

Inverse Conjecture for the Gowers norm is false

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Abstract

Let p be a fixed prime number, and N be a large integer. The 'Inverse Conjecture for the Gowers norm' states that if the " d -th Gowers norm" of a function $f : \mathbb{F}_p^N \rightarrow \mathbb{F}$ is non-negligible, that is larger than a constant independent of N , then f can be non-trivially approximated by a degree $d - 1$ polynomial. The conjecture is known to hold for $d = 2, 3$ and for any prime p . In this paper we show the conjecture to be false for $p = 2$ and for $d = 4$, by presenting an explicit function whose 4-th Gowers norm is non-negligible, but whose correlation any polynomial of degree 3 is exponentially small.

Essentially the same result (with different correlation bounds) was independently obtained by Green and Tao [5]. Their analysis uses a modification of a Ramsey-type argument of Alon and Beigel [1] to show inapproximability of certain functions by low-degree polynomials.

We observe that a combination of our results with the argument of Alon and Beigel implies the inverse conjecture to be false for any prime p , for $d = p^2$.

1 Introduction

We consider multivariate functions over finite fields. The main question of interest here would be whether these functions can be non-trivially approximated by a low-degree polynomial.

Fix a prime number p . Let $\mathbb{F} = \mathbb{F}_p$ be the finite field with p elements. Let $\xi = e^{\frac{2\pi i}{p}}$ be the primitive p -th root of unity. Denote by $e(x)$ the exponential function taking $x \in \mathbb{F}$ to $\xi^x \in \mathbb{C}$. For two functions $f, g : \mathbb{F}^N \rightarrow \mathbb{F}$, let $\langle f, g \rangle := \mathbb{E}_x e(f(x) - g(x))$.

Definition 1.1: A function f is non-trivially approximable by a degree- d polynomial if

$$|\langle f, g \rangle| > \epsilon$$

for some polynomial g of degree at most d in $\mathbb{F}[x_1 \dots x_N]$. ■

More precisely, in this definition we are looking at a sequence f_N of functions and of approximating low-degree polynomials g_N in N variables, and let N grow to infinity. In this paper, the remaining parameters, that is the field size p , the degree d and the offset ϵ are fixed, independent of N .

A counting argument shows that a generic function can not be approximated by a polynomial of low degree. The problems of showing a specific given function to have no non-trivial

approximation and of constructing an explicit non-approximable function have been extensively investigated, since solutions to these problems have many applications in complexity (cf. discussion and references in [1, 9, 2]).

This paper studies a technical tool that measures distance from low-degree polynomials. This is the Gowers norm, introduced in [3]. For a function $f : \mathbb{F}^N \rightarrow \mathbb{F}$ and a vector $y \in \mathbb{F}^n$, we take f_y to be the directional derivative of f in direction y by setting

$$f_y(x) = f(x + y) - f(x)$$

For a k -tuple of vectors $y_1 \dots y_k$ we take the iterated derivative in these directions to be

$$f_{y_1 \dots y_k} = (f_{y_1 \dots y_{k-1}})_{y_k}$$

It is easy to see that this definition does not depend on the ordering of $y_1 \dots y_k$.

The k -th Gowers "norm" $\|f\|_{U^k}$ of f is

$$(\mathbb{E}_{x, y_1 \dots y_k} [e(f_{y_1 \dots y_k}(x))])^{1/2^k}$$

More accurately, as shown in [3], this is indeed a norm of the associated complex-valued function $e(f)$ (for $k \geq 2$).

It is easy to see that $\|f\|_{U^{d+1}}$ is 1 iff f is a polynomial of degree at most d . This is just another way of saying that all order- $(d+1)$ iterative derivatives of f are zero if and only if f is a polynomial of degree at most d . It is also possible to see [4] that $|\langle f, g \rangle| > \epsilon$ for g of degree at most d , implies $\|f\|_{U^{d+1}} > \epsilon$. That is to say, if f is non-trivially close to a degree- d polynomial, this can be detectable via an appropriate Gowers norm.

This discussion naturally leads to the inverse conjecture [4, 7, 8], that is if $(d+1)$ -th Gowers norm of f is non-trivial, then f is non-trivially approximable by a degree- d polynomial. This conjecture is easily seen to hold for $d = 1$ and has been proved also for $d = 2$ [4, 7]. It is of interest to prove this conjecture for higher values of d .

In this paper we show this conjecture, which we will refer to as the 'Inverse Conjecture for the Gowers norm', or, informally, as ICGN, to be false. Let S_n be the elementary symmetric polynomial of degree n in N variables, that is

$$S_n(x) = \sum_{S \subseteq [N], |S|=n} \prod_{i \in S} x_i$$

We prove two claims about symmetric polynomials. Note that here and below a constant is *absolute* if it does not depend on N .

First, we show Gowers norms of some symmetric polynomials to be non-trivial.

Theorem 1.2: *There is an absolute positive constant ϵ such that for any prime p*

$$\|S_{2p}\|_{U^{p+2}} > \epsilon,$$

Here S_{2p} is viewed as a function over $\mathbb{F} = \mathbb{F}_p$.

Two versions of this result will be useful later.

- A special case $p = 2$.

$$\|S_4\|_{U^4} > \epsilon \tag{1}$$

- An easy generalization: for any $n \geq 2p$,

$$\|S_n\|_{U^{n-p+2}} > \epsilon \tag{2}$$

In the second claim we show a specific symmetric polynomial to have no non-trivial approximation by polynomials of lower degree.

Theorem 1.3: *Let $p = 2$. For any polynomial g of degree 3 holds*

$$|\langle S_4, g \rangle| < \exp\{-\alpha N\} \tag{3}$$

We conjecture the second claim of the theorem to be true for any prime number p , replacing 3 with $p + 1$ and 4 with $2p$.

The combination of (1) and (3) shows ICGN to be false for $p = 2$ and $d = 4$.

1.1 Related work

Our results have a large overlap with a recent work of Green and Tao [5].

The paper of Green and Tao has two parts. In the first part ICGN is shown to be true when f is itself a polynomial of degree less than p and $d < p$. In the second part, the conjecture is shown to be false in general. In particular the symmetric polynomial S_4 is shown to be a counterexample for $p = 2$ and $d = 4$.

To prove non-approximability of S_4 by lower-degree polynomials in [5] uses a modification of a Ramsey-type argument due to Alon and Beigel [1]. Very briefly, this argument shows that if a function over \mathbb{F}_2 has a non-trivial correlation with a multilinear polynomial of degree d , then its restriction to a subcube of smaller dimension has a non-trivial correlation with a symmetric polynomial of degree d . The problem of inapproximability by symmetric polynomials turns out to be easier to analyze.

This argument gives a somewhat weaker bounds for non-inapproximability of S_4 , in that it shows $\langle S_4, g \rangle < \log^{-c}(N)$ for any degree-3 polynomial g and for an absolute constant $c > 0$.

On the other hand, this argument is more robust than our inapproximability argument. We observe below that it can be readily extended to the case of general prime p and, combined with (2), show ICGN to be false for all p .

1.2 The case of a general prime field

We briefly observe here that a minor adaptation of the Alon-Beigel argument, together with (2), show the symmetric polynomial S_{p^2} to have a non-negligible (p^2)-nd Gowers norm over \mathbb{F}_p and to have no good approximation by lower-degree polynomials. In that, S_{p^2} provides a counterexample to ICGN for any prime p .

