Electronic Colloquium on Computational Complexity, Report No. 63 (2007)

Nearly Tight Bounds on the Number of Hamiltonian Circuits of the Hypercube and Generalizations

Tomás Feder* Carlos Subi

Abstract

We conjecture that for every perfect matching M of the d-dimensional n-vertex hypercube, $d \geq 2$, there exists a second perfect matching M' such that the union of M and M' forms a Hamiltonian circuit of the d-dimensional hypercube. We prove this conjecture in the case where there are two dimensions that do not get used by M. As a consequence, if M_d is the number of perfect matchings and H_d is the number of Hamiltonian circuits of the d-dimensional hypercube, then $M_{d-2}^4 \leq H_d \leq M_d^2/4$. By known bounds on the number of perfect matchings of the d-dimensional hypercube that show $M_d = (\frac{d}{e}(1+o(1)))^{n/2}$ and, in particular, $M_d \leq (d!)^{n/(2d)}$ we infer that $(\frac{d}{e}(1-o(1)))^{n/2} \leq H_d \leq (d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$. We finally strenthen this result to a nearly tight bound $((d\log 2/(e\log\log d))(1-o(1)))^n \leq H_d \leq ((d/e)(1+o(1)))^n$. We extend the results to graphs that are the Cartesian product of squares and arbitrary bipartite regular graphs that have a Hamiltonian cycle. We also study a labeling scheme related to matchings.

1 Introduction

We study properties of matchings and Hamiltonian cycles in various classes of graphs, including Cartesian products of graphs that generalize the hypercube and regular bipartite and non-bipartite graphs.

For a balanced bipartite graph $G = (U, V, E)$ where $|U| = |V| = n$, the bipartite adjacency matrix $A = A(G) = [a_{uv}]$ is the $n \times n$ matrix with $a_{uv} = 1$ if $uv \in E$ and $a_{uv} = 0$ if $uv \notin E$ for $u \in U, v \in V$.

Independently, Fisher[8] and Kastelyn[9] proved that the number of perfect matchings of G is the permanent of $A(G)$ when G is a balanced bipartite graph with adjacency matrix $A(G)$. Brègman[1] proved the conjecture of Minc[10] that for any $n \times n$ 0,1-matrix A with row sums $r_1, \ldots, r_n$, the permanent of A is at most $\prod_{i=1}^n (r_i!)^{1/r_i}$. In particular, a d-regular bipartite n-vertex graph has at most $(d!)^{n/(2d)} = (\frac{d}{e}(1+o(1)))^{n/2}$ perfect matchings.

Independently, Egoryčev [6] and Falikman [7] proved the conjecture of van der Waerden [12] that for any doubly stochastic $n \times n$ matrix A the permanent of A is at least $n!/n^n$. This was used by Clark, George and Porter [3] to show that the number of perfect matchings of a d-regular bipartite n-vertex graph is at least $(2d/n)^{n/2}(n/2)! = (\frac{d}{e}(1+o(1)))^{n/2}$.

2 Balanced Labeling Matching Partitions

Every d-regular bipartite graph is the union of d edge-disjoint perfect matchings. Suppose more generally that G is an n-vertex graph that is the union of d edge-disjoint perfect matchings

*268 Waverley Street, Palo Alto, CA 94301. Email: tomas@theory.stanford.edu

ISSN 1433-8092

$M_1, M_2, \ldots, M_d$. A *labeling orientation of* G is an assignment of directions to the edges of G and a corresponding assignment of labels $x = x_1x_2\cdots x_d$ to each vertex v of G, so that if the edge e of M_i incident to v is outgoing then $x_i = 0$ and if e is incoming then $x_i = 1$. A *balanced labeling orientation of* G is a labeling orientation of G such that the number of vertices having any given label x is either $\lfloor n/2^d \rfloor$ or $\lceil n/2^d \rceil$. Notice that if the d-dimensional hypercube, with $n = 2^d$ vertices, is decomposed into d perfect matchings corresponding to the d dimensions, and dimension i is oriented from 0 to 1 in the ith bit position, then we obtain a balanced labeling orientation of the hypercube that assigns to each vertex v is corresponding coordinates $x = x_1x_2\cdots x_d$.

