On the Rectangle Method in proofs of Robustness of Tensor Products

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July 2007

Abstract

Given linear two codes $R, C$, their tensor product $R \otimes C$ consists of all matrices whose rows are codewords of $R$ and whose columns are codewords of $C$. The product $R \otimes C$ is said to be robust if for every matrix $M$ that is far from $R \otimes C$ it holds that the rows and columns of $M$ are far from $R$ and $C$ respectively. Ben-Sasson and Sudan (ECCC TR04-046) have asked under which conditions the product $R \otimes C$ is robust. During the last few years, few important families of tensor products were shown to be robust, and a counter-example of a product that is not robust was also given. However, a precise characterization of codes whose tensor product is robust remains unknown.

In this note we highlight a common theme in the above papers, which we call “The Rectangle Method”. In short, we observe that all proofs of robustness in the above papers are done by constructing a “rectangle”, while in the counterexample no such rectangle can be constructed. We then show that a rectangle can be constructed if and only if the tensor product is robust, and therefore the proof strategy of constructing a rectangle is complete.

1 Introduction

An error correcting code is said to be locally testable if there is a test that can check whether a given string is a codeword of the code, or rather far from the code, by reading only a constant number of symbols of the string. Locally Testable Codes (LTCs) were first explicitly studied by Goldreich and Sudan [5] and since then few constructions of LTCs were suggested (See [4] for an extensive survey of those constructions).

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Ben-Sasson and Sudan [1] suggested using the tensor product operation for the construction of LTCs. Given two linear codes $R, C$, their tensor product $R \otimes C$ consists of all matrices whose rows are codewords of $R$ and whose columns are codewords of $C$. If $R$ and $C$ are locally testable, we would like $R \otimes C$ to be locally testable. Ben-Sasson and Sudan suggested using the following test for testing the tensor product $R \otimes C$.

**The Row/Column Test** Choose a random row (or column) and accept iff it is a codeword of $R$ ($C$, respectively).

In order to study the conditions under which $R \otimes C$ is locally testable, Ben-Sasson and Sudan introduced the notion of “robust” tensor product. The tensor product $R \otimes C$ is said to be robust if for every matrix $M$ that is far from $R \otimes C$ it holds that the rows and columns of $M$ are far from $R$ and $C$ respectively. It is not hard to see that if both $R$ and $C$ are locally testable and $R \otimes C$ is robust then $R \otimes C$ is locally testable.

This gives rise to the question in which cases the tensor product is robust. Ben-Sasson and Sudan managed to show that the tensor product of three codes or more ($C_1 \otimes C_2 \otimes C_3 \ldots \otimes C_k$) is robust under a more general notion of robustness. As for the tensor product of two codes, it was proved that for two important cases that the tensor product is robust: The case of Reed-Solomon codes (this is the bivariate low degree test of Polishchuk and Spielman [6]), and the case of “smooth” LDPC codes [3]. In addition, Paul Valiant gave an example of codes whose tensor product is not robust [7], and his example was extended in [2].

In this note we highlight a common theme in all of those papers. We observe that all proofs of robustness share a common proof strategy, which we call “The Rectangle Method” and consists of constructing a “large rectangle”. This gives rise to the question of whether this strategy is complete - that is, can it always be used to prove robustness. We answer this question positively, by showing that a “large rectangle” can be constructed for every robust tensor product.

We make the notation required to continue the discussion in Section 2. We describe the rectangle method and our results in detail in Section 3, and prove our results in Sections 4 and 5. We review the proof that the existence of a “large rectangle” implies robustness in Section 6.

## 2 Preliminaries

Let $R, C$ denote binary linear codes with block lengths $m, n$ and relative distances $\delta_R, \delta_C$. For any two binary strings $x, y (|x| = |y|)$, we denote by $\delta(x, y)$ the relative Hamming distance between $x$ and $y$. The Tensor Product $R \otimes C \subseteq \{0, 1\}^{n \times m}$ is the linear code that consists of all the binary $n \times m$ matrices whose rows are codewords of $R$ and whose columns are codewords of $C$.