Indeed, by monotonicity of the Gowers norms ([4]), and since $p \geq 2$, a direct implication of (2) gives

$$\|S_{p^2}\|_{U_{p^2}} > \epsilon$$

On the other hand, let g be a polynomial of degree less than p^2 in N variables such that $\langle S_{p^2}, g \rangle > \epsilon$. Note that the Alon-Beigel argument (as given in [1] and in [5]) does not seem to be immediately applicable in this case, since g does not have to be multilinear. A way around this obstacle, is to observe, via an averaging argument, that there is a copy of an N' -dimensional boolean cube $\{0, 1\}^{N'}$, such that restrictions S' and g' of S_{p^2} and of g on this subcube satisfy $\langle S', g' \rangle > \epsilon'$, and N', ϵ' depend linearly on N, ϵ . Without loss of generality assume the coordinates of the boolean cube to be $\{1 \dots N'\}$ and consider the functions S', g' as functions in variables $x_1, \dots, x_{N'}$ (with some fixed assignment of values to variables $x_i, i > N'$). Now, $S' = \sum_{i=0}^{p^2} a_i S_i$ is a symmetric polynomial of degree p^2 over $\mathbb{F}^{N'}$, with $a_i = 1$, and g' is a polynomial of a degree smaller than p^2 . Our gain is in that now g' can be replaced by a multilinear polynomial coinciding with g' on the boolean cube, and hence having a non-trivial correlation with S' on the boolean cube.

Now, the Alon-Beigel argument can be applied to show that the symmetric polynomial S_{p^2} has a non-trivial correlation with a symmetric polynomial h of a smaller degree over the boolean cube $\{0, 1\}^{N'}$ viewed as a subset of $\mathbb{F}^{N'}$. This, however, couldn't be true due to a theorem of Lucas, which implies that for a boolean vector x with Hamming weight $w = \sum_{i=1}^{N'} x_i$, the value $S_{p^2}(x)$ depends only on the 3-rd digit in the representation of w in base p , while the value of h depends only on the first 2 digits.

This completes the argument. We conclude with an observation that this argument directly extends to S_{p^k} for any $k > 1$.

Here is a brief overview of the rest of the paper. Section 2 defines relevant notions and contains proofs of several technical claims. Theorem 1.2 is proved in Section 3. Theorem 1.3 is proved in Section 4.

2 Some useful notions and claims

2.1 Some multilinear polynomials and their properties

In this sub-section we introduce and discuss certain polynomials over the finite field \mathbb{F} . These polynomials can be conveniently viewed as multi-linear functions on matrices whose entries are elements of \mathbb{F} , or formal variables with values in the field. A basic object we consider is a rectangular $n \times N$ matrix, $N \geq n$. A matrix M with rows $r_1 \dots r_n$ will be denoted by $M[r_1 \dots r_n]$.

Sometimes there will be repeated rows. In such a case we consider a partition $\lambda = (\lambda_1 \dots \lambda_k)$ of $[n]$, that is λ_i are (possibly empty) subsets of $[n]$, whose disjoint union is $[n]$. We denote by $M_\lambda[r_1 \dots r_k]$ the matrix whose rows in positions indexed by elements of λ_i equal r_i . Note that the partition λ is ordered, in that the ordering of the sets λ_i is relevant. We use the notation $\{\lambda_1 \dots \lambda_k\}$ for an unordered partition.

First, we introduce the "symmetric" function \mathcal{S} . We define $\mathcal{S}(M)$ to be the sum of all the permanent minors of M , that is

$$\mathcal{S}(M) := \sum_{C \subseteq [N], |C|=n} \text{Per}(M_C),$$

where M_C is an $n \times n$ submatrix of M which is obtained by deleting all the columns of M except these with indices in C .

Let $\lambda = (\lambda_1 \dots \lambda_k)$ be a partition of $[n]$, and set $\ell_i = |\lambda_i|$. Clearly $\mathcal{S}(M_\lambda)$ depends only on the cardinalities ℓ_i of λ_i . This leads to the notation $M[r_1^{(\ell_1)} \dots r_k^{(\ell_k)}]$ which denotes the matrix in which the row r_1 appears ℓ_1 times, followed by ℓ_2 appearances of the row r_2 and so on. In this notation, therefore

$$\mathcal{S}(M_{(\lambda_1 \dots \lambda_k)}[r_1 \dots r_k]) = \mathcal{S}\left(M[r_1^{(|\lambda_1|)} \dots r_k^{(|\lambda_k|)}]\right)$$

The second matrix function we consider is the "forward" function \mathcal{F} , with

$$\mathcal{F}(M[r_1 \dots r_n]) = \sum_{C \subseteq [N], |C|=\{j_1 < j_2 < \dots < j_n\}} \prod_{i=1}^n r_i(j_i)$$

Here $r_i(j)$ denote the j -th coordinate of the vector r .

To connect the two notions, observe that

$$\mathcal{S}(M[r_1 \dots r_n]) = \sum_{\sigma} \mathcal{F}(M[r_{\sigma_1} \dots r_{\sigma_n}])$$

where σ runs over all permutations on n items.

The last function we consider is a "hybrid" function \mathcal{H} which has some 'symmetric' and some 'forward' properties. Let $\lambda = (\lambda_1 \dots \lambda_k)$ be an ordered partition of $[n]$ with k terms. For another such partition $\theta = (\theta_1 \dots \theta_k)$ of $[n]$ write $\theta \sim \lambda$ if $|\theta_1| = |\lambda_1|, \dots, |\theta_k| = |\lambda_k|$. We define

$$\mathcal{H}(M_\lambda[r_1 \dots r_k]) = \sum_{C \subseteq [N], |C|=\{j_1 < j_2 < \dots < j_n\}} \sum_{\theta \sim \lambda} \prod_{t=1}^k \prod_{i \in \theta_t} r_t(j_i)$$

An alternative view of the functions \mathcal{S} , \mathcal{F} and \mathcal{H} might be helpful at this point. Consider the set of *paths* which are one-to-one functions from $[n]$ to $[N]$. Let us call a path ρ monotone on a subset $\{i_1 < i_2 < \dots < i_\ell\}$ of $[n]$ if $\rho(i_1) < \rho(i_2) < \dots < \rho(i_\ell)$. A path is (fully) monotone if it is monotone on $[n]$. Then, for a partition $\lambda = (\lambda_1 \dots \lambda_k)$ of $[n]$ and an $n \times N$ matrix $M = M_\lambda$,

$$\mathcal{S}(M) = \sum_{\text{all } \rho} \prod_{i=1}^n M_{i, \rho(i)}$$

$$\mathcal{F}(M) = \sum_{\text{monotone } \rho} \prod_{i=1}^n M_{i,\rho(i)}$$

$$\mathcal{H}(M) = \sum_{\rho \text{ monotone on } \lambda_1 \dots \lambda_k} \prod_{i=1}^n M_{i,\rho(i)}$$

Note that for the function \mathcal{H} , similarly to the symmetric function \mathcal{S} , holds

$$\mathcal{H}(M_{(\lambda_1 \dots \lambda_k)}[r_1 \dots r_k]) = \mathcal{H}\left(M \begin{bmatrix} r_1^{(|\lambda_1|)} & \dots & r_k^{(|\lambda_k|)} \end{bmatrix}\right)$$

Observe also that if $\lambda = (\{1\} \dots \{n\})$ then $\mathcal{S}(M) = \mathcal{H}(M)$. If $\lambda = (\{[n]\})$ then $\mathcal{F}(M) = \mathcal{H}(M)$ and $\mathcal{S}(M) = n! \cdot \mathcal{F}(M) = n! \cdot \mathcal{H}(M)$. For a general $\lambda = (\lambda_0 \dots \lambda_k)$

$$\mathcal{S}(M) = \left(\prod_{t=1}^k |\lambda_t|! \right) \cdot \mathcal{H}(M) \tag{4}$$

Note that this is an identity in \mathbb{F} . In particular, if one of the terms λ_i has cardinality at least p then $\mathcal{S}(M) = 0$ and (4) provides no information.

