

# Inverse Conjecture for the Gowers norm is false

Shachar Lovett <sup>\*</sup>      Roy Meshulam <sup>†</sup>      Alex Samorodnitsky <sup>‡</sup>

## Abstract

Let  $p$  be a fixed prime number, and  $N$  be a large integer. The 'Inverse Conjecture for the Gowers norm' states that if the " $d$ -th Gowers norm" of a function  $f : \mathbb{F}_p^N \rightarrow \mathbb{F}$  is non-negligible, that is larger than a constant independent of  $N$ , then  $f$  can be non-trivially approximated by a degree  $d - 1$  polynomial. The conjecture is known to hold for  $d = 2, 3$  and for any prime  $p$ . In this paper we show the conjecture to be false for  $p = 2$  and for  $d = 4$ , by presenting an explicit function whose 4-th Gowers norm is non-negligible, but whose correlation any polynomial of degree 3 is exponentially small.

Essentially the same result (with different correlation bounds) was independently obtained by Green and Tao [5]. Their analysis uses a modification of a Ramsey-type argument of Alon and Beigel [1] to show inapproximability of certain functions by low-degree polynomials.

We observe that a combination of our results with the argument of Alon and Beigel implies the inverse conjecture to be false for any prime  $p$ , for  $d = p^2$ .

## 1 Introduction

We consider multivariate functions over finite fields. The main question of interest here would be whether these functions can be non-trivially approximated by a low-degree polynomial.

Fix a prime number  $p$ . Let  $\mathbb{F} = \mathbb{F}_p$  be the finite field with  $p$  elements. Let  $\xi = e^{\frac{2\pi i}{p}}$  be the primitive  $p$ -th root of unity. Denote by  $e(x)$  the exponential function taking  $x \in \mathbb{F}$  to  $\xi^x \in \mathbb{C}$ . For two functions  $f, g : \mathbb{F}^N \rightarrow \mathbb{F}$ , let  $\langle f, g \rangle := \mathbb{E}_x e(f(x) - g(x))$ .

**Definition 1.1:** A function  $f$  is non-trivially approximable by a degree- $d$  polynomial if

$$|\langle f, g \rangle| > \epsilon$$

for some polynomial  $g$  of degree at most  $d$  in  $\mathbb{F}[x_1 \dots x_N]$ . ■

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<sup>\*</sup>Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100, Israel. Research supported by ISF grant 1300/05.

<sup>†</sup>Department of Mathematics, The Technion, Haifa 32000, Israel.

<sup>‡</sup>School of Engineering and Computer Science, The Hebrew University of Jerusalem, Jerusalem 91904, Israel. Research partially supported by ISF grant 039-7682 and by BSF grant 2006377.

More precisely, in this definition we are looking at a sequence  $f_N$  of functions and of approximating low-degree polynomials  $g_N$  in  $N$  variables, and let  $N$  grow to infinity. In this paper, the remaining parameters, that is the field size  $p$ , the degree  $d$  and the offset  $\epsilon$  are fixed, independent of  $N$ .

A counting argument shows that a generic function can not be approximated by a polynomial of low degree. The problems of showing a specific given function to have no non-trivial approximation and of constructing an explicit non-approximable function have been extensively investigated, since solutions to these problems have many applications in complexity (cf. discussion and references in [1, 9, 2]).

This paper studies a technical tool that measures distance from low-degree polynomials. This is the Gowers norm, introduced in [3]. For a function  $f : \mathbb{F}^N \rightarrow \mathbb{F}$  and a vector  $y \in \mathbb{F}^n$ , we take  $f_y$  to be the directional derivative of  $f$  in direction  $y$  by setting

$$f_y(x) = f(x + y) - f(x)$$

For a  $k$ -tuple of vectors  $y_1 \dots y_k$  we take the iterated derivative in these directions to be

$$f_{y_1 \dots y_k} = (f_{y_1 \dots y_{k-1}})_{y_k}$$

It is easy to see that this definition does not depend on the ordering of  $y_1 \dots y_k$ .

The  $k$ -th Gowers "norm"  $\|f\|_{U^k}$  of  $f$  is

$$(\mathbb{E}_{x, y_1 \dots y_k} [e(f_{y_1 \dots y_k}(x))])^{1/2^k}$$

More accurately, as shown in [3], this is indeed a norm of the associated complex-valued function  $e(f)$  (for  $k \geq 2$ ).

It is easy to see that  $\|f\|_{U^{d+1}}$  is 1 iff  $f$  is a polynomial of degree at most  $d$ . This is just another way of saying that all order- $(d+1)$  iterative derivatives of  $f$  are zero if and only if  $f$  is a polynomial of degree at most  $d$ . It is also possible to see [4] that  $|\langle f, g \rangle| > \epsilon$  for  $g$  of degree at most  $d$ , implies  $\|f\|_{U^{d+1}} > \epsilon$ . That is to say, if  $f$  is non-trivially close to a degree- $d$  polynomial, this can be detectable via an appropriate Gowers norm.

This discussion naturally leads to the inverse conjecture [4, 7, 8], that is if  $(d+1)$ -th Gowers norm of  $f$  is non-trivial, then  $f$  is non-trivially approximable by a degree- $d$  polynomial. This conjecture is easily seen to hold for  $d = 1$  and has been proved also for  $d = 2$  [4, 7]. It is of interest to prove this conjecture for higher values of  $d$ .

In this paper we show this conjecture, which we will refer to as the 'Inverse Conjecture for the Gowers norm', or, informally, as ICGN, to be false. Let  $S_n$  be the elementary symmetric polynomial of degree  $n$  in  $N$  variables, that is

$$S_n(x) = \sum_{S \subseteq [N], |S|=n} \prod_{i \in S} x_i$$

We prove two claims about symmetric polynomials. Note that here and below a constant is *absolute* if it does not depend on  $N$ .

First, we show Gowers norms of some symmetric polynomials to be non-trivial.

**Theorem 1.2:** *There is an absolute positive constant  $\epsilon$  such that for any prime  $p$*

$$\|S_{2p}\|_{U^{p+2}} > \epsilon,$$

Here  $S_{2p}$  is viewed as a function over  $\mathbb{F} = \mathbb{F}_p$ .

Two versions of this result will be useful later.

- A special case  $p = 2$ .

$$\|S_4\|_{U^4} > \epsilon \tag{1}$$

- An easy generalization: for any  $n \geq 2p$ ,

$$\|S_n\|_{U^{n-p+2}} > \epsilon \tag{2}$$

In the second claim we show a specific symmetric polynomial to have no non-trivial approximation by polynomials of lower degree.

**Theorem 1.3:** *Let  $p = 2$ . For any polynomial  $g$  of degree 3 holds*

$$|\langle S_4, g \rangle| < \exp\{-\alpha N\} \tag{3}$$

We conjecture the second claim of the theorem to be true for any prime number  $p$ , replacing 3 with  $p + 1$  and 4 with  $2p$ .

The combination of (1) and (3) shows ICGN to be false for  $p = 2$  and  $d = 4$ .

## 1.1 Related work

Our results have a large overlap with a recent work of Green and Tao [5].

The paper of Green and Tao has two parts. In the first part ICGN is shown to be true when  $f$  is itself a polynomial of degree less than  $p$  and  $d < p$ . In the second part, the conjecture is shown to be false in general. In particular the symmetric polynomial  $S_4$  is shown to be a counterexample for  $p = 2$  and  $d = 4$ .

To proof of non-approximability of  $S_4$  by lower-degree polynomials in [5] uses a modification of a Ramsey-type argument due to Alon and Beigel [1]. Very briefly, this argument shows that if a function over  $\mathbb{F}_2$  has a non-trivial correlation with a multilinear polynomial of degree  $d$ , then its restriction to a subcube of smaller dimension has a non-trivial correlation with a symmetric polynomial of degree  $d$ . The problem of inapproximability by symmetric polynomials turns out to be easier to analyze.

This argument gives a somewhat weaker bounds for non-inapproximability of  $S_4$ , in that it shows  $\langle S_4, g \rangle < \log^{-c}(N)$  for any degree-3 polynomial  $g$  and for an absolute constant  $c > 0$ .

On the other hand, this argument is more robust than our inapproximability argument. We observe below that it can be readily extended to the case of general prime  $p$  and, combined with (2), show ICGN to be false for all  $p$ .

## 1.2 The case of a general prime field

We briefly observe here that a minor adaptation of the Alon-Beigel argument, together with (2), show the symmetric polynomial  $S_{p^2}$  to have a non-negligible ( $p^2$ )-nd Gowers norm over  $\mathbb{F}_p$  and to have no good approximation by lower-degree polynomials. In that,  $S_{p^2}$  provides a counterexample to ICGN for any prime  $p$ .