For any binary $n \times m$ matrix $M$ we denote by $\delta(M)$ the relative distance of $M$ to $R \otimes C$, and by $\overline{M}$ the codeword of $R \otimes C$ nearest to $M$. We also denote by $\delta_{\text{row}}(M)$ the average relative
distance of a row of $M$ to $R$, and define $\delta_{\text{col}}$ similarly. Finally, we denote by $\rho(M)$ the average of $\delta_{\text{row}}(M), \delta_{\text{col}}(M)$, that is

$$\rho(M) = \frac{\delta_{\text{row}}(M) + \delta_{\text{col}}(M)}{2}.$$ 

The trivial observation that $\delta_{\text{row}}(M) \leq 2\rho(M)$ and $\delta_{\text{col}}(M) \leq 2\rho(M)$ will be useful later.

We say that $R \otimes C$ is $\alpha$-robust if for every $M$ it holds that $\rho(M) \geq \alpha \cdot \delta(M)$. We also denote by $M_R$ the matrix that is obtained by replacing each row of $M$ with the nearest codeword of $R$, and define $M_C$ similarly.

### 3 The Rectangle Method

Proofs for robustness of $R \otimes C$ usually go along the following line: They first prove that there exists a constant $\alpha_0$ such that for every matrix $M$ that satisfies $\rho(M) < \alpha_0$, the matrices $M_R$ and $M_C$ agree on a large rectangle. That is, for every matrix $M$ such that $\rho(M) < \alpha_0$ there exists sets $U \subseteq [m], V \subseteq [n], |U| > (1 - \frac{\delta_R}{2})m, |V| > (1 - \frac{\delta_C}{2})n$, such that $M_R$ and $M_C$ agree on every coordinate in $U \times V$. The argument then uses a known property of tensor products, asserting that if $M_R$ and $M_C$ agree on a such a large rectangle, then $\rho(M) \geq \frac{1}{8}\delta(M)$ (For completeness, a proof of this fact is given in Section 6). The conclusion is that $R \otimes C$ is $\alpha$-robust for $\alpha = \min\{\alpha_0, \frac{1}{8}\}$.

In this note we prove that this strategy is “complete”; that is, it can always prove the robustness of $R \otimes C$, provided that $R \otimes C$ is indeed robust. We prove this by showing that if the code is robust and a matrix $M$ has very small $\rho(M)$, then there exists a large rectangle on which $M_R$ and $M_C$ agree.

To summarize, consider the following two conditions on a tensor product of codes:

1. There exists a constant $\alpha_0$ such that for every matrix $M$ that satisfies $\rho(M) < \alpha_0$ there exist sets $U \subseteq [n], V \subseteq [m]$ such that $M_R$ and $M_C$ agree on every coordinate in $U \times V$, where $|U| > (1 - \frac{\delta_C}{2})n, |V| > (1 - \frac{\delta_R}{2})m$.

2. There exists a constant $\alpha$ such that $R \otimes C$ is $\alpha$-robust.

It is a known fact that the Condition 1 implies Condition 2, and the proofs of robustness usually work by proving that the tensor product satisfies Condition 1, and then conclude that Condition 2 holds. The possible novelty of this note is that we prove that Condition 2 implies Condition 1, so they are equivalent.

We prove that Condition 2 implies Condition 1 in Section 5. For the sake of completeness, we also review the proof that Condition 1 implies Condition 2 in Section 6.

The equivalence between the conditions holds also when $\alpha_0$ and $\alpha$ are sub-constants. The exact quantitative relation is that Condition 1 with $\alpha_0$ implies Condition 2 with $\alpha = \min\{\alpha_0, \frac{1}{8}\}$, and that Condition 2 with $\alpha$ implies Condition 1 with $\alpha_0 = \frac{1}{8}\delta_R\delta_C\alpha$.  