To simplify the notation we will usually write $\mathcal{S}(r_1 \dots r_n)$ for $\mathcal{S}(M[r_1 \dots r_n])$, $\mathcal{F}_\lambda(r_1 \dots r_k)$ for $\mathcal{F}(M_\lambda[r_1 \dots r_k])$ and so on.

2.2 Directional derivatives of symmetric polynomials

The functions we have defined are relevant to the discussion here for two reasons. First, the elementary symmetric polynomial $S_n(x)$ in N variables can be viewed as the forward function \mathcal{F} applied to the matrix $M[x \dots x]$, where M has n identical rows equal to x . In our notation,

$$S_n(x) = \mathcal{F}_{\{[n]\}}(x)$$

Second, it is possible to write a directional derivative $(S_n)_{y_1 \dots y_k}$ of S_n of any order as a combination of values of \mathcal{F} on explicitly defined matrices M whose rows are either the indeterminate x or the directions y_i .

The basic observation here is the following lemma which is straightforward from the definition of directional derivative.

Lemma 2.1: *Let a polynomial $P(x)$ in N variables be given by*

$$P(x) = \mathcal{F}_{(\lambda_0 \dots \lambda_k)}(x, y_1 \dots y_k)$$

Then

$$P_z(x) = \sum_{A \subset \lambda_0} \mathcal{F}_{(A, \lambda_0 \setminus A, \lambda_1 \dots \lambda_k)}(x, z, y_1 \dots y_k)$$

In words, when we take the derivative of such a polynomial in direction z , we replace some of the rows which contained x with z .

As a corollary we have a following expression for higher order derivatives of a symmetric polynomial.

Proposition 2.2: *Let $k \leq n$, then*

$$(S_n)_{y_1 \dots y_k}(x) = \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}\left(x^{(m)}, y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

Proof: Iterating Lemma 2.1,

$$(S_n)_{y_1 \dots y_k}(x) = \sum_{\lambda=(\lambda_0, \lambda_1 \dots \lambda_k)} \mathcal{F}_\lambda(x, y_1 \dots y_k)$$

where the summation is over partitions λ such that λ_i are not empty for $i = 1 \dots k$. Rearranging, this is

$$\begin{aligned} \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \sum_{\lambda: |\lambda_0|=m, |\lambda_1|=\ell_1 \dots |\lambda_k|=\ell_k} \mathcal{F}_\lambda(x, y_1 \dots y_k) = \\ \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}\left(x^{(m)}, y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right) \end{aligned}$$

■

We can give explicit expressions for the coefficients of $(S_n)_{y_1 \dots y_k}(x)$. Fix m indices $j_1 < j_2 < \dots < j_m$ for $0 \leq m \leq n - k$, and let a be the coefficient of $x_{j_1} \dots x_{j_m}$ in $(S_n)_{y_1 \dots y_k}$.

Corollary 2.3:

•

$$a = \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}^{\{j_1 \dots j_m\}}\left(y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

• If $k + m + p > n + 1$ then

$$a = \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \left(\prod_{i=1}^k \ell_i! \right)^{-1} \cdot \mathcal{S}^{\{j_1 \dots j_m\}}\left(y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

Here, for a subset of indices $T \subseteq [N]$, $\mathcal{H}^T(M)$ returns the value of the matrix function \mathcal{H} applied to the $n \times (N - |T|)$ matrix obtained from M by deleting columns in T . The function $\mathcal{S}^T(M)$ is defined similarly.

Proof: The first claim is immediate from Proposition 2.2. The second claim follows from the first claim, from (4), and from the simple observation that if $k + m + p > n + 1$ then $\ell_i < p$ for $i = 1 \dots k$ in the above summation, which means $\ell_i!$ is invertible in \mathbb{F}_p . ■

Example 2.4: The following "toy" example will be relevant for the case of the binary field. It is sufficiently simple to illustrate what's going on behind the cumbersome formulas. Consider $P = (S_4)_{y,z}$. Then P is a quadratic polynomial and for $1 \leq i < j \leq N$

$$\text{coef}_{x^{(i)}x^{(j)}}(P) = \sum_{k \neq l, k, l \notin \{i, j\}} y(k)z(l) = \mathcal{S}^{\{i, j\}}(y, z)$$

■

Continuing with the same example, note that it convenient to express the symmetric function $\mathcal{S}(y, z)$ via inner products of vectors $y, z, \mathbf{1}$, where $\mathbf{1}$ is the all-1 vector of length N .

$$\mathcal{S}(y, z) = \sum_{k \neq l} y(k)z(l) = \langle y, \mathbf{1} \rangle \cdot \langle z, \mathbf{1} \rangle - \langle yz, \mathbf{1} \rangle$$

Here we take yz to be the vector whose coordinates are point-wise inner products of the coordinates of y and z , that is $(yz)(i) = y(i)z(i)$. Of course, $\langle yz, \mathbf{1} \rangle$ is the same as $\langle y, z \rangle$.

Similarly, we can express the 'incomplete' symmetric function $\mathcal{S}^{\{i, j\}}(y, z)$ via the complete symmetric function $\mathcal{S}(y, z)$ minus forbidden terms, as follows

$$\mathcal{S}^{\{i, j\}}(y, z) = \mathcal{S}(y, z) - \left(z(i) + z(j) \right) \langle y, \mathbf{1} \rangle - \left(y(i) + y(j) \right) \langle z, \mathbf{1} \rangle + \left(y(i)z(j) + y(j)z(i) \right)$$

Note the "inclusion-exclusion" structure in the two expressions above. (To make it even clearer we use "+" and "-" notation, though in the binary field both are, of course, the same.) This structure becomes more evident as we pass to our next order of business, which is expressing, for general n and k , the coefficients of $(S_n)_{y_1 \dots y_k}$ via inner products of vectors $y_1 \dots y_k, \mathbf{1}$.

2.3 Inclusion-Exclusion formulas for symmetric functions

Some notation: Given m vectors $y_1 \dots y_m$ and a subset $\tau \subseteq [m]$, let y_τ to be vector whose coordinates are point-wise products of the corresponding coordinates of $y_i, i \in \tau$. Let $\mathcal{S}(y[\tau])$ for the value of the function \mathcal{S} on a matrix with $|\tau|$ rows $y_i, i \in \tau$. Let $\langle y_\tau, \mathbf{1} \rangle = \sum_{j=1}^N \prod_{i \in \tau} y_i(j)$.

We start with an auxiliary lemma expressing the incomplete symmetric function $\mathcal{S}^{\{k\}}(r_1 \dots r_n)$ as a polynomial in the k -th coordinate of the vectors r_i and in complete symmetric functions applied to sub-matrices of $M[r_1 \dots r_n]$.

Lemma 2.5:

$$\mathcal{S}^{\{k\}}(r_1 \dots r_n) = \sum_{\tau \subseteq [n]} (-1)^{|\tau|} (|\tau|)! \cdot r_\tau(k) \cdot \mathcal{S}\left(r\left[[n] \setminus \tau\right]\right)$$

From now on we assume r_\emptyset to be the all-1 vector, and $\mathcal{S}(r[\emptyset])$ to equal 1.

Proof: The proof is by induction on n . For $n = 1$ both sides equal $\sum_{j=1}^N r_1(j) - r_1(k)$.

