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A Combinatorial Geometric Approach to Linear Image Matching*

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Abstract. The problem of image matching is to find for two given digital images A and B an admissible transformation that converts image A as close as possible to B. This problem becomes hard if the space of admissible transformations is too complex. Consequently, in many real applications, like the ones allowing nonlinear elastic transformations, the known algorithms solving the problem either work in exponential worst-case time or can only guarantee to find a local optimum. In this paper we study the image matching problem for affine transformations, an important class of functions, from the image matching point of view, and we give a generic exhaustive search algorithm solving the problem in polynomial time. Next, we apply the algorithm for some important subclasses of affine transformations like translations, rotations, scalings, and linear transformations and we prove lower and upper bounds for the corresponding search spaces. Furthermore we extend the results to projective transformations which are a natural generalization of affine transformations.

1 Introduction

Image matching is a canonical problem in image processing and in many other related fields like computer vision, medical imaging, pattern recognition, and digital watermarking. In general, the Image Matching Problem (IMP, for short) is that of finding for two given digital images A and B and some space $\mathcal{F}$ of admissible transformations, a transformation $f \in \mathcal{F}$ that changes A closest to B under some image distortion measure. The problem was intensively studied both experimentally and theoretically by using different approaches ranging from discrete methods to techniques based on continuous analysis (for an overview we refer to [5, 4, 13, 17, 16, 1] and the references therein).

We model a digital image A in a standard way as a two dimensional array over the finite set of integers $\Sigma = \{0, 1, \ldots, \sigma\}$ where each item A_{ij} represents a gray value of the pixel with coordinates (i, j). For simplicity's sake, assume $-n \leq i, j \leq n$, and let $A_{ij} = 0$, if either $|i| > n$ or $|j| > n$. We let $\mathcal{N} = \{-n, \ldots, 0, \ldots, n\}$ and call $\mathcal{N} \times \mathcal{N}$ the *support of the image* A. The pixel (i, j) is a unit square in the real plane $\mathbb{R}^2$ with the geometric center point (i, j). Thus the pixels for A cover a square area of size $(2n+1) \times (2n+1)$ with the geometric center point $(0, 0)$. A transformation f of an image A is an arbitrary injective mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$. Transformations of particular importance from the image matching point of view fulfill some additional constraints like smoothness and elasticity. Specifically such functions as rotations, scalings, translations, affine and some nonlinear elastic transformations play an important role in this area. Applying a transformation f on A we get the image $f(A)$, which is a two dimensional array over Σ with indices ranging in the same interval as in A. The gray value of the pixel (i, j) in $f(A)$ is equal to the value of the pixel (i', j') of the image A such that $f^{-1}(i, j)$ lies in the unit square with the geometric center point (i', j') (for an example see Fig. 1). For two images A and B of the same size the distortion between A and B is measured by $\sum \delta(A_{ij}, B_{ij})$ where $\delta(a, b)$ is a function charging mismatches, for example, $\delta(a, b) = |a - b|$.

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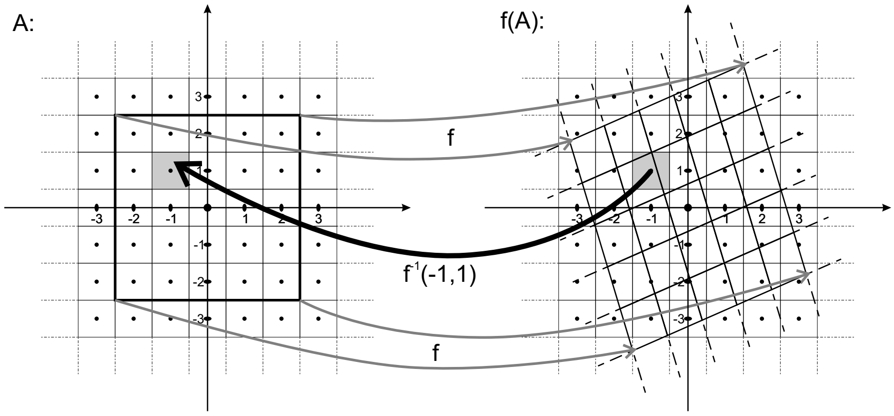

Fig. 1. Image A and the transformed image $f(A)$. The gray value of the pixel $(-1,1)$ in $f(A)$ is equal to the value of the pixel $(-1,1)$ of A since in $\mathbb{Z}^2$ the point $(-1,1)$ is the closest one to the point $f^{-1}(-1,1)$.

The Image Matching Problem is hard if the set of admissible transformations $\mathcal{F}$ is too complex. Subsequently, known image matching algorithms for optimal or approximate solutions for classes such as nonlinear elastic transformations (see e.g. [18]) use exponential resources. In [15] Keysers and Unger have proved that the decision problem corresponding to the Image Matching Problem for this important class of transformations is NP-complete, thus giving evidence that the known exponential time algorithms are justified.

On the other hand, allowing only translations, rotations and scalings the problem becomes tractable [16, 10, 11, 1–3, 12]. For example, restricting the problem to rotations the Image Matching Problem can be solved in time $O(n^4)$ which is quadratic with respect to to the input size [2]. The situation changes drastically if one enlarges the class of transformations to all affine transformations, which will be denoted $\mathcal{F}_\mathtt{a}$ in this paper. Image matching under affine transformations has been intensively studied both experimentally and theoretically. In image processing, for example, the classical approach to the problem is to transform the images A and B into a space where certain affine distortions correspond to simple transformations like translations. The advantage of such an approach is that image matching can be easily done by exhaustive search in the set of translations. Though the methods work quite efficiently for specific classes, they do not work for arbitrary affine transformations and it is still open what is the computational complexity of image matching for this class of transformations. Recently we have given a partial solution to this problem [12] proving that the image matching under affine transformations can be solved in polynomial time.

In this paper we investigate the Image Matching Problem for some classes of affine transformations. An affine transformation $f \in \mathcal{F}_\mathtt{a}$ is a function $f(x,y) = M \cdot (x,y)^T + t$ with M an invertible (2×2)-matrix and t a vector in $\mathbb{R}^2$. We give a generic exhaustive search algorithm solving the Image Matching Problem for $\mathcal{F}_\mathtt{a}$ in polynomial time and prove some combinatorial bounds on the size of the search space. Next, we apply the algorithm for other important subclasses of $\mathcal{F}_\mathtt{a}$ like translations ($\mathcal{F}_\mathtt{t}$), scalings ($\mathcal{F}_\mathtt{s}$), rotations ($\mathcal{F}_\mathtt{r}$), combined scalings and rotations ($\mathcal{F}_\mathtt{sr}$) and linear transformations ($\mathcal{F}_\mathtt{l}$) and we prove lower and upper bounds for the size of the corresponding search spaces. Finally we extend the polynomial search strategy to projective transformations $\mathcal{F}_\mathtt{p}$ which are a natural and important generalization of affine transformations.

The main contribution of this paper is a new general exhaustive search technique for image matching under some specific transformations like $\mathcal{F}_\mathtt{a}$, $\mathcal{F}_\mathtt{p}$ or its subclasses and a combinatorial method for the analysis of the size of the corresponding search spaces. For any injective function $f : \mathbb{R}^2 \to \mathbb{R}^2$ in some specific space $\mathcal{F} \subseteq \mathcal{F}_\mathrm{p}$ of admissible transformations, we define the corresponding coordinate mapping $\gamma : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ which for any coordinate (i, j) in $f(A)$ determines the coordinate (i', j') in A. Thus, the mapping γ is a discrete representation of f which uniquely describes $f(A)$. We denote the set of all coordinate mappings γ corresponding to functions $f \in \mathcal{F}$ by $\Gamma_n(\mathcal{F})$.

Our exhaustive search algorithm has to search the space $\Gamma_n(\mathcal{F})$ to solve IMP under transformations $\mathcal{F}$. But there are two difficulties in this approach:

1. how to enumerate efficiently all elements in $\Gamma_n(\mathcal{F})$ and
2. how to estimate the size of $\Gamma_n(\mathcal{F})$?

Both problems seem to be highly nontrivial even for such simple transformations like rotations (see e.g. [1]). In this paper we present a powerful geometrical approach to solve both problems. The presented techniques work for the classes of affine transformations, projective transformations as well as for important subclasses such as translations, scalings, rotations, combined scalings and rotations, and linear transformations. The table below summarizes our main results.

Admissible transformations	Notation	Cardinality of search space	Complexity of the IMP	Remarks
Projective	$\mathcal{F}_\mathrm{p}$	$\Omega(n^{12}) \cap O(n^{24})$	$O(n^{24})$	
Affine	$\mathcal{F}_\mathrm{a}$	$\Omega(n^{12}) \cap O(n^{18})$	$O(n^{18})$	(a)
Linear	$\mathcal{F}_\mathrm{l}$	$\Omega(n^{10}) \cap O(n^{12})$	$O(n^{12})$	
Scalings and rotations	$\mathcal{F}_\mathrm{sr}$	$\Omega(n^{5}) \cap O(n^{6})$	$O(n^{6})$	
Rotations	$\mathcal{F}_\mathrm{r}$	$\Theta(n^3)$	$O(n^3 \log n)$	(b)
Scalings	$\mathcal{F}_\mathrm{s}$	$\Theta(n^2)$	$O(n^3)$	
Translations	$\mathcal{F}_\mathrm{t}$	$\Theta(n^2)$	$O(n^4)$	

Table 1. The cardinalities of search spaces $\Gamma_n(\mathcal{F})$ and the complexities of our image matching algorithms for particular admissible transformations $\mathcal{F}$: from projective transformations to translations. Remarks: (a) In [12] it has been given the first algorithm solving IMP for $\mathcal{F}_\mathrm{a}$ in polynomial time. (b) The lower bound $\Omega(n^3)$ has been shown in [1]; In this paper we show a new proof for this bound and give an algorithm solving the IMP under rotations almost optimal.

The paper is organized as follows. In Section 2 some basic preliminaries and definitions are given. Section 3 discusses structural properties of the search spaces $\Gamma_n(\mathcal{F})$. In Section 4 we give the general exhaustive search algorithm for image matching and apply it to affine transformations. In the subsequent Sections 5, 6, and 7 we show how the algorithm may be used for the subclasses of affine transformations like translations, rotations, scalings, and linear transformations and we prove there lower and upper bounds for the corresponding search spaces.

2 Preliminaries

Let A be an image with support $\mathcal{N} \times \mathcal{N}$ and let f be an arbitrary injective function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Define for $g = f^{-1}$ the mapping $\gamma_g : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \times \mathcal{N} \cup \{\perp\}$ which determines for any pixel coordinate (i, j) in $f(A)$ the corresponding coordinate (i', j') in A:

$$\gamma_g(i,j) = \begin{cases} [g(i,j)] & \text{if } [g(i,j)] \in \mathcal{N} \times \mathcal{N}, \\ \perp & \text{otherwise.} \end{cases}$$

Here, $\perp$ is an extra symbol and $[(x, y)] := ([x], [y])$ denotes rounding all components of a vector $(x, y) \in \mathbb{R}^2$. We formally define the image $f(A)$ as follows: for any $(i, j) \in \mathcal{N} \times \mathcal{N}$ the gray value of the pixel (i, j) in $f(A)$ is equal to the value of the pixel $\gamma_g(i, j)$ in A if $\gamma_g(i, j) \neq \perp$; Otherwise, we let the value equal 0.

We assume that there is a function $\delta : \Sigma \times \Sigma \to \mathbb{N}$, measuring differences in gray values. In this paper we assume the evaluation of $\delta(a, b)$ has constant costs. Then for images A and B with support $\mathcal{N} \times \mathcal{N}$ we measure the *distortion* $\Delta(A, B)$ between A and B as the sum of pixel differences, i.e.,

$$\Delta(A, B) = \textstyle\sum_{(i,j)\in\mathcal{N}^2} \delta(A_{ij}, B_{ij}).$$

Obviously, the costs for the computation of $\Delta(A, B)$ is in $O(n^2)$. Furthermore, in this paper we will assume that elementary arithmetic operations have unit costs.

We call the following optimization problem the Image Matching Problem (IMP) for $\mathcal{F}$:

Problem 1. For a given reference image A and a distorted image B, both of the same size $(2n + 1) \times (2n + 1)$, find an injective transformation $f \in \mathcal{F}$ minimizing the distortion $\Delta(f(A), B)$.

In this paper we are mainly interested in affine transformations $\mathcal{F}_{\mathsf{a}}$ and some of their important subclasses as well as projective transformations. Affine transformations f have the form: $f(x, y) = M \cdot (x, y)^T + t$ where M is an invertible (2×2)-matrix and t a vector in $\mathbb{R}^2$. Clearly, for all affine transformations the matrix $M = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ contains the four parameters a_1 to a_4 and the vector $t = \left(\begin{smallmatrix} a_5 \\ a_6 \end{smallmatrix}\right)$ the two additional parameters a_5 and a_6. Hence, each affine transformation can be characterized by a six dimensional vector $(a_1, \ldots, a_6)^T$ and thus $\mathbb{R}^6$ can be seen as a parameter space for $\mathcal{F}_{\mathsf{a}}$. A six dimensional vector $u = (a_1, \ldots, a_6)^T$ defines in a natural way an affine transformation g_u if the matrix defined by a_1 to a_4 is invertible. To avoid problems with exceptional vectors we denote by $(\mathbb{R}^6)_{\mathsf{a}}$ the subset of $\mathbb{R}^6$ which contains only the parameter vectors encoding invertible matrices.

