



# The Tensor Product of Two Good Codes Is Not Necessarily Robustly Testable\*

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## Abstract

Given two codes  $R, C$ , their tensor product  $R \otimes C$  consists of all matrices whose rows are codewords of  $R$  and whose columns are codewords of  $C$ . The product  $R \otimes C$  is said to be robust if for every matrix  $M$  that is far from  $R \otimes C$  it holds that the rows and columns of  $M$  are far from  $R$  and  $C$  respectively. Ben-Sasson and Sudan (ECCC TR04-046) have asked under which conditions the product  $R \otimes C$  is robust.

Paul Valiant (APPROX-RANDOM 2005) constructed two linear codes with constant relative distance whose tensor product is not robust. However, one of those codes has a sub-constant rate. We show that this construction can be modified such that both codes have both constant rate and constant relative distance. We also provide an alternative proof for the non-robustness of the tensor product of the codes of Valiant, based on the inverse direction of the “rectangle method” that was presented by the second author (ECCC TR07-061). We believe that this proof gives an additional intuition for why this construction works.

## 1 Introduction

An error correcting code is said to be *locally testable* if there is a test that can check whether a given string is a codeword of the code, or rather far from the code, by reading only a constant number of symbols of the string. Locally Testable Codes (LTCs) were first systematically studied by Goldreich and Sudan [GS06] and since then several constructions of LTCs were suggested

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(See [Gol05] for an extensive survey of some constructions, as well as [Din07, BSS08, Mei09] for a few later constructions).

Ben-Sasson and Sudan [BSS06] suggested using the tensor product operation for the construction of LTCs. Given two linear error correcting codes  $R, C$ , their tensor product  $R \otimes C$  is the code that consists of all matrices whose rows are codewords of  $R$  and whose columns are codewords of  $C$ . If  $R$  and  $C$  are locally testable, we would like  $R \otimes C$  to be locally testable. [BSS06] suggested using the following test for testing the tensor product  $R \otimes C$ .

**The Row/Column Test.** Choose a random row (or column) and accept iff it is a codeword of  $R$  ( $C$ , respectively).

In order to study the conditions under which  $R \otimes C$  is locally testable, Ben-Sasson and Sudan introduced the notion of “robust” tensor product (which is a special case of the notion of robustness of [BSGH<sup>+</sup>06, DR06]). The tensor product  $R \otimes C$  is said to be robust if, for every matrix  $M$  that is far from  $R \otimes C$ , it holds that the rows and columns of  $M$  are far from  $R$  and  $C$  respectively. It is not hard to see that if both  $R$  and  $C$  are locally testable, and  $R \otimes C$  is robust, then  $R \otimes C$  is locally testable.

This gives rise to the question in which cases the tensor product is robust. Paul Valiant [Val05] constructed two linear codes whose tensor product is not robust, and this construction was extended in [CR05]. However, one of the codes constructed by [Val05] has sub-constant rate. In this note, we show the construction of [Val05] can be modified such that both codes have constant rate. Using Theorem 2 of [CR05], it follows that there exists a linear code of constant rate and relative distance, whose tensor product *with itself* is not robust.

In addition, we provide an alternative proof for the non-robustness of the tensor product of the codes of [Val05]. We believe that our alternative proof gives an additional intuition for why this construction works. Our proof is based on the inverse direction of the “rectangle method” that was presented in [Mei07].

**Organization.** The rest of this work is organized as follows. In Section 2, we provide the required preliminaries. In Section 3, we describe our construction of the aforementioned codes, and show that they have the required rate and relative distance. In Section 4, we adapt the proof of [Val05] to show that the tensor product of our codes is not robust. Finally, In Section 5, we provide the alternative proof that the tensor product of our codes is not robust.

## 2 Preliminaries

For any two binary strings  $x, y$  of the same length, we denote by  $\delta(x, y)$  the *relative* Hamming distance between  $x$  and  $y$ . We say that  $x$  and  $y$  are  $\tau$ -close if  $\delta(x, y) \leq \tau$ , and otherwise we say

that they are  $\tau$ -far.

## 2.1 Error Correcting Codes

We review the basics of error correcting codes [MS88]. A linear code  $C$  is linear subspace of  $\{0, 1\}^n$ , where  $n$  is called the **block length** of  $C$ . The vectors of  $C$  are called **codewords**. The code  $C$  has relative distance  $\delta_C$  if for any two distinct codewords  $c^1 \neq c^2 \in C$  it holds that  $\delta(c^1, c^2) \geq \delta_C$ . It is well-known that  $C$  has relative distance  $\delta_C$  if and only if every non-zero codeword  $c \in C$  has at least  $\delta_C$  fraction of non-zero coordinates.

