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Polynomial time algorithms to approximate mixed volumes within a simply exponential factor

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Abstract

Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. The Minkowski polynomial $V_{\mathbf{K}}$ is defined as $V_{\mathbf{K}}(x_1, ..., x_n) = V(\lambda_1 K_1 + ... + \lambda_n K_n)$ and the mixed volume $V(K_1, ..., K_n)$ as

$$V(K_1...K_n) = \frac{\partial^n}{\partial\lambda_1...\partial\lambda_n} V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n).$$

In this paper, we study randomized algorithms to approximate the mixed volume of well-presented convex compact sets. Our main result is a polynomial time algorithm which approximates $V(K_1, ..., K_n)$ with a multiplicative error of e^n and with better rates if the affine dimensions of most of the sets K_i are small.

Our approach is based on a particular convex relaxation of $\log(V(K_1, ..., K_n))$ via geometric programming. We prove the mixed volume analogues of the Van der Waerden and the Schrijver/Valiant conjectures for the permanent. These results, though interesting on their own, allow one to "justify" the above mentioned convex relaxation. This relaxation is solved with the ellipsoid method using a randomized polynomial time algorithm for the approximation of the volume of a convex set.

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1 Introduction

Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. It is a well known result by Herman Minkowski (see for instance [5]) that $V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n)$ is a homogeneous polynomial of degree n (called the Minkowski polynomial) in nonnegative variables $\lambda_1...\lambda_n$, where $''+''$ denotes Minkowski sum, and λK denotes the dilatation of K by coefficient λ. The coefficient $V(K_1...K_n)$ of $\lambda_1 \cdot \lambda_2 \ldots \cdot \lambda_n$ is called the *mixed volume* of $K_1, \ldots, K_n$. Equivalently,

$$V(K_1...K_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n).$$

Mixed volume is known to be monotone [5], i.e $K_i \subseteq L_i$, for $i = 1, ..., n$, implies $V(K_1...K_n) \leq V(L_1...L_n)$. In particular, it is always nonnegative and therefore all the coefficients of the Minkowski polynomial $V_{\mathbf{K}}$ are nonnegative real numbers.

The corresponding **Brunn-Minkowski theory**, which is the backbone of convex geometry and its numerous applications, is about various implications of the fact that the functional $(V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n))^{\frac{1}{n}}$ is concave on the nonnegative orthant $R^n_+ = \{(\lambda_1, ...\lambda_n) : \lambda_i \geq 0\}$. Its generalization, **Alexandrov-Fenchel theory**, is based on the fact that the functionals $(\frac{\partial^k}{\partial \lambda_1 ... \partial \lambda_k} V_{\mathbf{K}}(0, .., 0, \lambda_{k+1}, ..., \lambda_n)^{\frac{1}{n-k}}$ are concave on R^{n-k}_+ for all $1 \leq k \leq n-1$.

The problem of computing the mixed volume of convex bodies is also important for combinatorics and algebraic geometry [9]. For instance, the number of toric solutions to a generic system of n polynomial equations on $\mathbf{C}^n$ is equal (and in a general case is upper bounded by) to the mixed volume of the Newton polytopes of the the corresponding polynomials (see for instance [25] and [5]).

1.1 Previous Work

The remarkable connection between mixed volumes and solutions of polynomial equations, called **BKK Theorem**, created an "industry" of computing (exactly) the mixed volume of integer polytopes and its various generalizations, and most of algorithms in the area are of exponential runing time (see [19],[11], [13] and many more). Most of researchers in the "industry" don't bother to formally write down the complexity, they rather describe the actual amount of the computer time. Altough there was a substantial algorithmic activity on the mixed volume of polytopes prior to [9] , the paper [9] was first, to our knowledge, systematic complexity-theoretic study in the area. It followed (naturally) famous FPRAS algorithms [8] for volumes of convex bodies , solved several natural problems and posed many important hard questions.The existence of FPRAS for the mixed volume even for polytopes or ellipsoids is still an open problem.

Efficient polynomial-time probabilistic algorithms that approximate the mixed volume extremely tightly ($(1+\epsilon)$-factor) were developed for some classes of well-presented convex bodies [9]. The algorithms in [9] are based on the multivariate polynomial interpolation and work if and only if the number k of distinct convex sets in the tuple $\mathbf{K}$ is "small", i.e. $k = O(\log(n))$.

The first efficient probabilistic algorithm that provides a $n^{O(n)}$-factor approximation for *arbitrary well-presented proper convex bodies* was obtained by Barvinok in [2]. Barvinok's algorithms start with replacing convex bodies by ellipsoids. This first step already gives $n^{O(n)}$-factor in the worst case. After that the mixed volume of ellipsoids is being approximated with a simply exponential factor c^n by two randomized algorithms, one of which deals with the approximation of the mixed discriminant.

The question of existence of an efficient *deterministic* algorithm for approximating the mixed volume of arbitrary well-presented proper convex bodies with an error depending only on the dimension was posed in [9]. They quote a lower bound (**Barany-Furedi bound**) [1] of $\left(\Omega\left(\frac{n}{\log n}\right)\right)^{\frac{n}{2}}$ for the approximation factor of such an algorithm. (Notice that Barvinok's *randomized* algorithm [2] does not beat the **Barany-Furedi bound**.)

Deterministic polynomial-time algorithms that approximate the mixed volume with a factor of $n^{O(n)}$ were given, for a fixed number of distinct proper convex bodies in $\mathbf{K} = (K_1...K_n)$, in [2], [9]. Finally, a deterministic polynomial-time algorithm that approximates the mixed volume with a factor of $n^{O(n)}$ in the general case of well-presented compact convex sets was given in [15] ,[16]. Let $\mathbf{A} = (A_1, ..., A_n)$ be an n-tuple of $n \times n$ complex matrices; the corresponding determinantal polynomial is defined as $Det_{\mathbf{A}}(\lambda_1, ..., \lambda_n) = \det(\sum_{1 \leq i \leq n} \lambda_i A_i)$. The mixed discriminant is defined as $D(A_1, ..., A_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} Det_{\mathbf{A}}(\lambda_1, ..., \lambda_n)$. Similarly to the randomized algorithm from [2], the algorithm in [15], [16] reduced the approximation of the mixed volume of well-presented compact convex sets to the approximation of the mixed volume of ellipsoids; this first step gives $n^{O(n)}$-factor in the worst case. Next, the mixed volume of ellipsoids is approximated by $(D(A_1, ..., A_n))^{\frac{1}{2}}$ of the corresponding positive semidefinite matrices $A_i \succeq 0$. This second step adds $\sqrt{3}^n$ to the multiplicative approximation error (see inequality (17) below). Finally, the approximation of the mixed discriminant was relaxed by some convex optimization problem (geometric programming). In order to prove the accuracy of the convex relaxation, the author proved in [17] the mixed discriminant analogue of the Van der Waerden conjecture on permanents of doubly stochastic matrices [21], which was posed by R. V. Bapat in [3].

Summarizing: the interpolational approach from [9] is limited by the restriction that the number of distinct convex sets is $O(\log(n))$; the approaches from [2] and [15] ,[16] can't give the simply exponential approximation factor c^n because of the initial approximation of convex sets by ellipsoids.