Theorem 1 *Suppose that* G *is an* n*-vertex graph that is the union of* d *edge-disjoint perfect matchings* $M_1, M_2, \ldots, M_d$. *Then* G *has a balanced labeling orientation.*

Proof. Orient the edges of M_1 arbitrarily. This assigns to x_1 the value 0 to $n/2$ of the vertices and 1 to the other $n/2$ vertices. Suppose inductively that we have already oriented the first t matchings M_i, $1 \leq i \leq t$ in a balanced manner, so that each label occurs $\lfloor n/2^t \rfloor$ or $\lceil n/2^t \rceil$ times. We greedily select edges of M_{t+1} and orient them as follows. Select an edge $e = uv$ of M_{t+1} and orient it arbitrarily, say from u to v. Suppose u and v have labels x and y respectively for the first t bits of the label. If $x = y$, then $x' = xx_{t+1} = x0$ and $y' = xy_{t+1} = x1$, and we have made progress towards splitting the label x evenly. If $x \neq y$, then select another edge $e' = u'v'$ of M_{t+1} such that the label of u' is also y, if such an e' exists, and orient e' from u' to v', so in this case we have also made progress towards splitting y evenly. If no such e' exists, then the number k of vertices with label y was odd and we have split k as $\lfloor k/2 \rfloor$ and $\lceil k/2 \rceil$ for labels $y0$ and $y1$ respectively, as required. If e' exists, then either v' has the same label x as u in which case we have again made progress towards splitting x evenly as $x0$ and $x1$, or v' has label $z \neq x$ and we proceed inductively to look for an edge $e'' = u''v''$ with u'' having the same label z has v'. Eventually the process ends in some edge $e^i = u^iv^i$. If the label of v^i is x then we have made progress towards splitting x evenly as before, otherwise the number of vertices with the label of v^i was odd and split evenly as floor and ceiling as before. In this last case the only imbalance is at u with label $x0$, so we start looking for $e' = u'v'$ where u' has label x, and orient e' from v' to u', making progress towards splitting x evenly. We proceed with the imbalance at v' with label $y0$ as we just did for x to make progress towards splitting y similarly. In the end, each label x with k vertices will have been split into two labels $x0$ and $x1$ having one $\lfloor k/2 \rfloor$ vertices and the other one $\lceil k/2 \rceil$ vertices, completing the induction. □

Corollary 1 *If the* d*-dimensional hypercube with* $n = 2^d$ *vertices is decomposed into* d *edge-disjoint perfect matchings, then a balanced labeling orientation exists and assigns each label* $x = x_1x_2\cdots x_d$ *exactly once.*

Proof. The result follows from Theorem 1 and the fact that $\lfloor n/2^d \rfloor = \lceil n/2^d \rceil = 1$. □

3 Hamiltonian Circuits and Isomorphisms

Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = r = 2k$ having a Hamiltonian circuit C. We may label the vertices of each of U, V as $1, 2, \ldots, r$. We may also decompose C as the union of two perfect matchings M and M', and view M and M' as two permutations p and p' on $1, 2, \ldots, r$, so that $p(i) = j$ and $p'(i') = j'$ if vertex i in U is matched to j in V by M, and vertex i' in U is matched to j' in V by M'.

Theorem 2 *Of the two permutations p and p', one is odd and the other one even. Thus the graph joining pairs of matchings M and M' if they jointly form a Hamiltonian circuit is bipartite, and if M_G and H_G are the number of perfect matchings and Hamiltonian circuits of G, respectively, then $H_G \leq M_G^2/4$.*

Proof. We may relabel the vertices so that p and p' become q and q' with $q(i) = i$ and $q'(i) = i + 1$ modulo r. Thus there exist permuatations s and t such that $p = sqt = st$ and $p' = sq't$. The proof is completed by observing that $q' = (12 \cdots r) = (12)(13) \cdots (1r)$ is odd since $r - 1 = 2k - 1$. □

This improves by a factor of two the bound $M_G^2/2$ of Clark [2] on the number of Hamiltonian circuits, which is also an improvement on earlier results by Dixon and Goodman [4], Douglas [5], and Mollard [11].