A generator matrix for  $C$  is a matrix  $G$  whose rows form a basis of  $C$ . The dual code  $C^\perp$  is the code defined by

$$C^\perp \stackrel{\text{def}}{=} \{v : \langle v, c \rangle = 0 \text{ for every } c \in C\}$$

where the inner product is over  $\mathbf{GF}(2)$ .

We are usually interested in infinite families of codes. An infinite family of codes  $C = \{C_k\}_k$  is an infinite sequence of codes such that the code  $C_k$  has dimension  $k$ . We say that the family  $C$  has block length  $n(k)$  (where  $n : \mathbb{N} \rightarrow \mathbb{N}$ ) if the code  $C_k$  has block length  $n(k)$  for every  $k$ , and say that the family has rate  $r_C$  (for  $r_C \in (0, 1)$ ) if for every  $k$  it holds that  $\frac{k}{n(k)} \geq r_C$ . Finally, the family  $C$  has relative distance  $\delta_C$  (for  $\delta_C \in (0, 1)$ ) if for every  $k$  it holds that  $C_k$  has relative distance at least  $\delta_C$ .

## 2.2 Tensor Product Codes

Let  $R, C$  denote linear codes with block lengths  $m, n$  and relative distances  $\delta_R, \delta_C$  respectively. The tensor product  $R \otimes C \subseteq \{0, 1\}^{n \cdot m}$  is the linear code that consists of all the binary  $n \times m$  matrices whose rows are codewords of  $R$  and whose columns are codewords of  $C$ . It is well known that the relative distance of  $R \otimes C$  is  $\delta_R \cdot \delta_C$ .

For any binary  $n \times m$  matrix  $M$ , we denote by  $\delta_{R \otimes C}(M)$  the relative distance of  $M$  to  $R \otimes C$ . We also denote by  $\delta_{\text{row}}(M)$  the average relative distance of a row of  $M$  to  $R$ , and define  $\delta_{\text{col}}$  similarly. Finally, we denote by  $\rho(M)$  the average of  $\delta_{\text{row}}(M)$  and  $\delta_{\text{col}}(M)$ , that is,

$$\rho(M) \stackrel{\text{def}}{=} \frac{\delta_{\text{row}}(M) + \delta_{\text{col}}(M)}{2}$$

We can now state the definition of robustness.

**Definition 2.1.** We say that  $R \otimes C$  is  $\alpha$ -robust if for every  $M$  it holds that  $\rho(M) \geq \alpha \cdot \delta_{R \otimes C}(M)$ .

Now, let  $R = \{R_k\}_k, C = \{C_k\}_k$  be two infinite families of codes. We say that the infinite family  $\{R_k \otimes C_k\}_k$  is  $\alpha$ -robust (for  $\alpha \in (0, 1)$ ) if for every  $k \in \mathbb{N}$  it holds that  $R_k \otimes C_k$  is  $\alpha$ -robust.

### 3 The codes

Our main result in this work is the following.

**Theorem 3.1.** *There exists two infinite families of codes  $R = \{R_k\}_k$   $C = \{C_k\}_k$  such that both  $R$  and  $C$  have rate and relative distance at least  $\frac{1}{1000}$ , but the infinite family  $\{R_k \otimes C_k\}_k$  is not  $\alpha$ -robust for any constant  $\alpha \in (0, 1)$ .*

Fix  $k \in \mathbb{N}$ . The rest of this section is devoted to describing the construction of the codes  $R_k$  and  $C_k$ , and establishing their rate and relative distance. We will show that the infinite family  $\{R_k \otimes C_k\}_k$  is not  $\alpha$ -robust for any constant  $\alpha \in (0, 1)$  in Sections 4 and 5.

With a slight abuse of notation, we abbreviate  $R \stackrel{\text{def}}{=} R_k$  and  $C \stackrel{\text{def}}{=} C_k$ . Let  $n \stackrel{\text{def}}{=} 100 \cdot k$ . We choose the code  $C$  to be an arbitrary code with dimension  $k$ , block length  $n$ , and relative distance at least  $1/100$ , such that the dual code  $C^\perp$  has relative distance at least  $1/100$  as well. The existence of such code existence can be established rather easily using the probabilistic method. In order to construct the code  $R$  we define some additional notation.

