorithm , neglecting the log *terms, is bounded by* $O(n^9(n + \log(\prod_{1\leq i\leq n} d_i)))$.

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