1.2 Our Approach

Assume, modulo deterministic poly-time preprocessing [9], that the mixed volume $V(K_1...K_n)$ is positive. We define the capacity of the volume polynomial $V_{\mathbf{K}}$ as $Cap(V_{\mathbf{K}}) = \inf_{x_i > 0: 1 \leq i \leq n} \frac{V_{\mathbf{K}}(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i}$. Since the coefficients of the volume polynomial $V_{\mathbf{K}}$ are nonnegative real numbers we get the inequality $\frac{Cap(V_{\mathbf{K}})}{V(K_1...K_n)} \geq 1$. The trick is that $\log(Cap(V_{\mathbf{K}}))$ is a solution of the following convex minimization problem

$$\log(Cap(V_{\mathbf{K}})) = \inf_{y_1 + ... + y_n = 0} \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})). \tag{1}$$

We view $Cap(V_{\mathbf{K}})$ as an approximation for the mixed volume $V(K_1...K_n)$, and to justify this we prove the upper bound $\frac{Cap(V_{\mathbf{K}})}{V(K_1...K_n)} \leq \frac{n^n}{n!} \approx e^n$, which is the mixed volume analogue of the Van der Waerden conjecture. We also present a better upper bounds when "most" of the convex sets K_i have small affine dimension, which are analogues of Schrijver-Valiant conjecture [26], [27].

After establishing this, we present a randomized polynomial time algorithm to solve the problem (1) based on ellipsoid method and randomized polynomial time algorithms for volume approximation. Together with the just-mentioned Van Der Waerden conjecture for mixed volumes, this gives a randomized poly-time algorithm to approximate the mixed volume $V(K_1...K_n)$ within relative accuracy e^n. Notice that, in view of (**Barany-Furedi bound**), this can not be achieved by a deterministic poly-time oracle algorithm. We use the ellipsoid method because of its robustness: we deal essentially with a random oracle which computes $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}))$ with an additive "small" error ϵ; we use this oracle to get an approximation of the gradient of $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}))$.

The idea of our approach is very similar to our treatment of POS-hyperbolic polynomials in [18]. Recall that a homogeneous polynomial $p(x_1, ..., x_n)$ with nonnegative coefficients is called POS-hyperbolic if all the roots of the univariate polynomials $b(t) = p(x_1 - ty_1, ..., x_n - ty_n)$ are real for all real vectors $(x_1, ..., x_n)$ and positive real vectors $(y_1, ..., y_n)$. Not all Minkowski polynomials $V_{\mathbf{K}}$ are POS-hyperbolic: indeed, any univariate polynomial with nonnegative coefficients $S(x) = \sum_{0 \leq i \leq n} \binom{n}{i} a_i x^i; a_i^2 \geq a_{i-1}a_{i-1}, 1 \leq i \leq n-1$ can be represented as $S(x) = V(A + xB)$ for some convex compact subsets (simplexes) $A, B \subset R^n$ [28]. Fortunately, a modification of the induction in [18] works for Minkowski polynomials and presented in the next Section.

2 Van the Waerden conjecture for mixed volumes

*Consult Appendix **A** for the proofs of results in this section.*

Definition 2.1:

1. Let $n \geq k \geq 1$ be two integers. We define the univariate polynomial $sv_{n,k}(x) = 1 + \sum_{1 \leq i \leq k} (\frac{x}{n})^i \binom{n}{i}$. Notice that $sv_{n,n}(x) = (1 + \frac{x}{n})^n$. We define the following, important for what follows, functions :
$$\lambda(n, k) = (\min_{x>0}(\frac{sv_{n,k}(x)}{x}))^{-1} \tag{2}$$

Remark 2.2: It was observed in [18] that
$$\lambda(k, k) = g(k) =: (\frac{k-1}{k})^{k-1}, k \geq 1; \prod_{1 \leq k \leq n} g(k) = \frac{n!}{n^n} \tag{3}$$

The following inequalities are easy to prove :
$$\lambda(n, k) < \lambda(n, l) : n \geq k > l \geq 1; \lambda(m, k) > \lambda(n, k) : n > m \geq k. \tag{4}$$

It follows that
$$\lambda(\infty, k) =: \lim_{n \to \infty} \lambda(n, k) = (\min_{t>0} \frac{\sum_{0 \leq i \leq k} \frac{t^i}{i!}}{t})^{-1} \tag{5}$$

The equality $\lambda(n,2) = (1+\sqrt{2}\sqrt{\frac{n-1}{n}})^{-1} \geq (1+\sqrt{2})^{-1}$ follows from basic calculus. ∎

2. An univariate polynomial with nonnegative coefficients $R(t) = \sum_{0 \leq i \leq m} a_i t^i$ is called $(n-Newton)$, where $n \geq m$ if it satifies the following inequalities :

$$NIs : (\frac{a_i}{\binom{n}{i}})^2 \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} : 1 \leq i \leq m-1 \tag{6}$$

The following weak Newton's inequalities $WNIs$ follow from NIs if the coefficients are nonnegative:

$$WNIs : a_i a_0^{i-1} \leq (\frac{a_1}{n})^i \binom{n}{i} : 2 \leq i \leq m. \tag{7}$$

(Recall that the Newton's inequalities NIs with $n = k$ are satisfied if all the roots of p are real.)

∎

The main mathematical result in this paper is the next theorem :

Theorem 2.3: *Let $\mathbf{K} = (K_1 ... K_n)$ be a n-tuple convex compact subsets in the Euclidean space $\mathbf{R}^n$ and $af(i)$ be the affine dimension of K_i , $1 \leq i \leq n$. Then the following inequality holds :*

$$Cap(V_{\mathbf{K}}) \geq V(K_1, ..., K_n) \geq \prod_{1 \leq i \leq n} \lambda(i, \min(i, af(i))) Cap(V_{\mathbf{K}}) \tag{8}$$

Corollary 2.4:

1. **The mixed volume analogue of the Van der Waerden-Falikman-Egorychev inequality**

$$V(K_1, ..., K_n) \geq \frac{n!}{n^n} Cap(V_{\mathbf{K}}) \tag{9}$$

The equality in (9) is attained if and only if either the mixed volume $V(K_1, ..., K_n) = 0$ or $K_i = a_i K_1 + b_i : a_i > 0, b_i \in R^n; 2 \leq i \leq n$.

2. *Suppose that $af(i) \leq k : k+1 \leq i \leq n$ then*

$$V(K_1, ..., K_n) \geq \frac{k!}{k^k} \lambda(n,k)^{n-k} Cap(V_{\mathbf{K}}). \tag{10}$$

If $k = 2$ we get the inequality $V(K_1, ..., K_n) \geq \frac{1}{2}(1+\sqrt{2})^{2-n} Cap(V_{\mathbf{K}})$.