Though the image $f(A)$ is described by the transformation f the pixel values of $f(A)$ are determined by the inverse transformation f^{-1}. Similarly as in [12] this fact is stressed by using g for f^{-1}. Denote by $\mathcal{F}_{\mathsf{a}}^{-1}$ the set of all inverse transformation for $\mathcal{F}_{\mathsf{a}}$. Notice that, according to our definition, $\mathcal{F}_{\mathsf{a}}$ is closed under inversion, i.e., $\mathcal{F}_{\mathsf{a}}^{-1} = \mathcal{F}_{\mathsf{a}}$, which means that each inverse affine transformation is an affine transformation itself.

To solve IMP we have to search $\mathcal{F}_{\mathsf{a}}^{-1}$ for a transformation that minimizes the distortion. Hence we call $\mathcal{F}_{\mathsf{a}}^{-1}$ the search space of IMP. The search space cannot be enumerated though the set $\{f(A) \mid f \in \mathcal{F}_{\mathsf{a}}\}$ can be. Thus we give a discretization transferring $\mathcal{F}_{\mathsf{a}}^{-1}$ into a discrete counterpart, which we will denote by $\Gamma_n(\mathcal{F}_{\mathsf{a}})$. We define the discretization $\Gamma_n(\mathcal{F})$ for an arbitrary class $\mathcal{F}$ of injective functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ as follows. Let $\Gamma_n = \{\gamma : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \times \mathcal{N} \cup \{\perp\}\}$. Then

$$\Gamma_n(\mathcal{F}) = \{\gamma_{f^{-1}} \in \Gamma_n \mid f \in \mathcal{F}\}.$$

Because we have already established a connection between $\mathcal{F}_{\mathsf{a}}$ and $(\mathbb{R}^6)_{\mathsf{a}}$ it is straightforward to connect also $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ to $(\mathbb{R}^6)_{\mathsf{a}}$. For all u in $(\mathbb{R}^6)_{\mathsf{a}}$ let γ_x denote the mapping γ_g for the transformation g that corresponds to the point u.

For our geometrical approach we need some further definitions. Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{R}^d$. Any plane

$$H : a_1x_1 + a_2x_2 + \ldots + a_dx_d = b$$

in $\mathcal{H}$ divides $\mathbb{R}^d$ into two subspaces:

$$H^+ = \{x \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d \geq b\} \quad \text{and} \quad H^- = \{x \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d < b\}.$$

Furthermore, we let

$$h^0 = \{x \in \mathbb{R}^d \mid a_1x_1 + \ldots + a_dx_d = b\}$$

and define $h^+ = H^+ \setminus h^0$ and $h^- = H^-$.

For convenience we will allow, that hyperplanes of $\mathcal{H}$ can coincide with each other, i.e., that $\mathcal{H}$ can contain $H_1 : a_1x_1 + a_2x_2 + \ldots + a_dx_d = b$ and $H_2 : \hat{a}_1x_1 + \hat{a}_2x_2 + \ldots + \hat{a}_dx_d = \hat{b}$ which describe the same hyperplane in the space $\mathbb{R}^d$. Nevertheless, we will assume that $\mathcal{H}$ contains both representations of the hyperplane: H_1 and H_2. By $\tilde{\mathcal{H}}$ we mean the subset of $\mathcal{H}$ obtained by removing all duplicate hyperplanes. Particularly, the case that H_1 and H_2 (as defined above) both belong to $\tilde{\mathcal{H}}$ is not possible. We say that $\mathcal{H}$ is *simple* if every d hyperplanes of $\mathcal{H}$ have a unique point in common and any $d+1$ hyperplanes have no point in common.

For a finite set of hyperplanes $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ consider the following two kinds of partition of $\mathbb{R}^d$ into convex subspaces determined by the hyperplanes in $\mathcal{H}$:

1. $\mathcal{C}(\mathcal{H}) = \{C \subseteq \mathbb{R}^d \mid C = \bigcap_{i=1}^t H_i^{s_i} \text{ for some } s_1, \ldots, s_t \in \{+, -\}\}$.
2. $\mathcal{A}(\mathcal{H}) = \{C \subseteq \mathbb{R}^d \mid C = \bigcap_{i=1}^t h_i^{s_i} \text{ for some } s_1, \ldots, s_t \in \{+, -, 0\}\}$.

It is obviously the case that $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\tilde{\mathcal{H}})$ and $|\mathcal{C}(\mathcal{H})| \leq |\mathcal{A}(\tilde{\mathcal{H}})|$.

We call the elements of $\mathcal{C}(\mathcal{H})$ cells and the elements of $\mathcal{A}(\tilde{\mathcal{H}})$ faces. A cell (face) is called a k-cell (k-face) if its dimension is k. A face φ is said to be subface of another face φ' if the dimension of φ is one less than the dimension of φ' and φ is contained in the boundary of φ'. If φ is a subface of φ' then we also say that φ and φ' are incident and that φ' is a superface of φ.

The incidence graph $\mathcal{I}(\tilde{\mathcal{H}})$ of $\mathcal{A}(\tilde{\mathcal{H}})$ is defined as follows: each face φ of $\tilde{\mathcal{H}}$ contains a node $v(\varphi)$ in $\mathcal{I}(\tilde{\mathcal{H}})$ that represents φ. If two faces φ and φ' are incident upon each other then $v(\varphi)$ and $v(\varphi')$ are connected by an edge. The incidence graph is described in detail in [7] (see also [6]).

Let x be a point in $\mathbb{R}^d$. Then by $C_{\mathcal{H}}(x)$ we denote the cell in $\mathcal{C}(\mathcal{H})$ that contains x.

3 Structural Properties of the Search Space

In this section we will present our basic techniques to determine the structure of the search space for specific classes of transformations. The techniques are general and we will show how they work for translations, scalings, rotations, combined scalings and rotations, linear transformations, and affine transformations. Instead of proving the structural properties of search spaces for every class separately, in this section we will just give a proof for the most general case, i.e. for affine transformations $\mathcal{F}_{\mathsf{a}}$ and in the subsequent sections we will show how to adapt this generic proof for the specific subclasses. For projective transformations the structural properties of the search space are similar but not equal and we will handle this case in Section 7.

The following definition relates the set $\Gamma_n(\mathcal{F})$ with the set of cells defined by a set $\mathcal{H}$ of hyperplanes:

Definition 1. *Let $\mathcal{F} \subseteq \mathcal{F}_{\mathsf{a}}$ be a class of injective functions and let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{R}^d$. We say that*

$$\Gamma_n(\mathcal{F}) \cong \mathcal{C}(\mathcal{H})$$

if and only if for all $u, u' \in \mathbb{R}^d$ with $g_u, g_{u'} \in \mathcal{F}^{-1}$ it is true: $\gamma_{g_u} = \gamma_{g_{u'}} \Leftrightarrow C_{\mathcal{H}}(u) = C_{\mathcal{H}}(u')$.

To characterize the search space $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ we define the following hyperplanes. For all $i, j \in \mathcal{N}$ and $i', j' \in [-n-1, n+1]$ let

$$X_{iji'} : ix_1 + jx_2 + x_5 + (0.5 - i') = 0 \text{ and}$$
$$Y_{ijj'} : ix_3 + jx_4 + x_6 + (0.5 - j') = 0$$

be hyperplanes in $\mathbb{R}^6$. We denote by $\mathcal{H}_{\mathsf{a},n}$ the set of all planes $X_{iji'}$ and $Y_{ijj'}$ for all $i, j \in \mathcal{N}$ and $i', j' \in [-n-1, n+1]$. Now we are ready to give the main result of this section.

Theorem 1. $\Gamma_n(\mathcal{F}_{\mathsf{a}}) \cong \mathcal{C}(\mathcal{H}_{\mathsf{a},n})$.

Before we give the proof note that by the theorem it suffices to estimate the number of cells in $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$ to get a bound on the cardinality of $\Gamma_n(\mathcal{F}_{\mathsf{a}})$. To get an upper bound on $|\mathcal{C}(\mathcal{H}_{\mathsf{a},n})|$ we will estimate $|\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{a},n})|$. The limitations for i, j, i' and j' imply that the number of planes in $\tilde{\mathcal{H}}_{\mathsf{a},n}$ is in $O(n^3)$. Any set $\mathcal{H}$ of $O(n^3)$ hyperplanes partitions $(\mathbb{R}^6)_{\mathsf{a}}$ into at most $\sum_{k=0}^{6}\sum_{\ell=k}^{6}\binom{|\mathcal{H}|}{\ell}\binom{\ell}{k} = O(|\mathcal{H}|^6) = O(n^{18})$ faces. This gives that the cardinality of $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ is in $O(n^{18})$, which is polynomial in n. For detailed information on hyperplanes and the corresponding partitions of the space $\mathbb{R}^d$ we refer the reader to Edelsbrunner [6] or de Berg et al. [9] and to the next section.

Proof (of Theorem 1). To analyze the space $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ we define the set $\mathcal{R}_{\mathsf{a}}$ to be the following equivalence relation on $(\mathbb{R}^6)_{\mathsf{a}} \times (\mathbb{R}^6)_{\mathsf{a}}$:

$$\mathcal{R}_{\mathsf{a}} = \{(u_1, u_2) \mid u_1, u_2 \in (\mathbb{R}^6)_{\mathsf{a}} \text{ and } \gamma_{u_1} = \gamma_{u_2}\}.$$

Thus the relation $\mathcal{R}_{\mathsf{a}}$ partitions $\mathcal{F}_{\mathsf{a}}^{-1}$ into subsets of transformations of equal discrete counterparts. The following lemma gives the major structural property of $\mathcal{R}_{\mathsf{a}}$.

Lemma 1. *Two vectors $u, v \in (\mathbb{R}^6)_{\mathsf{a}}$ belong to the same equivalence class of $\mathcal{R}_{\mathsf{a}}$ if and only if for all $i, j \in \mathcal{N}$ and any $i', j' \in [-n-1, n+1]$ the vectors u and v belong to the same half-subspace according to the partition of $(\mathbb{R}^6)_{\mathsf{a}}$ with the hyperplane $X_{iji'}$, respectively $Y_{ijj'}$.*

Proof. Let $u = (a_1, \ldots, a_6)$ and $v = (b_1, \ldots, b_6)$ be two parameter vectors from $(\mathbb{R}^6)_{\mathsf{a}}$. By definition u and v belong to the same equivalence class in $\mathcal{R}_{\mathsf{a}}$, if and only if γ_u equals γ_v.

$\Longrightarrow$: Let $\gamma_u = \gamma_v$ hold but for the contradiction assume that there is at least one $(i, j) \in \mathcal{N} \times \mathcal{N}$ and $i', j' \in [-n-1, n+1]$ such that with respect to $X_{iji'}$ or $Y_{ijj'}$, u and v belong to different half-spaces of $(\mathbb{R}^6)_{\mathsf{a}}$. Let without loss of generality assume that $X_{iji'}(u) < 0$ and $X_{iji'}(v) \geq 0$. Then it holds that

1. $ia_1 + ja_2 + a_5 + (0.5 - i') < 0$ and
2. $ib_1 + jb_2 + b_5 + (0.5 - i') \geq 0$.

But this means that

1. $[a_1 i + a_2 j + a_5] < i'$ and
2. $[b_1 i + b_2 j + b_5] \geq i'$

which implies that γ_u and γ_v differ at least for the argument (i, j). An analogous proof holds for the case when u and v are separated by $Y_{ijj'}$.