Indeed, by monotonicity of the Gowers norms ([4]), and since  $p \geq 2$ , a direct implication of (2) gives

$$\|S_{p^2}\|_{U_{p^2}} > \epsilon$$

On the other hand, let  $g$  be a polynomial of degree less than  $p^2$  in  $N$  variables such that  $\langle S_{p^2}, g \rangle > \epsilon$ . Note that the Alon-Beigel argument (as given in [1] and in [5]) does not seem to be immediately applicable in this case, since  $g$  does not have to be multilinear. A way around this obstacle, is to observe, via an averaging argument, that there is a copy of an  $N'$ -dimensional boolean cube  $\{0, 1\}^{N'}$ , such that restrictions  $S'$  and  $g'$  of  $S_{p^2}$  and of  $g$  on this subcube satisfy  $\langle S', g' \rangle > \epsilon'$ , and  $N', \epsilon'$  depend linearly on  $N, \epsilon$ . Without loss of generality assume the coordinates of the boolean cube to be  $\{1 \dots N'\}$  and consider the functions  $S', g'$  as functions in variables  $x_1, \dots, x_{N'}$  (with some fixed assignment of values to variables  $x_i, i > N'$ ). Now,  $S' = \sum_{i=0}^{p^2} a_i S_i$  is a symmetric polynomial of degree  $p^2$  over  $\mathbb{F}^{N'}$ , with  $a_i = 1$ , and  $g'$  is a polynomial of a degree smaller than  $p^2$ . Our gain is in that now  $g'$  can be replaced by a multilinear polynomial coinciding with  $g'$  on the boolean cube, and hence having a non-trivial correlation with  $S'$  on the boolean cube.

Now, the Alon-Beigel argument can be applied to show that the symmetric polynomial  $S_{p^2}$  has a non-trivial correlation with a symmetric polynomial  $h$  of a smaller degree over the boolean cube  $\{0, 1\}^{N'}$  viewed as a subset of  $\mathbb{F}^{N'}$ . This, however, couldn't be true due to a theorem of Lucas, which implies that for a boolean vector  $x$  with Hamming weight  $w = \sum_{i=1}^{N'} x_i$ , the value  $S_{p^2}(x)$  depends only on the 3-rd digit in the representation of  $w$  in base  $p$ , while the value of  $h$  depends only on the first 2 digits.

This completes the argument. We conclude with an observation that this argument directly extends to  $S_{p^k}$  for any  $k > 1$ .

Here is a brief overview of the rest of the paper. Section 2 defines relevant notions and contains proofs of several technical claims. Theorem 1.2 is proved in Section 3. Theorem 1.3 is proved in Section 4.

## 2 Some useful notions and claims

### 2.1 Some multilinear polynomials and their properties

In this sub-section we introduce and discuss certain polynomials over the finite field  $\mathbb{F}$ . These polynomials can be conveniently viewed as multi-linear functions on matrices whose entries are elements of  $\mathbb{F}$ , or formal variables with values in the field. A basic object we consider is a rectangular  $n \times N$  matrix,  $N \geq n$ . A matrix  $M$  with rows  $r_1 \dots r_n$  will be denoted by  $M[r_1 \dots r_n]$ .

Sometimes there will be repeated rows. In such a case we consider a partition  $\lambda = (\lambda_1 \dots \lambda_k)$  of  $[n]$ , that is  $\lambda_i$  are (possibly empty) subsets of  $[n]$ , whose disjoint union is  $[n]$ . We denote by  $M_\lambda[r_1 \dots r_k]$  the matrix whose rows in positions indexed by elements of  $\lambda_i$  equal  $r_i$ . Note that the partition  $\lambda$  is ordered, in that the ordering of the sets  $\lambda_i$  is relevant. We use the notation  $\{\lambda_1 \dots \lambda_k\}$  for an unordered partition.

First, we introduce the "symmetric" function  $\mathcal{S}$ . We define  $\mathcal{S}(M)$  to be the sum of all the permanent minors of  $M$ , that is

$$\mathcal{S}(M) := \sum_{C \subseteq [N], |C|=n} \text{Per}(M_C),$$

where  $M_C$  is an  $n \times n$  submatrix of  $M$  which is obtained by deleting all the columns of  $M$  except these with indices in  $C$ .

Let  $\lambda = (\lambda_1 \dots \lambda_k)$  be a partition of  $[n]$ , and set  $\ell_i = |\lambda_i|$ . Clearly  $\mathcal{S}(M_\lambda)$  depends only on the cardinalities  $\ell_i$  of  $\lambda_i$ . This leads to the notation  $M \left[ r_1^{(\ell_1)} \dots r_k^{(\ell_k)} \right]$  which denotes the matrix in which the row  $r_1$  appears  $\ell_1$  times, followed by  $\ell_2$  appearances of the row  $r_2$  and so on. In this notation, therefore

$$\mathcal{S}(M_{(\lambda_1 \dots \lambda_k)}[r_1 \dots r_k]) = \mathcal{S}\left(M \left[ r_1^{(|\lambda_1|)} \dots r_k^{(|\lambda_k|)} \right]\right)$$

The second matrix function we consider is the "forward" function  $\mathcal{F}$ , with

$$\mathcal{F}(M[r_1 \dots r_n]) = \sum_{C \subseteq [N], |C|=\{j_1 < j_2 < \dots < j_n\}} \prod_{i=1}^n r_i(j_i)$$

Here  $r_i(j)$  denote the  $j$ -th coordinate of the vector  $r$ .

To connect the two notions, observe that

$$\mathcal{S}(M[r_1 \dots r_n]) = \sum_{\sigma} \mathcal{F}(M[r_{\sigma_1} \dots r_{\sigma_n}])$$

where  $\sigma$  runs over all permutations on  $n$  items.

The last function we consider is a "hybrid" function  $\mathcal{H}$  which has some 'symmetric' and some 'forward' properties. Let  $\lambda = (\lambda_1 \dots \lambda_k)$  be an ordered partition of  $[n]$  with  $k$  terms. For another such partition  $\theta = (\theta_1 \dots \theta_k)$  of  $[n]$  write  $\theta \sim \lambda$  if  $|\theta_1| = |\lambda_1|, \dots, |\theta_k| = |\lambda_k|$ . We define

$$\mathcal{H}(M_\lambda[r_1 \dots r_k]) = \sum_{C \subseteq [N], |C|=\{j_1 < j_2 < \dots < j_n\}} \sum_{\theta \sim \lambda} \prod_{t=1}^k \prod_{i \in \theta_t} r_t(j_i)$$

An alternative view of the functions  $\mathcal{S}$ ,  $\mathcal{F}$  and  $\mathcal{H}$  might be helpful at this point. Consider the set of *paths* which are one-to-one functions from  $[n]$  to  $[N]$ . Let us call a path  $\rho$  monotone on a subset  $\{i_1 < i_2 < \dots < i_\ell\}$  of  $[n]$  if  $\rho(i_1) < \rho(i_2) < \dots < \rho(i_\ell)$ . A path is (fully) monotone if it is monotone on  $[n]$ . Then, for a partition  $\lambda = (\lambda_1 \dots \lambda_k)$  of  $[n]$  and an  $n \times N$  matrix  $M = M_\lambda$ ,

$$\mathcal{S}(M) = \sum_{\text{all } \rho} \prod_{i=1}^n M_{i, \rho(i)}$$

$$\mathcal{F}(M) = \sum_{\text{monotone } \rho} \prod_{i=1}^n M_{i,\rho(i)}$$

$$\mathcal{H}(M) = \sum_{\rho \text{ monotone on } \lambda_1 \dots \lambda_k} \prod_{i=1}^n M_{i,\rho(i)}$$

Note that for the function  $\mathcal{H}$ , similarly to the symmetric function  $\mathcal{S}$ , holds

$$\mathcal{H}(M_{(\lambda_1 \dots \lambda_k)}[r_1 \dots r_k]) = \mathcal{H}\left(M \begin{bmatrix} r_1^{(|\lambda_1|)} & \dots & r_k^{(|\lambda_k|)} \end{bmatrix}\right)$$

Observe also that if  $\lambda = (\{1\} \dots \{n\})$  then  $\mathcal{S}(M) = \mathcal{H}(M)$ . If  $\lambda = (\{[n]\})$  then  $\mathcal{F}(M) = \mathcal{H}(M)$  and  $\mathcal{S}(M) = n! \cdot \mathcal{F}(M) = n! \cdot \mathcal{H}(M)$ . For a general  $\lambda = (\lambda_0 \dots \lambda_k)$

$$\mathcal{S}(M) = \left( \prod_{t=1}^k |\lambda_t|! \right) \cdot \mathcal{H}(M) \tag{4}$$

Note that this is an identity in  $\mathbb{F}$ . In particular, if one of the terms  $\lambda_i$  has cardinality at least  $p$  then  $\mathcal{S}(M) = 0$  and (4) provides no information.