3. **Reverse Alexandrov-Fenchel Inequality** *Define $C = V(K_1, K_2, K_3, ..., K_n)$, $A = V(K_1, K_1, K_3, ..., K_n)$, $B = V(K_2, K_3, K_3, ..., K_n)$ and $G = C + \sqrt{AB}$.*
(Notice that $G = Cap(R)$, where the polynomial $R(x_1, x_2) = Cx_1x_2 + \frac{A}{2}x_1^2 + \frac{B}{2}x_2^2$.)
Then the next inequality holds:

$$G \geq \prod_{3 \leq i \leq n} \lambda(i, \min(i, af(i))) Cap(V_{\mathbf{K}}) \geq 2\frac{n!}{n^n} Cap(V_{\mathbf{K}}) \tag{11}$$

2.1 Generalized Alexandrov-Fenchel inequalities

Define L_n as a set of all integer vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = n$. For an integer vector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \geq 0, \ \sum \alpha_i = N$$

we define a tuple of convex compact sets

$$\mathbf{K}^{(\alpha)} = (\underbrace{K_1, \ldots, K_1}_{\alpha_1}, \ldots, \underbrace{K_k, \ldots, K_k}_{\alpha_k}, \ldots, \underbrace{K_n, \ldots, K_n}_{\alpha_n})$$

i.e., convex set K_i has α_i entries in $\mathbf{K}^{(\alpha)}$.
We will use the notations $V^{(\alpha)}$ for the mixed discriminant $V(\mathbf{A}^{(\alpha)})$. The next result follows from the concavity of the Minkowski polynomials $\log(V_{\mathbf{K}}(x_1, ..., x_n)) : x_i > 0$ and inequalities (9), (10). Our proof follows directly the proof of Theorem 5.2 in [17].

Theorem 2.5: *Let $\mathbf{K} = (K_1...K_n)$ be a n-tuple convex compact subsets in the Euclidean space $\mathbf{R}^n$ and their affine dimensions $af(i) \leq k \leq n, 1 \leq i \leq n$.*
If vectors α, $\alpha^1, \ldots, \alpha^m$ belong to L_n and

$$\alpha = \sum_{1 \leq i \leq m} \gamma_i \alpha^i; \quad \gamma_i \geq 0\,, \quad \sum \gamma_i = 1$$

then the following inequality holds:

$$V^{(\alpha)} \geq (\frac{k!}{k^k} \lambda(n,k)^{n-k}) \prod_{1 \leq i \leq m} (V^{(\alpha_i)})^{\gamma_i} \geq (\frac{n!}{n^n}) \prod_{1 \leq i \leq m} (V^{(\alpha_i)})^{\gamma_i} \tag{12}$$

Remark 2.6: The inequality (9) is an analoque of the famous Van der Waerden conjecture [21], proved in [12], [10], on the permanent of doubly-stochastic matrices. Indeed , consider the "boxes" $K_i = \{(x_1, ..., x_n) : 0 \leq x_j \leq A(i,j), 1 \leq j \leq n\}$. Then the mixed volume $V(\mathbf{K}) = (V(K_1, ..., K_n) = Perm(A)$; and if the $n \times n$ matrix A is doubly-stochastic then $Cap(V_{\mathbf{K}}) = 1$.
The inequality (10) is an analoque of the Schrijver's lower on the number of perfect matchings in k-regular bipartite graphs: affine dimensions play role of the degrees of vertices.
The reader familiar with [18] can recognize the similarity between inequalities 8, 9, 10 and the corresponding inequalities in [18], proved for POS-hyperbolic polynomials. The method of proof in this paper is also similar to [18], inspite the fact that there exist non POS-hyperbolic Minkowski polynomials $V_{\mathbf{K}}$. But we get the worse constants: for instance ,if $k = 2$,in notations of (10) , then in POS-hyperbolic case one gets the factor 2^{-n+1} instead of $\frac{1}{2}(1+\sqrt{2})^{n-2}$ in this paper. Whether the latter factor is assymptoticaly sharp is an open problem. ∎

3 Convex Optimization Relaxation of the Mixed Volume

Inequalities (8, 9, 10) justify the following strategy for approximation of the mixed volume $V(K_1, ..., K_n)$ within simply exponential multiplicative factor: solve the convex optimization problem 1 with an additive $O(1)$ error. We follow here the approach from [15], [16] which dealt with the following problem:

$$\log(Cap(Det_{\mathbf{A}})) = \inf_{y_1+...+y_n=0} \log(\det(\sum_{1\leq i\leq n} e^{y_i}A_i)) : A_i \succeq 0. \tag{13}$$

The main difference between these problems is that the value and the gradient of determinantal polynomials can be exactly evaluated in deterministic polynomial time. The case of the Minkowski polynomials $V_{\mathbf{K}}$ requires some extra care. We overview below the main points :

3.1 A brief overview

1. **Representations of convex sets and a priori ball for the convex relaxation** : we deal in this paper with two types of representations. First, similarly to [9] we consider *well-presented* convex compact sets; second, motivated by algebraic applications and **BKK** theorem, we consider integer polytopes given as a list of extreme points. In both cases, we start with deterministic poly-time preprocessing which converts the initial tuple $\mathbf{K}$ into a collection of indecomposable tuples. The tuple $\mathbf{K}$ is indecomposable iff the minimum in (1) is attained and unique. After that we deal only with the indecomposable case and get a priori ball which guaranted to contain the unique minimizer of (1). The radius of such ball is expressed in terms of the complexity of the corresponding representation: $r \leq O(n^2(\log(n)+ <\mathbf{K}>))$, where $<\mathbf{K}>$ is the complexity of the initial tuple $\mathbf{K}$.
This part is farly similar to the analogous problem for (13) treated in [16].

2. **Lipschitz Property, Rounding** In the course of our algorithm we need to evaluate the volumes $Vol(e^{y_1}K_1 + ... + e^{y_n}K_n)$ and the mixed volumes $V(K_i, B, ..., B), B = e^{y_1}K_1 + ... + e^{y_n}K_n$. It requires a well-presentations of the Minkowski sum $B = e^{y_1}K_1 + ... + e^{y_n}K_n$. Given the well-presentation of $\mathbf{K}$ one gets well presentation of $a_1K_1 + ... + a_nK_n$ if the sizes of positive rational numbers a_i are bounded by $poly(n, <\mathbf{K}>)$ [9]. Therefore we need a rounding procedure, which requires to keep only "small" number of fractional bits of y_i. It can be done using the Lipschitz property (19) of $Vol(e^{y_1}K_1 + ... + e^{y_n}K_n)$ (just Euler's identity for homogeneous functionals) and its partial derivatives (3.3).

3. **Complexity of our Algorithm**: We give an upper bound on the number of calls to oracles for Minkowski sums $a_1K_1 + ... + a_nK_n$. The number of calls to oracles for the initial tuple $\mathbf{K}$ will be larger but still polynomial, see the discussion in [4] in the context of the surface area computation (which is, up to a constant, the mixed volume $V(Ball(0;1), A, ..., A)$).

4. **Ellipsoid Method with noisy first order oracle** : let $g(.)$ be a differentiable convex functional defined on the closed ball $Ball(0;r) = \{X \in R^n :< X, X >\leq r^2\}$ and $Var(g) = \max_{X\in Ball(0;r)} g(X) - \min_{X\in Ball(0;r)} g(X)$. The standard version of the ellipsoid requires exact values of the function and its gradient. Fortunatelly, there exists a noisy version [22] , which needs approximations of the value $\bar{g}(X)$ and of the gradient $\overline{(\bigtriangledown g)(X)}$ such that

$$sup_{Y,X\in Ball(0;r)}|(\bar{g}(X)+ <\overline{(\bigtriangledown f)}(X), Y-X>) - (g(X)+(\bigtriangledown g)(X), Y-X>)| \leq \delta Var(g).$$

In our case, $g(y_1,...,y_n) = \log(V_{\mathbf{K}}(e^{y_1},...,e^{y_n}))$. We get the additive approximation of $g(y_1,...,y_n)$ using a FPRAS for volume approximation and the additive approximation of $(\bigtriangledown g)(X)$ using FPRAS from [9] for approximating "simple" mixed volume (generalized surface area) $V(K_1, K_2, ..., K_2)$.