For $n > 1$, observe that

$$\mathcal{S}^{\{k\}}(r_1 \dots r_n) = \mathcal{S}(r_1 \dots r_n) - \sum_{i=1}^n r_i(k) \cdot \mathcal{S}^{\{k\}}\left(r\left[[n] \setminus \{i\}\right]\right)$$

and the claim is easily verified using the induction hypothesis. ■

Now we can state two main claims of this section. The first expresses the complete symmetric function $\mathcal{S}(r_1 \dots r_n)$ via inner products $\langle r_T \rangle$.

Proposition 2.6:

$$\mathcal{S}(r_1 \dots r_n) = \sum_{\lambda = \{\lambda_1 \dots \lambda_m\}} \prod_{t=1}^m \left((-1)^{|\lambda_t| - 1} (|\lambda_t| - 1)! \cdot \langle r_{\lambda_t} \rangle \right)$$

In this summation $\lambda = \{\lambda_1 \dots \lambda_m\}$ runs over all unordered partitions of $[n]$ with non-empty λ_i .

Proof: Again, the proof is by induction on n . For $n = 1$ both sides equal $\sum_{j=1}^N r_1(j)$. For $n > 1$ we have

$$\mathcal{S}(r_1 \dots r_n) = \sum_{k=1}^N r_n(k) \cdot \mathcal{S}^{\{k\}}(r_1 \dots r_{n-1})$$

Using Lemma 2.5 and the induction hypothesis,

$$\begin{aligned} \mathcal{S}(r_1 \dots r_n) &= \sum_{k=1}^N r_n(k) \cdot \sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot r_\tau(k) \cdot \mathcal{S}\left(r\left[[n-1] \setminus \tau\right]\right) = \\ &\sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot \langle r_{\tau \cup [n]} \rangle \cdot \mathcal{S}\left(r\left[[n-1] \setminus \tau\right]\right) \end{aligned}$$

Consider the summand corresponding to $\tau = [n-1]$. Recall the boundary assumption $\mathcal{S}(r[\emptyset]) = 1$. Hence this summand is $(-1)^{n-1} (n-1)! \cdot \langle r_{[n]} \rangle$. This summand therefore corresponds to the partition $\lambda = \{[n]\}$ in the claim of the proposition.

For τ a proper subset of $[n-1]$, we use the induction hypothesis to obtain

$$\begin{aligned} \mathcal{S}(r_1 \dots r_n) &= \sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot \langle r_{\tau \cup [n]} \rangle \cdot \sum_{\theta = \{\theta_1 \dots \theta_l\}} \prod_{t=1}^l \left((-1)^{|\theta_t| - 1} (|\theta_t| - 1)! \cdot \langle r_{\theta_t} \rangle \right) + \\ &(-1)^{n-1} (n-1)! \cdot \langle r_{[n]} \rangle \end{aligned}$$

Here θ runs over all the unordered partitions of $[n-1] \setminus \tau$ with non-empty θ_i . Observe that each pair (τ, θ) leads to a unique partition $\lambda = \{\lambda_1 \dots \lambda_{l+1}\} = \{\theta_1 \dots \theta_l, \tau \cup [n]\}$ of $[n]$. Rearranging the terms, the last summation can be written as

$$\sum_{\lambda = (\lambda_1 \dots \lambda_m)} \prod_{t=1}^m \left((-1)^{|\lambda_t| - 1} (|\lambda_t| - 1)! \cdot \langle r_{\lambda_t} \rangle \right)$$

completing the proof of the proposition. ■

The second claim expresses the incomplete symmetric function $\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n)$ as a polynomial in the missing coordinates $j_1 \dots j_k$ of the vectors r_i and in complete symmetric functions applied to sub-matrices of $M[r_1 \dots r_n]$. Note that Lemma 2.5 is a special case $k = 1$ of this claim.

Proposition 2.7:

$$\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n) = \sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left((-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left(r \left[[n] \setminus \cup_t \tau_t \right] \right)$$

Here the summation is on all ordered set systems τ such that the terms τ_t are disjoint subsets of $[n]$. The terms may also be empty.

Proof: The proof is by induction on k and n . The case $k = 1$ is treated in Lemma 2.5.

Consider the case $n = 1$. On one hand $\mathcal{S}^{\{j_1 \dots j_k\}}(r_1) = \sum_{j=1}^N r_1(j) - \sum_{t=1}^k r_1(j_t)$. We claim that this value can be also represented as

$$\sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left((-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left(r \left[[1] \setminus \cup_t \tau_t \right] \right)$$

Here τ_i are disjoint subsets of $[1]$. Observe that there are $k + 1$ summands in this expression, corresponding to different set systems τ . Let $\tau^{(0)}$ denote the set system with k empty terms, and let $\tau^{(t)}$, for $t = 1 \dots k$ denote the set system with $\tau_t = \{1\}$ and all the remaining terms are empty. The summand corresponding to $\tau^{(0)}$ is $\mathcal{S}(r_1) = \sum_{j=1}^N r_1(j)$. The summand corresponding to $\tau^{(t)}$ is $(-r_1(j_t)) \cdot \mathcal{S}(r_\emptyset) = -r_1(j_t)$, and we are done in this case.

For $k, n > 1$, we have

$$\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n) = \mathcal{S}^{\{j_1 \dots j_{k-1}\}}(r_1 \dots r_n) - \sum_{i=1}^n r_i(j_k) \cdot \mathcal{S}^{\{j_1 \dots j_k\}} \left(r \left[[n] \setminus \{i\} \right] \right)$$

By the induction hypothesis, this is

$$\begin{aligned} & \sum_{\theta=(\theta_1 \dots \theta_{k-1})} \prod_{t=1}^{k-1} \left((-1)^{|\theta_t|} (|\theta_t|)! \cdot r_{\theta_t}(j_t) \right) \cdot \mathcal{S} \left(r \left[[n] \setminus \cup_t \theta_t \right] \right) - \\ & \sum_{i=1}^n r_i(j_k) \cdot \sum_{\mu^{(i)}=(\mu_1^{(i)} \dots \mu_k^{(i)})} \prod_{u=1}^k \left((-1)^{|\mu_u^{(i)}|} (|\mu_u^{(i)}|)! \cdot r_{\mu_u^{(i)}}(j_u) \right) \cdot \mathcal{S} \left(r \left[[n] \setminus \cup_t \mu_t^{(i)} \setminus \{i\} \right] \right) \end{aligned}$$

Here the summation is on all ordered set systems θ such that the terms θ_t are disjoint subsets of $[n]$ and on ordered set systems $\mu^{(i)}$, $i = 1 \dots n$ such that the terms $\mu_u^{(i)}$ are disjoint subsets of $[n] \setminus \{i\}$.

Given a set system $\theta = (\theta_1 \dots \theta_{k-1})$ we define a set system $\tau = (\tau_1 \dots \tau_k)$ by setting $\tau_t = \theta_t$, $t = 1 \dots k-1$ and $\tau_k = \emptyset$. Given a set system $\mu^{(i)} = (\mu_1^{(i)} \dots \mu_k^{(i)})$ we define a set system $\tau = (T_1 \dots T_k)$ by setting $\tau_u = \mu_u^{(i)}$, $u = 1 \dots k-1$ and $\tau_k = \mu_k^{(i)} \cup \{i\}$. In both cases we have obtained a set system of the type we want, that is an ordered family of k disjoint subsets of $[n]$. Moreover, each such system with empty k -th term is obtained exactly once, from the corresponding θ -system, and each system with non-empty k -th term τ_k is obtained exactly $|\tau_k|$ times, from systems $\mu^{(i)}$ with $i \in \tau_k$. Rearranging the terms and the signs, the last expression is precisely

$$\sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left((-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left(r \left[[n] \setminus \cup_t \tau_t \right] \right),$$

completing the proof. ■

2.4 Some properties of Gowers' norms

The main result in this subsection shows that if a function from \mathbb{F}^N to \mathbb{F} is fixed on a subset of \mathbb{F}^N defined by low-degree polynomial constraints, then it has a non-trivial Gowers norm of an appropriate order.