To simplify the notation we will usually write  $\mathcal{S}(r_1 \dots r_n)$  for  $\mathcal{S}(M[r_1 \dots r_n])$ ,  $\mathcal{F}_\lambda(r_1 \dots r_k)$  for  $\mathcal{F}(M_\lambda[r_1 \dots r_k])$  and so on.

## 2.2 Directional derivatives of symmetric polynomials

The functions we have defined are relevant to the discussion here for two reasons. First, the elementary symmetric polynomial  $S_n(x)$  in  $N$  variables can be viewed as the forward function  $\mathcal{F}$  applied to the matrix  $M[x \dots x]$ , where  $M$  has  $n$  identical rows equal to  $x$ . In our notation,

$$S_n(x) = \mathcal{F}_{\{[n]\}}(x)$$

Second, it is possible to write a directional derivative  $(S_n)_{y_1 \dots y_k}$  of  $S_n$  of any order as a combination of values of  $\mathcal{F}$  on explicitly defined matrices  $M$  whose rows are either the indeterminate  $x$  or the directions  $y_i$ .

The basic observation here is the following lemma which is straightforward from the definition of directional derivative.

**Lemma 2.1:** *Let a polynomial  $P(x)$  in  $N$  variables be given by*

$$P(x) = \mathcal{F}_{(\lambda_0 \dots \lambda_k)}(x, y_1 \dots y_k)$$

Then

$$P_z(x) = \sum_{A \subset \lambda_0} \mathcal{F}_{(A, \lambda_0 \setminus A, \lambda_1 \dots \lambda_k)}(x, z, y_1 \dots y_k)$$

In words, when we take the derivative of such a polynomial in direction  $z$ , we replace some of the rows which contained  $x$  with  $z$ .

As a corollary we have a following expression for higher order derivatives of a symmetric polynomial.

**Proposition 2.2:** *Let  $k \leq n$ , then*

$$(S_n)_{y_1 \dots y_k}(x) = \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}\left(x^{(m)}, y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

**Proof:** Iterating Lemma 2.1,

$$(S_n)_{y_1 \dots y_k}(x) = \sum_{\lambda=(\lambda_0, \lambda_1 \dots \lambda_k)} \mathcal{F}_\lambda(x, y_1 \dots y_k)$$

where the summation is over partitions  $\lambda$  such that  $\lambda_i$  are not empty for  $i = 1 \dots k$ . Rearranging, this is

$$\begin{aligned} \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \sum_{\lambda: |\lambda_0|=m, |\lambda_1|=\ell_1 \dots |\lambda_k|=\ell_k} \mathcal{F}_\lambda(x, y_1 \dots y_k) = \\ \sum_{m=0}^{n-k} \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}\left(x^{(m)}, y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right) \end{aligned}$$

■

We can give explicit expressions for the coefficients of  $(S_n)_{y_1 \dots y_k}(x)$ . Fix  $m$  indices  $j_1 < j_2 < \dots < j_m$  for  $0 \leq m \leq n - k$ , and let  $a$  be the coefficient of  $x_{j_1} \dots x_{j_m}$  in  $(S_n)_{y_1 \dots y_k}$ .

**Corollary 2.3:**

•

$$a = \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \mathcal{H}^{\{j_1 \dots j_m\}}\left(y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

• If  $k + m + p > n + 1$  then

$$a = \sum_{\ell_1 \dots \ell_k \geq 1, \sum_i \ell_i = n-m} \left( \prod_{i=1}^k \ell_i! \right)^{-1} \cdot \mathcal{S}^{\{j_1 \dots j_m\}}\left(y_1^{(\ell_1)} \dots y_k^{(\ell_k)}\right)$$

Here, for a subset of indices  $T \subseteq [N]$ ,  $\mathcal{H}^T(M)$  returns the value of the matrix function  $\mathcal{H}$  applied to the  $n \times (N - |T|)$  matrix obtained from  $M$  by deleting columns in  $T$ . The function  $\mathcal{S}^T(M)$  is defined similarly.

**Proof:** The first claim is immediate from Proposition 2.2. The second claim follows from the first claim, from (4), and from the simple observation that if  $k + m + p > n + 1$  then  $\ell_i < p$  for  $i = 1 \dots k$  in the above summation, which means  $\ell_i!$  is invertible in  $\mathbb{F}_p$ . ■

**Example 2.4:** The following "toy" example will be relevant for the case of the binary field. It is sufficiently simple to illustrate what's going on behind the cumbersome formulas. Consider  $P = (S_4)_{y,z}$ . Then  $P$  is a quadratic polynomial and for  $1 \leq i < j \leq N$

$$\text{coef}_{x^{(i)}x^{(j)}}(P) = \sum_{k \neq l, k, l \notin \{i, j\}} y(k)z(l) = \mathcal{S}^{\{i, j\}}(y, z)$$

■

Continuing with the same example, note that it is convenient to express the symmetric function  $\mathcal{S}(y, z)$  via inner products of vectors  $y, z, \mathbf{1}$ , where  $\mathbf{1}$  is the all-1 vector of length  $N$ .

$$\mathcal{S}(y, z) = \sum_{k \neq l} y(k)z(l) = \langle y, \mathbf{1} \rangle \cdot \langle z, \mathbf{1} \rangle - \langle yz, \mathbf{1} \rangle$$

Here we take  $yz$  to be the vector whose coordinates are point-wise inner products of the coordinates of  $y$  and  $z$ , that is  $(yz)(i) = y(i)z(i)$ . Of course,  $\langle yz, \mathbf{1} \rangle$  is the same as  $\langle y, z \rangle$ .

Similarly, we can express the 'incomplete' symmetric function  $\mathcal{S}^{\{i, j\}}(y, z)$  via the complete symmetric function  $\mathcal{S}(y, z)$  minus forbidden terms, as follows

$$\mathcal{S}^{\{i, j\}}(y, z) = \mathcal{S}(y, z) - (z(i) + z(j)) \langle y, \mathbf{1} \rangle - (y(i) + y(j)) \langle z, \mathbf{1} \rangle + (y(i)z(j) + y(j)z(i))$$

Note the "inclusion-exclusion" structure in the two expressions above. (To make it even clearer we use "+" and "-" notation, though in the binary field both are, of course, the same.) This structure becomes more evident as we pass to our next order of business, which is expressing, for general  $n$  and  $k$ , the coefficients of  $(S_n)_{y_1 \dots y_k}$  via inner products of vectors  $y_1 \dots y_k, \mathbf{1}$ .

### 2.3 Inclusion-Exclusion formulas for symmetric functions

Some notation: Given  $m$  vectors  $y_1 \dots y_m$  and a subset  $\tau \subseteq [m]$ , let  $y_\tau$  to be vector whose coordinates are point-wise products of the corresponding coordinates of  $y_i, i \in \tau$ . Let  $\mathcal{S}(y[\tau])$  for the value of the function  $\mathcal{S}$  on a matrix with  $|\tau|$  rows  $y_i, i \in \tau$ . Let  $\langle y_\tau, \mathbf{1} \rangle = \sum_{j=1}^N \prod_{i \in \tau} y_i(j)$ .

We start with an auxiliary lemma expressing the incomplete symmetric function  $\mathcal{S}^{\{k\}}(r_1 \dots r_n)$  as a polynomial in the  $k$ -th coordinate of the vectors  $r_i$  and in complete symmetric functions applied to sub-matrices of  $M[r_1 \dots r_n]$ .

**Lemma 2.5:**

$$\mathcal{S}^{\{k\}}(r_1 \dots r_n) = \sum_{\tau \subseteq [n]} (-1)^{|\tau|} (|\tau|)! \cdot r_\tau(k) \cdot \mathcal{S}\left(r\left[[n] \setminus \tau\right]\right)$$

From now on we assume  $r_\emptyset$  to be the all-1 vector, and  $\mathcal{S}(r[\emptyset])$  to equal 1.