3.2 Representations of convex compact sets

. Following [9] we consider the following well-presentation of convex compact set $K_i \subset R^n, 1 \leq i \leq n$: A weak membership oracle for K and a rational $n \times n$ matrix A_i , a rational vector $y_i \in R^n$ such that

$$y_i + A_i(Ball_n) \subset K_i \subset y + n\sqrt{n+1}A_i(Ball_n) \tag{14}$$

Here $Ball_n = \{x \in R^n : ||x||_2 \leq 1\}$ is a standard unit ball in R^n. We define the size $< \mathbf{K} >$ as the maximum of bit sizes of entries of matrices $A_i, 1 \leq i \leq n$. Since the mixed volume $V(K_1,...,K_n) = V(K_1 + \{-y_1\},...,K_n + \{-y_n\}$, we will assume WLOG that $y_i = 0, 1 \leq i \leq n$. This assumption implies that the following identity for affine dimensions

$$af(\sum_{i \in S} K_i) = Rank(\sum i \in SA_iA_I^T), S \subset \{1,...,n\}. \tag{15}$$

Definition 3.1: A n-tuple $\mathbf{K} = (K_1...K_n)$ of convex compact subsets in R^n is called indecomposable if $af(\sum i \in SK_i) > Card(S) : S \subset \{1,...,n\}, 1 \leq Card(S) < n$.
We consider , similarly to [16] , $n(n-1)$ auxiliary n-tuples $\mathbf{K}^{ij}$, where $\mathbf{K}^{ij}$ is obtained from $\mathbf{K}$ by substituting K_i instead of K_j. Notice that

$$V(x_1K_1 + ... + x_nK_n) = x_1x_2...x_n(V(\mathbf{K}) + \frac{1}{2}\sum_{1 \leq i,j \leq n} \frac{x_i}{x_j} V(\mathbf{K}^{ij})) + ... \tag{16}$$

∎

It follows from 15 that the n-tuple $\mathbf{K} = (K_1...K_n)$ of well-presented convex sets is indecomposable iff the n-tuple of positive semidefinite matrices $\mathbf{Q} = (Q_1...Q_n) : Q_i = A_iA_I^T$ is fully indecomposable as defined in [16] , which implies that indecomposability of $\mathbf{K}$ is equivalent to the inequalities $V(\mathbf{K}^{ij})) > 0 : 1 \leq i,j \leq n$. Again , using Theorem 1.9 , Lemma 2.3 , Lemma 2.4 from [16] and "decomposition lemma" from [5] , we can , by deterministic poly-time preproprocessing , to check either the tuple $\mathbf{K}$ is indecomposable or to factor the mixed volume $V(\mathbf{K}) = \prod_{1 \leq j \leq m \leq n} V(\mathbf{K_j})$. Here the $n(j)$-tuple $\mathbf{K_j} = (K_{j,1},...,K_{j,n(j)} \subset R^{n(j)}$ is well presented and indecomposable ; $\sum_{1 \leq j \leq m} n(j) = n$ and the sizes $< \mathbf{K_j} > \leq < \mathbf{K} > +poly(n)$.
Based on the above remarks , we will deal from now on only with indecomposable well-presented tuples of convex compact sets. Moreover , to simplify the exposition , we assume WLOG that the matrices A_i in (14) are integer.
Let $\mathcal{E}_A$ be the ellipsoid $A(Ball_n)$ in R^n. The next inequality , proved in [2], connects the mixed volume of ellipsoids and the corresponding mixed discriminant

$$3^{-\frac{n+1}{2}} v_n D^{\frac{1}{2}}(A_1(A_1)^T,...,A_n(A_n)^T) \leq V(\mathcal{E}_{A_1}...\mathcal{E}_{A_n}) \leq v_n D^{\frac{1}{2}}(A_1(A_1)^T,...,A_n(A_n)^T). \tag{17}$$

Here v_n is the volume of the unit ball in $\mathbf{R}^n$.

3.3 Properties of Volume polynomials : Lipschitz , bound on the second derivative , a priori ball

Proposition 3.2:

1. **Lipschitz Property**.
   Let $p(x_1, ..., x_n)$ be a nonzero homogeneous polynomial of degree n with nonnegative coefficients , $x_i = e^{y_i}$. Then

$$\frac{\partial}{\partial y_i} \log(p(e^{y_1}, ..., e^{y_n})) = \frac{\frac{\partial}{\partial x_i} p(x_1, ..., x_n) e^{y_i}}{p(x_1, ..., x_n)} \tag{18}$$

   It follows from the Eyler's identity that $\sum_{1 \leq i \leq n} \frac{\partial}{\partial y_i} \log(p(e^{y_1}, ..., e^{y_n})) = n$, therefore the functional $f(y_1, ..., y_n) = \log(p(e^{y_1}, ..., e^{y_n}))$ is Lipschitz on R^n :

$$|f(y_1 + \delta_1, ..., y_n + \delta_n) - f(y_1, ..., y_n)| \leq n||\Delta||_\infty \leq n||\Delta||_2 \tag{19}$$

2. **Upper bound on second derivatives**.
   Let us fix real numbers $y_1, ..., y_{i-1}, y_{i+1}, ..., y_n$ and define univariate function $q(y_i) = \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_i}, ..., e^{y_n}))$.
   Notice that $e^{q(y_i)} = \sum_{0 \leq j \leq af(K_i)} a_j e^{jy}, a_j \geq 0$.

   ***Proposition 3.3*:**

$$0 \leq q''(y) \leq af(K_i) \tag{20}$$

   *. (Lemma(B.3) in Appendix **B** proves more general inequality.)*

3. **A Priori Ball result from [16]**.
   Let $p \in Hom_+(n, n)$, $p(x_1, ..., x_n) = x_1 x_2 ... x_n (a + \frac{1}{2} \sum_{1 \leq i \neq j \leq n} b^{i,j} \frac{x_i}{x_j}) + ...$
   Assume that $min_{1 \leq i \neq j \leq n} b^{i,j} = Stf(p) > 0$. Then there exists an unique minimizer $(z_1, ..., z_n) = Argmin(\log(p(e^{y_1}, ..., y_1)) : \sum_{1 \leq i \leq n} y_i = 0$. Moreover ,

$$|z_i - z_j| \leq \log(\frac{2Cap(p)}{Stf(p)}) \tag{21}$$

The next proposition adapts Lemma 4.1 from [16] to the "mixed volume situation" , using the Barvinok's inequality (17).