**Notation 3.2.** Let  $S \subseteq \{0, 1\}^n$  be the set of vectors that consist of  $\frac{n}{10}$  "homogenous" blocks of 10 bits (recall that  $n$  is divisible by 10). By "homogenous" we mean that in each block all the bits are equal.

In order to construct the code  $R$ , we use the following code  $D$ , whose existence can be established rather easily using the probabilistic method.

**Fact 3.3.** *There exists a code  $D$  with dimension  $k$  and block length  $n$ , such that every non-zero codeword of  $D$  is  $\frac{1}{100}$ -far from every vector in  $S$ .*

Now, let  $G_C$  and  $G_D$  be  $k \times n$  generator matrices of  $C$  and  $D$  respectively. We define  $J \stackrel{\text{def}}{=} G_C^T \cdot G_D$ . Let  $J^{10}$  be the  $n \times 10n$  matrix that consists of 10 consecutive copies of  $J$  and let  $I_n^{10}$  be the  $n$ -rank identity matrix with each column duplicated to appear 10 times consecutively. That is,  $I_n^{10}$  is a matrix of the form

$$n \left\{ \begin{pmatrix} \overbrace{1 \ 1 \ \dots \ 1}^{10} & \overbrace{0 \ 0 \ \dots \ 0}^{10} & \overbrace{0 \ 0 \ \dots \ 0}^{10} \\ 0 \ 0 \ \dots \ 0 & \overbrace{1 \ 1 \ \dots \ 1}^{10} & \overbrace{0 \ 0 \ \dots \ 0}^{10} \\ 0 \ 0 \ \dots \ 0 & 0 \ 0 \ \dots \ 0 & \dots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \ 0 \ \dots \ 0 \\ 0 \ 0 \ 0 \ 0 & 0 \ 0 \ 0 \ 0 & \dots & 1 \ 1 \ \dots \ 1 \end{pmatrix} \right.$$

We define the code  $R$  to be the space spanned by the rows of the matrix  $M = J^{10} + I_n^{10}$ . It remains to show that  $R$  has dimension  $k$  and relative distance at least  $1/1000$ . We will actually show a better bound, namely, that  $R$  has relative distance at least  $1/100$ . The following two claim and corollaries will be useful for both ends.

**Claim 3.4.** *The matrix  $J$  has rank  $k$ .*

**Proof.** On one hand, the columns of  $J$  are linear combinations of rows of  $G_C$ , so its rank can be at most  $k$ . On the other hand, both  $G_C$  and  $G_D$  are matrices of rank  $k$  and have  $k$  rows, so each of them contains a full rank  $k \times k$  submatrix, denote those submatrices  $K_C, K_D$  respectively. Observe that the matrix  $K_C^T \cdot K_D$  has full rank, i.e., has rank  $k$ . Now, note that  $K_C^T \cdot K_D$  is a submatrix of  $G_C^T \cdot G_D$ , so the rank of  $J$  is at least  $k$ . It follows that the rank of  $J$  is exactly  $k$ . ■

**Corollary 3.5.** *The columns of  $J$  span the code  $C$ .*

**Proof.** The corollary follows immediately from the facts that the columns of  $J$  are linear combinations of rows of  $G_C$  and that  $J$  has rank  $k$ . ■

**Corollary 3.6.** *The matrix  $J^{10}$  has rank  $k$ .*

### 3.1 The dimension of $R$

We first show that  $R$  has dimension  $k$ . Since  $J^{10}$  has rank  $k$ , it has  $k$  independent rows  $u_1, \dots, u_k \in \{0, 1\}^{10n}$ . Let  $w_1, \dots, w_k$  be the corresponding rows of  $M$ , and let  $v_1, \dots, v_k$  be the corresponding rows of  $I_n^{10}$ . We prove that the rows  $w_1, \dots, w_k$  of  $M$  are linearly independent, and this will imply that  $R$  has dimension at least  $k$  (since  $R$  is spanned by the rows of  $M$ ).

Let  $w_{i_1}, \dots, w_{i_m}$  for  $m \leq k$  be a subset of  $w_1, \dots, w_k$ . We show that the sum  $w_{i_1}, \dots, w_{i_m}$  is non-zero. Since this will hold for any choice of  $w_{i_1}, \dots, w_{i_m}$ , this will show that  $w_1, \dots, w_k$  are independent. We first observe that

$$\sum_{j=1}^m w_{i_j} = \sum_{j=1}^m u_{i_j} + \sum_{j=1}^m v_{i_j}$$

We now make two observations:

**Claim 3.7.** *The sum  $\sum_{j=1}^m u_{i_j}$  consists of 10 concatenated copies of a non-zero codeword of the code  $D$ .*