Theorem 3 *Let G be a d-regular bipartite graph $G = (U, V, E)$ with $|U| = |V| = n/2$. Then for $k \leq d$, the number of sequences $M_1, \ldots, M_k$ of k edge-disjoint perfect matchings for G is at most $\prod_{i=0}^{j-1} ((d-i)!)^{n/(2(d-i))}$ and at least $\prod_{i=0}^{j-1} (2(d-i)/n)^{n/2}(n/2)!$. In particular, the number of Hamiltonian circuits of G is at most $(d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$.*

Proof. The result follows on the bound on the permanent of Brègman [1] and Clark, George, and Potter [3] mentioned in the introduction for G and the successive sugraphs obtained from G by removing perfect matchings $M_1, \ldots, M_k$ one at a time, thus successively reducing the degree by one. For Hamiltonian circuits, the bound follows by choosing M_1 and M_2 forming the circuit in the two possible orders and dividing by two. □

We may ask whether there exists in general an isomorphism of G sending M to M' for a Hamiltonian circuit C. We answer this in the case of the hypercube.

Theorem 4 *An isomorphism sending M to M' for a Hamiltonian circuit C in the d-dimensional hypercube exists only if $d = 2$.*

Proof. It is clear that such an isomorphism of the 2-dimensional hypercube mapping one perfect matching to the other exists. Suppose $d \geq 3$, and choose the $r = 2^{d-1} = 2k$ labels $1, 2, \ldots, r$ by labeling two vertices that differ only in the first dimension the same. Of p and p', one is odd and the other one even. If the isomorphism exists, we may write $p' = qpq'$, where q and q' define the isomorphism. The isomorphism given by q and q' is a composition of two types of isomorphisms, either flipping bit i or exchanging bits i and j. The case $d = 3$ can be verified directly as the Hamiltonian circuit is in that case essentially, unique. If $d \geq 4$, then each subcube determined by dimensions $1, i$ in the case of a flip, or by dimensions $1, i, j$ in the case of an exchange, involves some number $r \leq 3$ of dimensions and t transpositions, for a total of $2^{d-r}t$ transpositions, which is even. Thus q and q' are both even, so p and $p' = qpq'$ are either both even or both odd, a contradiction to the fact that p and p' have different parity. □

4 Number of Hamiltonian Circuits in Products by a Square

Theorem 5 *Let G be the product of a graph G' that has an even number of vertices and a Hamiltonian path, and a square Q_2 (the 2-dimensional hypercube). Then any perfect matching of G that does not use the edges of Q_2 can be extended to a Hamiltonian circuit of G. Thus if M is the number of perfect matchings of G' and H is the number of Hamiltonian circuits of G, then $H \geq M^4$.*

Proof. Let $G'_{00}, G'_{01}, G'_{10}, G'_{11}$ be the four copies of G' in G, and let $M_{00}, M_{01}, M_{10}, M_{11}$ be the corresponding perfect matchings forming M. Let M_0 be the union of M_{00} and M_{01} in G', and let M_1 be the union of M_{10} and M_{11} in G'. Each component of M_i can be viewed as an alternating cycle combining alternating edges across dimension 2 of Q_2 and edges of M_{i0} and M_{i1} taken alternatively. It remains to combine these cycles into a single cycle. Let M be the union of M_0 and M_1 in G'. Each component of M consists in G of cycles corresponding to components in M_0 and in M_1. These cycles can be pairwise combined by replacing at a shared vertex of two cycles corresponding to M_0 and to M_1 the two edges across dimension 2 by two edges across dimension 1 of Q_2. It remains to combine the resulting cycles corresponding to components of M into a single cycle. Join the components of M with a minimal number of edges from the Hamiltonian path of G' to form a single component in G'. This gives a tree-like structure to the components of M. Any component C of M has at some vertex u of C at most two child components C_1 and C_2 at vertices v and w respectively. If at u we are using dimension 1 (resp. 2) of Q_2 for C, we may use dimension 1 (resp. 2) at C_1 and C_2 as well, since one can choose to combine or not to combine (changing dimension 1 for 2) at one chosen place for a component of M, as the number of meeting places across dimension 2 is even. We may then replace each of the two edges across dimension 1 (resp. 2) at C to exchange with the edges (u, v) and (u, w) respectively in corresponding squares, thus joining C_1 and C_2 with C as desired, forming the Hamiltonian circuit. Clearly we have M^4 choices of possible M_{ij}, which proves $H \geq M^4$. ☐