Proposition 3.4: *Consider an indecomposable n-tuple of convex compact sets $\mathbf{K} = (K_1, .., K_n)$ with the well-presentation $A_i(Ball_n) \subset K_i \subset y + n\sqrt{n+1} A_i(Ball_n), 1 \leq i \leq n$ with integer $n \times n$ matrices A_i. Then the minimum in the convex optimization problem (1) is attained and unique. The unique minimizer vector*
$(z_1, ..., z_n), \sum_{1 \leq i \leq n} z_i = 0$ satisfies the following inequalities

$$|z_i - z_j| \leq O(n^{\frac{3}{2}}(\log(n) + < \mathbf{K} >)); ||z_1, ..., z_n)||_2 \leq O(n^2(\log(n) + < \mathbf{K} >)) \tag{22}$$

In other words the convex optimization problem (1) can be solved on the following ball in R^{n-1}:

$$Apr(\mathbf{K}) = \{(z_1, ..., z_n) : ||z_1, ..., z_n)||_2 \leq O(n^2(\log(n) + < \mathbf{K} >)), \sum_{1 \leq i \leq n} z_i = 0\} \tag{23}$$

The following inequality follows from the Lipschitz property (19):

$$|\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}) - \log(V_{\mathbf{K}}(e^{l_1}, ..., e^{l_n})| \leq O(n^3(\log(n) + < \mathbf{K} >)) : Y, L \in Apr(\mathbf{K}). \tag{24}$$

3.4 Ellipsoid method with noisy first order oracles

. We recall the following fundamental result [22]:
Let $f(Y)$ be differentiable convex functional defined on the ball $Ball_n(r) = \{Y \in R^n :< Y, Y > \leq r^2\}$ of radius r. Let $Var(f) = \max_{Y \in Ball_n(r)} f(Y) - \min_{Y \in Ball_n(r)} f(Y)$. Assume that at each vector $Y \in Ball_n(r)$ we have an oracle evaluating a value $g(Y)$ such that $|g(Y) - f(Y)| \leq 0.2\delta Var(f)$ and the vector $gr(Y) \in R^n$ such that $||gr(Y) - (\bigtriangledown f)(Y)||_2 \leq 0.2\delta r^{-1} Var(f)$ (here $(\bigtriangledown f)(Y)$ is the gradient of f evaluated at Y).
Then the **Ellipsoid method** *finds a vector $Z \in Ball_n(r)$ such that $f(Z) \leq \min_{Y \in Ball_n(r)} f(Y) + \epsilon Var(f), \epsilon > \delta$. The method requires $O(n^2 \log(\frac{1}{\epsilon - \delta}))$ oracle calls plus $O(n^2)$ elementary operations to run the algorithm itself.*

3.5 Putting things together

We take advantage here of randomized algorithms which can evaluate $\log(Vol(K))$,for a well-presented convex set K, with an additive error ϵ and failure probability δ in $O(\epsilon^{-k} n^l \log(\frac{1}{\delta}))$ oracle calls. For instance , the best current algorithm [20] gives $k = 2, l = 4$. We will need below to evaluate volumes $V(\sum_{1 \leq i \leq n} x_i K_i)$. In our case the functional $f = \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})$ defined on ball $Apr(\mathbf{K})$ of radius $O(n^2(\log(n) + < \mathbf{K} >))$ with the variance $Var(f) \leq O(n^3(\log(n) + < \mathbf{K} >))$. Theorem 2.3 gives the bound :
$\log(V(\mathbf{K})) \leq (\min_{Y \in Apr(\mathbf{K})} f(Y) \leq \log(V(\mathbf{K})) + \log(\frac{n^n}{n!}) \approx \log(V(\mathbf{K}) + n$.
Therefore , to get e^n approximation of the mixed volume $V(\mathbf{K}$ it is sufficient to find out $Z \in Apr(\mathbf{K})$ such that $f(Z) \leq \min_{Y \in Apr(\mathbf{K})} f(Y) + O(1)$. In order to get that via the Ellipsoid method we need to approximate $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}))$ with the additive error $O(Var(f)^{-1}) = O(n^{-3}(\log(n) + < \mathbf{K} >)^{-1})$ and its gradient with the additive l_2 error $O(n^{-2}(\log(n) + < \mathbf{K} >)^{-1})$.

1. **Approximation of** $\log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n}))$ **with failure probability** δ. The complexity is $O(n^{10}(\log(n) + < \mathbf{K} >)^2 \log(\delta^{-1}))$

2. **Approximation of the partial derivatives**. Let $x_i = e^{y_i}$ and recall that the partial derivatives
$$\beta_i = \frac{\partial}{\partial y_i} \log(V_{\mathbf{K}}(e^{y_1}, ..., e^{y_n})) = \frac{\frac{\partial}{\partial x_i} V_{\mathbf{K}}(x_1, ..., x_n) e^{y_i}}{V_{\mathbf{K}}(x_1, ..., x_n)}$$
Suppose that $0 \leq 1 - a \leq \frac{\gamma_i}{\beta_i} \leq 1 + a$. It follows from the Eyler's identity that $\sum_{1 \leq i \leq n} |\gamma_i - \beta_i| \leq a$. If $a = O(n^{-2}(\log(n) + < \mathbf{K} >)^{-1})$, then such vector $(\gamma_1, ..., \gamma_n)$ is the needed

approximation of the gradient.
Notice that $\Gamma_i = \frac{\partial}{\partial x_i} V_{\mathbf{K}}(x_1,...,x_n) = \frac{1}{(n-1)!} V(A,B,...,B)$, where the convex sets $A = K_i$ and $B = \sum_{1 \leq i \leq n} e^{y_j} K_j$. The randomized algorithm from [9] approximates $V(A,B,...,B)$ with the complexity $O(n^{4+o(1)} \epsilon^{-(2+o(1)} \log(\delta)$. This gives the needed approximation of the gradient with the complexity $nO(n^{8+o(1)}(\log(n)+ < \mathbf{K} >)^{2+o(1)} \log(\delta^{-1}))$.

3. **Controlling the failure probability** δ . We need to approximate $O(n^2 \log(Var(f)))$ values and gradients. To achieve a probability of success $\frac{3}{4}$ we need $\delta \approx \frac{1}{4}(n^2(n^{\frac{5}{2}}(\log(n)+ < \mathbf{K} >)))^{-1}$. Which gives $\log((\delta)^{-1}) \approx O(\log(n) + \log(\log(n)+ < \mathbf{K} >))$.

Theorem 3.5: *Given n-tuple* $\mathbf{K}$ *of well-presented convex compact sets in* R^n *there is a poly-time algorithm which computes the number* $AV(\mathbf{K})$ *such that*

$$Prob\{1 \leq \frac{AV(\mathbf{K})}{V(\mathbf{K})} \leq 2 \prod_{1 \leq i \leq n} \lambda(i, \min(i, af(i))) \leq 2\frac{n^n}{n!}\} \geq .25$$

The complexity of the algorithm , neglecting the log *terms, is bounded by* $O(n^{12}(\log(n)+ < \mathbf{K} >)^2)$.

Next, we focus on the case of the Newton polytopes, in other words ,polytopes with integer vertices. I.e. we will consider the mixed volumes $V(\mathbf{P}) = V(P_1,...,P_n)$, where $P_i = CO(v_{i,j} : 1 \leq j \ m(i), v_{i,j} \in Z^n_+$. We define $d(i) = \min\{k : P_i \subset kCO(0, e_1,...,e_n)\}$, i.e. $d(i)$ is the maximum coordinate sum attained on P_i. It follows from the monotonicity of the mixed volume that $V(P_1,...,P_n) \leq \prod_{1 \leq i \leq k} d(i)$. Such polytopes are well-presented if, for instance, they are given as a list of $poly(n)$ vertices. This case corresponds to the system of sparse polynomial equations. Notice that the value $V(P_1,...,P_n)$ is either zero or integer number (**BKK** Theorem) and the capacity $Cap(V_{\mathbf{P}} \leq \frac{n^n}{n!} \prod_{1 \leq i \leq k} d(i)$ (inequality (9)).
The next theorem is proved in the same way as Theorem (3.5).