$\Longleftarrow$: Let for all $(i, j) \in \mathcal{N} \times \mathcal{N}$ and $i', j' \in [-n-1, n+1]$ the vectors u and v belong to the same half-space of $(\mathbb{R}^6)_{\mathsf{a}}$ with respect to $X_{iji'}$ and $Y_{ijj'}$ but assume that there exist $(i, j) \in \mathcal{N} \times \mathcal{N}$ such that $\gamma_u(i, j) = (i'_1, j'_1) \neq (i'_2, j'_2) = \gamma_v(i, j)$. Without loss of generalization assume that i'_1 and i'_2 differ and

1. $a_1 i + a_2 j + a_5 < i'_1 - 0.5$,
2. $b_1 i + b_2 j + b_5 \geq i'_2 - 0.5$, and
3. $i'_1 < i'_2$ which implies that
4. $a_1 i + a_2 j + a_5 < i'_2 - 0.5$

Notice that either i'_1 or i'_2 must be in $\mathcal{N}$ since otherwise $\gamma_u(i,j) = \gamma_v(i,j) = \perp$. Hence, if $i'_2 \leq n+1$ then u and v belong to different subspaces according to plane $X_{iji'_2}$ and else u and v are still separated by $X_{iji'}$ with $i' = (n+1)$. Analogously proof holds for the case when j'_1 and j'_2 differ.□

Obviously the points u which represent inverse affine transformations $g = f^{-1}$ with equal discretizations γ_g fall together in cells of $(\mathbb{R}^6)_{\mathsf{a}}$ which are defined by the set $\mathcal{H}_{\mathsf{a},n}$ of hyperplanes. Thus, every such cell of $(\mathbb{R}^6)_{\mathsf{a}}$ gives a unique discrete affine transformation. This concludes the proof of the theorem. □

4 The Polynomial Time Exhaustive Search Algorithm

In the previous section we have shown that the set of all coordinate mappings $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ corresponding to affine transformations f is isomorph to $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$, the set of all cells determined by the set $\mathcal{H}_{\mathsf{a},n}$ of hyperplanes. To solve the IMP for $\mathcal{F}_{\mathsf{a}}$ we will use this characterization and instead of searching $\Gamma_n(\mathcal{F}_{\mathsf{a}})$ directly, we will perform an exhaustive search of $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$ choosing for each convex cell in $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$ one representative point $(a_1, \ldots, a_6)$ of $(\mathbb{R}^6)_{\mathsf{a}}$ encoded by rational numbers of length $O(\log n)$. The point represents an inverse transformation g of f. Hence, using the mapping γ_g the distortion between $f(A)$ and B can be computed easily. The property above guarantees that in this way all coordinate mappings corresponding to affine transformations will be tested.

To search the set $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$ one can in fact search $\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{a},n})$ since for each cell C in $\mathcal{C}(\mathcal{H}_{\mathsf{a},n})$ there exists at least on face φ of $\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{a},n})$ which is contained in C. Our algorithm performs the searching of $\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{a},n})$ traversing the corresponding incidence graph $\mathcal{I}(\tilde{\mathcal{H}})$ described in Section 2.

The algorithm below solves the image matching problem for affine transformations but, as will be shown in the next sections, it also enables an efficient image matching for some other classes of transformations. Therefore, the description of the algorithm is general in the sense that it can handle any class of transformations which can be characterized by an appropriate set $\mathcal{H}$ of hyperplanes in $\mathbb{R}^d$. In the algorithm below, we will assume that the set $\mathcal{H}$ consists of hyperplanes $X_{iji'}$ and $Y_{ijj'}$, for all $i,j \in \mathcal{N} \times \mathcal{N}$, and $-n-1 \leq i', j' \leq n+1$ and that every $X_{iji'}$ and $Y_{ijj'}$ is a hyperplane in $\mathbb{R}^d$ that depends on parameters i, j, i' and i, j, j', respectively.

Theorem 2 (see e.g. [6]). *The incidence graph $\mathcal{I}(\tilde{\mathcal{H}})$ for the set $\tilde{\mathcal{H}}$ of m hyperplanes in $\mathbb{R}^d$ contains $O(m^d)$ nodes and edges. Moreover, $\mathcal{I}(\tilde{\mathcal{H}})$ can be constructed in $O(m^d)$ time.*

In a standard implementation of $\mathcal{I}(\tilde{\mathcal{H}})$ each node $v(\varphi)$ is a record containing some auxiliary information and two lists containing pointers to all subfaces and superfaces. In our setting the following additional auxiliary information are stored for every node $v(\varphi)$: coordinates of a representative point $p(\varphi)$ of φ and a set *Planes*(φ) of hyperplanes in $\mathcal{H}$.

The coordinates of a representative point $p(\varphi)$ of a 0-face φ is just the vertex φ itself, i.e., we get $p(\varphi) := \varphi$ in this case. If $\varphi_1, \varphi_2, \ldots, \varphi_t$ are the subfaces of φ and $t \geq 2$, then $p(\varphi) := \frac{1}{t}\sum_{\ell=1}^{t} p(\varphi_\ell)$. Without loss of generality, we will assume that $t \geq 2$ for every φ; if not we add some artificial border hyperplanes. Note that $p(\varphi)$ can be encoded by rational numbers of length $O(\log n)$. The sets *Planes* are defined as follows. For all d-faces φ we let *Planes*$(\varphi) := \emptyset$. If φ is a $(d-1)$-face determined by the hyperplane h then *Planes*$(\varphi) := \{X_{iji'}, Y_{ijj'} \in \mathcal{H} \mid X_{iji'} = h \text{ and } Y_{ijj'} = h\}$. If $\varphi_1, \varphi_2, \ldots, \varphi_t$ are the superfaces of a k-face φ, with $k < d-1$, then *Planes*$(\varphi) := \bigcup_{\ell=1}^{t}$ *Planes*(φ_ℓ).

To solve the image matching problem for given images A and B under $\mathcal{F}_{\mathsf{a}}$ an exhaustive search algorithm can work as follows: visit systematically all faces of the incidence graph $\mathcal{I}(\tilde{\mathcal{H}})$, for $\mathcal{H} = \mathcal{H}_{\mathsf{a},n}$ and for each face φ estimate the distortion between $f(A)$ and B for some f determined by φ. Notice that for all points u of φ the discretizations γ_u are equal to each other. Particularly, they are equal for all points $u \in \varphi$ which represent inverse affine transformations $g = f^{-1}$ we are looking for. Thus to compute pixelvalues of $f(A)$ and to estimate the distortion between $f(A)$ and B the algorithm can use γ_u for an arbitrary $u \in \varphi$, e.g., $u = p(\varphi)$. The result of the algorithm is the function f of minimum distortion. The time complexity of such a method is at least the size of the graph $\mathcal{I}(\tilde{\mathcal{H}})$ times $O(n^2)$, where the last term describes the cost of distortion estimation. However, using our approach we can improve this complexity for specific transformations. In fact, when considering $\mathcal{F}_{\mathtt{sr}}$, $\mathcal{F}_{\mathtt{l}}$, $\mathcal{F}_{\mathtt{a}}$ or $\mathcal{F}_{\mathtt{p}}$ our algorithm works in time linear in the size of $\mathcal{I}(\tilde{\mathcal{H}})$ and for $\mathcal{F}_{\mathtt{s}}$ the advantage is $O(n)$. For $\mathcal{F}_{\mathtt{t}}$ no improvement is achieved.

Our algorithm performs depth first search of the graph $\mathcal{I}(\tilde{\mathcal{H}})$. The algorithm works as follows. Visiting a node $v(\varphi)$ the algorithm stores the current distortion value between $f(A)$ and B for f determined by $p(\varphi)$. Next, traversing from φ to an incident (sub or super)face φ' the algorithm updates only the pixel values of $f(A)$, coordinates of which correspond to the parameters of hyperplanes we have just left or entered when traversing from φ to φ'. Speaking more precisely, if f' is a function determined by $p(\varphi')$ then the only difference between $f'(A)$ and $f(A)$ are the pixel values of coordinates (i, j) such that i, j are the parameters of hyperplanes we have just left or entered. Thus, to compute the distortion between $f'(A)$ and B, it is enough to update only the pixel values for those coordinates.

Now we are ready to give a complete algorithm solving IMP for $\mathcal{F}_{\mathtt{a}}$. For input images A and B we call the procedure IM(A, B) below for $\mathcal{F} := \mathcal{F}_{\mathtt{a}}$ and with the set of hyperplanes $\mathcal{H} := \mathcal{H}_{\mathtt{a},n}$.

Procedure IM(A, B); /* Image Matching for the admissible transformations $\mathcal{F}$ */
Input: Images A and B of size $(2n + 1) \times (2n + 1)$.
Output: Transformation $f = \arg\min_{f' \in \mathcal{F}}\{\Delta(f'(A), B)\}$.
Parameter: The set of hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$.

Procedure SEARCH$(v(\varphi))$; /* Depth first searching */
begin
  mark node $v(\varphi)$ as visited;
  for each neighbor $v(\varphi')$ of $v(\varphi)$ **do**
  if $v(\varphi')$ not visited **then begin**
    UPDATE(φ, φ'); SEARCH$(v(\varphi'))$; UPDATE(φ', φ);
  end;
end;
begin /* Procedure IM */
  let $\tilde{\mathcal{H}} := \{h | h = X_{i,j,i'}$ or $h = Y_{i,j,j'}$ for some $i, j, i'j'\}$;
  construct the incidence graph $\mathcal{I}(\tilde{\mathcal{H}})$;
  let φ_0 be a face of $\mathcal{A}(\tilde{\mathcal{H}})$ which corresponds to the identity mapping;
  let $\varphi_{opt} := \varphi_0$; $err, min\_err := \Delta(A, B)$ and let $T := A$;
  for all $-n \le i, j \le n$ **do** $x[i, j] := i$; $y[i, j] := j$;
  set all nodes in $\mathcal{I}(\tilde{\mathcal{H}})$ as not visited;
  call SEARCH$(v(\varphi_0))$; /* find φ_{opt} */
  $g :=$SELECT$_{\mathcal{F}}(\varphi_{opt})$; /* get $g \in \mathcal{F}$ with representative in φ_{opt} */
  return $f := g^{-1}$;
end.

The procedure UPDATE is called with two incident faces φ_1 and φ_2 and starts with global variables err, A, B and T such that for the representative point $u = p(\varphi_1)$ of φ_1, $T_{ij} = A_{\gamma_u(i,j)}$ for all $-n \le i, j \le n$, and $\Delta(T, B) = err$. The procedure updates T and err for the face φ_2 and if $err < min\_err$ then it modifies $min\_err := err$ and $\varphi_{opt} := \varphi_2$. The time complexity of UPDATE's execution with parameters φ_1 and φ_2 is proportional to the cardinality of $Planes(\varphi_2)$

if φ_2 is a subface of φ_1 and to the cardinality of $Planes(\varphi_1)$, otherwise. Below we give the complete algorithm for the procedure UPDATE.

Procedure UPDATE(φ_1, φ_2);
Input: Two incident faces φ_1 and φ_2.
Global Variables: $\varphi_{opt}, min\_err, err, A, B$ and T such that for the representative point $u = p(\varphi_1)$ of φ_1, $T[i,j] = A[\gamma_u(i,j)]$ for all $-n \le i, j \le n$, and $\Delta(T, B) = err$.
Task: Update T and err for the face φ_2 and if $err < min\_err$ then update min_err and φ_{opt}.

---

begin
  initialize empty stack S;
  if φ_2 is a subface of φ_1 **then begin**
    for all $X_{iji'} \in Planes(\varphi_2)$ **do**
      if $X_{iji'}(p(\varphi_1)) < 0$ **then** $x[i,j] := i'$; PUSH($(i,j), S$);
    for all $Y_{ijj'} \in Planes(\varphi_2)$ **do**
      if $Y_{ijj'}(p(\varphi_1)) < 0$ **then** $y[i,j] := j'$; PUSH($(i,j), S$);
  end
  else if φ_2 is a superface of φ_1 **then begin**
    for all $X_{iji'} \in Planes(\varphi_1)$ **do**
      if $X_{iji'}(p(\varphi_2)) < 0$ **then** $x[i,j] := i' - 1$; PUSH($(i,j), S$);
    for all $Y_{ijj'} \in Planes(\varphi_1)$ **do**
      if $Y_{ijj'}(p(\varphi_2)) < 0$ **then** $y[i,j] := j' - 1$; PUSH($(i,j), S$);
  end;
  while not EMPTY(S) **do begin**
    $(i,j) :=$POP(S); $temp := T[i,j]$;
    if $-n \le x[i,j], y[i,j] \le n$ **then** $T[i,j] := A[x[i,j], y[i,j]]$ **else** $T[i,j] := \perp$;
    $err := err - \delta(temp, B[i,j]) + \delta(T[i,j], B[i,j])$;
  end;
  if $err < min\_err$ **then** $min\_err := err$; $\varphi_{opt} := \varphi_2$;
end.

Remark 1. To stress that we mean an instruction and not the mathematical notation we used $A[i,j]$ instead of A_{ij} for accessing the pixel (i,j) of an image A in the UPDATE procedure.

The function SELECT$_{\mathcal{F}}(\varphi_{opt})$ of the main algorithm returns an (invertible) $g \in \mathcal{F}$ whose representative is in φ_{opt}. Thus, in case $\mathcal{F} = \mathcal{F}_{\mathsf{a}}$ the function returns $g \in \varphi_{opt} \cap (\mathbb{R}^6)_{\mathsf{a}}$. A natural candidate for such a representative is the point $(a_1, \ldots, a_6) = p(\varphi_{opt})$. But a problem occurs when the matrix $M = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ is singular. In Section 5.4 we will discuss how to solve this problem for linear transformations. The same method can be used for affine transformations in a straightforward way.

