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Equilibria of Graphical Games with Symmetries*

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Abstract

We study graphical games where the payoff function of each player satisfies one of four types of symmetries in the actions of his neighbors. We establish that deciding the existence of a pure Nash equilibrium is NP-hard in graphical games with each of the four types of symmetry. Using a characterization of games with pure equilibria in terms of even cycles in the neighborhood graph, as well as a connection to a generalized satisfiability problem, we identify tractable subclasses of the games satisfying the most restrictive type of symmetry. In the process, we characterize a satisfiability problem that remains NP-hard in the presence of a matching, a result that may be of independent interest. Finally, games with symmetries of two of the four types are shown to possess a symmetric *mixed* equilibrium which can be computed in polynomial time. We have thus identified a class of games where the pure equilibrium problem is computationally harder than the mixed equilibrium problem, unless P=NP.

1 Introduction

The idea underlying graphical games (Kearns et al., 2001) is that in games with a large number of players, the payoff of any particular player will often depend only on the actions of a small number of other players in a local neighborhood. A graphical game is any game in strategic form such that there exists a (directed or undirected) graph on the set of players, and the payoff of each player depends only on the actions of his neighbors in this graph. If neighborhood sizes are bounded, graphical games can be represented using space polynomial in the number of players.

One of the best-known solution concepts for strategic games is Nash equilibrium (Nash, 1951). A vector of strategies, one for each player, is called Nash equilibrium if no player can increase his (expected) payoff by unilaterally changing his strategy.

Problem and Related Work The computational problem of finding Nash equilibria in graphical games with degree bounded by $d \geq 3$ has recently been shown equivalent to the same problem for general n-player games, $n \geq 4$ (Goldberg and Papadimitriou, 2006), and thus complete for the complexity class PPAD (Daskalakis et al., 2006). It is not surprising that the graph structure plays an important role for the complexity of the equilibrium problem. PPAD-hardness holds even if the underlying graph has constant pathwidth, but becomes tractable for graphs of degree 2, *i.e.*, for *paths* (Elkind et al., 2006). All known

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algorithms for the more general case of *trees* have exponential worst-case running time even on trees with bounded-degree and pathwidth 2.

A different line of research has investigated the problem of deciding the existence of *pure* Nash equilibria, *i.e.*, equilibria where the support of each strategy contains only a single action. The pure equilibrium problem has been shown NP-complete for graphical games on directed graphs with outdegree bounded by $d \geq 2$ and with only two actions for each player and two different payoffs, and tractable for graphs with bounded treewidth (Gottlob et al., 2005; Fischer et al., 2006). Unlike Nash equilibria in *mixed* strategies, *i.e.*, probabilistic combinations of actions, pure Nash equilibria are not guaranteed to exist. They nevertheless form an interesting subset of equilibria for three reasons. First, requiring randomization in order to reach a stable outcome has been criticized on various grounds. In multi-player games, where action probabilities in equilibrium can be irrational numbers, randomization is particularly questionable. Secondly, the computation of pure equilibria, if they exist, may be tractable in cases where that of mixed ones is not. Finally, pure equilibria as computational objects are usually much smaller in size than mixed ones.

Symmetric games are characterized by the fact that players can not, or need not, distinguish between other players. Brandt et al. (2007) analyze four different classes of symmetric games, and show that the pure equilibrium problem is tractable if the number of actions is a constant, and complete for NP or PLS, respectively, if the number of actions grows logarithmically in the number of players. One of the classes is guaranteed to possess a symmetric equilibrium, *i.e.*, one where all players play the same strategy. This equilibrium is not necessarily pure, but can be found efficiently if the number of actions is not too large compared to the number of players. A strictly larger class has recently been found to still admit a PTAS, *i.e.*, an efficient way to compute approximate equilibria (Daskalakis and Papadimitriou, 2007).

This fuels hope that tractability results can be obtained for larger classes of games satisfying some kind of symmetry. In this regard, Daskalakis and Papadimitriou (2005) consider games on a d-dimensional undirected torus or grid with payoff functions that are identical for all players and symmetric in the actions of the players in the neighborhood (a condition that will be called *strong symmetry* in this paper). In particular, they show that deciding the existence of a pure Nash equilibrium in such a game is NL-complete when $d = 1$ and NEXP-complete for $d \geq 2$. In this paper, we investigate the pure equilibrium problems in graphical games that satisfying the kinds of symmetries considered by Brandt et al. (2007). Our work can thus be seen as a refinement of the work of Gottlob et al. (2005) and of Daskalakis and Papadimitriou (2005).