Theorem 3.6: *Given n-tuple of* $\mathbf{P} = (P_1,...,P_n)$ *of well-presented integer polytopes in* R^n *there is a poly-time algorithm which computes the number* $AV(\mathbf{P})$ *such that*

$$Prob\{1 \leq \frac{AV(\mathbf{P})}{V(\mathbf{P})} \leq 2 \prod_{1 \leq i \leq n} \lambda(i, \min(i, af(P_i))) \leq 2\frac{n^n}{n!}\} \geq .25$$

The complexity of the algorithm , neglecting the log *terms, is bounded by* $O(n^9(n+\log(\prod_{1 \leq i \leq n} d_i))^2)$.

4 Conclusion and Further Work

1. This paper provides a new approach to approximation of the mixed volume with nontrivial improvement of previous results. The practicality of our approach is limited by two factors: **A**: e^n is rather large, **B**: the worst case complexity is rather high.

   Addressing **A**, it is not impossible that for random convex sets/random integer polytopes the approximation is much better. Consider,for instance, "simple" zonotopes $K_i = \{aX_i :$

$0 \le a \le 1\} + \{aY_i : 0 \le a \le 1\} : X_i, Y_i \in R^n, 1 \le i \le n$. If vectors X_i form the standard basis, i.e. $X_i = e_i$, then the mixed volume $V(K_1, ..., K_n) = \sum_{S \subset \{1,...,n\}} |\det A_{S,S}|$, where $n \times n$ matrix $A = [Y_1, ..., Y_n]$. This is already $NP - HARD$ problem (and most likely $\#P - COMPLETE$): all principal minors $\det A_{S,S}$ are nonnegative (P-property) iff $V(K_1, ..., K_n) = \det(I + A)$ and checking P-property is NP-COMPLETE [6]. On the other hand $D^{\frac{1}{2}} \le V(K_1, ..., K_n) \le \sqrt{2}^n D^{\frac{1}{2}}$. The number D is the mixed discriminant of rank two PSD matrices: $D = D(X_1X_1^T + Y_1Y_1^T, ..., X_nX_n^T) + Y_nY_n^T)$. Such mixed discriminant can be approximated in deterministic poly-time within a factor $\frac{2^n}{n^m}$ for any integer m [18], which gives a deterministic algorithm to approximate $V(K_1, ..., K_n)$ with the same rate. Computer simulations by Alex Olshevsky, using Algorithm (32) from Appendix **B**, suggest approximation within $O(1)$ on random integer vectors X_i, Y_i.

Addressing **B**: the optimization algorithm should be improved; one can try black box oriented randomized algorithms like center of gravity and its modifications. In the case of integer polytopes, computing Minkowski sums boils down to linear programming. Again it should be done in the most efficient way.

But what would be really deep and interesting is to come up with new efficiently solvable convex relaxations of the mixed volume and the mixed discriminant.

2. An important open problem is regarding the assymptotic sharpness of our generalization of the Schrijver-Gurvits bound (10).

3. The proofs in "mathematical" part of this paper (Section 2 and Appendix **A**) are, we hope, quite transparent if not simple. This simplicity had become possible after the inductive approach, first introduced in [18]. Still, we use the very nontrivial *Alexandrov-Fenchel Inequalities*. Is there a *"Brunn-Minkowski"* based proof of results in Section 2?

4. The most important question is whether exists a FPRAS algorithm for the mixed volume (or for the mixed discriminant).

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A Proof of Theorem 2.3 and its corollaries

The next definition is adapted from [18].

Definition A.1:

1. Let $p \in Hom_+(n,m), p(x_1,...,x_n) = \sum_{(r_1,...,r_n) \in I_{n,m}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be a homogeneous polynomial with nonnegative real coefficients of degree m in n variables. Here $I_{n,m}$ stands for the set of vectors $r = (r_1,...,r_n)$ with nonnegative integer components and $\sum_{1 \leq i \leq n} r_i = m$.

   The support of the polynomial p as above is defined as $supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p.
   For a subset $A \subset \{1,...,n\}$ we define $S_p(A) = \max_{(r_1,...,r_n) \in supp(p)} \sum_{i \in A} r_i$. *(If $p = V_{\mathbf{K}}$ then $S_p(A)$ is equal to the affine dimension of the Minkowski sum $\sum_{i \in A} K_i$.)*

   The following linear differential operator maps $Hom(n,n)$ onto $Hom(n-1,n-1)$:

   $$p_{x_1}(x_2,...,x_n) = \frac{\partial}{\partial x_1} p(0,x_2,...,x_n).$$

   We define $p_{x_i}, 2 \leq i \leq n$ in the same way . We will use the notation $p_{i_1,...,i_k}$ for the composition $((p_{x_{i_1}})_{x_{i_2}}...)_{x_{i_k}}$. Notice that the operator $p_{i_1,...,i_k}$ maps $H(n,n)$ onto $Hom(n-k,n-k)$.

   The following inequality follows straight from the definition :

   $$S_{p_{x_1}}(A) \leq \min(n-1, S_p(A)) : A \subset \{2,...,n\}, p \in Hom_+(n,n). \quad (25)$$

2. A homogeneous polynomial $p \in Hom_+(n,m)$ is called $Newton$ if for all vectors $X, Y \in R^n_+$ the polynomial $R(t) = p(tX + Y)$ is $(m - Newton)$. A notion of the $Weak - Newton$ is defined analogously.
   Finally, a homogeneous polynomial $p \in Hom_+(n,n)$ is called AF-polynomial if the polynomials $p_{i_1,...,i_k}$ are $Newton$ for all $1 \leq i_1 < ... < i_k \leq n$.

3. We define **Capacity** of a homogeneous polynomial $p \in Hom_+(n,n)$ as

$$Cap(p) = \inf_{x_i > 0 : 1 \leq i \leq n} \frac{p(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i} \tag{26}$$

∎

We recall here the fundamental Alexandrov-Fenchel inequality for the mixed volume of n convex sets in R^n :

$$V(K_1, K_2, K_3, ..., K_n)^2 \geq V(K_1, K_1, K_3, ..., K_n) V(K_2, K_2, K_3, ..., K_n) \tag{27}$$

Using our definition, the inequality (27) can be stated in the following equivalent form (implicit in [5]) :

Proposition A.2: *Let* $\mathbf{K} = (K_1 ... K_n)$ *be a* n*-tuple of convex compact subsets* R^n *and* $V_{\mathbf{K}}$ *is the corresponding Minkowski polynomial. Then for all* $1 \leq i_1, ..., i_k \leq n, k \leq n-1$ *the polynomials* $(V_{\mathbf{K}})_{i_1,...,i_k}$ *are* $Newton$*. Or , in other words , the Minkowski polynomials* $V_{\mathbf{K}}$ *are* AF*-polynomials.*

We need the next (elementary) result:

Lemma A.3:

1. *Let* $R(t) = \sum_{0 \leq i \leq k} a_i t^i$ *be* $(n - Newton)$ *polynomial,* $n \geq k$*. Then*

$$a_1 = R'(0) \geq \lambda(n,k) \inf_{t > 0} \frac{R(t)}{t} \tag{28}$$

   The inequality (28) is attained if and only if $R(t) = R(0)(1 + \sum_{1 \leq i \leq k} (\frac{t}{n})^i \binom{n}{i})$*. If* $n = k$ *, it attained iff* $R(t) = R(0)(1 + \frac{t}{n})^n$*.*

2. *Let* $p \in Hom_+(n,n)$ *be Newton polynomial of degree* n *in* $n \geq 2$ *variables. Then* $Cap(p_{x_i}) \geq \lambda(n, S_p(\{i\})) Cap(p)$*.*

Proof: The first part of Lemma A.3 is a minor modification of Lemma 2.7 in [18].
To prove the second part, let us consider WLOG the case $i = 1$ and fix $n - 1$ positive numbers $x_2, .., x_{n-1}$ such that $\prod_{2 \leq i \leq n} x_i = 1$. Then $p(t, x_2, ..., x_n) = R(t) = \sum_{0 \leq i \leq k} a_i t^i$; the degree k of R is equal to $S_p(\{1\})$ and $R'(0) = a_1 = p_{x_1}(x_2, ..., x_n)$. Since the polynomial p is $Newton$ hence the univariate polynomial r is $(n - Newton)$. It follows from the definition (26) of the **Capacity** that $R(t) \geq Cap(p) t x_2, .., x_{n-1} = Cap(p)$. Therefore, using inequality (28) ,we get

that
$p_{x_1}(x_2,...,x_n) = R^{'}(0) \geq \lambda(n,k)Cap(p)$ or equivalently that $Cap(p_{x_1}) \geq \lambda(n, S_p(\{1\}))Cap(p)$.
∎

The next theorem follows from Lemma A.3 by a direct induction , using the inequalities (25) , (4) and the fact $Cap(L_a) = a$, where the univariate polynomial $L_a(x) = ax, a \geq 0$.

Theorem A.4: *Let $p \in Hom_+(n,n)$ be AF polynomial. Then*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \geq \prod_{1 \leq i \leq n} \lambda(i, \min(i, S_p(\{i\}))Cap(p) \geq \prod_{1 \leq i \leq n} \lambda(i,i)Cap(p) = \frac{n!}{n^n} Cap(p). \tag{29}$$

Let $V_{\mathbf{K}}(x_1,...,x_n)$ be Minkowski polynomial. Then $S_{V_{\mathbf{K}}}(A) = af(\sum_{i \in A} K_i)$. Also $V_{\mathbf{K}}$ is AF-polynomial. Therefore , Theorem (2.3) and Corollary (**??**) follow from Theorem (A.4).

We sketch below the uniqueness part of the first part of Corollary (2.4).
Proof: (Proof of the uniqueness part of the Van der Waerden conjecture for the mixed volume)

1. Assume that $Cap(V_{\mathbf{K}}) > 0$. If there is equality in (9) then necessary the affine dimensions $af(K_i) = n, 1 \leq i \leq n$. This fact implies that all coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive.

2. **Scaling**.
As all coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive, hence there exist unique positive numbers $a_1,...,a_n$ such that the scaled polynomial $p = V_{\{a_1 K_1,...,a_n K_n\}}$ is doubly stochastic (see [16]) : $\frac{\partial}{\partial x_i} p(1,1,...,1) = 1, 1 \leq i \leq n$. We will deal , without loss of generality, only with this doubly stochastic case.

3. **Brunn-Minkowski**.
Let $(z_2,...,z_n))$ be the unique minimizer of the problem
$\min_{x_i > 0, 2 \leq i \leq n; \prod_{2 \leq i \leq n} x_i = 1} p_{x_1}(x_2,...,x_n)$. Such unique minimizer exists for all the coefficients of p_{x_1} are positive. It follows from the proof of second part of Lemma A.3 , that $V_{\mathbf{K}}(t, z_2, ..., z_n) = (at+b)^n$ for some positive numbers a, b. It follows from the equality case of the Brunn-Minkowski inequality [5] that $K_1 = \sum_{2 \leq j \leq n} A(1,j)K_j + \{T_1\}$, where $A_{1,j} > 0$ and $T_1 \in R^n$. In the same way , we get that there exist a $n \times n$ matrix A , with the zero diagonal and positive off-diagonal part , and vectors $T_1,...,T_n \in R^n$ such that $K_i = \sum_{j \neq i} A(1,j)K_j + \{T_i\}$. It follows from the doubly-stochasticity that all row sums of the matrix A are equal to one.

4. Associate with the convex compact set $K_i \subset R^n$ its support function
$\gamma_i(X) = \max_{Y \in K_i} < X, Y >, X \in R^n$. We get that
$\gamma_i(X) = \sum_{j \neq i} A(1,j)\gamma_j(X) + < X, T_i >, X \in R^n$.
As the kernel $\{Y \in R^n : Y = AY\} = \{c(1,1,...,1), c \in R\}$, we get finally that
$\gamma_i(X) = \alpha(X) + < X, L_j >, X \in R^n$ for some functional $\alpha(X)$ and vectors $L_1,...,L_n \in R^n$. Which means , in the doubly-stochastic case , that $K_i = K_1 + \{L_i - L_i\}, 2 \leq i \leq n$.

∎

B Inequalities for Minkowski (like) Polynomials

Recall that $\sqrt[n]{V_{\mathbf{K}}(x_1, ..., x_n)}$ is concave on R^n_{++}.
Let $f : R^n_{++} \rightarrow R_{++}$ be differentiable positive valued functional defined on the strictly positive orthant R^n_{++}. We assume that f is n-homogenewous, i.e. that $f(ax_1, ..., ax_n) = a^n f(x_1, ..., x_n)$ and the partial derivatives $\frac{\partial}{\partial x_i} f(x_1, ..., x_n) > 0 : (x_1, ..., x_n) \in R^n_{++}$. We denote a set of such homogeneous functional as $PoH(n)$.

We define the capacity as

$$Cap(f) = \inf_{x_i > 0} \frac{f(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i} = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} f(x_1, ..., x_n).$$

We define two subsets of $PoH(n)$:
$Cav(n)$ - consisting of $f \in PoH(n)$ such that $f^{\frac{1}{n}}$ is concave on all half-lines $\{X + tY : t \geq 0\} : X, Y \in R^n_{++}$; $Vex(n)$ - consisting of $f \in PoH(n)$ such that $f^{\frac{1}{n}}$ is convex on all half-lines $\{X + tY : t \geq 0\} : X, Y \in R^n_{++}$.

Recall Brunn-Minkowski theorem : the Minkowski polynomial $V_{\mathbf{K}}(x_1, ..., x_n) \in Cav(n)$. Therefore the results in this Appendix apply to the Minkowski polynomials.