Lemma 2. *Let $\tau_{\mathcal{H}}(n)$ denote the worst-case time complexity of the procedure IM on input images of support $\{-n, \ldots, n\} \times \{-n, \ldots, n\}$ working with the parameter $\mathcal{H}$. Then for all sets of hyperplanes $\mathcal{H}, \mathcal{H}'$ in $\mathbb{R}^d$ such that $|\mathcal{H}| = |\mathcal{H}'|$ and $\mathcal{H}'$ is simple it is true $\tau_{\mathcal{H}}(n) \le \tau_{\mathcal{H}'}(n)$.*

Proof (Sketch). Consider any simple set of planes $\mathcal{H}'$ over $\mathbb{R}^d$. Notice that $\mathcal{H}'$ has no coincideing planes. It is well known (see [6]) that for any set $\mathcal{H}$ of planes without coinciding planes and $|\mathcal{H}| = |\mathcal{H}'|$ it is true $|\mathcal{A}(\mathcal{H})| \le |\mathcal{A}(\mathcal{H}')|$. Hence, simple plane sets are responsible for the maximum sets $\mathcal{A}(\mathcal{H})$. Furthermore, let $S_{\mathcal{H}}$ be an arbitrary schedule for the depth first search of IM for $\mathcal{I}(\tilde{\mathcal{H}})$, i.e., the order in which the faces of $\mathcal{A}(\tilde{\mathcal{H}})$ are visited. We denote by $c(S_{\mathcal{H}})$ the computational costs for the specific schedule $S_{\mathcal{H}}$. It easy to see that in fact $c(S_{\mathcal{H}})$ grows linearly with respect to the number of pixels updated during the depth first search. If $\mathcal{H}$ contains no coinciding planes then the costs $c(S_{\mathcal{H}})$ are in $O(|\mathcal{A}(\mathcal{H})|)$ since with every visited face exactly two pixel have to be updated during the depth first search.

Now assume that $\mathcal{H}$ contains coinciding planes. Imagine that we change $\mathcal{H}$ slightly to $\hat{\mathcal{H}}$ by converting all coinciding planes to a set of parallel non-coinciding planes which have pairwise distance of at most δ. We choose δ in a way such that each set of the new parallel planes intersects exactly and in the same order the planes that have been intersected by the original coinciding planes. Obviously $\hat{\mathcal{H}}$ does not contain coinciding planes anymore. Also we modify the schedule $S_{\mathcal{H}}$ to $S_{\hat{\mathcal{H}}}$ in the following way: If there are three consecutive φ_1, φ_2 and φ_3 in $S_{\mathcal{H}}$ and in $\mathbb{R}^d$ the faces φ_1 and φ_3 are separated by a set of k coinciding planes of $\mathcal{H}$ represented by the face φ_2, then by the separation of $\mathbb{R}^d$ with $\hat{\mathcal{H}}$ between φ_1 and φ_3 there are $2k+1$ new faces. We insert into $S_{\hat{\mathcal{H}}}$ between φ_1 and φ_3 all the $2k+1$ associated faces instead of φ_2.

The new schedule is at most as expensive as the former. For that see that by traversing in depth first search manner in $S_{\mathcal{H}}$ from φ_1 to φ_3 over φ_2 gives k pixel updates. Equivalently traversing from φ_1 to φ_3 over the k new into $S_{\hat{\mathcal{H}}}$ inserted planes also gives k updates. Subsequently, $c(S_{\mathcal{H}}) \leq c(S_{\hat{\mathcal{H}}})$. Furthermore, since $\hat{\mathcal{H}}$ contains no coinciding planes it must be true by the above observations that $c(S_{\hat{\mathcal{H}}}) \leq c(\mathcal{S}_{\mathcal{H}'})$ for any simple set $\mathcal{H}'$ with optimal travering schedule $c(\mathcal{S}_{\mathcal{H}'})$.

Consequently, for all schedules $S_{\mathcal{H}}$ and for all simple sets $\mathcal{H}'$ of planes it holds that $c(S_{\mathcal{H}}) \leq c(\mathcal{S}_{\mathcal{H}'})$. Since in worst case the running time of IM is affected exclusively by the depth first search of $\mathcal{A}(\tilde{\mathcal{H}})$ the statement follows. □

Thus, to give an upper bound on the time complexity of the image matching procedure IM for $\mathcal{F}_{\mathtt{a}}$ it is enough to analyze the time complexity for IM working with any simple set $\mathcal{H}'$ of the cardinality $|\mathcal{H}'| = |\mathcal{H}_{\mathtt{a},n}| = O(n^3)$. Using Theorem 2 one can construct the incidence graph for $\tilde{\mathcal{H}}'$ in time $O(|\tilde{\mathcal{H}}'|^6) = O(n^{18})$. Moreover, the size of $\mathcal{I}(\tilde{\mathcal{H}}')$ is in $O(n^{18})$ and thus, the depth first search can be done in time $O(n^{18})$, too. Since for the simple set $\mathcal{H}'$ each single execution of the procedure UPDATE can be done in constant time, this proves the following.

Theorem 3. *The* IMAGE MATCHING PROBLEM *for affine transformations can be solved in time* $O(n^{18})$.

5 Image Matching under some Special Affine Transformations

In this section we will analyse subsets $\mathcal{F}_\star$ of affine transformations, for $\star \in \{\mathtt{t}, \mathtt{s}, \mathtt{sr}, \mathtt{l}\}$, with respect to the complexity of their search spaces $\mathcal{C}(\mathcal{H}_{\star,n})$. Generally an affine transformation $f(x,y) = M \cdot (x,y)^T + t$ with an invertible (2×2)-matrix M and $\mathbb{R}^2$-vector t is completely described by the components a_1 to a_6 of the matrix M and vector t. In the following the subsets of affine transformation are described by limitations to the parameters a_1 to a_6.

In each case we identify the subset of parameters for inverse transformations, the form of the associated discretizations, and construct the relevant set of hyperplanes $\mathcal{H}_{\star,n}$. Afterwards we use the generic proof of Theorem 1 to provide the equivalence $\Gamma_n(\mathcal{F}_\star) \cong \mathcal{C}(\mathcal{H}_{\star,n})$ and next we give bounds for the cardinalities $|\mathcal{A}(\tilde{\mathcal{H}}_{\star,n})|$ that play a crucial role in estimating the computational complexity of the exhaustive search algorithm of Section 4 used for the set $\mathcal{F}_\star$ and $\mathcal{H}_{\star,n}$.

In the following subsections we will for the sake of space savings move the proofs and auxilliary lemmas to the appendix and instead provide only the main results. To get an efficient searching strategy for rotations we modify the approach above and discuss it in detail in the next section. Likewise we delegate projective transformations to a separate section.

5.1 Translation

The easiest case we consider is when the set $\mathcal{F}_{\mathtt{t}} \subset \mathcal{F}_{\mathtt{a}}$ contains all translations with M the identity matrix, $t = \binom{t_1}{t_2}$ and $t_1, t_2 \in \mathbb{R}$. Hence, we consider the two dimensional subspace $(\mathbb{R}^2)_{\mathtt{t}}$ of $(\mathbb{R}^6)_{\mathtt{a}}$ which has $a_1 = a_4 = 1$, $a_2 = a_3 = 0$, $a_5 = t_1$, $a_6 = t_2$, and $t_1, t_2 \in \mathbb{R}$. We will use the simple case of translations to illustrate how the definitions of affine transformations can be transfered to a specific subclass. In the following subsections we will reduce this technical transfer. Like in each considered case it holds that $\mathcal{F}_{\mathtt{t}}^{-1} = \mathcal{F}_{\mathtt{t}}$, since we only consider subclasses of $\mathcal{F}_{\mathtt{a}}$ which are closed under inversion. From the restricted versions of M and t we get the following form of discretizations γ_g for $g \in \mathcal{F}_{\mathtt{t}}^{-1}$

$$\gamma_g(i,j) = \begin{cases} ([i+t_1],[j+t_2]) & \text{if } ([i+t_1],[j+t_2]) \in \mathcal{N} \times \mathcal{N}, \\ \perp & \text{otherwise.} \end{cases}$$

In the subspace of $(\mathbb{R}^2)_{\mathtt{t}}$ the planes in $\mathcal{H}_{\mathtt{a},n}$ have a simpler description and thus we regard the set $\mathcal{H}_{\mathtt{t},n}$. For all $i, j \in \mathcal{N}$ and all $i', j' \in [-n-1, n+1]$ we have in $\mathcal{H}_{\mathtt{t},n}$ the planes

$$X_{iji'} : x_1 + i + 0.5 - i' = 0 \text{ and}$$
$$Y_{ijj'} : x_2 + j + 0.5 - j' = 0$$

For this set of planes we get the following characterization.

Lemma 3. *$\Gamma_n(\mathcal{F}_{\mathtt{t}}) \cong \mathcal{C}(\mathcal{H}_{\mathtt{t},n})$.*

Proof. Since we only consider points in $(\mathbb{R}^2)_{\mathtt{t}}$ the relation $\mathcal{R}_{\mathtt{a}}$ given in the proof of Theorem 1 is restricted to the relation $\mathcal{R}_{\mathtt{t}}$ on $(\mathbb{R}^2)_{\mathtt{t}} \times (\mathbb{R}^2)_{\mathtt{t}}$ defined as follows: a pair $u = (t_1, t_2)$ and $v = (t'_1, t'_2)$ of points in $(\mathbb{R}^2)_{\mathtt{t}}$ is in $\mathcal{R}_{\mathtt{t}}$ iff the pair of $U = (1,0,0,1,t_1,t_2)$ and $V = (1,0,0,1,t'_1,t'_2)$ is in $\mathcal{R}_{\mathtt{a}}$. By applying the generic proof of Lemma 1 with the restrictions to $\mathcal{R}_{\mathtt{t}}$ and $\mathcal{H}_{\mathtt{t},n}$ we can conclude the following. Two vectors $u, v \in (\mathbb{R}^2)_{\mathtt{t}}$ belong to the same equivalence class of $\mathcal{R}_{\mathtt{t}}$ if and only if for all $i, j \in \mathcal{N}$ and any $i', j' \in [-n-1, n+1]$ the vectors u and v belong to the same half-subspace according to the partition of $(\mathbb{R}^2)_{\mathtt{t}}$ with the hyperplane $X_{iji'}$, respectively $Y_{ijj'}$. □

Lemma 4. *$|\mathcal{C}(\mathcal{H}_{\mathtt{t},n})| \in \Theta(n^2)$.*

Proof. Consider the planes in $\mathcal{H}_{\mathtt{t},n}$ and let $\mathcal{H}^1_{\mathtt{t},n} = \{X_{iji'} | i \in \mathcal{N}, i' \in [-n-1, n+1]\}$ and $H^2_{\mathtt{t},n} = \{Y_{ijj'} | j \in \mathcal{N}, j' \in [-n-1, n+1]\}$. Obviously, two planes from $\mathcal{H}_{\mathtt{t},n}$ are parallel if they are either both in $\mathcal{H}^1_{\mathtt{t},n}$ or in $\mathcal{H}^2_{\mathtt{t},n}$ and otherwise orthogonal. Hence, $\tilde{\mathcal{H}}_{\mathtt{t},n}$ forms a two dimensional grid $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{t},n})$. It is easy to see that $|\tilde{\mathcal{H}}^1_{\mathtt{t},n}| = |\tilde{\mathcal{H}}^2_{\mathtt{t},n}| = 4n + 2$. Hence, the number of 2-, 1- and 0-faces is in $O(n^2)$. For a lower bound see that the number of 2-faces and by that the number of 2-cells is also in $\Omega(n^2)$. Hence, the number of cells is in $\Theta(n^2)$. □

Theorem 4. *The* Image Matching Problem *for translations can be solved in time $O(n^4)$.*

Proof. We get the running time of $O(n^4)$ applying the generic exhaustive search algorithm of Section 4 for $\mathcal{H}_{\mathtt{t},n}$, i.e., running the procedure $\mathrm{IM}(A, B)$ with parameter $\mathcal{H} := \mathcal{H}_{\mathtt{t},n}$. The statement follows from the following facts

1. The structure $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{t},n})$ can be constructed in time $O(n^2)$ (in fact, one can construct this set straightforwardly, without using the general approach of Theorem 2).
2. The DFS search of $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{t},n})$ visits $O(n^2)$ faces. Each time a new face is visited $O(n^2)$ planes have to be considered and hence, an equal number of pixels are changed in the UPDATE procedure.
3. Translations are always invertible and by that each point in $(\mathbb{R}^2)_{\mathtt{t}}$ represents an invertible transformation. Hence, $\mathrm{SELECT}_{\mathcal{F}_{\mathtt{t}}}(\varphi_{opt})$ simply computes in constant time the transformation g associated to the point $p(\varphi_{opt})$. □

5.2 Scaling

In the case when the set of transformations is restricted to scalings $\mathcal{F}_{\mathsf{s}}$, we have $M = \left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ for any $s \in \mathbb{R}$. We get the search space of inverse scaling transformations $\mathcal{F}_{\mathsf{s}}^{-1} = \mathcal{F}_{\mathsf{s}}$. Hence, if we consider s as the only parameter for g, then we have a one dimensional subspace $\mathbb{R}_{\mathsf{s}}$ of $(\mathbb{R}^6)_{\mathsf{a}}$. The discretizations for $g \in \mathcal{F}_{\mathsf{s}}^{-1}$ are of the form

$$\gamma_g(i,j) = \begin{cases} ([is],[js]) & \text{if } ([is],[js]) \in \mathcal{N} \times \mathcal{N}, \\ \perp & \text{otherwise.} \end{cases}$$

The planes in $\mathcal{H}_{\mathsf{a},n}$ have a simpler description and we regard the set $\mathcal{H}_{\mathsf{s},n}$: For all $i,j \in \mathcal{N}$ and all $i',j' \in [-n-1,n+1]$ we have in $\mathcal{H}_{\mathsf{s},n}$ the planes

$$X_{iji'} : ix + 0.5 - i' = 0 \text{ and}$$
$$Y_{ijj'} : jx + 0.5 - j' = 0$$

Modifying the proof Theorem 1 for $\mathcal{F}_{\mathsf{s}}$ (we replace there the relation $\mathcal{R}_{\mathsf{a}}$ by the intersection of $\mathcal{R}_{\mathsf{a}}$ and $\mathbb{R}_{\mathsf{s}} \times \mathbb{R}_{\mathsf{s}}$ and the space $\mathbb{R}_{\mathsf{a}}$ by $\mathbb{R}_{\mathsf{s}}$) the following holds:

Lemma 5. $\Gamma_n(\mathcal{F}_{\mathsf{s}}) \cong \mathcal{C}(\mathcal{H}_{\mathsf{s},n})$.