4 Triangle inequality for robustness

We will begin with reviewing the following useful inequality: For every two matrices $M_1, M_2$ we have that

$$\rho(M_1) \leq \rho(M_2) + \delta(M_1, M_2)$$  \hspace{1cm} (1)

Inequality 1 results from averaging the two following inequalities:

$$\delta_{\text{row}}(M_1) \leq \delta_{\text{row}}(M_2) + \delta(M_1, M_2)$$  \hspace{1cm} (2)

$$\delta_{\text{col}}(M_1) \leq \delta_{\text{col}}(M_2) + \delta(M_1, M_2)$$  \hspace{1cm} (3)

We will only prove Inequality 2, and the proof of Inequality 3 is similar. Inequality 2 follows by applying the usual triangle inequality of distances for each row separately, details follow.

For every $1 \leq i \leq n$ we denote by $M_1^i$ and $M_2^i$ the $i$-th row of $M_1$ and $M_2$ respectively. With a slight abuse of notation, we denote by $\delta_R(M^i)$ the relative distance of the $i$-th row of $M$ to $R$, and by $\delta(M_1^i, M_2^i)$ the relative distance between the $i$-th rows of $M_1$ and $M_2$. We now have that

$$\delta_{\text{row}}(M_1) = \frac{1}{n} \sum_{i=1}^{n} \delta_R(M_1^i)$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \left[ \delta_R(M_2^i) + \delta(M_1^i, M_2^i) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \delta_R(M_2^i) + \frac{1}{n} \sum_{i=1}^{n} \delta(M_1^i, M_2^i)$$

$$= \delta_{\text{row}}(M_2) + \delta(M_1, M_2)$$

This completes the proof of Inequality 1.

We mention that Inequality 1 yields the following useful conclusion:

**Fact 4.1.** For every matrix $M$ we have

$$\rho(M_R) \leq 3\rho(M)$$

$$\rho(M_C) \leq 3\rho(M)$$

**Proof** We prove only for $M_R$:

$$\rho(M_R) \leq \rho(M) + \delta(M, M_R) = \rho(M) + \delta_{\text{row}}(M) \leq \rho(M) + 2\rho(M) = 3\rho(M)$$

Where the first equality is due to the definition of $M_R$. \hspace{1cm} \blacksquare
5 Proving that a large rectangle can always be found

We prove that if $R \otimes C$ is robust then for every matrix $M$ whose rows and columns are close enough to $R$ and $C$, there exists a large rectangle on which $M_R$ and $M_C$ agree. The proof proceeds in two steps: We first show that $M = M_R = M_C$ (see Claim 5.1). We then show that this implies the existence of a large rectangle (see Claim 5.2): The idea is that $M = M_R = M_C$ implies that $M_R$ and $M_C$ must be very close to $M$. Thus $M_R$ and $M$ must agree on many rows, and $M_C$ and $M$ must agree on many columns. We can therefore choose the rectangle to be the intersection of the rows on which $M_R$ and $M$ agree with the columns on which $M_C$ and $M$ agree.

Claim 5.1. If $R \otimes C$ is $\alpha$-robust, there exists a constant $\alpha_0$ such that for every matrix $M$ with $\rho(M) < \alpha_1$, we have that $M = M_R = M_C$. Furthermore, we can choose $\alpha_0 = \frac{1}{6} \delta_R \delta_C \alpha$.