Recall that for a vector $x \in \mathbb{F}^N$, x^i stands for a vector in \mathbb{F}^N whose coordinates are i -th powers of the coordinates of x .

Proposition 2.8: *Let K be an absolute constant. Let $y_{i,j}$, $i = 1 \dots p-1$, $j = 1 \dots K$, be $K(p-1)$ vectors in \mathbb{F}^N . Let M be a subset of \mathbb{F}^N defined by the constraints $\langle x^i, y_{i,j} \rangle = 0$ for all i, j .*

Let f be a function from \mathbb{F}^N to \mathbb{F} . Assume that f is fixed on M . Then

$$\|f\|_{U^p} > \left(\frac{|M|}{2^N} \right)^2 =: Pr^2\{M\}$$

Proof: Let $f|_M \equiv c_0$.

Consider a subspace V of polynomials of degree at most $p-1$ in $\mathbb{F}[x_1 \dots x_N]$ spanned by the polynomials $\langle x^i, y_{i,j} \rangle$, for all i, j . We will first find a polynomial $g \in V$ such that $|\langle f, g \rangle| \geq Pr\{M\}$. This, combined with a lemma from [4], will imply the claim of the proposition.

Let $\mathbf{b} = (b_{i,j})$, $i = 1 \dots p-1$, $j = 1 \dots K$, be a matrix with entries in \mathbb{F} . Let $c \in \mathbb{F}$. Set

$$\mu(\mathbf{b}, c) = Pr \left\{ x : f(x) = c \wedge \langle x^i, y_{i,j} \rangle = b_{i,j} \text{ for all } i, j \right\}$$

Note that, by assumption, for a zero matrix \mathbf{b} holds $\mu(\mathbf{b}, c_0) = Pr\{M\}$. In other words, $\mu(\mathbf{b}, c) = 0$ and for $\mathbf{b} = 0$ any $c \neq c_0$.

Now, for any $g(x) = \sum_{i,j} a_{i,j} \langle x^i, y_{i,j} \rangle$ in V holds

$$\langle f, g \rangle = \mathbb{E} e(f - g) = \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c - \langle \mathbf{a}, \mathbf{b} \rangle)$$

where $\mathbf{a} = (a_{i,j})_{i,j}$ and $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i,j} a_{i,j} b_{i,j}$. Averaging over V , we have

$$\begin{aligned} \mathbb{E}_{g \in V} \langle f, g \rangle &= \frac{1}{|V|} \sum_{\mathbf{a}} \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c - \langle \mathbf{a}, \mathbf{b} \rangle) = \frac{1}{|V|} \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c) \sum_{\mathbf{a}} e(-\langle \mathbf{a}, \mathbf{b} \rangle) = \\ &= \sum_c \mu(0, c) \cdot e(c) = \mu(0, c_0) \cdot e(c_0) = Pr\{M\} \cdot e(c_0) \end{aligned}$$

This means, there is $g \in V$ with $|\langle f, g \rangle| \geq Pr\{M\}$. We conclude the proof of the proposition by quoting a lemma from [4], which states that $|\langle f, g \rangle| \geq \epsilon$ implies $\|f\|_{U^p} \geq \epsilon$. ■

2.5 Asymptotic uniformity and independence of some random variables

In this subsection we deal with another property of multivariate polynomials. Let n be fixed integer and let N be an integer parameter growing to infinity. Let $r_1 \dots r_n$ be n vectors in \mathbb{F}^N . Let $\kappa = (k_1 \dots k_n)$ be a non-zero sequence of integers $0 \leq k_i < p$. For each such sequence define a polynomial $X_\kappa(r_1, \dots, r_n) = \sum_{j=1}^N \prod_{i=1}^n r_i^{k_i}(j)$.

Now, let $r_1 \dots r_n$ be chosen uniformly and independently from \mathbb{F}^N . We claim that for a large N the random variables $X_\kappa(r_1, \dots, r_n)$ are nearly independent and uniformly distributed over \mathbb{F} . Let $X = (X_\kappa)_\kappa$, and let $K = p^n$.

Proposition 2.9: *Let U be the uniform distribution on \mathbb{F}^K . Let P be distribution of X on \mathbb{F}^K . Let $\|\cdot\|$ denote the statistical (l_1) distance between distributions.*

Then there is a constant $c > 0$ depending on n, p but not on N such that

$$\|P - U\| \leq \exp\{-cN\}$$

Proof: We start from a simple observation that Fourier transform of a uniform distribution is the delta function at 0. In addition, the two following statements are equivalent up to constants: 'a distribution is exponentially close to uniform' and 'all non-zero Fourier coefficients of the distribution are exponentially close to zero'. Accordingly, we will show that all the non-zero Fourier coefficients of P tend exponentially fast in N to zero.

Consider a character $\chi(y) = \xi^{\langle y, a \rangle}$, corresponding to a non-zero vector $a = (a_\kappa)_\kappa \in \mathbb{F}^K$. (Recall that $\xi = e^{2\pi i/p}$ is the p -th primitive root of unity.) Then, normalizing appropriately,

$$\widehat{P}(\chi) = \sum_y P(y) \bar{\chi}(y) = \sum_y Pr\{X = y\} \cdot \xi^{-\sum_\kappa a_\kappa y_\kappa} = \mathbb{E} \xi^{-\sum_\kappa a_\kappa X_\kappa}$$

Let P_a denote the distribution of the random variable $X_a = \sum_\kappa a_\kappa X_\kappa$. Then we have shown $\widehat{P}(\chi) = \widehat{P}_a(1)$. We will show the non-zero Fourier coefficients of P_a to be exponentially small, completing the proof of the proposition.

We have

$$X_a(r_1, \dots, r_n) = \sum_\kappa a_\kappa P_\kappa(r_1, \dots, r_n) = \sum_{j=1}^N \sum_{\kappa=(k_1 \dots k_n)} a_\kappa \prod_{i=1}^n r_i^{k_i}(j)$$

Let x_i be elements of the field \mathbb{F} . Consider an n -variate polynomial

$$Q(x_1 \dots x_n) = \sum_{\kappa=(k_1 \dots k_n)} a_\kappa \prod_{i=1}^n x_i^{k_i}$$

Since not all of the coefficients a_κ are zero, and since all κ are non-zero sequences, Q is a multi-variate polynomial of degree at least 1 in $\mathbb{F}[x_1 \dots x_n]$, and therefore attains at least two values with probability bounded away from zero. Now, $X_a = \sum_{j=1}^N Q(r_1(j) \dots r_n(j))$ is a sum of N independent copies of Q . Let μ denote the distribution of Q on \mathbb{F} . Then the distribution P_a of X_a is μ^{*N} , the N -wise convolution of μ with itself. Since p is prime, $\widehat{\mu}(0) = 1$, and $|\widehat{\mu}| < 1$ everywhere else. Therefore, $\widehat{P}_a = (\widehat{\mu})^N$ tends to the delta function at 0 exponentially fast in N , completing the proof. ■

2.6 Estimates on the number of common zeroes of some families of polynomials

The main claim of this subsection is the following proposition.