Paper Structure and Results We begin by formally defining the necessary game-theoretic concepts in Section 2. In Section 3, we then investigate the computational complexity of the pure equilibrium problem in graphical games satisfying four different types of symmetries. The question for tractable classes of graphical games is answered mostly in the negative. For three of the four symmetry classes, deciding the existence of a pure equilibrium is NP-hard already for the case of two actions, two payoffs, and neighborhoods of size two. Assuming the most restricted type of symmetry, the problem becomes NP-hard when there are three different payoffs, or neighborhoods of size four. The latter class has some interesting connections to the problem of finding even cycles in a directed graph, and to generalized satisfiability. In particular, we identify tractable classes of games by showing that they correspond to graphs with even cycles, or to tractable satisfiability instances. As a corollary, we a satisfiability problem that remains NP-hard in the presence of a matching, *i.e.*, a bijective mapping between variables and clauses. We present this result, which may be of independent interest, in Section 4. Finally, in Section 5, we show that mixed equilibria in games with two of the above symmetry types can be found in polynomial time if the number of actions grows only slowly in the neighborhood size. Quite interestingly, there exists a class of games where deciding the existence of a pure equilibrium problem is likely to be harder than finding a mixed equilibrium.

We assume the reader to be familiar with the well-known complexity classes P and NP and the notion of polynomial-time reducibility (see, *e.g.*, Papadimitriou, 1994). P and NP are the classes of problems that can be solved in polynomial time by deterministic and nondeterministic Turing machines, respectively. Furthermore, #P is the class of counting problems associated with polynomially balanced polynomial-time decidable relations.

2 Preliminaries

An accepted way to model situations of strategic interaction is by means of a normal-form game (see, *e.g.*, Luce and Raiffa, 1957).

Definition 1 (normal-form game) *A* game in normal-form *is a tuple* $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ *where* N *is a set of* players *and for each player* $i \in N$, A_i *is a nonempty set of* actions *available to player* i, *and* $p_i : (\bigtimes_{i\in N} A_i) \to \mathbb{R}$ *is a function mapping each action profile of the game,* i.e.*, combination of actions, to a real-valued* payoff *for player* i.

A vector $s \in \bigtimes_{i\in N} A_i$ of actions is also called a profile of *pure strategies*. This concept can be generalized to *(mixed) strategy profiles* $s \in S = \bigtimes_{i\in N} S_i$, by letting players randomize over their actions. We have S_i denote the set of probability distributions over player i's actions, or *(mixed) strategies* available to player i. We further write $n = |N|$ for the number of players in a game, s_i for the ith strategy in profile s, and s_C for the vector of strategies for all players in a subset $C \subseteq N$.

A graphical game is given by a graph on the set of players, such that the payoff of a player only depends only on his own action, and on the actions of his neighbors in the graph. In the following definition, the underlying graph is directed, corresponding to a neighborhood relation that is not necessarily symmetric.

Definition 2 (graphical game) *Let* $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ *be a normal-form game,* $\nu : N \to 2^N$. Γ *is a* graphical game *with neighborhood* ν *if for all* $i \in N$, $s, s' \in A^N$, $p_i(s) = p_i(s')$ *whenever* $s_{\hat{\nu}(i)} = s'_{\hat{\nu}(i)}$, *where* $\hat{\nu}(i) = \nu(i) \cup \{i\}$.

A game Γ *has* k*-bounded neighborhoods if there exists* $\nu : N \to 2^N$ *such that* Γ *is a graphical game with neighborhood* ν *and for all* $i \in N$, $|\nu(i)| \leq k$.

Let us now turn to symmetries in games as considered by Brandt et al. (2007). A normal-form game is symmetric if payoffs depend on the number of players playing the different actions. For ease of exposition, we will assume that all players have the same set of actions at their disposal, and not only those that are neighbors of the same player. It should be noted that this does not restrict the expressiveness of our results. The following definition uses a notion introduced by Parikh (1966) in the context of context-free languages. Let A be a set of actions. The *commutative image* of an action profile $s \in A^N$ is given by $\#(s) = (\#(a, s))_{a\in A}$ where $\#(a, s) = |\{ i \in N \mid s_i = a \}|$.