Lemma 6. $|\mathcal{C}(\mathcal{H}_{\mathsf{s},n})| \in \Theta(n^2)$.

Proof. To prove the quadratic bound on the cardinality of $\mathcal{C}(\mathcal{H}_{\mathsf{s},n})$ we use the following:

Fact 1 *Let $i \in \mathcal{N}$ and $i' \in \mathcal{N}$. The number of coprime pairs $(2i'-1, 2i)$ is in $\Theta(n^2)$.*

Proof. The absolute value of $2i'-1$ or $2i$ are natural numbers between 1 and $2n$. By Theorem 330 in [8] there are already at least $\frac{24}{\pi^2}n^2$ coprime pairs in $[1,2n]$. Anyway, since $2i'-1$ is always odd and $2i$ even we consider only a quarter of all pairs over $[1,2n]$. Because pairs of even numbers cannot be coprime on the other hand only three quarter of all pairs are candidates to be coprime. Even if any remaining not coprime pair is included in the quarter of pairs we consider there are more than $\frac{2}{5}n^2$ coprime pairs left. Hence, the number is in $\Theta(n^2)$. □

Notice that $X_{iji'} = Y_{jii'}$. Hence, for $\tilde{\mathcal{H}}_{\mathsf{s},n}$ we may consider only the planes $X_{iji'}, i,j \in \mathcal{N}, i' \in [-n-1,n+1]$. Furthermore, for all $i \in \mathcal{N}$, all $i' \in [n-1,n+1]$ it holds that for all $j_1, j_2 \in \mathcal{N}$ the plane $X_{ij_1i'}$ equals the plane $X_{ij_2i'}$. Denote by $s_{ii'}$ the point in $\mathbb{R}$ with $X_{iji'}(s_{ii'}) = 0$ for all $i,j \in \mathcal{N}$ and all $i' \in [-n-1,n+1]$, hence, $s_{ii'} = \frac{2i'-1}{2i}$. The number of points $s \in \mathbb{R}_+$ with the property that there exist $i \in \mathcal{N}$ and $i' \in [-n-1,n+1]$ such that $s = s_{ii'}$ is one less the number of 1-faces and equal to the number of 0-faces in $\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{s},n})$. We can determine that number if we only consider the number of points $s_{ii'}$ where numerator and denominator are coprime. By Fact 1 this number is in Thus, the upper bound for the number of cells is the number of 1- and 0-faces and the lower bound the number of 1-faces which proves the lemma.$\Theta(n^2)$. □

Theorem 5. *The* Image Matching Problem *for scalings can be solved in time $O(n^3)$.*

Proof. We solve the problem running the exhaustive search procedure IM(A,B) with the parameter $\mathcal{H} := \mathcal{H}_{\mathsf{s},n}$. Because the considered space is one dimensional and we have $\Theta(n^2)$ planes the set $\mathcal{A}(\tilde{\mathcal{H}}_{\mathsf{s},n})$ can be constructed easily in time $O(n^2)$. The DFS search strategy degrades to simple linear search. During the search each plane in $\mathcal{H}_{\mathsf{s},n}$ is considered once and thus altogether $O(n^3)$ pixels are updated. SELECT$_{\mathcal{F}_{\mathsf{s}}}$ simply computes $s = p(\varphi_{opt})$. If $s \neq 0$ SELECT$_{\mathcal{F}_{\mathsf{s}}}$ computes in constant time g the transformation associated to s. If otherwise $s = 0$ then φ_{opt} is an interval of positive length because $s_{ii'} \neq 0$ holds for all $i \in \mathcal{N}$ and $i' \in [-n-1,n+1]$. SELECT$_{\mathcal{F}_{\mathsf{s}}}$ chooses for $\hat{\varphi}_{opt}$ the positive subinterval of φ_{opt} and computes $\hat{s} = p(\hat{\varphi}_{opt})$ in constant time. Afterwards SELECT$_{\mathcal{F}_{\mathsf{s}}}$ proceeds like before and sets $\hat{g}$ the transformation associated to $\hat{s}$. □

5.3 Scaling and Rotation

Considering scalings and rotations together we get $\mathcal{F}_{\mathtt{sr}} \subset \mathcal{F}_{\mathtt{a}}$ where $M = \left(\begin{smallmatrix} s\cos\phi & s\sin\phi \\ -s\sin\phi & s\cos\phi \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ with $s, \phi \in \mathbb{R}$. The search space is $\mathcal{F}_{\mathtt{sr}}^{-1} = \mathcal{F}_{\mathtt{sr}}$ and we consider the two dimensional parameter subspace with parameters ϕ and s. We substitute $x = s\cos\phi$ and $y = s\sin\phi$ and by that we get the two dimensional parameter space $(\mathbb{R}^2)_{\mathtt{sr}}$ over $x, y \in \mathbb{R}$. The corresponding discretizations are of the form

$$\gamma_g(i,j) = \begin{cases} ([ix+jy],[jx-iy]) & \text{if } ([ix+jy],[jx-iy]) \in \mathcal{N}\times\mathcal{N}, \\ \perp & \text{otherwise.} \end{cases}$$

The set $\mathcal{H}_{\mathtt{sr},n}$ contains the planes

$$X_{iji'} : ix + jy + 0.5 - i' = 0 \text{ and}$$
$$Y_{ijj'} : jx - iy + 0.5 - j' = 0$$

for all $i, j \in \mathcal{N}$ and all $i', j' \in [-n-1, n+1]$. For visualization see in Figure 2 the space $(\mathbb{R}^2)_{\mathtt{sr}}$ partitioned by the planes in $\tilde{\mathcal{H}}_{\mathtt{sr},2}$ (for the moment ignore the circle). The following can be shown by an appropriate modification of the proof of Theorem 1 (we replace there the relation $\mathcal{R}_{\mathtt{a}}$ by $\mathcal{R}_{\mathtt{sr}}$ as follows: any pair $u, v \in (\mathbb{R}^2)_{\mathtt{sr}}$ is in $\mathcal{R}_{\mathtt{sr}}$ iff the pair (U, V) with $U = (x, y, -y, x, 0, 0)$ and $V = (x', y', -y', x', 0, 0)$ is in $\mathcal{R}_{\mathtt{a}}$).

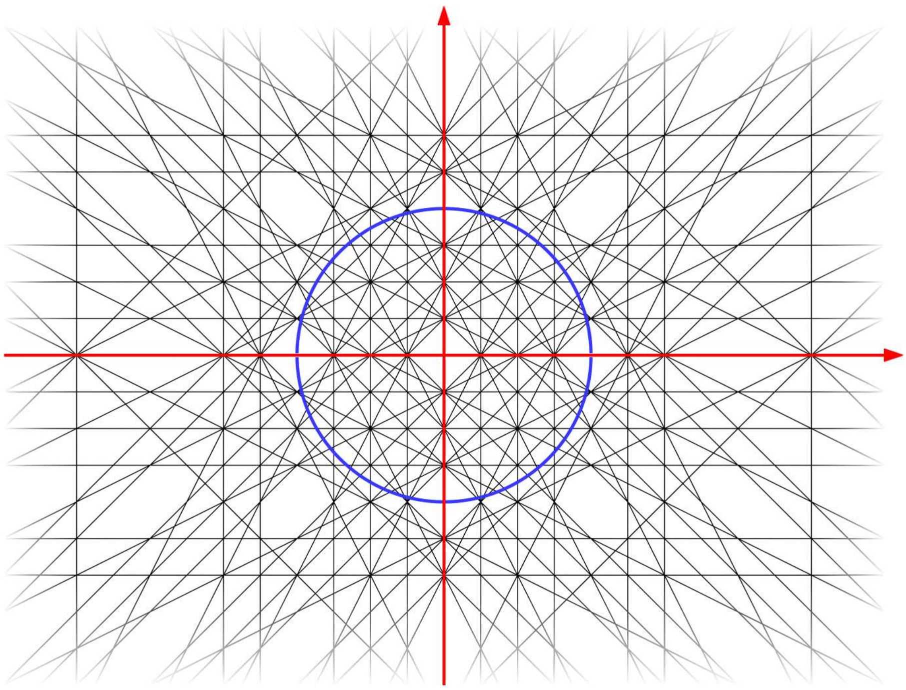

Fig. 2. The space $\mathbb{R}^2$ partitioned by $\mathcal{H}_{\mathrm{sr},2}$ or $\mathcal{H}_{\mathrm{r},2}$ respectively. In the first case the complexity of IMP is determined by the number of cells the space $\mathbb{R}^2$ is cut into. In the second case the number of segments on the unit circle determines the problem's complexity.

Theorem 6. $\Gamma_n(\mathcal{F}_{\mathtt{sr}}) \cong \mathcal{C}(\mathcal{H}_{\mathtt{sr},n})$.

Lemma 7. $|\mathcal{C}(\mathcal{H}_{\mathtt{sr},n})| \in \Omega(n^5) \cap O(n^6)$.

Proof. To deliver the proof we introduce the following two facts:

Fact 2 *Let $\mathcal{H}_n = \{H|H : ix + jx + 0.5 - i' = 0, i, j, i' \in \mathcal{N}\}$ be a set of planes in $\mathbb{R}^2$ over x and y. Then the cardinality of $\tilde{\mathcal{H}}_n$ is in $\Theta(n^3)$.*

Proof. The upper bound is trivial by the choice of $i, j, i' \in \mathcal{N}$. For the lower bound see that two choices i_1, j_i, i'_1 and i_2, j_2, i'_2 represent the same plane if and only if there is $\alpha \in \mathbb{R}$ with $i_1 = \alpha i_2$, $j_1 = \alpha j_2$ and $i'_1 - 0.5 = \alpha(i'_2 - 0.5)$. This is never the case if we consider only those triples i, j, i' with i and j coprime. By Theorem 330 in [8] there are at least $\frac{6}{\pi^2}n^2$ pairs i and j that are coprime. If i and j are coprime, i' can be chosen arbitrarily and hence, the statement follows. □

Fact 3 *Let $\mathcal{H}_n = \{H|H : ix + jy + 0.5 - i' = 0, i, j, i' \in \mathcal{N}\}$ be a set of planes. The space $\mathbb{R}^2$ is partitioned by $\mathcal{H}_n$ into $\tau(n)$ cells such that $\tau(n) \in \Omega(n^5) \cap O(n^6)$.*

Proof. Since $\tilde{\mathcal{H}}_n$ is a set of two dimensional planes, the number of faces in $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{sr},n})$ is bounded from above by $\sum_{k=0}^{2}\sum_{\ell=k}^{2}\binom{|\mathcal{H}|}{\ell}\binom{\ell}{k}$ which is in $O(|\tilde{\mathcal{H}}_n|^2) = O(n^6)$ by Fact 2. See [6] for this result. Hence, the upper bound for the number of cells is also $O(n^6)$.