**Claim 3.8.** *The sum  $\sum_{j=1}^m v_{i_j}$  consists of 10 concatenated vectors from the set  $S$ .*

Since by Fact 3.3, every non-zero codeword of  $D$  is  $1/100$ -far from every vector in  $S$ , it holds that the sum

$$\sum_{j=1}^m w_{i_j} = \sum_{j=1}^m u_{i_j} + \sum_{j=1}^m v_{i_j}$$

has  $1/100$  fraction of non-zero coordinates, and in particular it is a non-zero vector, as required. It remains to prove Claims 3.7 and 3.8. The proof of Claim 3.8 is trivial. To see that Claim 3.7 holds, observe that:

1. Every row of  $J \stackrel{\text{def}}{=} G_C^T \cdot G_D$  is a linear combination of rows of  $G_D$ .
2. Therefore, every row of  $J$  is a codeword of  $D$ .
3. Hence, every row of  $J^{10}$  consists of 10 concatenated copies of a codeword of  $D$ .

This concludes the proof that  $R$  has dimension  $k$ .

### 3.2 The distance of $C_2$

We show that every non-zero codeword of  $R$  has at least  $1/100$  fraction of non-zero coordinates, and this will imply that  $R$  has relative distance at least  $1/100$ . Let  $c$  be any nonzero codeword of  $R$ . Since  $R$  is spanned by the rows of  $M$ , there exists a non-zero vector  $v \in \{0, 1\}^n$  such that  $c = v \cdot M$ . Now, recall that  $M = J^{10} + I_n^{10}$ . We consider two cases: the case where  $v \cdot J^{10} = 0$ , and the case where  $v \cdot J^{10} \neq 0$ .

**The case where  $v \cdot J^{10} = 0$ .** If  $v \cdot J^{10} = 0$ , then in particular it holds that  $v \cdot J = 0$ . Recall that by Corollary 3.5, the columns of  $J$  span  $C$ . Thus, the assumption that  $v \cdot J = 0$  implies that  $v$  is orthogonal to all the codewords in  $C$ , or in other words,  $v \in C^\perp$ .

Since, by assumption,  $C^\perp$  has relative distance at least  $1/100$ , it follows that  $v$  has at least  $1/100$  fraction of non-zero coordinates. This implies, in turn, that  $v \cdot I_n^{10}$  has at least  $1/100$  fraction of non-zero coordinates. Finally, it holds that

$$c = v \cdot M = v \cdot J^{10} + v \cdot I_n^{10} = v \cdot I_n^{10},$$

and therefore  $c$  has  $1/100$  fraction of non-zero coordinates, as required.

**The case where  $v \cdot J^{10} \neq 0$ .** Denote  $d \stackrel{\text{def}}{=} v \cdot J$ . We proceed as in the proof that  $C_2$  has dimension  $k$ . Observe that

1.  $d$  is a non-zero codeword of  $D$ .

2.  $v \cdot J^{10}$  is a concatenation of 10 copies of  $d$
3.  $v \cdot I_n^{10}$  is the concatenation of 10 elements of  $S$ .

It follows that

$$c = v \cdot M = v \cdot J^{10} + v \cdot I_n^{10}$$

consists of 10 blocks, each of them is the sum of  $d$  with an element of  $S$ . Since  $d$  is a non-zero codeword of  $D$ , so by Fact 3.3 it holds that  $d$  is  $1/100$  far from every vector of  $S$ . It follows that  $c$  has at least  $1/100$  fraction of non-zero coordinates..

## 4 The non-robustness of $R \otimes C$

In this section, we prove that  $R \otimes C$  is not  $\alpha$ -robust for any constant  $\alpha \in (0, 1)$  using a variant of the proof of [Val05]. To this end, we show that the matrix  $M$  that was constructed as part of the definition of  $R$  in Section 3 is a counter-example to the robustness of  $R \otimes C$ . In other words,  $M$  is a  $n \times 10n$  matrix that it is far from  $R \otimes C$ , while the rows and columns of  $M$  are close to  $R$  and  $C$ .

Every row of  $M$  is a codeword of  $R$ , so  $\delta_{\text{row}}(M) = 0$ . Furthermore, every column of  $J^{10} = M - I_n^{10}$  is a codeword of  $C$ , so  $\delta_{\text{col}}(M) = \frac{1}{n}$ . We thus have that  $\rho(G_2) \leq \frac{1}{2n}$ . We now observe the following.