**Proof:** The proof is by induction on  $n$ . For  $n = 1$  both sides equal  $\sum_{j=1}^N r_1(j) - r_1(k)$ .

For  $n > 1$ , observe that

$$\mathcal{S}^{\{k\}}(r_1 \dots r_n) = \mathcal{S}(r_1 \dots r_n) - \sum_{i=1}^n r_i(k) \cdot \mathcal{S}^{\{k\}}\left(r\left[[n] \setminus \{i\}\right]\right)$$

and the claim is easily verified using the induction hypothesis. ■

Now we can state two main claims of this section. The first expresses the complete symmetric function  $\mathcal{S}(r_1 \dots r_n)$  via inner products  $\langle r_T \rangle$ .

**Proposition 2.6:**

$$\mathcal{S}(r_1 \dots r_n) = \sum_{\lambda = \{\lambda_1 \dots \lambda_m\}} \prod_{t=1}^m \left( (-1)^{|\lambda_t|-1} (|\lambda_t| - 1)! \cdot \langle r_{\lambda_t} \rangle \right)$$

In this summation  $\lambda = \{\lambda_1 \dots \lambda_m\}$  runs over all unordered partitions of  $[n]$  with non-empty  $\lambda_i$ .

**Proof:** Again, the proof is by induction on  $n$ . For  $n = 1$  both sides equal  $\sum_{j=1}^N r_1(j)$ . For  $n > 1$  we have

$$\mathcal{S}(r_1 \dots r_n) = \sum_{k=1}^N r_n(k) \cdot \mathcal{S}^{\{k\}}(r_1 \dots r_{n-1})$$

Using Lemma 2.5 and the induction hypothesis,

$$\begin{aligned} \mathcal{S}(r_1 \dots r_n) &= \sum_{k=1}^N r_n(k) \cdot \sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot r_\tau(k) \cdot \mathcal{S}\left(r\left[[n-1] \setminus \tau\right]\right) = \\ &\sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot \langle r_{\tau \cup [n]} \rangle \cdot \mathcal{S}\left(r\left[[n-1] \setminus \tau\right]\right) \end{aligned}$$

Consider the summand corresponding to  $\tau = [n-1]$ . Recall the boundary assumption  $\mathcal{S}(r[\emptyset]) = 1$ . Hence this summand is  $(-1)^{n-1} (n-1)! \cdot \langle r_{[n]} \rangle$ . This summand therefore corresponds to the partition  $\lambda = \{[n]\}$  in the claim of the proposition.

For  $\tau$  a proper subset of  $[n-1]$ , we use the induction hypothesis to obtain

$$\begin{aligned} \mathcal{S}(r_1 \dots r_n) &= \sum_{\tau \subseteq [n-1]} (-1)^{|\tau|} (|\tau|)! \cdot \langle r_{\tau \cup [n]} \rangle \cdot \sum_{\theta = \{\theta_1 \dots \theta_l\}} \prod_{t=1}^l \left( (-1)^{|\theta_t|-1} (|\theta_t| - 1)! \cdot \langle r_{\theta_t} \rangle \right) + \\ &(-1)^{n-1} (n-1)! \cdot \langle r_{[n]} \rangle \end{aligned}$$

Here  $\theta$  runs over all the unordered partitions of  $[n-1] \setminus \tau$  with non-empty  $\theta_i$ . Observe that each pair  $(\tau, \theta)$  leads to a unique partition  $\lambda = \{\lambda_1 \dots \lambda_{l+1}\} = \{\theta_1 \dots \theta_l, \tau \cup [n]\}$  of  $[n]$ . Rearranging the terms, the last summation can be written as

$$\sum_{\lambda = (\lambda_1 \dots \lambda_m)} \prod_{t=1}^m \left( (-1)^{|\lambda_t|-1} (|\lambda_t| - 1)! \cdot \langle r_{\lambda_t} \rangle \right)$$

completing the proof of the proposition. ■

The second claim expresses the incomplete symmetric function  $\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n)$  as a polynomial in the missing coordinates  $j_1 \dots j_k$  of the vectors  $r_i$  and in complete symmetric functions applied to sub-matrices of  $M[r_1 \dots r_n]$ . Note that Lemma 2.5 is a special case  $k = 1$  of this claim.

**Proposition 2.7:**

$$\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n) = \sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left( (-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left( r \left[ [n] \setminus \cup_t \tau_t \right] \right)$$

Here the summation is on all ordered set systems  $\tau$  such that the terms  $\tau_t$  are disjoint subsets of  $[n]$ . The terms may also be empty.

**Proof:** The proof is by induction on  $k$  and  $n$ . The case  $k = 1$  is treated in Lemma 2.5.

Consider the case  $n = 1$ . On one hand  $\mathcal{S}^{\{j_1 \dots j_k\}}(r_1) = \sum_{j=1}^N r_1(j) - \sum_{t=1}^k r_1(j_t)$ . We claim that this value can be also represented as

$$\sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left( (-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left( r \left[ [1] \setminus \cup_t \tau_t \right] \right)$$

Here  $\tau_i$  are disjoint subsets of  $[1]$ . Observe that there are  $k + 1$  summands in this expression, corresponding to different set systems  $\tau$ . Let  $\tau^{(0)}$  denote the set system with  $k$  empty terms, and let  $\tau^{(t)}$ , for  $t = 1 \dots k$  denote the set system with  $\tau_t = \{1\}$  and all the remaining terms are empty. The summand corresponding to  $\tau^{(0)}$  is  $\mathcal{S}(r_1) = \sum_{j=1}^N r_1(j)$ . The summand corresponding to  $\tau^{(t)}$  is  $(-r_1(j_t)) \cdot \mathcal{S}(r_\emptyset) = -r_1(j_t)$ , and we are done in this case.

For  $k, n > 1$ , we have

$$\mathcal{S}^{\{j_1 \dots j_k\}}(r_1 \dots r_n) = \mathcal{S}^{\{j_1 \dots j_{k-1}\}}(r_1 \dots r_n) - \sum_{i=1}^n r_i(j_k) \cdot \mathcal{S}^{\{j_1 \dots j_k\}} \left( r \left[ [n] \setminus \{i\} \right] \right)$$

By the induction hypothesis, this is

$$\begin{aligned} & \sum_{\theta=(\theta_1 \dots \theta_{k-1})} \prod_{t=1}^{k-1} \left( (-1)^{|\theta_t|} (|\theta_t|)! \cdot r_{\theta_t}(j_t) \right) \cdot \mathcal{S} \left( r \left[ [n] \setminus \cup_t \theta_t \right] \right) - \\ & \sum_{i=1}^n r_i(j_k) \cdot \sum_{\mu^{(i)}=(\mu_1^{(i)} \dots \mu_k^{(i)})} \prod_{u=1}^k \left( (-1)^{|\mu_u^{(i)}|} (|\mu_u^{(i)}|)! \cdot r_{\mu_u^{(i)}}(j_u) \right) \cdot \mathcal{S} \left( r \left[ [n] \setminus \cup_t \mu_t^{(i)} \setminus \{i\} \right] \right) \end{aligned}$$

Here the summation is on all ordered set systems  $\theta$  such that the terms  $\theta_t$  are disjoint subsets of  $[n]$  and on ordered set systems  $\mu^{(i)}$ ,  $i = 1 \dots n$  such that the terms  $\mu_u^{(i)}$  are disjoint subsets of  $[n] \setminus \{i\}$ .

Given a set system  $\theta = (\theta_1 \dots \theta_{k-1})$  we define a set system  $\tau = (\tau_1 \dots \tau_k)$  by setting  $\tau_t = \theta_t$ ,  $t = 1 \dots k-1$  and  $\tau_k = \emptyset$ . Given a set system  $\mu^{(i)} = (\mu_1^{(i)} \dots \mu_k^{(i)})$  we define a set system  $\tau = (T_1 \dots T_k)$  by setting  $\tau_u = \mu_u^{(i)}$ ,  $u = 1 \dots k-1$  and  $\tau_k = \mu_k^{(i)} \cup \{i\}$ . In both cases we have obtained a set system of the type we want, that is an ordered family of  $k$  disjoint subsets of  $[n]$ . Moreover, each such system with empty  $k$ -th term is obtained exactly once, from the corresponding  $\theta$ -system, and each system with non-empty  $k$ -th term  $\tau_k$  is obtained exactly  $|\tau_k|$  times, from systems  $\mu^{(i)}$  with  $i \in \tau_k$ . Rearranging the terms and the signs, the last expression is precisely

$$\sum_{\tau=(\tau_1 \dots \tau_k)} \prod_{t=1}^k \left( (-1)^{|\tau_t|} (|\tau_t|)! \cdot r_{\tau_t}(j_t) \right) \cdot \mathcal{S} \left( r \left[ [n] \setminus \cup_t \tau_t \right] \right),$$

completing the proof. ■

## 2.4 Some properties of Gowers' norms

The main result in this subsection shows that if a function from  $\mathbb{F}^N$  to  $\mathbb{F}$  is fixed on a subset of  $\mathbb{F}^N$  defined by low-degree polynomial constraints, then it has a non-trivial Gowers norm of an appropriate order.