For a lower bound on the number of cells we give a lower bound on the number of 2-faces in $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{sr},n})$. Consider the subset $\mathcal{H}'_n$ of planes in $\mathcal{H}_n$ with $i = 0$. Obviously for the plane $H \in \mathcal{H}'_n$ with $i', j \in \mathcal{N}$ it holds that $y = \frac{2i'-1}{2j}$. Hence, the planes in $\mathcal{H}'_n$ are parallel to the x-axis and by Fact 1 $|\tilde{\mathcal{H}}'_n| = \Omega(n^2)$. Hence, $\tilde{\mathcal{H}}'_n$ divides $\mathbb{R}^2$ into $\Omega(n^2)$ slices $S_1, \ldots, S_t$ such that the sum of the 2-faces which are contained in the slices gives the number of 2-faces in $\mathbb{R}^2$. We show that every slice contains $\Omega(n^3)$ 2-faces. Therefore imagine for $k \in \{1, \ldots, t\}$ a plane ℓ_k which is parallel to the x-axis, contained in S_k and intersecting the y-axis at an irrational coordinate. The plane ℓ_k intersects a certain subset of the 2-faces in the slice S_k and no 2-face of any other slice. The number of 2-faces intersected by ℓ_k is one bigger than the number of intersection points between ℓ_k and planes from $\tilde{\mathcal{H}}_n$. Intersetion between planes in $\tilde{\mathcal{H}}_n$ happens only at rational coordinates and intersection between planes in $\tilde{\mathcal{H}}_n$ and ℓ_k only at irrational coordinates and thus, both kinds of intersection cannot fall together. Subsequently, each plane in $\tilde{\mathcal{H}}_n$ intersects ℓ_k at a unique point. It is easy to see that all but $\Theta(n^2)$ (those which are parallel to ℓ_k) planes in $\tilde{\mathcal{H}}_n$ intersect ℓ_k. Hence, by Fact 2 the number of intersection points on ℓ_k and by that the number of 2-faces in S_k is in $\Omega(n^3)$. By the quadratical number of slices this gives the lower bound $\Omega(n^5)$ of 2-faces. □

Opposite to the case where only rotations are considered (see Section 6) here *each* two dimensional point defines a valid transformation containing both rotation and scaling parameter. Thus, we have to estimate the number of cells in $(\mathbb{R}^2)_{\mathtt{sr}}$ defined by the planes $\mathcal{H}_{\mathtt{sr},n}$ (see Figure 2). Like before $X_{iji'} = Y_{j(-i)i'}$ and we only have to consider the planes $X_{iji'}, i, j \in \mathcal{N}, i' \in [-n-1, n+1]$. For this subset Fact 3 fits and the statement follows. □

Theorem 7. *The* Image Matching Problem *for $\mathcal{F}_{\mathtt{sr}}$ can be solved in time $O(n^6)$.*

Proof. We apply the exhaustive search procedure $\mathrm{IM}(A, B)$ with parameter $\mathcal{H} := \mathcal{H}_{\mathtt{sr},n}$. The considered space is two dimensional and we have $\Theta(n^3)$ planes. Hence, $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{sr},n})$ can be constructed in time $O(n^6)$ by Theorem 2. Then the procedure $\mathrm{IM}(A, B)$ performs DFS search on $\mathcal{A}(\tilde{\mathcal{H}}_{\mathtt{sr},n})$ and visits each of the $O(n^6)$ faces. Like described in Section 4 the number of pixels updated during the whole searching is $O(n^6)$. The argumentation for this follows again from the fact that the worst case for pixel updates would occure when $\mathcal{H}_{\mathtt{sr},n}$ was simple. In that case we would have one pixel update for each visited face and in such a simple partition are $O(n^6)$ faces.

Like in scalings alone any transformation can be inverted if and only if $s \neq 0$. Hence, in the case of $s \neq 0$ SELECT$_{\mathcal{F}_{\mathtt{sr}}}$ simply computes g the transformations asociated to s and ϕ. Else, since no plane intersects the origin, φ_{opt} is a face arround the origin of positive area. SELECT$_{\mathcal{F}_{\mathtt{sr}}}$ sets $\hat{\varphi}_{opt}$ to be the positive subarea of φ_{opt} and computes in constant time another representative $\hat{g}$ for $\hat{\varphi}_{opt}$. □

5.4 Linear Transformations

In linear transformations, which differ from affine transformation only by restricting t to the zero vector, we have $\mathcal{F}_{\mathtt{l}} \subset \mathcal{F}_{\mathtt{a}}$. However, we already have the matrix $M = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ with $a_1, a_2, a_3, a_4 \in \mathbb{R}$. In that case the search space is $\mathcal{F}_{\mathtt{l}}^{-1} = \mathcal{F}_{\mathtt{l}}$ and we consider the subspace $(\mathbb{R}^4)_{\mathtt{l}}$ of $(\mathbb{R}^6)_{\mathtt{a}}$. The restrictions imply discretizations of the form

$$\gamma_g(i,j) = \begin{cases} ([ia_1 + ja_2], [ia_3 + ja_4]) & \text{if } ([ia_1 + ja_2], [ia_3 + ja_4]) \in \mathcal{N} \times \mathcal{N}, \\ \perp & \text{otherwise,} \end{cases}$$

and $\mathcal{H}_{\mathtt{l},n}$ is the set which contains the planes

$$X_{iji'} : ix_1 + jx_2 + 0.5 - i' = 0 \text{ and}$$
$$Y_{ijj'} : ix_3 + jx_4 + 0.5 - j' = 0$$

for all $i, j \in \mathcal{N}$ and $i', j' \in [-n-1, n+1]$. Again, modifying the proof of Theorem 1 for for linear transformations we get:

Lemma 8. *$\Gamma_n(\mathcal{F}_{\mathtt{l}}) \cong \mathcal{C}(\mathcal{H}_{\mathtt{l},n})$.*

Lemma 9. *$|\mathcal{C}(\mathcal{H}_{\mathtt{l},n})| \in \Omega(n^{10}) \cap O(n^{12})$.*

Proof. We consider the four dimensional parameter space $(\mathbb{R}^4)_{\mathtt{l}}$ with parameters a_1, a_2, a_3 and a_4 and its partition into cells by $\mathcal{H}_{\mathtt{l},n}$. Notice that for all $i, j \in \mathcal{N}$ and $i', j' \in [-n-1, n+1]$ the plane $X_{iji'}$ is independent from a_3 and a_4 and the plane $Y_{ijj'}$ independent from a_1 and a_2. Let

$$\mathcal{H}^1_{\mathtt{l},n} = \{X_{iji'} | i, j \in \mathcal{N}, i' \in [-n-1, n+1]\} \text{ and}$$
$$\mathcal{H}^2_{\mathtt{l},n} = \{Y_{ijj'} | i, j \in \mathcal{N}, j' \in [-n-1, n+1]\}.$$

Obviously each plane in $\mathcal{H}^1_{\mathtt{l},n}$ is orthogonal to each plane in $\mathcal{H}^2_{\mathtt{l},n}$ and vica versa. We can also look at $\mathcal{H}^1_{\mathtt{l},n}$ and $\mathcal{H}^2_{\mathtt{l},n}$ as plane sets for the two dimensional space $\mathbb{R}^2$ and consider the correponding partitions. Because $\mathcal{H}^1_{\mathtt{l},n}$ and $\mathcal{H}^2_{\mathtt{l},n}$ are orthogonal it follows that there is a one-to-one correpondence between cells in $(\mathbb{R}^4)_{\mathtt{l}}$ defined by $\mathcal{H}_{\mathtt{l},n}$ and pairs (C_1, C_2) of cells in $\mathbb{R}^2$ where C_1 is defined by the planes in $\mathcal{H}^1_{\mathtt{l},n}$ and C_2 by $\mathcal{H}^2_{\mathtt{l},n}$.

Both $\mathcal{H}^1_{\mathtt{l},n}$ and $\mathcal{H}^2_{\mathtt{l},n}$ fit Lemma 3. Hence, the number of cells in each two dimensional setting is in $\Omega(n^5) \cap O(n^6)$ and thus, $\mathcal{H}_{\mathtt{l},n}$ partitions $(\mathbb{R}^4)_{\mathtt{l}}$ into $\Omega(n^{10}) \cap O(n^{12})$ cells. □

Theorem 8. *In the case of linear transformation the running time of exhaustive search IM(A, B) with parameter $\mathcal{H} := \mathcal{H}_{\mathtt{l},n}$ is in $O(n^{12})$.*

Proof. In this case the considered space is four dimensional and we have $\Theta(n^3)$ planes. This implies that the incidence graph $\mathcal{I}(\tilde{\mathcal{H}}_{\mathtt{l},n})$ can be constructed in time $O(n^{12})$ by Theorem 2. The DFS search of the graph is done in time $O(n^{12})$ and during the whole searching $O(n^{12})$ pixels are updated which follows from the argumentation of Section 4.

Some of the faces φ of the arrangement $\mathcal{A}(\tilde{\mathcal{H}}_{1,n})$ contain vectors (a_1, a_2, a_3, a_4) which do not correpond to an invertible transformation and thus SELECT$_{\mathcal{F}_1}$ cannot just return g simply as $p(\varphi_{opt})$ (as usually we mean that (a_1, a_2, a_3, a_4) corresponds to linear transformation $g = Mx$ iff $M = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$). Obviously g is singular if the associated matrix M contains rows which are linearly dependent to each other. Thus, g can be inverted iff it is not part of the three dimensional hypersurface $Z : a_1a_4 - a_2a_3 = 0$.

We will first show that each face φ in $\mathcal{A}(\tilde{\mathcal{H}}_{1,n})$ contains a representative g which can be inverted iff φ is at least one dimensional. Notice that Z is only three dimensional and thus every 4-face trivially fulfills the statement. Let φ be at least a 1-face. Since a_1 and a_2 are independent from a_3 and a_4 we can assume without loss of generality that a_3 and a_4 are fixed and there is at least one degree of freedom in a_1 and a_2. Assume that $v = (a_1, a_2, a_3, a_4)^T$ is in Z and a_1 and a_2 can be chosen independently in the face φ. Then even a small change to either a_1 or a_2 moves v out of Z. Otherwise, assume that the choice of a_1 determines a_2 and φ is completely contained in Z. Then obviously $a_1 = \alpha a_2$ with $\alpha = \frac{a_3}{a_4}$. Since in this case φ must be contained in a plane this implies that there must be at least one plane which containes the origin. However, $\mathcal{H}_{1,n}$ does not contain such a plane and hence, φ cannot be contained in Z.

If φ_{opt} is a 0-face and the only representative cannot be inverted then SELECT$_{\mathcal{F}_1}$ would fail. However, we can filter out such face already during the depth first search of the incidence graph. Even more, we can do this in constant time since the test whether g is invertible or not can be done in constant time. This test step does not occur explicitely in the listing in Section 4 for the sake of comprehensibility.

Now, assume that φ_{opt} is a d-face, with $d > 0$ and let g be the transformation associated to $p(\varphi_{opt})$. If g is invertible then we are done. If not, SELECT$_{\mathcal{F}_1}$ searches a new representative corresponding to an invertible $\hat{g}$ solving the following linear programs.

Let $h_1, \ldots, h_r$ be planes which determine the boundary of φ_{opt}, i.e., let $h_1, \ldots, h_r$ be a minimum set such that $\varphi_{opt} = \bigcap_{i=1}^{r} h_i^{s_i}$ for some appropriate values $s_i \in \{+, -, 0\}$. For example, if φ_{opt} is a 4-face and $\varphi_1, \ldots, \varphi_r$ denote all subfaces of φ_{opt} then this set is defined by $h_1, \ldots, h_r$ such that for every i with $1 \le i \le r$, h_i is a plane from $Planes(\varphi_i)$. To determine $h_1, \ldots, h_r$ in the general case, one has to consider additionally planes form $Planes(\varphi_{opt})$. Now, let

$$
\begin{aligned}
&c_{1,1}a_1 + c_{1,2}a_2 + c_{1,3}a_3 + c_{1,4}a_4 \le d_1 \\
&\ldots \\
&c_{t,1}a_1 + c_{t,2}a_2 + c_{t,3}a_3 + c_{t,4}a_4 \le d_t
\end{aligned}
$$

be a system of linear inequalities that describes the intersection $\bigcap_{i=1}^{r} h_i^{s_i}$. We define the system as follows: for any $h^0 = \{x \in \mathbb{R}^4 \mid c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = d\}$ we get two inequalities: $c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4 \le d$ and $-c_1a_1 - c_2a_2 - c_3a_3 - c_4a_4 \le -d$, for $h^- = \{x \in \mathbb{R}^4 \mid c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 < d\}$ we get the inequality $c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4 \le d - \delta$ and for $h^+ = \{x \in \mathbb{R}^4 \mid c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 > d\}$ we get $-c_1a_1 - c_2a_2 - c_3a_3 - c_4a_4 \le -d - \delta$, where δ is an appropriate small constant (such a constant exists since we consider φ_{opt} of dimension at least 1). By the definition of $\mathcal{H}_{1,n}$ it is true that for every i with $1 \le i \le t$ either $c_{i,1} = 0$ and $c_{i,2} = 0$ or $c_{i,3} = 0$ and $c_{i,4} = 0$.