Proposition 2.10: *Let M be the ring of \mathbb{F} -valued functions on \mathbb{F}^N , that is $M = \mathbb{F}[x_1 \dots x_N]/I$, where I is the ideal $(x_1^p - x, \dots, x_N^p - x)$. Let $f_1 \dots f_K$ be polynomials in M . Let S be the set of common zeroes of $f_1 \dots f_K$, that is*

$$S = \left\{ u \in \mathbb{F}^N : f_1(u) = \dots = f_K(u) = 0 \right\}$$

Then

$$|S| \leq \dim(M/J)$$

where J is the ideal generated by $\{f_i\}$, and $\dim(M/J)$ denotes the dimension of M/J , viewed as a vector space over \mathbb{F} .

Proof: For each $u \in S$, let $q_u \in M$ be defined by $q_u(u) = 1$ and $q_u(v) = 0$ for all $v \neq u$. We will show that the family $\{q_u + J\}_{u \in S}$ is linearly independent in M/J . This will immediately imply the claim of the proposition.

Consider a linear combination $q = \sum_{u \in S} \lambda_u q_u$ such that $q \in J$. Let $v \in S$. We compute $q(u)$ in two ways. First, since $q \in J$, we have $q(v) = 0$. On the other hand, $q(v) = \sum_{u \in S} \lambda_u q_u(v) = \lambda_v$. This shows $\lambda_v = 0$ for all $v \in S$, completing the proof. ■

In some cases, the dimension of M/J is easy to estimate.

Lemma 2.11: *Let $p = 2$, let $K = \binom{N}{k}$, and let $\{f_I\}$ be indexed by k -subsets I of $[N]$. Assume that for any such subset I holds*

$$\deg \left(f_I(x) - \prod_{i \in I} x_i \right) \leq k - 1 \tag{5}$$

Then,

$$\dim(M/J) \leq \sum_{j=0}^{k-1} \binom{N}{j}$$

Proof: We will construct a generating subset of the vector space M/J of cardinality at most $\sum_{j=0}^{k-1} \binom{N}{j}$. We start from a trivial generating set $\{m + J\}$, where m runs through all the 2^N multi-linear monomials in N variables. Now, in the factor space M/J , we can replace any product of k variables, $\prod_{i \in I} x_i$, by a polynomial of degree smaller than k . Iterating this procedure, we arrive to a generating set spanned by $\{s + J\}$, where s now runs through $\sum_{j=0}^{k-1} \binom{N}{j}$ monomials of degree at most $k - 1$. ■

3 Proof of Theorem 1.2

We need to show that

$$\|S_{2p}\|_{U^{p+2}} > \epsilon$$

for an absolute constant ϵ .

We remark that (2) can be shown exactly in the same way, replacing $2p$ with n and $p + 2$ with $n - p + 2$ throughout.

Recall ([4]) that $\|f\|_{U^{p+2}} = \mathbb{E}_{y,z}^{1/2^{p+2}} \|f_{y,z}\|_{U^p}^{2^p}$. Since the Gowers' norms are nonnegative, it will suffice to show that $\|f_{y,z}\|_{U^p}$ is non-negligible for a non-negligible fraction of directions y, z .

Let

$$A = \left\{ (y, z) : \langle y^a, z^b \rangle = 0 \text{ for all } 0 \leq a, b < p \right\}$$

By Proposition 2.9, for uniformly and independently chosen directions y, z , and for a sufficiently large N , the probability of A is very close to p^{-p^2} . Therefore, A is a non-negligible event. We will now show that for any $(y, z) \in A$ holds $\|f_{y,z}\|_{U^p} > \epsilon'(y, z)$, for an appropriate function ϵ' .

Fix (y, z) in A . Let $f = (S_{2p})_{y,z}$. Let

$$M = M(y, z) = \left\{ x : \langle x^i, y^a z^b \rangle = 0 \text{ for all } 1 \leq i \leq p - 1, 0 \leq a, b < p \right\}$$

We will show that f is fixed on M . Assuming this, by Proposition 2.8, we have $\|f_{y,z}\|_{U^p} > Pr^2\{M\}$, and therefore

$$\begin{aligned} \|f\|_{U^{p+2}}^{2^{p+2}} &= \mathbb{E}_{y,z} \|f_{y,z}\|_{U^p}^{2^p} \geq Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr^{2^{p+1}}\{M(y, z)\} \geq \\ Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr\{M(y, z)\} &\geq (Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr\{M(y, z)\})^{2^{p+1}} = \\ Pr^{2^{p+1}} \left\{ x : \langle x^i y^a z^b \rangle = 0 \text{ for all } 0 \leq a, b, i \leq p - 1 \right\} &\geq \Omega\left(p^{-p^3 \cdot 2^{p+1}}\right) \end{aligned}$$

The last inequality follows from Proposition 2.9, since random variables $\langle x^i y^a z^b \rangle$ are asymptotically uniform and independent.

It remains to prove the following fact.

Lemma 3.1: Let x, y, z be three vectors in \mathbb{F}^N satisfying $\langle x^i y^a z^b \rangle = 0$ for all $0 \leq a, b, i \leq p-1$. Then

$$(S_{2p})_{y,z}(x) = \mathcal{H}(y^{(p)}, z^{(p)})$$

Proof: By Proposition 2.2,

$$(S_{2p})_{y,z}(x) = \sum_{m=0}^{2p-2} \sum_{a,b \geq 1, a+b=2p-m} \mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$$

We claim that all of the summands on the right, except (possibly) $\mathcal{H}(y^{(p)}, z^{(p)})$ are 0.

There are two possible cases to consider. The easier case is when $a, b, m < p$. In such a case, by (4), $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$ is proportional to $\mathcal{S}(x^{(m)}, y^{(a)}, z^{(b)})$. By Proposition 2.6, the symmetric function $\mathcal{S}(x^{(m)}, y^{(a)}, z^{(b)})$ is a polynomial in $\langle x^i y^a z^b \rangle$, which vanishes when all of these inner products are 0.

In the second case, one of the indices a, b, m is at least p . Note, that there could be at most one such index (barring the case $a = b = p$). We may assume this index is m . We claim that in this case $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$ can be written as a linear combination of hybrid functions $\mathcal{H}(x^{(\ell)}, r_1, \dots, r_{m-\ell})$, where $\ell < m$ and the vectors r_i are of the form $x^\alpha y^\beta z^\gamma$. Note that this will suffice to prove the lemma, since iterating this step will express $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$ as a linear combination of symmetric functions in r_i , and these functions vanish.

Consider $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$. For notational convenience, let $w_1 \dots w_{a+b}$ stand for the vectors $y \dots y, z \dots z$ (y taken a times and z taken b times). Note that both a and b are smaller than p . Using Corollary 2.3 and Proposition 2.7,

$$\begin{aligned} \mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)}) &= (a! \cdot b!)^{-1} \cdot \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \mathcal{S}^{\{i_1 \dots i_m\}}(y^{(a)}, z^{(b)}) = \\ &(a! \cdot b!)^{-1} \cdot \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \cdot \sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m \left((-1)^{|\tau_t|} (|\tau_t|)! \cdot w_{\tau_t}(i_t) \right) \cdot \mathcal{S}(w[[a+b] \setminus \cup_t \tau_t]) \end{aligned}$$

Here the inner summation is on all ordered set systems τ such that the terms τ_t are disjoint subsets of $[a+b]$. The terms may also be empty.