Let M_d be the number of perfect matchings of the d-dimensional hypercube. Let H_d be the number of Hamiltonian circuits of the d-dimensional hypercube. It is known that as d tends to infinity, $M_d^{2/n}$ is asymptotic to d/e [3].

Corollary 2 *As d tends to infinity, $H_d \leq (d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$ (with $(d!)^{1/d}$ asymptotic to d/e), and H_d is at least asymptotic to $((d/e)(1-o(1)))^{n/2}$.*

Proof. The lower bound follows from Theorem 5 since $H_d \geq M_{d-2}^4$. The upper bound follows from the bounds mentioned in the introduction since we may select a matching in a graph of degree d, remove it, and select a matching in a graph of degree $d-1$. ☐

These upper and lower bounds improve the results by Dixon and Goodman [4], Douglas [5], Mollard [11], and Clark [2].

Corollary 3 *Let $G' = (U, V, E)$ be a regular bipartite graph, and let G be an n-vertex, d-regular graph that is the product of G' by a 2-dimensional cube Q_2. Suppose that G' has a Hamiltonian path. When d tends to infinity, the number of Hamiltonian circuits of G is at least $((d/e)(1-o(1)))^{n/2}$.*

Proof. Follows from Theorem 5 the remarks in the Introduction. ☐

The upper and lower bounds in Corollary 2 differ essentially by a square (a factor of two in the exponent). We can reduce this gap by considering decompositions into cycles instead of Hamiltonian circuits, where the cycles are required to be of length a multiple of 2^k for some k.

Theorem 6 *The number of decompositions of the d-dimensional hypercube into cycles of length a multiple of 2^k for $k = O(\sqrt{d})$ is at least $((d/(ek))(1-o(1)))^n$.*

Proof. Partition the d dimensions into k groups R_i of about d/k dimensions. If we combine together the dimensions in each group R_i and replace it by the parity of the bits in the group R_i, we obtain a k-dimensional hypercube that has a Hamiltonian cycle C of length 2^k. Back in the original hypercube, each edge in the reduced cycle C corresponds to choosing matchings in smaller cubes corresponding to the d/k dimensions of each cube in one group R_i of dimensions, where by the remarks in the Introduction the number of choices per vertex is about $(d/(ek))(1-o(1))$. Combining these choices of matchings of subcubes over all n vertices gives the stated bound. □

We now combine the approach of Theorems 5 and 6 to infer the following.

Theorem 7 *Let G be an n-vertex graph that is the Cartesian product of a square Q_2 and k regular bipartite graphs G_i of degrees $d_i \geq f$ that have a Hamiltonian path. Then G has at least $((f/e)(1+o(1)))^{n(1-(k+1)/2^k)}$ Hamiltonian circuits as d and k tend to infinity.*

Proof. If we do not take into account Q_2 and replace each bipartite graph G_i by two adjacent vertices v_0^i and v_1^i representing both sides of the bipartition, we obtain a k-dimensional hypercube that has a Hamiltonian cycle C that takes edges corresponding to (v_0^1, v_1^1) in alternation. For each occurrence of an edge (v_0^i, v_1^i) in C we may take a perfect matching in G_i, so by the remarks in the Introduction we have $((f/e)(1+o(1)))^n$ possible choices of matchings that give a decomposition of G into cycles, as n vertices have each about f/e choices.