Proof Suppose that $R \otimes C$ is $\alpha$-robust for some constant $\alpha > 0$. We first show that there exists a constant $\alpha_0 > 0$ such that $\rho(M) < \alpha_0$ implies that $M = M_R$. It can be shown similarly for $M = M_C$, and by choosing $\alpha_0$ to be small enough we will obtain $M = M_R = M_C$ whenever $\rho(M) < \alpha_0$. So, assume that $\rho(M) < \alpha_0$ for some constant $\alpha_1$ that will be chosen later. We have the following:

$$
\delta(M, M_R) \leq \delta(M, M_R) + \delta(M_R, M) + \delta(M_R, M_R)
= \delta(M, M_R) + \delta_{\text{row}}(M_R) + \delta(M_R)
\leq \frac{1}{\alpha} \rho(M) + 2 \rho(M) + \frac{1}{\alpha} \rho(M_R)
\leq \frac{1}{\alpha} \rho(M) + 2 \rho(M) + \frac{3}{\alpha} \rho(M)
= \left(2 + \frac{4}{\alpha}\right) \cdot \rho(M)
< \left(2 + \frac{4}{\alpha}\right) \cdot \alpha_0
$$

Where the first equality uses $\delta(M, M_R) = \delta_{\text{row}}(M_R)$ (by definition of $M_R$), the second inequality uses $\delta_{\text{row}}(M) \leq 2 \rho(M)$ and $\rho(M) \geq \alpha \delta(M)$, $\rho(M_R) \geq \alpha \delta(M_R)$ (By the $\alpha$-robustness of $R \otimes C$) and the third inequality uses $\rho(M_R) \leq 3 \rho(M)$ (Fact 4.1). Now, if we choose $\alpha_1$ such that $\left(2 + \frac{4}{\alpha}\right) \cdot \alpha_0 \leq \delta_{R \otimes C} = \delta_R \delta_C$, we shall obtain that $M = M_R$ for any matrix $M$ such that $\rho(M) < \alpha_0$. Note that choosing $\alpha_0 = \frac{1}{6} \delta_R \delta_C \alpha$ satisfies $\left(2 + \frac{4}{\alpha}\right) \cdot \alpha_0 \leq \delta_{R \otimes C}$.

Claim 5.2. If $R \otimes C$ is $\alpha$-robust, then for every matrix $M$ that satisfies $\rho(M) < \alpha_0 = \frac{1}{6} \delta_R \delta_C \alpha$, there exist sets $U \subseteq [n]$, $V \subseteq [m]$ such that $M_R$ and $M_C$ agree on every coordinate in $U \times V$, and $|U| > (1 - \frac{1}{2} \delta_C)n$, $|V| > (1 - \frac{1}{2} \delta_R)m$. 

5
Proof Let $M$ be such that $\rho(M) < \frac{1}{6}\delta_R\delta_C\alpha$. By Lemma 5.1, we get that $\overline{M} = \overline{M}_R = \overline{M}_C$. We also know that

$$\delta(M_R, \overline{M}) = \delta(M_R, \overline{M}_R) = \delta(M_R) \leq \frac{1}{\alpha} \rho(M_R) \leq \frac{3}{\alpha} \rho(M) < \frac{1}{2}\delta_R\delta_C$$

where the first inequality uses $\rho(M_R) \geq \alpha \cdot \delta(M_R)$ and the second inequality uses $\rho(M_R) \leq 3\rho(M)$ (by Fact 4.1).

Observe that every row of $M_R$ and every row of $\overline{M}$ are codewords of $R$, and therefore every row of $M_R$ is either equal to the corresponding row of $\overline{M}$ or they differ on at least $\delta_R$-fraction of the coordinates. It follows that $M_R$ and $\overline{M}$ must agree on more than $\left(1 - \frac{1}{2}\delta_C\right)$-fraction of the rows, or otherwise they would disagree on $\frac{1}{2}\delta_R\delta_C$ of the coordinates. Similarly it can be shown $M_C$ and $\overline{M}$ must agree on more than $\left(1 - \frac{1}{2}\delta_R\right)$-fraction of the columns. Now take $U$ to be the set of rows on which $M_R$ agrees with $\overline{M}$, and take $V$ to be the set of columns on which $M_C$ agrees with $\overline{M}$.
We have that $M_R$ and $M_C$ both agree with $\overline{M}$ on $U \times V$, and therefore they agree with each other on $U \times V$.