Recall that for a vector  $x \in \mathbb{F}^N$ ,  $x^i$  stands for a vector in  $\mathbb{F}^N$  whose coordinates are  $i$ -th powers of the coordinates of  $x$ .

**Proposition 2.8:** *Let  $K$  be an absolute constant. Let  $y_{i,j}$ ,  $i = 1 \dots p-1$ ,  $j = 1 \dots K$ , be  $K(p-1)$  vectors in  $\mathbb{F}^N$ . Let  $M$  be a subset of  $\mathbb{F}^N$  defined by the constraints  $\langle x^i, y_{i,j} \rangle = 0$  for all  $i, j$ .*

*Let  $f$  be a function from  $\mathbb{F}^N$  to  $\mathbb{F}$ . Assume that  $f$  is fixed on  $M$ . Then*

$$\|f\|_{U^p} > \left( \frac{|M|}{2^N} \right)^2 =: Pr^2\{M\}$$

**Proof:** Let  $f|_M \equiv c_0$ .

Consider a subspace  $V$  of polynomials of degree at most  $p-1$  in  $\mathbb{F}[x_1 \dots x_N]$  spanned by the polynomials  $\langle x^i, y_{i,j} \rangle$ , for all  $i, j$ . We will first find a polynomial  $g \in V$  such that  $|\langle f, g \rangle| \geq Pr\{M\}$ . This, combined with a lemma from [4], will imply the claim of the proposition.

Let  $\mathbf{b} = (b_{i,j})$ ,  $i = 1 \dots p-1$ ,  $j = 1 \dots K$ , be a matrix with entries in  $\mathbb{F}$ . Let  $c \in \mathbb{F}$ . Set

$$\mu(\mathbf{b}, c) = Pr \left\{ x : f(x) = c \wedge \langle x^i, y_{i,j} \rangle = b_{i,j} \text{ for all } i, j \right\}$$

Note that, by assumption, for a zero matrix  $\mathbf{b}$  holds  $\mu(\mathbf{b}, c_0) = Pr\{M\}$ . In other words,  $\mu(\mathbf{b}, c) = 0$  and for  $\mathbf{b} = 0$  any  $c \neq c_0$ .

Now, for any  $g(x) = \sum_{i,j} a_{i,j} \langle x^i, y_{i,j} \rangle$  in  $V$  holds

$$\langle f, g \rangle = \mathbb{E} e(f - g) = \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c - \langle \mathbf{a}, \mathbf{b} \rangle)$$

where  $\mathbf{a} = (a_{i,j})_{i,j}$  and  $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i,j} a_{i,j} b_{i,j}$ . Averaging over  $V$ , we have

$$\begin{aligned} \mathbb{E}_{g \in V} \langle f, g \rangle &= \frac{1}{|V|} \sum_{\mathbf{a}} \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c - \langle \mathbf{a}, \mathbf{b} \rangle) = \frac{1}{|V|} \sum_{\mathbf{b}, c} \mu(\mathbf{b}, c) \cdot e(c) \sum_{\mathbf{a}} e(-\langle \mathbf{a}, \mathbf{b} \rangle) = \\ &= \sum_c \mu(0, c) \cdot e(c) = \mu(0, c_0) \cdot e(c_0) = \text{Pr}\{M\} \cdot e(c_0) \end{aligned}$$

This means, there is  $g \in V$  with  $|\langle f, g \rangle| \geq \text{Pr}\{M\}$ . We conclude the proof of the proposition by quoting a lemma from [4], which states that  $|\langle f, g \rangle| \geq \epsilon$  implies  $\|f\|_{U^p} \geq \epsilon$ . ■

## 2.5 Asymptotic uniformity and independence of some random variables

In this subsection we deal with another property of multivariate polynomials. Let  $n$  be fixed integer and let  $N$  be an integer parameter growing to infinity. Let  $r_1 \dots r_n$  be  $n$  vectors in  $\mathbb{F}^N$ . Let  $\kappa = (k_1 \dots k_n)$  be a non-zero sequence of integers  $0 \leq k_i < p$ . For each such sequence define a polynomial  $X_\kappa(r_1, \dots, r_n) = \sum_{j=1}^N \prod_{i=1}^n r_i^{k_i}(j)$ .

Now, let  $r_1 \dots r_n$  be chosen uniformly and independently from  $\mathbb{F}^N$ . We claim that for a large  $N$  the random variables  $X_\kappa(r_1, \dots, r_n)$  are nearly independent and uniformly distributed over  $\mathbb{F}$ . Let  $X = (X_\kappa)_\kappa$ , and let  $K = p^n$ .

**Proposition 2.9:** *Let  $U$  be the uniform distribution on  $\mathbb{F}^K$ . Let  $P$  be distribution of  $X$  on  $\mathbb{F}^K$ . Let  $\|\cdot\|$  denote the statistical ( $l_1$ ) distance between distributions.*

*Then there is a constant  $c > 0$  depending on  $n, p$  but not on  $N$  such that*

$$\|P - U\| \leq \exp\{-cN\}$$

**Proof:** We start from a simple observation that Fourier transform of a uniform distribution is the delta function at 0. In addition, the two following statements are equivalent up to constants: 'a distribution is exponentially close to uniform' and 'all non-zero Fourier coefficients of the distribution are exponentially close to zero'. Accordingly, we will show that all the non-zero Fourier coefficients of  $P$  tend exponentially fast in  $N$  to zero.

Consider a character  $\chi(y) = \xi^{\langle y, a \rangle}$ , corresponding to a non-zero vector  $a = (a_\kappa)_\kappa \in \mathbb{F}^K$ . (Recall that  $\xi = e^{2\pi i/p}$  is the  $p$ -th primitive root of unity.) Then, normalizing appropriately,

$$\widehat{P}(\chi) = \sum_y P(y) \bar{\chi}(y) = \sum_y \text{Pr}\{X = y\} \cdot \xi^{-\sum_\kappa a_\kappa y_\kappa} = \mathbb{E} \xi^{-\sum_\kappa a_\kappa X_\kappa}$$

Let  $P_a$  denote the distribution of the random variable  $X_a = \sum_\kappa a_\kappa X_\kappa$ . Then we have shown  $\widehat{P}(\chi) = \widehat{P}_a(1)$ . We will show the non-zero Fourier coefficients of  $P_a$  to be exponentially small, completing the proof of the proposition.

We have

$$X_a(r_1, \dots, r_n) = \sum_\kappa a_\kappa P_\kappa(r_1, \dots, r_n) = \sum_{j=1}^N \sum_{\kappa=(k_1 \dots k_n)} a_\kappa \prod_{i=1}^n r_i^{k_i}(j)$$

Let  $x_i$  be elements of the field  $\mathbb{F}$ . Consider an  $n$ -variate polynomial

$$Q(x_1 \dots x_n) = \sum_{\kappa=(k_1 \dots k_n)} a_{\kappa} \prod_{i=1}^n x_i^{k_i}$$

Since not all of the coefficients  $a_{\kappa}$  are zero, and since all  $\kappa$  are non-zero sequences,  $Q$  is a multi-variate polynomial of degree at least 1 in  $\mathbb{F}[x_1 \dots x_n]$ , and therefore attains at least two values with probability bounded away from zero. Now,  $X_a = \sum_{j=1}^N Q(r_1(j) \dots r_n(j))$  is a sum of  $N$  independent copies of  $Q$ . Let  $\mu$  denote the distribution of  $Q$  on  $\mathbb{F}$ . Then the distribution  $P_a$  of  $X_a$  is  $\mu^{*N}$ , the  $N$ -wise convolution of  $\mu$  with itself. Since  $p$  is prime,  $\widehat{\mu}(0) = 1$ , and  $|\widehat{\mu}| < 1$  everywhere else. Therefore,  $\widehat{P}_a = (\widehat{\mu})^N$  tends to the delta function at 0 exponentially fast in  $N$ , completing the proof. ■

## 2.6 Estimates on the number of common zeroes of some families of polynomials

The main claim of this subsection is the following proposition.