Now for each choice of v_j^i, $i \geq 2$, that chooses v_0^i for all but at most one of the $i \geq 2$, replace the edge corresponding to (v_0^1, v_1^1) by the edge on dimension 1 of Q_2. This corresponds to $k+1$ choices of the 2^k edges of C, so the bound on the number of choices is reduced to $((f/e)(1+o(1)))^{n(1-(k+1)/2^k)}$. Now all cycles in such a choice go through dimension 1 of Q_2 with the choice v_0^i for $i \geq 2$. We may combine such cycles as in the proof of Theorem 5 by alternating dimensions 1 and 2 of Q_2 at the places that choose v_0^i for all but at most one of the $i \geq 2$. In this combination of cycles, we may choose one place to combine by switching dimensions 1 and 2 by a parity argument, as if two collections of such cycles meet in one place they must meet in another place, because an even number meet across dimension 2. Finally, we may consider the Hamiltonian path for each G_i and remove the last vertex with v_1^i. Combining these paths by traversing two edges at a time in G_1 for the whole path, then two edges in G_2, then G_1 path backwards, then two edges in G_2, and so on, we obtain a path P that visits alternatingly all vertices that have all v_0^i. We use this path as in the proof of Theorem 5 to finally combine all the cycles into a single cycle, where each square Q_2 that has two uses of dimension 1 or two uses of dimension 2 gets joined to at most two adjacent such squares on P. □

Corollary 4 *Let G be the n-vertex d-dimensional hypercube, with $n = 2^d$, and H_d be the number of Hamiltonian circuits of G. Then $((d \log 2/(e \log\log d))(1-o(1)))^n \leq H_d \leq ((d/e)(1+o(1)))^n$.*

Proof. The upper bound is from Theorem 2. For the lower bound, we apply Theorem 7 with $f = \lfloor (d-2)/k \rfloor$ and choose k such that $2^k/(k+1)^2 = \log d$. □

5 Matchings and Hamiltonian Circuits in Grids

Theorem 8 *Let G be an n-vertex d-dimensional grid, which is the Cartesian product of paths $P_1, P_2, \ldots, P_d$, where P_i has $r_i \geq 2$ vertices, with r_1 even. When d tends to infinity, the graph G has at least $((d/(2e))(1-o(1)))^{n/2}$ perfect matchings and at most $((2d)!)^{n/(4d)}$ perfect matchings.*

Proof. The upper bound follows from the remarks in the Introduction and the bound $2d$ on the degree.

For the lower bound, divide each path P_i of length r_i into $r'_i = \lfloor r_i/2 \rfloor$ matched pairs and at most one single additional vertex. This divides the grid into hypercubes of various dimensions. Suppose the r_i for $1 \leq i \leq k$ are even and the r_i for $k+1 \leq i \leq d$ are odd. Suppose we divide the $d-k$ odd dimensions into t dimensions for which we choose the r'_i matched edges and $d-k-t$ dimensions for which we choose the additional vertex. This gives cubes of $k+t$ dimensions with $(((k+t)/e)(1-o(1)))^{2^{k+t}/2}$ perfect matchings, and the number of such cubes is the product of $k+t$ factors r'_i. When we multiply these terms, the exponents add up to terms in the product of terms $2r'_i$ for $i \leq k$ and terms $2r'_i + 1$ for $i \geq k+1$, divided by 2, and this product is the product of the r_i divided by 2, which is $n/2$. This expression is significantly dominated by the terms with $k+t \geq d(1-o(1))/2$, giving the expression $(d(1-o(1))/(2e)^{n/2}$. □

Theorem 9 *Let G be the d-dimensional grid, the Cartesian product of paths $P_1, P_2, \ldots, P_d$, where P_i has $r_i \geq 2$ vertices. If $d = 1$, or all r_i are odd, then G does not have a Hamiltonian circuit. Otherwise ($d \geq 2$ and some r_i is even) G has a Hamiltonian circuit.*

Proof. If $d = 1$ then G is a path P_1 and does not have a Hamiltonian circuit. If all P_i have an odd number of vertices r_i, then G has an odd number of vertices $r = r_1 r_2 \cdots r_d$. A bipartite graph with an odd number of vertices cannot have a Hamiltonian circuit.