By combining the two last claims, we get that $\rho(M) < \alpha_0$ implies the existence of a large rectangle for $M$, as required.

6 Proving that a large rectangle implies robustness

For the sake of completeness, we review the proof (given in [3]) that the existence of a large rectangle implies robustness. Let $M$ be matrix and suppose there exist $U \subseteq [n], V \subseteq [m]$ such that $M_R$ and $M_C$ agree on the coordinates in $U \times V$ and such that $|U| > (1 - \frac{1}{2}\delta_C)n, |V| > (1 - \frac{1}{2}\delta_R)m$.
We shall show this implies that $\delta(M) < 8\rho(M)$.

Claim 6.1 ([1]). There exists a legal codeword $N$ that agrees with $M_R$ and $M_C$ on $U \times V$.

Proof Let $R'$ and $C'$ denote the codes obtained from projecting the codewords of $R,C$ to the coordinates in $V$ and $U$ respectively. Let $M'$ denote the matrix obtained by projecting $M_R$ or $M_C$ to $U \times V$ (Since the two matrices agree on $U \times V$, the matrix $M'$ is well defined).

Observe that the projection from $R$ to $R'$ is actually a bijection: The projection is surjective by the definition of $R'$. As for the projection being injective, note that if two codewords of $R$ are projected to the same codeword of $R'$, then they must agree on more than $\left(1 - \frac{1}{2}\delta_R\right)$-fraction of the coordinates and therefore must be equal. Similarly the projection from $C$ to $C'$ is a bijection.

This implies that the projection from $R \otimes C$ to $R' \otimes C'$ (obtained by projecting codewords of $R \otimes C$ to $U \times V$) is one-to-one. Since $R \otimes C$ and $R' \otimes C'$ are linear spaces of the same dimension $(\dim R \cdot \dim C)$, it follows that this projection is a bijection between $R \otimes C$ to $R' \otimes C'$. Now, note that $M'$ is a legal codeword of $R' \otimes C'$, so it follows that there exists a codeword $N$ of $R \otimes C$ whose projection to $U \times V$ equals $M'$. ■
Claim 6.2 ([3]). $\delta(M_R, N) = \delta_{\text{col}}(M_R)$.

**Proof** We shall show that if we decode each of the columns of $M_R$ to the nearest codeword of $C$, we will get $N$. For each row, $M_R$ and $N$ either agree on it or differ on at least $\delta_R$-fraction of its coordinates. Since we know $N$ and $M_R$ agree on $U \times V$, they must agree on all the rows in $U$. This implies that in every column, $M_R$ and $N$ agree on more than $(1 - \frac{1}{2}\delta_C)$-fraction of the coordinates, and therefore the decoding of each column of $M_R$ to the nearest codeword of $C$ must result in the corresponding column of $N$. That is, we have that $(M_R)_C = N$. It follows that $\delta(M_R, N) = \delta_{\text{col}}(M_R)$, as required.

The inequality $\delta(M) < 8\rho(M)$ now follows by

$$\delta(M, N) \leq \delta(M, M_R) + \delta(M_R, N) \leq \delta_{\text{row}}(M) + 2\rho(M_R) \leq 2\rho(M) + 6\rho(M) = 8\rho(M)$$

where the second inequality uses $\delta(M, M_R) = \delta_{\text{row}}(M)$ (by definition of $\delta_{\text{row}}$) and $\delta(M_R, N) = \delta_{\text{col}}(M_R) \leq 2\rho(M_R)$ (by Claim 6.2). The third inequality follows from $\delta_{\text{row}}(M) \leq 2\rho(M)$ and from $\rho(M_R) \leq 3\rho(M)$ (By fact 4.1).

**Acknowledgement.** The author wishes to thank Oded Goldreich for his helpful comments on this note. The author also wishes to thank Changyuan Yu for pointing out an error in a previous version of this note.

**References**