Let us attempt to simplify the double summation we obtained. First, we may disregard the constant term $(a! \cdot b!)^{-1}$. Next, observe that, as before, all symmetric functions of the form $\mathcal{S}(w[T])$ vanish, unless T is empty, in which case they equal 1. Therefore, we may consider the double summation

$$\sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \cdot \sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m \left((-1)^{|\tau_t|} (|\tau_t|)! \cdot w_{\tau_t}(i_t) \right)$$

Here the inner summation is on all ordered partitions τ of $[a+b]$. The terms τ_t may also be empty. Changing the order of summation, and ignoring the constant term $(-1)^{a+b}$, we get

$$\sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m (|\tau_t|)! \cdot \sum_{i_1 < i_2 < \dots < i_m} \prod_{t=1}^m (x \cdot w_{\tau_t})(i_t) = \sum_{\tau=(\tau_1 \dots \tau_m)} \left(\prod_{t=1}^m (|\tau_t|)! \right) \cdot \mathcal{F}(xw_{\tau_1}, xw_{\tau_2}, \dots, xw_{\tau_m})$$

Consider the last expression. Let us use some more notation. For an ordered partition $\tau = (\tau_1 \dots \tau_m)$, let $n = n(\tau)$ be the number of empty terms. Let $\{\tau_1 \dots \tau_m\}$ denote the unordered version of this partition, where the first $n(\tau)$ terms are taken, by agreement, to be the empty ones. Then we can rewrite this expression as

$$\sum_{\tau=\{\tau_1 \dots \tau_m\}} \left(\prod_{t=1}^m (|\tau_t|)! \right) \cdot \mathcal{H} \left(x^{(n)}, xw_{\tau_{n+1}}, \dots, xw_{\tau_m} \right)$$

Now, clearly not all the terms in the partition are empty and, therefore, $n(\tau) < m$ for all τ , completing the proof of our last claim, of the lemma, and of the theorem. ■

4 Proof of Theorem 1.3

Let $p = 2$. We will show there is an absolute constant $\alpha > 0$ such that for any polynomial g of degree at most 3 in N variables holds

$$\langle S_4, g \rangle < \exp\{-\alpha N\}$$

A first step is to observe that there is a relation between the inner product of two functions and the average inner product of their derivatives.

Lemma 4.1: *For any two functions f and g holds*

$$\langle f, g \rangle^4 \leq \mathbb{E}_y \langle f_y, g_y \rangle^2$$

Proof: This is an immediate corollary of a lemma in [7], but we give the elementary proof for completeness. By the Cauchy-Schwarz inequality,

$$\mathbb{E}_y \langle f_y, g_y \rangle^2 \geq \mathbb{E}_y^2 \langle f_y, g_y \rangle = \mathbb{E}_{x,y}^2 (-1)^{f(x)+f(x+y)+g(x)+g(x+y)} = \mathbb{E}^4 (-1)^{f(x)+g(x)} = \langle f, g \rangle^4$$

■

Corollary 4.2:

$$\langle f, g \rangle^8 \leq \mathbb{E}_{y,z} \langle f_{y,z}, g_{y,z} \rangle^2$$

We will show that for any polynomial g of degree at most 3 holds $\mathbb{E}_{y,z} \left\langle (S_4)_{y,z}, g_{y,z} \right\rangle^2 \leq \exp\{-\alpha N\}$. First, here is a brief overview of the argument.

The point is that taking second derivatives makes life easier, since a second derivative of g is a linear function, and a second derivative of S_4 is a quadratic. We therefore need to show that for the large majority of directions y, z , the quadratic function $(S_4)_{y,z}$ has a small inner product with the linear function $(-1)^{g_{y,z}}$. In this we will be helped by a theorem of Dixon giving a structural description of quadratic polynomials, which, in particular, characterizes the Fourier transform of functions of the type $(-1)^Q$, where Q is a quadratic. In fact, setting

$Q = (S_4)_{y,z}$ we will see that for many of the directions y, z the Fourier coefficients of $(-1)^Q$ will be exponentially small. For the remaining directions, these Fourier coefficients will be supported on an explicit easy to describe 3-dimensional affine subspace depending on y, z . We will then argue that for any fixed polynomial g of lower degree, the support of the character $(-1)^{g_{y,z}}$ lies in this affine subspace with exponentially small probability over y, z .

We proceed with computing the second derivative $Q = (S_4)_{y,z}$.

4.1 Second derivatives of S_4

Write $Q(x) = \sum_{i<j} q_{i,j}x(i)x(j) + \sum_i \ell_i x(i) + c$.

By Proposition 2.2 or by Example 2.4.

$$q_{i,j} = \mathcal{S}(y, z) - \langle y, \mathbf{1} \rangle \cdot (z(i) + z(j)) + \langle z, \mathbf{1} \rangle \cdot (y(i) + y(j)) + (y(i)z(j) + y(j)z(i))$$

At this point we invoke (a corollary of) a theorem of Dixon [6]:

Theorem 4.3: *Let $Q(x) = \sum_{i<j} q_{i,j}x(i)x(j) + \sum_i \ell_i x(i) + c$ be a quadratic polynomial over \mathbb{F}_2 . Consider the symmetric matrix with zeros on the diagonal and off-diagonal entries given by $S_{i,j} = S_{j,i} = q_{i,j}$. Let the rank of $B = 2h$ (it is always even). Then the function $(-1)^Q$ has 2^{2h} non-zero Fourier coefficients of absolute value 2^{-h} . Moreover, all these coefficients lie in an $2h$ -dimensional affine subspace of \mathbb{F}_2^n .*

Consider the matrix B in our case. Some notation: let J be the matrix with 0 on the diagonal and 1 off the diagonal. Let $u \otimes v$ denote the outer product uv^t . Then,

$$B = \mathcal{S}(y, z) \cdot J + \langle y, \mathbf{1} \rangle \cdot (z \otimes \mathbf{1} + \mathbf{1} \otimes z) + \langle z, \mathbf{1} \rangle \cdot (y \otimes \mathbf{1} + \mathbf{1} \otimes y) + (y \otimes z + z \otimes y)$$

Since the rank of J is at least $N - 1$ and the rank of the remaining matrices is at most 2, the matrix B is almost of full rank if $\mathcal{S}(y, z) = 1$. In this case, by Theorem 4.3, the Fourier coefficients of $(-1)^Q$ are exponentially small.

We therefore may assume $\mathcal{S}(y, z) = 0$. In this case the quadratic part of Q may be written as

$$\sum_{i<j} q_{i,j}x(i)x(j) = \langle y, \mathbf{1} \rangle \cdot \langle x, \mathbf{1} \rangle \langle x, z \rangle + \langle z, \mathbf{1} \rangle \cdot \langle x, \mathbf{1} \rangle \langle x, y \rangle + (\langle x, y \rangle \langle x, z \rangle + \langle x, yz \rangle)$$

Recall that yz denotes the pointwise product of vectors y and z .

This implies the non-zero Fourier coefficients of $\sum_{i<j} q_{i,j}x(i)x(j)$ lie in a 3-dimensional affine subspace of \mathbb{F}_2^n . The linear part of this subspace is spanned by the vectors $y, z, \mathbf{1}$ and it is shifted by a vector yz .