Suppose instead $d \geq 2$ and some P_i has r_i even, say P_1 has r_1 even. If $d = 2$, then a Hamiltonian circuit is obtained by going down P_1 at the left end of P_2, then going in the direction of backwards P_1 one vertex at a time, each time traversing P_2 back and forth while avoinding the left vertex of P_2 that was already visited. Since r_1 is even, the last time P_2 will be traversed backwards to its leftmost vertex where the circuit was started. If $d \geq 3$, assume inductively the result without P_d for the product G' of $P_1, P_2, \ldots, P_{d-1}$, giving a Hamiltonian circuit C'. Place the even edges of C' at one end of P_d and the odd edges of C' at the other end of P_d in G, and add all copies of the path P_d to obtain the Hamiltonian circuit. □

Theorem 10 *Let G be a d-dimensional grid that has a Hamiltonian circuit as in Theorem 9. The number of Hamiltonian circuits of G is at least $((d \log 2/(2e \log\log d))(1-o(1)))^n$ and at most $((2d)!)^{n/(4d)}((2d-1)!)^{n/(4d-2)}/2$. when d tends to infinity.*

Proof. The upper bound follows by the remarks in the Introduction by choosing a matching in a graph of degree at most $2d$, removing it, and choosing a matching in a graph of degree at most $2d-1$.

For the lower bound, decompose the grid into cubes as in Theorem 8, find Hamiltonian circuits in each subcube, with the asymptotics larger than for matchings by Corollary 4. To interconnect these subcubes, we contract the subcubes, replacing r_i by $\lceil r_i/2 \rceil$, while keeping r_1 even as r_i, and use on this smaller grid the cycle from Theorem 9. This requires entering and exiting each subcube at adjacent vertices x and y that have the edge (x, y) in the Hamiltonian cycle for the subcube. We choose x and y with all coordinates even except for at most two coordinates, and say x with an even number of odd coordinates. If the subcube must be exited in two dimensions in the direction that has an odd value in the dimension, then this determines the two odd value dimensions for x and the one odd value dimension for y. After exiting at such an x, there is one odd value dimension carried over from x into the next subcube, until we exit through an even value dimension, in which case the corresponding x will have all even values. This completes combining the cycles of the various subcubes, with the bounds following from Corollary 4 as in Theorem 8. □

References

[1] L.M. Brègman, Certain properties of nonnegative matrices and their permanents, Soviet Math. Dok., 14 (1973), 945–949.

[2] L.H. Clark, A new upper bound for the number of Hamiltonian cycles in the n-cube, J. Comb. Infor. System Sci. 25 (2000), 35–37.

[3] L.H. Clark, J.C. George, and T.D. Porter, On the number of 1-factors in the n-cube, Congr. Number 127 (1997), 67–69.

[4] E. Dixon and S. Goodman, On the number of Hamiltonian circuits in the n-cube, Proc. Amer. Math. Soc., 50 (1975), 500–504.

[5] R.J. Douglas, Bounds on the number of Hamiltonian cirucits in the n-cube, Discrete Math., 17 (1977), 143–146.

[6] C.P. Egoryčev, A solution of van der Waerden's permanent problem, Dokl. Akad. Nauk SSSR 258 (1981), 1041–1044.

[7] D.I. Falikman, A proof of van der Waerden's conjecture on the permanent of a doubly stochastic matrix, Mat. Zametki 29 (1981), 931–938.

[8] M.E. Fisher, Statistical mechanics of dimers on a plain lattice, Phys. Rev., 124 (1961), 278–286.

[9] P.W. Kasteleyn, The statistics of dimers on a lattice I: The number of dimer arrangements on a quadratic lattice', Physica, 27 (1961), 1209–1225.

[10] H. Minc, Upper bound for permanents of (0,1)-matrices, Bull. Amer. Math. Soc., 69 (1963), 789–791.

[11] M. Mollard, Un nouvel encadrement du nombre de cycles hamiltoniens du n-cube, European J. Combinatorics, 9 (1988), 49–52.

[12] B.L. van der Waerden, Aufgabe 45, Jber. Deutsch. Math.-Verein. 35 (1926), 117.

ECCC ISSN 1433-8092

http://eccc.hpi-web.de/