We also define the following Generalized Sinkhorn Scaling :

$$SH(x_1, ..., x_n) = (y_1, ..., y_n) : y_i = \frac{f(x_1, ..., x_n)}{\frac{\partial}{\partial x_i} f(x_1, ..., x_n)} = \frac{x_i}{\gamma_i}, \gamma_i = \frac{x_i \frac{\partial}{\partial x_i} f(x_1, ..., x_n)}{f(x_1, ..., x_n)}.$$

Theorem B.1: *If $f \in Cav(n)$ then the following inequality holds:*

$$f(SH(x_1, ..., x_n)) \leq f(x_1, ..., x_n) \tag{30}$$

If $f \in Vex(n)$ then the reverse inequality holds:

$$f(SH(x_1, ..., x_n)) \geq f(x_1, ..., x_n) \tag{31}$$

Proof: Let $X = (x_1, ..., x_n) \in R^n_{++}$ and $Y = SH(x_1, ..., x_n)$. We can assume without loss of generality that $f(X) = 1$. If $f \in Cav(n)$ then the univariate function $g(t) = (f(X + tY))^{\frac{1}{n}}$ is concave for $t \geq 0$. Therefore

$$g(t) \leq (g(0) + \frac{g'(0)}{n} t)^n = (1 + \frac{g'(0)}{n} t)^n.$$

We get, by the standard calculus, that

$$g'(0) = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial x_i} f(x_1, ..., x_n)) y_i = n.$$

Since the functional f is n-homogeneous, hence

$g(t) = t^n f(Y + t^{-1}X) \leq (1+t)^n$, and finally
$f(Y + t^{-1}X) \leq (\frac{1+t}{t})^n$.
Taking limit $t \to \infty$ we get $f(SH(x_1, ..., x_n)) \leq 1 = f(x_1, ..., x_n)$.
The convex case is proved in the very same way. ∎

Theorem (B.1) suggest the following algorithm to approximate $Cap(f)$:

$$X_{n+1} = Nor(SH(X_n)) : Nor(x_1, ..., x_n) = (\frac{x_1}{a}, ..., \frac{x_n}{a}), a = \sqrt[n]{(\prod_{1 \leq i \leq n} x_i)}. \tag{32}$$

Corollary B.2 : *Consider $f \in Cav(n)$. Supposed that $Cap(f) > 0$ and $\log(Cap(f)) \leq \log(f(x_1, ..., x_n)) \leq \log(Cap(f)) + \epsilon$; $\epsilon \leq \frac{1}{10}$ and $\prod_{1 \leq i \leq n} x_i = 1$. Then*

$$\sum_{1 \leq i \leq n} (1 - \frac{x_i \frac{\partial}{\partial x_i} f(x_1, ..., x_n)}{f(x_1, ..., x_n)})^2 \leq 10\epsilon \tag{33}$$

Proof: Let $\gamma_i = \frac{x_i \frac{\partial}{\partial x_i} f(x_1, ..., x_n)}{f(x_1, ..., x_n)}$. It follows from the Eyler's identity that $\sum_{1 \leq i \leq n} \gamma_i = n$ and thus $\log(\prod_{1 \leq i \leq n} \gamma_i) \leq 0$.
Inequality (30) can be rewritten as

$$f(\frac{x_1}{\gamma_1}, ..., \frac{x_n}{\gamma_n}) \leq f(x_1, ..., x_n).$$

Therefore $\log(Cap(f)) \leq Cap(f) + \epsilon + \log(\prod_{1 \leq i \leq n} \gamma_i)$. Which gives the inequality

$$-\epsilon \leq \log(\prod_{1 \leq i \leq n} \gamma_i) \leq 0.$$

And finally : $\sum_{1 \leq i \leq n} (1 - \gamma_i)^2 \leq 10\epsilon$ ∎

Corollary (B.2) generalizes (with much trasparent proof) corresponding results from [16] and [23].

The next Lemma (B.3) proves inequality (20).

Lemma B.3:

1. *Let $p(t) = \sum_{0 \leq i \leq n} a_i t^i, a_i \geq 0$ be a polynomial with nonnegative coefficients. Assume that $\log(p(t))$ is concave on R_{++} and define $q(x) = \log(p(e^x))$. Then $q(x)$ is convex on R and its second derivative satisfies the next inequality*

$$0 \leq q''(x) \leq n \tag{34}$$

2. *Let $p(t) = \sum_{0 \leq i \leq n} a_i t^i, a_i \geq 0$ be a polynomial with nonnegative coefficients. Assume that $p(t)^{\frac{1}{m}}, m \geq n$ is concave on R_+. Then*

$$0 \leq q''(x) \leq f(n, m), \tag{35}$$

where $f(n, m) = n - \frac{n^2}{m}$ if $n \leq \frac{m}{2}$; and $f(n, m) = \frac{m}{4}$ otherwise. If $n = m$ then $f(n, m) = \frac{n}{4}$ and the upper bound (35) is attained on polynomials $p(t) = (a + tb)^n; a, b > 0$.

Proof:

1. The convexity of $q(x)$ is well known. The concavity of $\log(p(t))$ is equivalent to the inequality $(p'(t))^2 \geq p(t)p''(t) : t \geq 0$.
   Putting $y = e^x$, we get that
   $$q''(x) = \frac{p''(y)y^2}{p(y)} + \frac{p'(y)y}{p(y)} - (\frac{p'(y)y}{p(y)})^2$$
   The concavity of $\log(p(t))$ gives that $\frac{p''(y)y^2}{p(y)} - (\frac{p'(y)y}{p(y)})^2 \leq 0$.
   Since the coefficients of the polynomial p are nonnegative hence $\frac{p'(y)y}{p(y)} \leq n$. This last observation proves that
   $$q''(x) \leq \frac{p'(y)y}{p(y)} \leq n.$$
2. Our proof of (35) is a direct adaptation of the above proof of (34), we use the following characterization of the concavity of $p(t)^{\frac{1}{m}}, m \geq n$:
   $(p'(t))^2 \geq \frac{m}{m-1}p(t)p''(t) : t \geq 0$.

∎

Just the nonnegativity of the coefficients implies the quadratic bound $q''(x) \leq .25n^2$.

C A note on the Newton Inequalities

We give here a simple convex-volume proof of the next result.

Proposition C.1: *Let $p(t), q(t)$ be two polynomials with nonnegative coefficients. Then the following statements hold:*

1. *If p is n-Newton and q is m-Newton then pq is $n+m$-Newton.*
2. *If p is n-Newton then $\sqrt[n]{p(t)}$ is concave on R_+.*
3. *If p is n-Newton then $p(t+a)$ is also n-Newton for all $a \geq 0$.*

The first part of Proposition (C.1) is a slight generalization of one Davenport-Polya result [7].

Proof: Let p is n-Newton and q is m-Newton. It follows from Sheppard's theorem [28] that there exists convex sets $K_1, K_2 \subset R^n$ and $C_1, C_2 \subset R^n$ such that $p(t) = Vol_n(K_1 + tK_2)$ and $q(t) = Vol_m(C_1 + tC_2)$. Define two convex sets in R^{n+m}: $Z_1 = K_1 \oplus C_1$ and $Z_2 = K_2 \oplus C_2$. Here the direct sum $K_1 \oplus C_1 = \{(X, Y) : X \in K_1, Y \in K_2\}$.
Since $Vol_{n+m}(A \oplus B) = Vol_n(A)Vol_m(B)$ hence $Vol_{n+m}(Z_1 + tZ_2) = p(t)q(t)$. It follows from Alexandrov-Fenchel Inequalities that the polynomial pq is $n+m$-Newton.
The other two statements are proved following similar application of Sheppard's theorem. ∎

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