**Proposition 2.10:** *Let  $M$  be the ring of  $\mathbb{F}$ -valued functions on  $\mathbb{F}^N$ , that is  $M = \mathbb{F}[x_1 \dots x_N]/I$ , where  $I$  is the ideal  $(x_1^p - x, \dots, x_N^p - x)$ . Let  $f_1 \dots f_K$  be polynomials in  $M$ . Let  $S$  be the set of common zeroes of  $f_1 \dots f_K$ , that is*

$$S = \left\{ u \in \mathbb{F}^N : f_1(u) = \dots = f_K(u) = 0 \right\}$$

Then

$$|S| \leq \dim(M/J)$$

where  $J$  is the ideal generated by  $\{f_i\}$ , and  $\dim(M/J)$  denotes the dimension of  $M/J$ , viewed as a vector space over  $\mathbb{F}$ .

**Proof:** For each  $u \in S$ , let  $q_u \in M$  be defined by  $q_u(u) = 1$  and  $q_u(v) = 0$  for all  $v \neq u$ . We will show that the family  $\{q_u + J\}_{u \in S}$  is linearly independent in  $M/J$ . This will immediately imply the claim of the proposition.

Consider a linear combination  $q = \sum_{u \in S} \lambda_u q_u$  such that  $q \in J$ . Let  $v \in S$ . We compute  $q(u)$  in two ways. First, since  $q \in J$ , we have  $q(v) = 0$ . On the other hand,  $q(v) = \sum_{u \in S} \lambda_u q_u(v) = \lambda_v$ . This shows  $\lambda_v = 0$  for all  $v \in S$ , completing the proof. ■

In some cases, the dimension of  $M/J$  is easy to estimate.

**Lemma 2.11:** *Let  $p = 2$ , let  $K = \binom{N}{k}$ , and let  $\{f_I\}$  be indexed by  $k$ -subsets  $I$  of  $[N]$ . Assume that for any such subset  $I$  holds*

$$\deg \left( f_I(x) - \prod_{i \in I} x_i \right) \leq k - 1 \tag{5}$$

Then,

$$\dim(M/J) \leq \sum_{j=0}^{k-1} \binom{N}{j}$$

**Proof:** We will construct a generating subset of the vector space  $M/J$  of cardinality at most  $\sum_{j=0}^{k-1} \binom{N}{j}$ . We start from a trivial generating set  $\{m + J\}$ , where  $m$  runs through all the  $2^N$  multi-linear monomials in  $N$  variables. Now, in the factor space  $M/J$ , we can replace any product of  $k$  variables,  $\prod_{i \in I} x_i$ , by a polynomial of degree smaller than  $k$ . Iterating this procedure, we arrive to a generating set spanned by  $\{s + J\}$ , where  $s$  now runs through  $\sum_{j=0}^{k-1} \binom{N}{j}$  monomials of degree at most  $k - 1$ . ■

### 3 Proof of Theorem 1.2

We need to show that

$$\|S_{2p}\|_{U^{p+2}} > \epsilon$$

for an absolute constant  $\epsilon$ .

We remark that (2) can be shown exactly in the same way, replacing  $2p$  with  $n$  and  $p + 2$  with  $n - p + 2$  throughout.

Recall ([4]) that  $\|f\|_{U^{p+2}} = \mathbb{E}_{y,z}^{1/2^{p+2}} \|f_{y,z}\|_{U^p}^{2p}$ . Since the Gowers' norms are nonnegative, it will suffice to show that  $\|f_{y,z}\|_{U^p}$  is non-negligible for a non-negligible fraction of directions  $y, z$ .

Let

$$A = \left\{ (y, z) : \langle y^a, z^b \rangle = 0 \text{ for all } 0 \leq a, b < p \right\}$$

By Proposition 2.9, for uniformly and independently chosen directions  $y, z$ , and for a sufficiently large  $N$ , the probability of  $A$  is very close to  $p^{-p^2}$ . Therefore,  $A$  is a non-negligible event. We will now show that for any  $(y, z) \in A$  holds  $\|f_{y,z}\|_{U^p} > \epsilon'(y, z)$ , for an appropriate function  $\epsilon'$ .

Fix  $(y, z)$  in  $A$ . Let  $f = (S_{2p})_{y,z}$ . Let

$$M = M(y, z) = \left\{ x : \langle x^i, y^a z^b \rangle = 0 \text{ for all } 1 \leq i \leq p-1, 0 \leq a, b < p \right\}$$

We will show that  $f$  is fixed on  $M$ . Assuming this, by Proposition 2.8, we have  $\|f_{y,z}\|_{U^p} > Pr^2\{M\}$ , and therefore

$$\begin{aligned} \|f\|_{U^{p+2}}^{2^{p+2}} &= \mathbb{E}_{y,z} \|f_{y,z}\|_{U^p}^{2p} \geq Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr^{2^{p+1}}\{M(y, z)\} \geq \\ Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr\{M(y, z)\} &\geq (Pr\{A\} \cdot \mathbb{E}_{(y,z) \in A} Pr\{M(y, z)\})^{2^{p+1}} = \\ Pr^{2^{p+1}} \left\{ x : \langle x^i y^a z^b \rangle = 0 \text{ for all } 0 \leq a, b, i \leq p-1 \right\} &\geq \Omega\left(p^{-p^3 \cdot 2^{p+1}}\right) \end{aligned}$$

The last inequality follows from Proposition 2.9, since random variables  $\langle x^i y^a z^b \rangle$  are asymptotically uniform and independent.

It remains to prove the following fact.

**Lemma 3.1:** Let  $x, y, z$  be three vectors in  $\mathbb{F}^N$  satisfying  $\langle x^i y^a z^b \rangle = 0$  for all  $0 \leq a, b, i \leq p-1$ . Then

$$(S_{2p})_{y,z}(x) = \mathcal{H}(y^{(p)}, z^{(p)})$$

**Proof:** By Proposition 2.2,

$$(S_{2p})_{y,z}(x) = \sum_{m=0}^{2p-2} \sum_{a,b \geq 1, a+b=2p-m} \mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$$

We claim that all of the summands on the right, except (possibly)  $\mathcal{H}(y^{(p)}, z^{(p)})$  are 0.

There are two possible cases to consider. The easier case is when  $a, b, m < p$ . In such a case, by (4),  $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$  is proportional to  $\mathcal{S}(x^{(m)}, y^{(a)}, z^{(b)})$ . By Proposition 2.6, the symmetric function  $\mathcal{S}(x^{(m)}, y^{(a)}, z^{(b)})$  is a polynomial in  $\langle x^i y^a z^b \rangle$ , which vanishes when all of these inner products are 0.

In the second case, one of the indices  $a, b, m$  is at least  $p$ . Note, that there could be at most one such index (barring the case  $a = b = p$ ). We may assume this index is  $m$ . We claim that in this case  $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$  can be written as a linear combination of hybrid functions  $\mathcal{H}(x^{(\ell)}, r_1, \dots, r_{m-\ell})$ , where  $\ell < m$  and the vectors  $r_i$  are of the form  $x^\alpha y^\beta z^\gamma$ . Note that this will suffice to prove the lemma, since iterating this step will express  $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$  as a linear combination of symmetric functions in  $r_i$ , and these functions vanish.

Consider  $\mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)})$ . For notational convenience, let  $w_1 \dots w_{a+b}$  stand for the vectors  $y \dots y, z \dots z$  ( $y$  taken  $a$  times and  $z$  taken  $b$  times). Note that both  $a$  and  $b$  are smaller than  $p$ . Using Corollary 2.3 and Proposition 2.7,

$$\begin{aligned} \mathcal{H}(x^{(m)}, y^{(a)}, z^{(b)}) &= (a! \cdot b!)^{-1} \cdot \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \mathcal{S}^{\{i_1 \dots i_m\}}(y^{(a)}, z^{(b)}) = \\ &(a! \cdot b!)^{-1} \cdot \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \cdot \sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m \left( (-1)^{|\tau_t|} (|\tau_t|)! \cdot w_{\tau_t}(i_t) \right) \cdot \mathcal{S}(w[[a+b] \setminus \cup_t \tau_t]) \end{aligned}$$

Here the inner summation is on all ordered set systems  $\tau$  such that the terms  $\tau_t$  are disjoint subsets of  $[a+b]$ . The terms may also be empty.