Next, consider the linear part $\sum_i \ell(i)x(i)$ of Q . By Proposition 2.2,

$$\ell(i) = \mathcal{H}^{\{i\}}(y^{(2)}, z) + \mathcal{H}^{\{i\}}(y, z^{(2)}) =$$

$$\sum_{j < k < l \neq i} \left(y(k)y(l)z(j) + y(j)y(l)z(k) + y(j)y(k)z(l) \right) + \left(y(j)z(k)z(l) + y(k)z(j)z(l) + y(l)z(j)z(k) \right)$$

This can be directly verified to be equal to

$$\begin{aligned} & \left(\mathcal{S}(y, z) + \mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot y(i) + \left(\mathcal{S}(y, z) + \mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot z(i) + \\ & \left(\mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle + \langle y, z \rangle \cdot \langle y + z, \mathbf{1} \rangle \right) \end{aligned}$$

By assumption, $\mathcal{S}(y, z) = \langle y, \mathbf{1} \rangle \cdot \langle z, \mathbf{1} \rangle + \langle y, z \rangle = 0$. Note that this also implies $\langle y, z \rangle \cdot \langle y + z, \mathbf{1} \rangle = 0$, implying

$$\ell(i) = \left(\mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot y(i) + \left(\mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot z(i) + \left(\mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle \right)$$

Consequently, the linear part of Q may be written as

$$\sum_i \ell(i)x(i) =$$

$$\left(\mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot \langle x, y \rangle + \left(\mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot \langle x, z \rangle + \left(\mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle \right) \cdot \langle x, \mathbf{1} \rangle$$

This means that the non-zero Fourier coefficients of the polynomial $Q = \sum_{i < j} q_{i,j}x(i)x(j) + \sum_i \ell(i)x(i) + c$ lie in the affine subspace $AF_{y,z} = yz + \text{Span}(y, z, \mathbf{1})$.

4.2 Second derivatives of a fixed polynomial of degree 3

Let

$$g(x) = \sum_{i < j < k} a_{i,j,k} x(i)x(j)x(k)$$

be a polynomial of degree 3. For directions $y, z \in \mathbb{F}^N$, consider the second derivative $g_{y,z} = \sum_i v_{y,z}(i)x(i) + c_{y,z}$. We need to show that the probability of the vector $v_{y,z}$ falling in the affine space $AF_{y,z} = yz + \text{Span}(y, z, \mathbf{1})$ is exponentially small.

First, some notation. For $1 \leq i \leq N$, let G_i be a symmetric $N \times N$ matrix over \mathbb{F} with $(G_i)_{j,k} = (G_i)_{k,j} = a_{i,j,k}$ for all $j \neq k$. (Here we think about $\{i, j, k\}$ as an unordered subset of $[N]$.) The diagonal entries of G_i are set to 0. For future use note the important property $(G_i)_{j,k} = (G_j)_{i,k} = (G_k)_{i,j}$.

These matrices are relevant because they describe the vector $v_{y,z}$.

Lemma 4.4:

•

$$v_{y,z}(i) = \text{coef}_{x(i)}(g_{y,z}(x)) = \langle y, G_i z \rangle$$

• An alternative representation of $v_{y,z}$ will be more convenient for us. For $z \in \mathbb{F}^N$, let $G(z) = \sum_{i=1}^N z(i)G_i$. Then

$$v_{y,z} = G(z) \cdot y$$

Proof: For the first claim of the lemma, by linearity of the derivative, it suffices to consider the monomial $g(x) = x(i)x(j)x(k)$. This case can be easily verified directly.

For the second claim, note that

$$(G(z) \cdot y)(l) = \sum_{k=1}^N (G(z))_{k,l} y(k) = \sum_{k=1}^N y(k) \cdot \sum_{i=1}^N z(i) (G_i)_{k,l} = \sum_{k=1}^N y(k) \cdot \sum_{i=1}^N (G_l)_{k,i} z(i) = \langle y, G_l z \rangle$$

■

Consider the event $\{v_{y,z} \in AF_{y,z}\}$. This means $v_{y,z} = yz + u_{y,z}$, for some vector $u_{y,z} \in \text{Span}(y, z, \mathbf{1})$. There are only 8 possible choices for $u_{y,z}$. For convenience, let us assume, without loss of generality (as can be easily seen from the proof), that $u_{y,z} = y + z + \mathbf{1}$ is the most popular one. By the lemma, the event $\{v_{y,z} = yz + u_{y,z}\}$ is the same as $\{G(z) \cdot y = yz + u_{y,z}\}$. To simplify things some more, let $A_i = G_i + e_i \otimes e_i$, $i = 1 \dots N$. That is, $A_i = G_i$ but for $(A_i)_{i,i} = 1$. Let $A(z) = \sum_{i=1}^N z(i)A_i$. Note that $A(z) \cdot y = G(z) \cdot y + yz$. Hence $\{G(z) \cdot y = yz + u_{y,z}\}$ is the same as $\{A(z) \cdot y = u_{y,z} = y + z + \mathbf{1}\}$

We conclude the proof by a technical claim.

Proposition 4.5: *Let $\{A_i\}$, $i = 1 \dots N$ be a family of symmetric $N \times N$ matrices over \mathbb{F} with $A_i(k, k) = \delta_{ik}$. Then, for y, z uniformly at random and independently from \mathbb{F}^N ,*

$$Pr_{y,z} \left\{ (A(z)) \cdot y = y + z + \mathbf{1} \right\} \leq \left(\frac{3}{4} \right)^N$$

The proof of the proposition is based on the claim that the rank of a matrix $A(z)$ is typically large.

Lemma 4.6: *Let matrices $\{A_i\}$ be as in the proposition. Let C be any fixed symmetric $N \times N$ matrix. Then*

$$Pr_z \left\{ \text{rank}(A(z) + C) \leq k - 1 \right\} \leq \frac{1}{2^N} \cdot \sum_{i=0}^{k-1} \binom{N}{i}.$$

Proof: Consider a family of $\binom{N}{k}$ polynomials f_I on \mathbb{F}^N . These polynomials are indexed by k -subsets of $[N]$. For a k -subset I , let $f_I(z)$ be the determinant of the $I \times I$ minor of $A(z) + C$. Clearly, rank of $A(z) + C$ is smaller than k if and only if z is a joint zero of $\{f_I\}$.

We now claim that the coefficient of $\prod_{i \in I} z_i$ in $f_I(z)$ is 1. If this is true, $\deg(f_I - \prod_{i \in I} z_i) \leq k - 1$, and the claim of the lemma will follow from Lemma 4.6.

Let $B(z) = A(z) + C$. Since we are working in characteristic two, the symmetry of $B(z)$ implies that

$$\begin{aligned} \det B(z) &= \sum_{\sigma \in S_N: \sigma = \sigma^{-1}} \prod_{i=1}^N B_{i\sigma(i)}(z) = \\ &= \sum_{\sigma \in S_N: \sigma = \sigma^{-1}} \prod_{\{i: \sigma(i)=i\}} (z_i + C_{i,i}) \cdot \prod_{\{i: i < \sigma(i)\}} B_{i\sigma(i)}(z) = \prod_{i \in I} z_i + \text{lower order terms.} \end{aligned}$$

In the second equality we use the identity $B_{i\sigma(i)}^2(z) = B_{i\sigma(i)}(z)$ in \mathbb{F} . ■

Let I denote the identity $N \times N$ matrix.

Let $p(z) = Pr_y\{A(z) \cdot y = y + z + \mathbf{1}\}$. Clearly $p(z) \leq 2^{-\text{rank}(A(z)+I)}$. By Lemma 4.6,

$$Pr_{y,z}\{(A(z)) \cdot y = y + z + \mathbf{1}\} = \mathbb{E}_z p_z \leq \mathbb{E}_z 2^{-\text{rank}(A(z)+I)} \leq \frac{1}{2^N} \sum_{k=0}^N \binom{N}{k} 2^{-k} = \left(\frac{3}{4}\right)^N$$

This concludes the proof of the proposition, and of Theorem 1.3.

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