Electronic Colloquium on Computational Complexity, Comment 1 on Report No. 46 (2007)

Comments on "An Exponential Time/Space Speedup For Resolution"

Jakob Nordström

Royal Institute of Technology (KTH)
SE-100 44 Stockholm, Sweden
jakobn@kth.se

September 28, 2007

Abstract

We present a greatly simplified proof of the length-space trade-off result for resolution in Hertel and Pitassi (2007), and point out two important ingredients needed for our proof to work. Our key trick is to look at formulas of the type $F = G \wedge H$, where G and H are over disjoint sets of variables and have very different length-space properties with respect to resolution. This trick is not present in the proof of Hertel and Pitassi, and thus their techniques can likely be used to prove results not obtainable by our methods.

In these notes, we present a simplification of the length-space trade-off result for resolution in [4, 5]. The simplified proof is given in Section 1. In Section 2, we highlight the two building blocks in our construction. Definitions, notation and some known facts are given in Appendix A for reference.

In a separate paper [6], we elaborate on our ideas in somewhat more detail, and prove a couple of other trade-off results for resolution of a similar flavour.

1 A Simplified Proof of the Length-Space Trade-off

Using the notation of Appendix A, the length-variable space trade-off theorem of Hertel and Pitassi can be expressed as follows.

Theorem 1.1 ([4]). *There is a family of CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\Theta(n^3)$ such that:*

- *The minimal variable space of refuting F_n in resolution is $\mathit{VarSp}(F_n \vdash 0) = \Theta(n)$.*
- *Any resolution refutation $\pi : F_n \vdash 0$ in minimal variable space has length $\exp(\Omega(n))$.*
- *Adding just 3 extra units of storage, it is possible to obtain a resolution refutation π' in variable space $\mathit{VarSp}(\pi') = \mathit{VarSp}(F_n \vdash 0) + 3 = \Theta(n)$ and length $L(\pi') = \mathrm{O}(n^3)$, i.e., linear in the formula size.*

Our simplified proof follows. We note that the CNF formulas used by Hertel and Pitassi, as well as those in our proof, have clauses of width $\Theta(n)$.

Proof of Theorem 1.1. Let G_n be CNF formulas as in Theorem A.2 having size $\Theta(n)$, refutation length $L(G_n \vdash 0) = \exp(\Omega(n))$ and refutation clause space $\mathit{Sp}(G_n \vdash 0) = \Theta(n)$. Let us define $g(n) = \mathit{VarSp}(G_n \vdash 0)$ to be the refutation variable space. We know that $\Omega(n) = g(n) = \mathrm{O}(n^2)$.

Let H_m be the formulas

$$H_m = y_1 \wedge \dots \wedge y_m \wedge (\overline{y}_1 \vee \dots \vee \overline{y}_m) \ . \tag{1}$$

It is not hard to see that $L(H_m \vdash 0) = 2m + 1$ and $\mathit{VarSp}(H_m \vdash 0) = 2m$ (the first derivation step must download the wide clause and some unit clause, and then resolve).

ISSN 1433-8092

Now define

$$F_n = G_n \wedge H_{\lfloor g(n)/2 \rfloor + 1} \tag{2}$$

where G_n and $H_{\lfloor g(n)/2 \rfloor + 1}$ have disjoint sets of variables. By Observation A.3, any resolution refutation of F_n refutes either G_n or $H_{\lfloor g(n)/2 \rfloor + 1}$. We have

$$\mathit{VarSp}\big(H_{\lfloor g(n)/2 \rfloor + 1} \vdash 0\big) = 2 \cdot (\lfloor g(n)/2 \rfloor + 1) > g(n) = \mathit{VarSp}(G_n \vdash 0) \ , \tag{3}$$

so a resolution refutation in minimal variable space must refute G_n in length $\exp(\Omega(n))$. However, allowing at most two more literals in memory, the resolution refutation can disprove the formula $H_{\lfloor g(n)/2 \rfloor + 1}$ instead in length linear in the (total) formula size.

Thus, we have a formula family $\{F_n\}_{n=1}^{\infty}$ of size $\Omega(n) = S(F_n) = \mathrm{O}\big(n^2\big)$ refutable in length and variable space both linear in the formula size, but where any minimum variable space refutation must have length $\exp(\Omega(n))$. Adjusting the indices as needed, we get a formula family with a trade-off of the form stated in Theorem 1.1 (or actually slightly stronger). □

2 Two Key Ingredients in the Proof

Our proof of Theorem 1.1 comes very easily; in fact almost *too* easily. What is it that makes this possible? We want to make explicit two important points.

Firstly, the formula family $\{F_n\}_{n=1}^{\infty}$ used in the proof is clearly "redundant" in the sense that each formula F_n is the conjunction of two formulas G_n and H_m which are themselves already unsatisfiable. Formally, we say that a formula F is *minimally unsatisfiable* if F is unsatisfiable, but removing any clause $C \in F$, the remaining subformula $F \setminus \{C\}$ is satisfiable. We note that if we would add the requirement in Theorem 1.1 that the formulas under consideration should be minimally unsatisfiable, our proof idea would fail completely.

Secondly, an important reason why our proof gives sharp results is that we are allowed to use CNF formulas of growing width. It is precisely because of this that we can easily construct the formulas H_m that are hard with respect to variable space but easy with respect to length. If we would have to restrict ourselves to k-CNF formulas for k fixed, it becomes much more difficult to find such examples. Although there are formulas that could be plugged in to give a slightly weaker trade-off, we are not aware of any family of k-CNF formulas that can provably give the very sharp result in Theorem 1.1.