Let us attempt to simplify the double summation we obtained. First, we may disregard the constant term  $(a! \cdot b!)^{-1}$ . Next, observe that, as before, all symmetric functions of the form  $\mathcal{S}(w[T])$  vanish, unless  $T$  is empty, in which case they equal 1. Therefore, we may consider the double summation

$$\sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \cdot \sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m \left( (-1)^{|\tau_t|} (|\tau_t|)! \cdot w_{\tau_t}(i_t) \right)$$

Here the inner summation is on all ordered partitions  $\tau$  of  $[a+b]$ . The terms  $\tau_t$  may also be empty. Changing the order of summation, and ignoring the constant term  $(-1)^{a+b}$ , we get

$$\sum_{\tau=(\tau_1 \dots \tau_m)} \prod_{t=1}^m (|\tau_t|)! \cdot \sum_{i_1 < i_2 < \dots < i_m} \prod_{t=1}^m (x \cdot w_{\tau_t})(i_t) = \sum_{\tau=(\tau_1 \dots \tau_m)} \left( \prod_{t=1}^m (|\tau_t|)! \right) \cdot \mathcal{F}(xw_{\tau_1}, xw_{\tau_2}, \dots, xw_{\tau_m})$$

Consider the last expression. Let us use some more notation. For an ordered partition  $\tau = (\tau_1 \dots \tau_m)$ , let  $n = n(\tau)$  be the number of empty terms. Let  $\{\tau_1 \dots \tau_m\}$  denote the unordered version of this partition, where the first  $n(\tau)$  terms are taken, by agreement, to be the empty ones. Then we can rewrite this expression as

$$\sum_{\tau=\{\tau_1 \dots \tau_m\}} \left( \prod_{t=1}^m (|\tau_t|)! \right) \cdot \mathcal{H} \left( x^{(n)}, xw_{\tau_{n+1}}, \dots, xw_{\tau_m} \right)$$

Now, clearly not all the terms in the partition are empty and, therefore,  $n(\tau) < m$  for all  $\tau$ , completing the proof of our last claim, of the lemma, and of the theorem. ■

## 4 Proof of Theorem 1.3

Let  $p = 2$ . We will show there is an absolute constant  $\alpha > 0$  such that for any polynomial  $g$  of degree at most 3 in  $N$  variables holds

$$\langle S_4, g \rangle < \exp\{-\alpha N\}$$

A first step is to observe that there is a relation between the inner product of two functions and the average inner product of their derivatives.

**Lemma 4.1:** *For any two functions  $f$  and  $g$  holds*

$$\langle f, g \rangle^4 \leq \mathbb{E}_y \langle f_y, g_y \rangle^2$$

**Proof:** This is an immediate corollary of a lemma in [7], but we give the elementary proof for completeness. By the Cauchy-Schwarz inequality,

$$\mathbb{E}_y \langle f_y, g_y \rangle^2 \geq \mathbb{E}_y^2 \langle f_y, g_y \rangle = \mathbb{E}_{x,y}^2 (-1)^{f(x)+f(x+y)+g(x)+g(x+y)} = \mathbb{E}^4 (-1)^{f(x)+g(x)} = \langle f, g \rangle^4$$

■

**Corollary 4.2:**

$$\langle f, g \rangle^8 \leq \mathbb{E}_{y,z} \langle f_{y,z}, g_{y,z} \rangle^2$$

We will show that for any polynomial  $g$  of degree at most 3 holds  $\mathbb{E}_{y,z} \left\langle (S_4)_{y,z}, g_{y,z} \right\rangle^2 \leq \exp\{-\alpha N\}$ . First, here is a brief overview of the argument.

The point is that taking second derivatives makes life easier, since a second derivative of  $g$  is a linear function, and a second derivative of  $S_4$  is a quadratic. We therefore need to show that for the large majority of directions  $y, z$ , the quadratic function  $(S_4)_{y,z}$  has a small inner product with the linear function  $(-1)^{g_{y,z}}$ . In this we will be helped by a theorem of Dixon giving a structural description of quadratic polynomials, which, in particular, characterizes the Fourier transform of functions of the type  $(-1)^Q$ , where  $Q$  is a quadratic. In fact, setting



$Q = (S_4)_{y,z}$  we will see that for many of the directions  $y, z$  the Fourier coefficients of  $(-1)^Q$  will be exponentially small. For the remaining directions, these Fourier coefficients will be supported on an explicit easy to describe 3-dimensional affine subspace depending on  $y, z$ . We will then argue that for any fixed polynomial  $g$  of lower degree, the support of the character  $(-1)^{g_{y,z}}$  lies in this affine subspace with exponentially small probability over  $y, z$ .

We proceed with computing the second derivative  $Q = (S_4)_{y,z}$ .

#### 4.1 Second derivatives of $S_4$

Write  $Q(x) = \sum_{i<j} q_{i,j}x(i)x(j) + \sum_i \ell_i x(i) + c$ .

By Proposition 2.2 or by Example 2.4.

$$q_{i,j} = \mathcal{S}(y, z) - \langle y, \mathbf{1} \rangle \cdot (z(i) + z(j)) + \langle z, \mathbf{1} \rangle \cdot (y(i) + y(j)) + (y(i)z(j) + y(j)z(i))$$

At this point we invoke (a corollary of) a theorem of Dixon [6]:

**Theorem 4.3:** *Let  $Q(x) = \sum_{i<j} q_{i,j}x(i)x(j) + \sum_i \ell_i x(i) + c$  be a quadratic polynomial over  $\mathbb{F}_2$ . Consider the symmetric matrix with zeros on the diagonal and off-diagonal entries given by  $S_{i,j} = S_{j,i} = q_{i,j}$ . Let the rank of  $B = 2h$  (it is always even). Then the function  $(-1)^Q$  has  $2^{2h}$  non-zero Fourier coefficients of absolute value  $2^{-h}$ . Moreover, all these coefficients lie in an  $2h$ -dimensional affine subspace of  $\mathbb{F}_2^n$ .*

Consider the matrix  $B$  in our case. Some notation: let  $J$  be the matrix with 0 on the diagonal and 1 off the diagonal. Let  $u \otimes v$  denote the outer product  $uv^t$ . Then,

$$B = \mathcal{S}(y, z) \cdot J + \langle y, \mathbf{1} \rangle \cdot (z \otimes \mathbf{1} + \mathbf{1} \otimes z) + \langle z, \mathbf{1} \rangle \cdot (y \otimes \mathbf{1} + \mathbf{1} \otimes y) + (y \otimes z + z \otimes y)$$

Since the rank of  $J$  is at least  $N - 1$  and the rank of the remaining matrices is at most 2, the matrix  $B$  is almost of full rank if  $\mathcal{S}(y, z) = 1$ . In this case, by Theorem 4.3, the Fourier coefficients of  $(-1)^Q$  are exponentially small.

We therefore may assume  $\mathcal{S}(y, z) = 0$ . In this case the quadratic part of  $Q$  may be written as

$$\sum_{i<j} q_{i,j}x(i)x(j) = \langle y, \mathbf{1} \rangle \cdot \langle x, \mathbf{1} \rangle \langle x, z \rangle + \langle z, \mathbf{1} \rangle \cdot \langle x, \mathbf{1} \rangle \langle x, y \rangle + (\langle x, y \rangle \langle x, z \rangle + \langle x, yz \rangle)$$

Recall that  $yz$  denotes the pointwise product of vectors  $y$  and  $z$ .

This implies the non-zero Fourier coefficients of  $\sum_{i<j} q_{i,j}x(i)x(j)$  lie in a 3-dimensional affine subspace of  $\mathbb{F}_2^n$ . The linear part of this subspace is spanned by the vectors  $y, z, \mathbf{1}$  and it is shifted by a vector  $yz$ .

Next, consider the linear part  $\sum_i \ell(i)x(i)$  of  $Q$ . By Proposition 2.2,

$$\ell(i) = \mathcal{H}^{\{i\}}(y^{(2)}, z) + \mathcal{H}^{\{i\}}(y, z^{(2)}) =$$

$$\sum_{j < k < l \neq i} \left( y(k)y(l)z(j) + y(j)y(l)z(k) + y(j)y(k)z(l) \right) + \left( y(j)z(k)z(l) + y(k)z(j)z(l) + y(l)z(j)z(k) \right)$$

This can be directly verified to be equal to

$$\begin{aligned} & \left( \mathcal{S}(y, z) + \mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot y(i) + \left( \mathcal{S}(y, z) + \mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot z(i) + \\ & \left( \mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle + \langle y, z \rangle \cdot \langle y + z, \mathbf{1} \rangle \right) \end{aligned}$$

By assumption,  $\mathcal{S}(y, z) = \langle y, \mathbf{1} \rangle \cdot \langle z, \mathbf{1} \rangle + \langle y, z \rangle = 0$ . Note that this also implies  $\langle y, z \rangle \cdot \langle y + z, \mathbf{1} \rangle = 0$ , implying

$$\ell(i) = \left( \mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot y(i) + \left( \mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot z(i) + \left( \mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle \right)$$

Consequently, the linear part of  $Q$  may be written as

$$\sum_i \ell(i)x(i) =$$

$$\left( \mathcal{S}(z, z) + \langle z, \mathbf{1} \rangle \right) \cdot \langle x, y \rangle + \left( \mathcal{S}(y, y) + \langle y, \mathbf{1} \rangle \right) \cdot \langle x, z \rangle + \left( \mathcal{S}(y, y) \cdot \langle z, \mathbf{1} \rangle + \mathcal{S}(z, z) \cdot \langle y, \mathbf{1} \rangle \right) \cdot \langle x, \mathbf{1} \rangle$$

This means that the non-zero Fourier coefficients of the polynomial  $Q = \sum_{i < j} q_{i,j}x(i)x(j) + \sum_i \ell(i)x(i) + c$  lie in the affine subspace  $AF_{y,z} = yz + \text{Span}(y, z, \mathbf{1})$ .