We will define four similar linear programs which differ only in single relation symbols denoted by $\bowtie_1$, $\bowtie_2$ and $\bowtie_3$. Define the symbols for each program according to the following table:

# linear program	$\bowtie_1$	$\bowtie_2$	$\bowtie_3$
1	$\le$	$>$	$<$
2	$\le$	$>$	$>$
3	$\ge$	$<$	$<$
4	$\ge$	$<$	$>$

The goal funtions of the programs are empty since we look only for a valid solution. The constraints of the linear programs are as follows:

I $\forall i \in \{1, \ldots, t\}$ with $c_{i,3} = c_{i,4} = 0$: $c_{i,1}a'_1 + c_{i,2}a'_2 - \alpha d_i \bowtie_1 0$,
II $\forall i \in \{1, \ldots, t\}$ with $c_{i,1} = c_{i,2} = 0$: $c_{i,3}a_3 + c_{i,4}a_4 \leq d_i$,
III $\alpha \bowtie_2 0$,
IV $a'_1 - a_3 = 0$, $a'_2 - a'_4 = 0$ and $a_4 - a'_4 \bowtie_3 0$.

At least one of the programs has a solution $(a'_1, a'_2, a_3, a_4, a'_4, \alpha)$ and the vector (a_1, a_2, a_3, a_4) with $a_1 = \frac{a'_1}{\alpha}$ and $a_2 = \frac{a'_2}{\alpha}$

1. belongs to the face φ_{opt} (see equations in (I), (II) and (III)) and
2. is not in Z (by the equations in (IV) $\alpha a_1 = a_3$ but $\alpha a_2 \neq a_4$).

Thus, the transformation $\hat{g}$ associated to (a_1, a_2, a_3, a_4) is valid and invertible.

The four linear programs can be solved, e.g., by Karmarkar's algorithm [14]. The algorithm's runtime is $O(m^{3.5}L)$ where m is the number of variables and L the number of bits to store the input. Since each face is bounded by at most $O(n^3)$ planes we have $t \in O(n^3)$. This implies that in our case $L = O(n^3 \log n)$ and furthermore m is constant. Subsequently, the total runtime of the procedure SELECT$_{\mathcal{F}_1}$ is in $O(n^3 \log n)$. In fact, since only a valid and no optimal solution is searched, the runtime is even much smaller. □

Remark 2. Since affine transformations contain all combinations of linear transformation and translations, this analysis gives also a corresponding lower bound for affine transformations that is $\Omega(n^{12})$.

6 Rotation

In this section we consider a set of transformation which is in a certain sense different to the previously introduced classes. In the case of rotations we have the set $\mathcal{F}_{\mathtt{r}} \subset \mathcal{F}_{\mathtt{a}}$ with $M = \left(\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\phi \in \mathbb{R}$. The search space is $\mathcal{F}_{\mathtt{r}}^{-1} = \mathcal{F}_{\mathtt{r}}$ over the only parameter ϕ. Thus we consider only a one dimensional subspace $\mathbb{R}_{\mathtt{r}}$ of $(\mathbb{R}^6)_{\mathtt{a}}$.

The major problem with rotations is that $\mathcal{F}_{\mathtt{r}}^{-1}$ is not partitioned by planes. If we define $x = \cos\phi$ and $y = \sin\phi$ then the discretizations have the form

$$\gamma_g(i,j) = \begin{cases} ([ix + jy], [jx - iy]) & \text{if } ([ix + jy], [jx - iy]) \in \mathcal{N} \times \mathcal{N}, \\ \bot & \text{otherwise.} \end{cases}$$

This gives us for $\mathcal{H}_{\mathtt{r},n}$ the set of surfaces

$$X_{iji'} : ix + jy + 0.5 - i' = 0 \text{ and}$$
$$Y_{ijj'} : jx - iy + 0.5 - j' = 0$$

for all $i, j \in \mathcal{N}$ and all $i', j' \in [-n-1, n+1]$. But these are no planes because they depend on the sin and cos of ϕ. However, if we regard x and y for parameters of $\mathcal{F}_{\mathtt{r}}^{-1}$ instead of ϕ then $\mathcal{H}_{\mathtt{r},n}$ contains only planes. But this introduces a number of problems which led us to consider rotation separately. One is that we have to consider only vectors (x, y) with $x^2 + y^2 = 1$, hence, the subspace of interest is the unit circle.

Concerning this subspace the mechanism of the planes remains the same. Two vectors (x, y) and (x', y') on the unit circle correspond to the same discrete version of a rotation iff they are not separated by any plane $X_{iji'}$ or $Y_{ijj'}, i, j \in \mathcal{N}, i', j' \in [-n-1, n+1]$. This leads to the question how many segments the unit circle is cut into by the planes in $\mathcal{H}_{\mathbf{r},n}$. We show the space $\mathbb{R}^2$ together with the unit circle and the planes $\mathcal{H}_{\mathbf{r},2}$ in Figure 2. See how the planes cut the circle into segments.

Lemma 10. *The set $\mathcal{H}_{\mathbf{r},n}$ of planes in $\mathbb{R}^2$ partitions the unit circle into $\Theta(n^3)$ segments.*

Proof. Consider the planes in $\tilde{\mathcal{H}}_{\mathbf{r},n}$. We have $X_{iji'} = Y_{j(-i)i'}$ and thus, have in $\tilde{\mathcal{H}}_{\mathbf{r},n}$ only planes $X_{iji'}, i, j \in \mathcal{N}, i' \in [-n-1, n+1]$. For all $i, j \in \mathcal{N}$ and all $i' \in [-n-1, n+1]$ denote by $s_{iji'} = (x, y)$ the point in $\mathbb{R}^2$ at which $X_{iji'}$ intersects the unit circle, if such a point exists. Hence, $X_{iji'}(s_{iji'}) = 0$ and $x^2 + y^2 = 1$. The coordinates (x, y) of point $s_{iji'}$ are as follows:

$$x = i(i' - 0.5) \pm \frac{j\sqrt{i^2 + j^2 - (i' - 0.5)^2}}{i^2 + j^2} \qquad y = j(i' - 0.5) \pm \frac{i\sqrt{i^2 + j^2 - (i' - 0.5)^2}}{i^2 + j^2}.$$

The number of points $s \in \mathbb{R}^2$ on the unit circle with the property that there exist $i, j \in \mathcal{N}$ and $i' \in [-n-1, n+1]$ such that $s_{iji'} = s$ equals the number $S(n)$ of segments the unit circle is cut into.

The cardinality of the subset of $\tilde{\mathcal{H}}_{\mathbf{r},n}$ containing the planes intersecting the unit circle gives the upper bound for $S(n)$. Hence, $S(n)$ is in $O(n^3)$. For all $i, j \in \mathcal{N}$ and $i' \in [-n-1, n+1]$ consider the coordinates (x, y) of $s_{iji'}$. The plane $X_{iji'}$ intersects the unit circle, if and only if the term under the square root is positive or zero, i.e., if the inequality $i^2 + j^2 \geq (i' - 0.5)^2$ holds. We count the number of planes which intersect the unit circle. For each $i' \in [-n-1, n+1]$ we consider only those pairs $i, j \in [1, n]$ which are coprime and $\max\{i, j\} \geq i'$. By this we assure that we do not count a plane twice and that $i^2 + j^2 \geq (i' - 0.5)^2$. By Theorem 330 in [8] for each i' there are at least $\frac{6}{\pi^2}n^2 - \frac{6}{\pi^2}i'^2 - O(i' \log i')$ coprime pairs for (i, j) which correspond to intersecting planes. This gives that at least $\frac{6}{\pi^2}n^3 - O(n^2 \log n) - \frac{6}{\pi^2}\sum_{i'=1}^{n} i'^2 \in \Omega(n^3)$ planes intersect the unit circle.

It remains to show that the planes intersect the unit circle at different points. Since the planes have natural numbers as coefficients they intersect each other at coordinates with rational components. We show that intersection between unit circle and planes occurs only at coordinates with irrational components and hence, each plane intersects the circle at two unique points. The intersection coordinates are irrational if and only if the term $\sqrt{i^2 + j^2 - (i' - 05)^2}$ is irrational.

Assume that there are natural numbers p and q such that $\frac{p}{q} = \sqrt{i^2 + j^2 - (i' - 05)^2}$. We let b be the natural number for which $b = i^2 + j^2 - i'^2 - i'$, and hence $p = q\sqrt{b - 0.25}$. Subsequently, q has to be a multiple of 2 and we let $q = 2q'$. It follows that $p = q'\sqrt{4b - 1}$. The square root of a natural number is either irrational or itself a natural number and thus we can assume $q' = 1$. By that it follows that $p^2 \equiv -1 \mod 4$. This contradicts the assumption of p and q being natural. □

From the lemma follows that there exist $\Theta(n^3)$ possible rotations of a $n \times n$ image. However, due to the fact that there are subspaces which contain only one representative with irrational coordinates IMP for $\mathcal{F}_{\mathbf{r}}$ cannot be solved in the usual way. For this setting, we will, instead of computing the rotation angle, compute the image $f(A)$ which is a rotation of A and most similar to B. Also, we cannot use the algorithm of Sections 4 but propose a new strategy:

We compute the segments on the unit circle in their clockwise order. To omit the precision problems with the intersection points we can compute in a clockwise order which plane is responsible for the end of the current segment and the start of the next one. This gives the sequence $\mathcal{S}_{\mathbf{r},n} = (\mathcal{H}_1, \ldots, \mathcal{H}_t)$. Each $\mathcal{H}_i, 1 \leq i \leq t$ represents a subset of planes which coincide in $\mathbb{R}^2$.

Lemma 11. *The computation of a sequence $\mathcal{S}_{\mathbf{r},n} = (\mathcal{H}_1, \ldots, \mathcal{H}_t)$ listing in clockwise order the subsets of planes in $\mathcal{H}_{\mathbf{r},n}$ intersecting the unit circle is feasible in time $O(n^3 \log n)$.*

Proof (Sketch). The central concept in computing $\mathcal{S}_{\mathtt{r},n}$ is to sort the planes of $\mathcal{H}_{\mathtt{r},n}$ that intersect the unit circle $\mathcal{C}$ one by one into the list $\mathcal{S}_{\mathtt{r},n}$. The test for intersection between a plane $H_{iji'}$ and $\mathcal{C}$ can be archieved by computing whether $i^2 + j^2 \geq (i' - 0.5)^2$ holds. Since we assume arithmetical operations to have unit costs this needs only contant time.

Notice that in $\mathcal{H}_{\mathtt{r},n}$ are no tangents of $\mathcal{C}$ because $i^2 + j^2 = (i' - 0.5)^2$ never holds. Hence, each intersecting plane has two occurences in $\mathcal{S}_{\mathtt{r},n}$. To determine the order of the intersecting planes in $\mathcal{S}_{\mathtt{r},n}$ we apply in each step binary search. During the binary search we recursively have to decide the following situation: Let $\mathcal{S}_{\mathtt{r},n} = (\mathcal{H}_1, \mathcal{H}_2, \ldots)$ be an initial order of planes intersecting $\mathcal{C}$ and let H be the plane which should be inserted. Assume we know that there is an intersection p of plane H with $\mathcal{C}$ between the intersections p_a and p_b of $\mathcal{H}_a$ and $\mathcal{H}_b$ with $\mathcal{C}$. Furthermore let $c = \left\lfloor \frac{a+b}{2} \right\rfloor$. Does H intersect $\mathcal{C}$ between p_a and p_c the intersection of planes in $\mathcal{H}_c$ and $\mathcal{C}$ or between p_c and p_b?

The test can basically be implemented by checking the side of $\mathcal{H}_c$ for a specific point on H. However, the determination of that point and some special cases introduce a lot of case differentiations and make the presentation very technical. Nevertheless, the test can be done in constant time. The special case of H coinciding with the planes in $\mathcal{H}_c$ should be mentioned. Depending on whether H and the planes in $\mathcal{H}_c$ add the vectors (x, y) in $\mathbb{R}^2$ with $H(x, y) = 0$ to the same side distinguishes the cases when H is simply inserted into $\mathcal{H}_c$ or starts a new set in $\mathcal{S}_{\mathtt{r},n}$.

By binary search we can insert one plane into $\mathcal{S}_{\mathtt{r},n}$ with $O(\log n)$ tests. Since we have $\Theta(n^3)$ planes intersecting $\mathcal{C}$ this gives a preprocessing time of $O(n^3 \log n)$. □

After $\mathcal{S}_{\mathtt{r},n}$ has been built the actual optimization can be done very easy. In fact one simply has to traverse the segments on the unit circle and check each time whether the current image $f(A)$ is closer to B then the previous ones. Like in the algorithm of Sections 4 it is possible to compute $f(A)$ gradually.

Theorem 9. Image Matching *on $\mathcal{F}_{\mathtt{r}}$ is feasible in time $O(n^3)$ with $O(n^3 \log n)$ preprocessing steps.*

Proof. For the preprocessing see Lemma 11. Like in Section 4 we will now give a more efficient way to gradually obtain $f(A)$ by traversing the segments on the unit circle in a $\mathcal{S}_{\mathtt{r},n}$ manner.

By Lemma 1 all vectors on the unit circle within one segment correpond to the same $f(A)$. Consider two neighbouring segments s_1 and s_2 on the unit circle and let $f_1(A)$ and $f_2(A)$ be the corresponding rotated versions of A. Notice that s_1 and s_2 might be separated from eachother by multiple planes. Thus, let $\mathcal{H}_{s_1,s_2}$ be the set of planes separating s_1 from s_2.