We refer to [6] for a more detailed discussion of these issues.

Acknowledgements

We are most grateful to Philipp Hertel and Toniann Pitassi for patiently answering several technical questions about Theorem 1.1. Also, we want to thank Jan Krajíček, Johan Håstad, Per Austrin and Mikael Goldmann for useful discussions and insightful comments.

A Definitions, Notation and Some Useful Facts

A *literal* is either a propositional logic variable x or its negation $\overline{x}$. A *clause* $C = a_1 \vee \ldots \vee a_k$ is a set of literals. The *width* $W(C)$ of a clause C is the number of literals appearing in it. A clause containing at most k literals is called a *k-clause*. A *CNF formula* $F = C_1 \wedge \ldots \wedge C_m$ is a set of clauses. A *k-CNF formula* is a CNF formula consisting of k-clauses. We let $\mathit{Vars}(C)$ denote the set of variables in a clause C, and extend this notation to formulas by taking unions over clauses. Also, the width $W(F)$ of a CNF formula F is the width of its largest clause. We define the *size* $S(F)$ of a CNF formula F to be the numbers of literals in it, counted with repetitions.

As in [4], we use the "configuration-style" definition of resolution. We employ the standard notation $[n] = \{1, 2, \ldots, n\}$.

Definition A.1 (Resolution [1])**.** A *clause configuration* $\mathbb{C}$ is a set of clauses. A sequence of clause configurations $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ is a *resolution derivation* from a CNF formula F if $\mathbb{C}_0 = \emptyset$ and for all $t \in [\tau]$, $\mathbb{C}_t$ is obtained from $\mathbb{C}_{t-1}$ by one of the following rules:

Axiom Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for a clause $C \in F$ (an *axiom*).

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for some clause $C \in \mathbb{C}_{t-1}$.

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for a clause $C \vee D$ inferred by the *resolution rule* from clauses $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$.

A resolution derivation $\pi : F \vdash A$ of a clause A from a CNF formula F is a derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $A \in \mathbb{C}_\tau$. A *resolution refutation* of F is a derivation $\pi : F \vdash 0$ of the empty clause 0 (the clause with no literals) from F.

In these notes, we are interested in the following complexity measures:

- The *length* $L(\pi)$ of a derivation π is the number of distinct clauses in π.
- The *clause space* $Sp(\pi)$ of a resolution derivation π is the maximal number of clauses in any clause configuration $\mathbb{C}_t \in \pi$.
- The *variable space* $VarSp(\pi)$ of a resolution derivation π is the maximal number of literals, counted with repetitions, in any clause configuration $\mathbb{C}_t \in \pi$.

The length of refuting F is $L(F \vdash 0) = \min_{\pi: F \vdash 0}\{L(\pi)\}$, where the minimum is taken over all resolution refutations of F. The clause space $Sp(F \vdash 0)$ and variable space $VarSp(F \vdash 0)$ of refuting F is defined wholly analogously.

We will need the fact that there are polynomial-size k-CNF formulas that are very hard with respect to length and clause space.

Theorem A.2 ([2, 3])**.** *There are arbitrarily large unsatisfiable 3-CNF formulas F_n with $\Theta(n)$ clauses and $\Theta(n)$ variables for which it holds that $L(F_n \vdash 0) = \exp(\Theta(n))$ and $Sp(F_n \vdash 0) = \Theta(n)$.*

Clearly, for these formulas F_n it also holds that $\Omega(n) = VarSp(F_n \vdash 0) = \mathrm{O}(n^2)$. We note in passing that the exact variable space complexity was mentioned as an open problem in [1], and to the best of our knowledge this problem is still unsolved.

We will also need the following easy observation.

Observation A.3. *Suppose that $F = G \wedge H$ where G and H are unsatisfiable CNF formulas over disjoint sets of variables. Then any resolution refutation $\pi : F \vdash 0$ must contain a refutation of either G or H.*

Proof. By induction, we can never resolve a clause derived from G with a clause derived from H, since the sets of variables of the two clauses are disjoint. ☐

References

[1] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, 2002.

[2] Paul Beame, Richard Karp, Toniann Pitassi, and Michael Saks. The efficiency of resolution and Davis-Putnam procedures. *SIAM Journal on Computing*, 31(4):1048–1075, 2002.

[3] Eli Ben-Sasson and Nicola Galesi. Space complexity of random formulae in resolution. *Random Structures and Algorithms*, 23(1):92–109, August 2003.

[4] Philipp Hertel and Toniann Pitassi. *An Exponential Time/Space Speedup For Resolution.* Technical Report TR07-046, Electronic Colloquium on Computational Complexity (ECCC), May 2007.

[5] Philipp Hertel and Toniann Pitassi. Exponential time/space speedups for resolution and the PSPACE-completeness of black-white pebbling. In *Proceedings 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS '07)*, October 2007. To appear.

[6] Jakob Nordström. *A Simplified Way of Proving Trade-off Results for Resolution.* Technical report, Royal Institute of Technology (KTH), Stockholm, Sweden, September 2007. Available at `http://www.csc.kth.se/~jakobn/publications/`.

ECCC ISSN 1433-8092
http://eccc.hpi-web.de/