## 4.2 Second derivatives of a fixed polynomial of degree 3

Let

$$g(x) = \sum_{i < j < k} a_{i,j,k} x(i)x(j)x(k)$$

be a polynomial of degree 3. For directions  $y, z \in \mathbb{F}^N$ , consider the second derivative  $g_{y,z} = \sum_i v_{y,z}(i)x(i) + c_{y,z}$ . We need to show that the probability of the vector  $v_{y,z}$  falling in the affine space  $AF_{y,z} = yz + \text{Span}(y, z, \mathbf{1})$  is exponentially small.

First, some notation. For  $1 \leq i \leq N$ , let  $G_i$  be a symmetric  $N \times N$  matrix over  $\mathbb{F}$  with  $(G_i)_{j,k} = (G_i)_{k,j} = a_{i,j,k}$  for all  $j \neq k$ . (Here we think about  $\{i, j, k\}$  as an unordered subset of  $[N]$ .) The diagonal entries of  $G_i$  are set to 0. For future use note the important property  $(G_i)_{j,k} = (G_j)_{i,k} = (G_k)_{i,j}$ .

These matrices are relevant because they describe the vector  $v_{y,z}$ .

**Lemma 4.4:**

•

$$v_{y,z}(i) = \text{coef}_{x(i)}(g_{y,z}(x)) = \langle y, G_i z \rangle$$

• An alternative representation of  $v_{y,z}$  will be more convenient for us. For  $z \in \mathbb{F}^N$ , let  $G(z) = \sum_{i=1}^N z(i)G_i$ . Then

$$v_{y,z} = G(z) \cdot y$$

**Proof:** For the first claim of the lemma, by linearity of the derivative, it suffices to consider the monomial  $g(x) = x(i)x(j)x(k)$ . This case can be easily verified directly.

For the second claim, note that

$$(G(z) \cdot y)(l) = \sum_{k=1}^N (G(z))_{k,l} y(k) = \sum_{k=1}^N y(k) \cdot \sum_{i=1}^N z(i) (G_i)_{k,l} = \sum_{k=1}^N y(k) \cdot \sum_{i=1}^N (G_l)_{k,i} z(i) = \langle y, G_l z \rangle$$

■

Consider the event  $\{v_{y,z} \in AF_{y,z}\}$ . This means  $v_{y,z} = yz + u_{y,z}$ , for some vector  $u_{y,z} \in \text{Span}(y, z, \mathbf{1})$ . There are only 8 possible choices for  $u_{y,z}$ . For convenience, let us assume, without loss of generality (as can be easily seen from the proof), that  $u_{y,z} = y + z + \mathbf{1}$  is the most popular one. By the lemma, the event  $\{v_{y,z} = yz + u_{y,z}\}$  is the same as  $\{G(z) \cdot y = yz + u_{y,z}\}$ . To simplify things some more, let  $A_i = G_i + e_i \otimes e_i$ ,  $i = 1 \dots N$ . That is,  $A_i = G_i$  but for  $(A_i)_{i,i} = 1$ . Let  $A(z) = \sum_{i=1}^N z(i) A_i$ . Note that  $A(z) \cdot y = G(z) \cdot y + yz$ . Hence  $\{G(z) \cdot y = yz + u_{y,z}\}$  is the same as  $\{A(z) \cdot y = u_{y,z} = y + z + \mathbf{1}\}$

We conclude the proof by a technical claim.

**Proposition 4.5:** *Let  $\{A_i\}$ ,  $i = 1 \dots N$  be a family of symmetric  $N \times N$  matrices over  $\mathbb{F}$  with  $A_i(k, k) = \delta_{ik}$ . Then, for  $y, z$  uniformly at random and independently from  $\mathbb{F}^N$ ,*

$$Pr_{y,z} \left\{ (A(z)) \cdot y = y + z + \mathbf{1} \right\} \leq \left( \frac{3}{4} \right)^N$$

The proof of the proposition is based on the claim that the rank of a matrix  $A(z)$  is typically large.

**Lemma 4.6:** *Let matrices  $\{A_i\}$  be as in the proposition. Let  $C$  be any fixed symmetric  $N \times N$  matrix. Then*

$$Pr_z \left\{ \text{rank}(A(z) + C) \leq k - 1 \right\} \leq \frac{1}{2^N} \cdot \sum_{i=0}^{k-1} \binom{N}{i}.$$

**Proof:** Consider a family of  $\binom{N}{k}$  polynomials  $f_I$  on  $\mathbb{F}^N$ . These polynomials are indexed by  $k$ -subsets of  $[N]$ . For a  $k$ -subset  $I$ , let  $f_I(z)$  be the determinant of the  $I \times I$  minor of  $A(z) + C$ . Clearly, rank of  $A(z) + C$  is smaller than  $k$  if and only if  $z$  is a joint zero of  $\{f_I\}$ .

We now claim that the coefficient of  $\prod_{i \in I} z_i$  in  $f_I(z)$  is 1. If this is true,  $\deg(f_I - \prod_{i \in I} z_i) \leq k - 1$ , and the claim of the lemma will follow from Lemma 4.6.

Let  $B(z) = A(z) + C$ . Since we are working in characteristic two, the symmetry of  $B(z)$  implies that

$$\begin{aligned} \det B(z) &= \sum_{\sigma \in S_N: \sigma = \sigma^{-1}} \prod_{i=1}^N B_{i\sigma(i)}(z) = \\ &= \sum_{\sigma \in S_N: \sigma = \sigma^{-1}} \prod_{\{i: \sigma(i)=i\}} (z_i + C_{i,i}) \cdot \prod_{\{i: i < \sigma(i)\}} B_{i\sigma(i)}(z) = \prod_{i \in I} z_i + \text{lower order terms.} \end{aligned}$$

In the second equality we use the identity  $B_{i\sigma(i)}^2(z) = B_{i\sigma(i)}(z)$  in  $\mathbb{F}$ . ■

Let  $I$  denote the identity  $N \times N$  matrix.

Let  $p(z) = Pr_y\{A(z) \cdot y = y + z + \mathbf{1}\}$ . Clearly  $p(z) \leq 2^{-\text{rank}(A(z)+I)}$ . By Lemma 4.6,

$$Pr_{y,z}\{(A(z)) \cdot y = y + z + \mathbf{1}\} = \mathbb{E}_z p_z \leq \mathbb{E}_z 2^{-\text{rank}(A(z)+I)} \leq \frac{1}{2^N} \sum_{k=0}^N \binom{N}{k} 2^{-k} = \left(\frac{3}{4}\right)^N$$

This concludes the proof of the proposition, and of Theorem 1.3.

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## References

- [1] N. Alon, R. Beigel *Lower Bounds for Approximations by Low Degree Polynomials over  $Z_m$* , SCT: Annual Conference on Structure in Complexity Theory, 2001.
- [2] A. Bogdanov, E. Viola *Pseudorandom bits for polynomials via the Gowers norm*, FOCS'07.
- [3] W. T. Gowers, *A new proof of Szemerédi's theorem*, GAFA Vol. 11(2001), pp. 465-588.
- [4] B. Green, T. Tao, *An inverse theorem for the Gowers  $U^3$  norm*, Proc. Edinburgh Math. Soc., to appear.
- [5] B. Green, T. Tao, *The distribution of polynomials over finite fields, with applications to the Gowers norms*, preprint, 2007.
- [6] J. MacWilliams and N. J. A. Sloane, **The Theory of Error Correcting Codes**, Amsterdam, North-Holland, 1977.
- [7] A. Samorodnitsky, *Low degree tests at large distances*, STOC '07.
- [8] T. Tao, *Structure and randomness in combinatorics*, FOCS '07.
- [9] E. Viola, A. Wigderson, *Norms, XOR lemmas, and lower bounds for  $GF(2)$  polynomials and multiparty protocols*, CCC '07.