**Claim 4.1.** *It holds that*

$$\delta_{R \otimes C}(M) \geq \frac{99}{100}$$

**Proof.** Consider an arbitrary  $N \in R \otimes C$ . Every row of  $M - N$  is a codeword of  $R$ . Furthermore, each column of  $M - I_n^{10} - N$  is a codeword of  $C$ , so the rank of  $M - I_n^{10} - N$  is at most  $k$ . This implies that the rank of  $M - N$  must be at least  $n - k$ : Otherwise, the rank of  $-I_n^{10} = (M - I_n^{10} - N) + (N - M)$  would have been less than  $n$  (since rank is sub-additive and the ranks of  $M - N$  and  $N - M$  are equal).

Thus, there are at least  $n - k$  non-zero rows in  $M - N$ , each of which is a codeword of  $R$ . Each of those non-zero rows of  $M - N$  has at least  $1/100$  fraction of non-zero coordinates, i.e., at least  $n/10$  non-zero coordinates. It follows that  $M$  and  $N$  differ on at least  $(n - k) \cdot n/10$  coordinates, so

$$\delta_{R \otimes C}(M) = \min_{N \in R \otimes C} \{\delta(M, N)\} \geq \frac{(n - k) \cdot n/10}{10n^2} \geq \frac{99}{100},$$

as required. ■

It follows that

$$\rho(M) \leq \frac{100}{198 \cdot n} \cdot \delta_{R \otimes C}(M),$$

and therefore  $R \otimes C$  is not  $\alpha$ -robust for any constant  $\alpha \in (0, 1)$ .

## 5 An alternative proof for the non-robustness of $R \otimes C$

In this section, we prove that  $R \otimes C$  is not  $\alpha$ -robust for any constant  $\alpha \in (0, 1)$  using the rectangle method of [Mei07]. For any  $n \times 10n$  matrix  $N$ , let  $N_R$  denote the matrix obtained from decoding every row of  $N$  to nearest codeword of  $R$ , and let  $N_C$  be defined similarly for the columns and  $C$ . We say that  $N_R$  and  $N_C$  agree on a large rectangle if there exist sets  $U \subseteq [n]$ ,  $V \subseteq [10n]$  such that  $|U| \geq 0.99 \cdot n$ ,  $|V| \geq 9.9 \cdot n$ , and such that  $N_R$  and  $N_C$  agree on all the entries in  $U \times V$ . The following fact is a corollary of [Mei07].

**Fact 5.1.**  *$R \otimes C$  is  $\alpha$ -robust only if for every  $n \times 10n$  matrix  $N$  that satisfies  $\rho(M) < \frac{1}{60000} \cdot \alpha$  it holds that  $N_R$  and  $N_C$  agree on a large rectangle.*

Thus, in order to prove that  $R \otimes C$  is not  $\alpha$ -robust for every constant  $\alpha \in (0, 1)$ , it suffices to prove that for every constant  $\alpha_0 \in (0, 1)$  there exists a matrix  $N$  with  $\rho(N) \leq \alpha_0$  such that  $N_R$  and  $N_C$  do not agree on a large rectangle. As in Section 4, the matrix  $N$  we will use will be the matrix  $M$  that was constructed as part of the definition of  $R$  in Section 3. Recall that  $\rho(M) \leq \frac{1}{2n}$ . We now prove the following claim, which will conclude the proof that  $R \otimes C$  is not  $\alpha$ -robust.

**Claim 5.2.**  *$M_R$  and  $M_C$  do not agree on a large rectangle.*

**Proof.** Let  $U$  and  $V$  be sets such that  $|U| \geq 0.99 \cdot n$ ,  $|V| \geq 9.9 \cdot n$ . Observe that  $M_R = M$  and  $M_C = J^{10} = M - I_n^{10}$ , so  $M_R - M_C = I_n^{10}$ . We show that  $I_n^{10}$  has an entry with value 1 in  $U \times V$ , and therefore  $M_R$  and  $M_C$  do not agree on  $U \times V$ .

We know that every column of  $I_n^{10}$  contains exactly one entry with value 1, so the total number of entries with value 1 contained in the columns of  $V$  is  $|V|$ . We also know that every row of  $I_n^{10}$  contains exactly ten entries with value 1, so there are at least  $\frac{1}{10} |V|$  rows that contain 1's in their intersection with  $V$ . Now, note that  $\frac{1}{10} |V| > n - |U|$ , and thus the rows containing entries with value 1 in their intersection with  $V$  can not all be in  $[n] \setminus U$ . It follows that  $I_n^{10}$  has 1 on at least one of the coordinates in  $U \times V$ , as required. ■

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