For simplicities sake denote by γ_{s_1} and γ_{s_2} the discreted representations of the rotations shared by all vectors on s_1 or s_2 respectively. It is easy to see that $f_1(A)_{ij}$ and $f_2(A)_{ij}$ can potentially be different iff $\gamma_{s_1} \neq \gamma_{s_2}$ which is only the case iff there exits $c \in [-n-1, n+1]$ and $X_{ijc} \in \mathcal{H}_{s_1,s_2}$ or $Y_{ijc} \in \mathcal{H}_{s_1,s_2}$.

Subsequently, if we knew $f_1(A)$ and $\mathcal{H}_{s_1,s_2}$ we could compute $f_2(A)$ efficiently by only updating the set of pixels $P = \{(ij) \mid \exists c \in [-n-1, n+1], X_{ijc} \in \mathcal{H}_{s_1,s_2} \vee Y_{ijc} \in \mathcal{H}_{s_1,s_2}\}$ in $f_1(A)$. It is easy to see that by the definition of Δ also the difference $\Delta(f_2(A), B)$ can be computed just by updating $\Delta(f_1(A), B)$ on the set of pixels P.

We will now see that during the transversal of all segments there is a total of only $O(n^3)$ pixel updates. Let $\mathcal{S}_{\mathtt{r},n} = (\mathcal{H}_1, \ldots, \mathcal{H}_t)$ be the order of planes computed in preprocessing. Obviously each plane in $\mathcal{H}_{\mathtt{r},n}$ can occur at most in two sets of $\mathcal{S}_{\mathtt{r},n}$. Furthermore, for any set $\mathcal{H}_i \in \mathcal{S}_{\mathtt{r},n}, 1 \leq i \leq t$ each plane of $\mathcal{H}_i$ stands for exactly one pixel to update. Since the number of planes is in $\Theta(n^3)$ the number of updates is in $O(n^3)$ when traversing $\mathcal{S}_{\mathtt{r},n}$. Since the complexity of updating one pixel in $f(A)$ and in $\Delta(f(A), B)$ is constant, the whole optimization procedure is in $O(n^3)$. □

7 Projective Transformations

Projective transformations $\mathcal{F}_\mathtt{p}$ are a generalization of affine transformations. For projective transformation it holds that, $f \in \mathcal{F}_\mathtt{a}$ if $f(x,y) = (\frac{a}{c}, \frac{b}{c})^T$ with $(a,b,c)^T = M \cdot (x,y,1)^T$ and $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{pmatrix}$ an invertible (3×3)-matrix. Like in all previously considered case here also $\mathcal{F}_\mathtt{p}^{-1} = \mathcal{F}_\mathtt{p}$. For M we can choose eight parameters a_1 to a_8 and thus have an eight dimensional parameter space $(\mathbb{R}^8)_\mathtt{p}$ with the exceptional vectors where either the corresponding matrix M is not invertible or there exist $i,j \in \mathcal{N}$ such that $a_7 i + a_8 j + 1 = 0$. From M we get the following form of discretizations γ_g for $g \in \mathcal{F}_\mathtt{p}^{-1}$

$$\gamma_g(i,j) = \begin{cases} \left(\left[\frac{a_1 i + a_2 j + a_3}{a_7 i + a_8 j + 1}\right], \left[\frac{a_4 i + a_5 j + a_6}{a_7 i + a_8 j + 1}\right]\right) & \text{if } \left(\left[\frac{a_1 i + a_2 j + a_3}{a_7 i + a_8 j + 1}\right], \left[\frac{a_4 i + a_5 j + a_6}{a_7 i + a_8 j + 1}\right]\right) \in \mathcal{N} \times \mathcal{N}, \\ \perp & \text{otherwise.} \end{cases}$$

Like in all previous sections we can define a set of planes $\mathcal{H}_{\mathtt{p},n}$ which contains for all $i,j \in \mathcal{N}, i', j' \in [-n-1, n+1]$ the planes:

$$X_{iji'} : ix_1 + jx_2 + x_3 + (0.5i - ii')x_7 + (0.5j - ji')x_8 + (0.5 - i')x_9 = 0 \quad \text{and}$$
$$Y_{ijj'} : ix_4 + jx_5 + x_6 + (0.5i - ij')x_7 + (0.5j - jj')x_8 + (0.5 - j')x_9 = 0$$

However, an analogous version of Theorem 1 for projective transformations is not true in this setting. Anyway, the following weaker version of the Theorem can be established for projective transformation. We say that

$$\Gamma_n(\mathcal{F}_\mathtt{p}) \prec \mathcal{C}(\mathcal{H}_{\mathtt{p},n})$$

if and only if for all $u, u' \in \mathbb{R}^8$ with $g_u, g_{u'} \in \mathcal{F}_\mathtt{p}^{-1}$ it is true: $C_\mathcal{H}(u) = C_\mathcal{H}(u') \Rightarrow \gamma_{g_u} = \gamma_{g_{u'}}$.

Theorem 10. *$\Gamma_n(\mathcal{F}_\mathtt{p}) \prec \mathcal{C}(\mathcal{H}_{\mathtt{a},n})$.*

Proof. We again define the set $\mathcal{R}_\mathtt{p}$ to be the following equivalence relation on $(\mathbb{R}^8)_\mathtt{p} \times (\mathbb{R}^8)_\mathtt{p}$

$$\mathcal{R}_\mathtt{p} = \{(u_1, u_2) \mid u_1, u_2 \in (\mathbb{R}^8)_\mathtt{p} \text{ and } \gamma_{u_1} = \gamma_{u_2}\}.$$

Thus the relation $\mathcal{R}_\mathtt{p}$ partitions $\mathcal{F}_\mathtt{p}^{-1}$ into subsets of transformations of equal discrete counterparts. The following lemma gives the major structural property of $\mathcal{R}_\mathtt{p}$.

Lemma 12. *Two vectors $u, v \in (\mathbb{R}^8)_\mathtt{p}$ belong to the same equivalence class of $\mathcal{R}_\mathtt{p}$ if for all $i,j \in \mathcal{N}$ and any $i', j' \in [-n-1, n+1]$ the vectors u and v belong to the same half-subspace according to the partition of $(\mathbb{R}^8)_\mathtt{p}$ with the hyperplane $X_{iji'}$, respectively $Y_{ijj'}$.*

Proof. Let $u = (a_1, \ldots, a_8)$ and $v = (b_1, \ldots, b_8)$ be two parameter vectors from $(\mathbb{R}^8)_\mathtt{p}$. By definition γ_u equals γ_v, iff u and v belong to the same equivalence class in $\mathcal{R}_\mathtt{p}$. We show that if u and v belong to the same half-subspace then $\gamma_u = \gamma_v$.

Let $\gamma_u \neq \gamma_v$ hold but for the contradiction assume that for all $(i,j) \in \mathcal{N} \times \mathcal{N}$ and $i', j' \in [-n-1, n+1]$ u and v belong to the same half-space of $(\mathbb{R}^8)_\mathtt{p}$ with respect to $X_{iji'}$ and $Y_{ijj'}$. There exist $(i,j) \in \mathcal{N} \times \mathcal{N}$ such that $\gamma_u(i,j) = (i'_1, j'_1) \neq (i'_2, j'_2) = \gamma_v(i,j)$. Without loss of generalization assume that $i'_1 \leq i'_2 - 1$. This means that

1. $\frac{a_1 i + a_2 j + a_3}{a_7 i + a_8 j + 1} < i'_1 + 0.5$ and
2. $i'_2 - 0.5 \leq \frac{b_1 i + b_2 j + b_3}{b_7 i + b_8 j + 1}$.

It is easy to see that all planes $X_{iji'}$ with fixed i and j have a common subspace Z_{ij}. Hence any two planes $X_{iji'_1}$ and $X_{iji'_2}$ intersect in the linear subspace where $x_7 i + x_8 j + 1 = 0$.

Lets first consider the easier case when u and v are both on the same side of the linear subspace Z_{ij}, hence, either $a_7 i + a_8 j + 1 > 0$ and $b_7 i + b_8 j + 1 > 0$ or $a_7 i + a_8 j + a_9 < 0$ and $b_7 i + b_8 j + b_9 < 0$. In that case we substitute i'_1 by $i'_2 - 1$ in the first inequality and get

1. $ia_1 + ja_2 + a_3 + (0.5i - ii'_2)a_7 + (0.5j - ji'_2)a_8 + (0.5 - i'_2) < 0$ as well as
2. $ib_1 + jb_2 + b_3 + (0.5i - ii'_2)b_7 + (0.5j - ji'_2)b_8 + (0.5 - i'_2) \geq 0$

or with switched relation symbols, respectively. Thus, the plane $X_{iji'_2}$ separates u and v.

Now consider the case of u and v being on opposite sides of $Z_{ij} : x_7 i + x_8 j + 1 = 0$. Then there exist only two possibilities for u and v being placed in the same half-subspace of the plane $X_{iji'}$ for all $i' \in [-n-1, n+1]$. If $X_{iji'}(u) \geq 0$ and $X_{ij(i'+1)}(u) < 0$ then it is not possible to place v on the opposite side of the space Z_{ij} such that neither $X_{iji'}$ nor $X_{ij(i'+1)}$ separates u and v because $X_{iji'}$ and $X_{ij(i'+1)}$ "cross" in Z_{ij}.

Hence, the only way to place u and v properly is when $X_{iji'}(u) < 0$ and $X_{iji'}(v) < 0$ or $X_{iji'}(u) \geq 0$ and $X_{iji'}(v) \geq 0$ for all $i' \in [-n-1, n+1]$. Obviously, in that case it is not possible that $\gamma_u(i,j) = (i', j')$ with $i' \in \mathcal{N}$ because this would imply that u is in the space between $X_{iji'}$ and $X_{ij(i'+1)}$. The same holds for v. Subsequently, $\gamma_u(i,j) = \perp = \gamma_v(i,j)$ must hold which is a contradiction.

An analogous proof holds for the case when j'_1 and j'_2 differ. □

□

Like in affine transformations the planes in $(\mathbb{R}^8)_{\mathtt{p}}$ separate classes of different projective transformations. However, there may be several cells which represent the same projective transformation. Despite this by the theorem it still suffices to estimate the number of cells in $\mathcal{C}(\mathcal{H}_{\mathtt{p},n})$ to get bound on the cardinality of $\Gamma_n(\mathcal{F}_{\mathtt{p}})$.

Lemma 13. $|\mathcal{C}(\mathcal{H}_{\mathtt{p},n})| \in O(n^{24})$.

Proof. The limitations for i, j, i' and j' imply that the number of planes in $\tilde{\mathcal{H}}_{\mathtt{p},n}$ is in $O(n^3)$. Any set $\mathcal{H}$ of $O(n^3)$ planes partitions $(\mathbb{R}^8)_{\mathtt{p}}$ into at most $\sum_{k=0}^{8} \sum_{\ell=k}^{8} \binom{|\mathcal{H}|}{\ell}\binom{\ell}{k} = O(|\mathcal{H}|^8) = O(n^{24})$ cells. □

Finally, we can also solve IMP efficiently for the general class of projective transformation. Anyway, the algorithm presented in Section 4 calls as a last step $\text{SELECT}_{\mathcal{F}_{\mathtt{p}}}$ which should computes for the optimum class φ_{opt} a representative g with an invertible 3×3 matrix. This is much harder then in linear and affine transformation and we cannot give any efficient solution for that problem. Instead we will assume that $\text{SELECT}_{\mathcal{F}_{\mathtt{p}}}$ simply returns g even if it is not invertible.

Theorem 11. *The* Image Matching Problem *for projective transformations can be solved in time $O(n^{24})$.*

8 Conclusions and Future Work

In this work we analyzed the Image Matching Problem with respect to several subclasses of affine transformations as well as the more general case of projective transformations. We introduced a general polynomial time searching strategy which takes advantage of the search space structure common among the covered classes of transformations. To provide precise bounds for the running time of the searching algorithm we examined the complexity of the search space structure for each class of transformations. As a consequence we showed sharp bounds for translations $\Theta(n^2)$, scalings $\Theta(n^2)$ and rotations $\Theta(n^3)$. We also gave narrow bounds for combined scalings and rotations $\Omega(n^5) \cap O(n^6)$, and for linear transformations $\Omega(n^{10}) \cap O(n^{12})$. We did not improve the upper bound for affine transformations $O(n^{18})$ but could give a nontrivial lower bound of $\Omega(n^{12})$. Finally, we analyzed projective transformation and showed $O(n^{24})$ as an upper bound.

We conjecture that the lower bound for the structural complexity of the search space of combined scalings and rotations is $\Omega(n^6)$. This would also imply the lower bound of $\Omega(n^{12})$ for the search space complexity of linear transformations. The other great challenge in this area is to close the gap between the lower and the upper bound for the search space complexity of affine and projective transformations.

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