Definition 3 (symmetries) *Let* $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ *be a graphical game,* A *a set of actions such that for all* $i \in N$, $A_i = A$. Γ *is said to satisfy*

- weak symmetry *if for all* $i \in N$ *and all* $s, s' \in A^N$, $p_i(s) = p_i(s)$ *whenever* $s_i = s'_i$ *and for all* $a \in A$, $\#(s_{\nu(i)}, a) = \#(s'_{\nu(i)}, a)$;
- strong symmetry *if for all* $i, j \in N$ *and all* $s, s' \in A^N$, $p_i(s) = p_j(s')$ *whenever* $s_i = s'_j$ *and for all* $a \in A$, $\#(s_{\nu(i)}, a) = \#(s'_{\nu(j)}, a)$;
- weak anonymity *if for all* $i \in N$ *and all* $s, s' \in A^N$, $p_i(s) = p_i(s')$ *whenever for all* $a \in A$, $\#(s_{\hat{\nu}(i)}, a) = \#(s'_{\hat{\nu}(i)}, a)$; *and*

- strong anonymity *if for all* $i, j \in N$ *and all* $s, s' \in A^N$, $p_i(s) = p_j(s')$ *whenever for all* $a \in A$, $\#(s_{\hat{\nu}(i)}, a) = \#(s'_{\hat{\nu}(j)}, a)$.

Weak symmetry has recently been referred to as anonymity by some computer scientists. We stick with the distinction between weak and strong symmetry used in the game theory literature, and use the term anonymity to refer to the remaining two classes. When talking about games with anonymity, we write $\mathbf{p}_i(j) = p_i(s)$ where $\#(s_{\hat{\nu}(i)}, 1) = j$ for the payoff of player i when j players in his neighborhood, including i himself, play action 1, and $\mathbf{p}_i = (\mathbf{p}_i(j))_{0 \leq j \leq \hat{\nu}(i)}$ for the vector of payoffs for the possible values of j.

One of the best-known solution concepts for strategic games is Nash equilibrium (Nash, 1951). In a Nash equilibrium, no player is able to increase his payoff by *unilaterally* changing his strategy.

Definition 4 (Nash equilibrium) *A strategy profile* $s \in S$ *is called a* Nash equilibrium *if for each player* $i \in N$ *and each strategy* $s'_i \in S_i$, $p_i(s) \geq p_i((s_{N\setminus\{i\}}, s'_i))$. *A Nash equilibrium is called* pure *if it is a pure strategy profile.*

3 Complexity of the Pure Equilibrium Problem

For graphical games with neighborhoods of size one, symmetries do not impose any restrictions. The pure equilibrium problem for such games can be decided in polynomial time (see, *e.g.*, Fischer et al., 2006). On the other hand, the game used by Schoenebeck and Vadhan (2006) to show NP-completeness of the pure equilibrium problem in general graphical games satisfies weak symmetry. We thus have the following initial result.

Corollary 1 *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two neighbors, two actions, and two different payoffs, and when restricted to games with weak symmetry.*

3.1 Strong Symmetry and Strong Anonymity

We now consider a more restrictive kind of symmetry. In particular, the following theorem concerns games where the utility functions of all players are identical. The proof of this theorem is similar to a construction used by Schoenebeck and Vadhan (2006) where each gate of a Boolean circuit corresponds to a player in a graphical game. Depending on the output of the circuit additional players either play a game with or without a pure equilibrium. The greatest difficulty in our case is to model the circuit using only a single payoff function.

Theorem 1 *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two actions and two different payoffs, and when restricted to games with strong symmetry and two different payoffs, or to games with strong anonymity and three different payoffs.*

Proof: Membership in NP is obvious. We can simply guess an action profile and verify that the action of each player is a best response to the actions of the players in his neighborhood.

For *hardness*, we provide a reduction from the NP-complete problem circuit satisfiability (CSAT) (see, *e.g.*, Papadimitriou, 1994). For a set N of players with appropriately defined neighborhoods, let $\Gamma(N) = (N, \{0,1\}^N, (p_i)_{i \in N})$ be a graphical game with neighborhood ν and payoffs satisfying strong symmetry or strong anonymity as given by Figure 1.[1] We observe the following:

[1] It was shown by Brandt et al. (2007) that every symmetric game with two actions per player can be reduced to an anonymous game while preserving pure equilibria.

$\#(s_{\nu(i)},1)$	0	1	2
0	0	0	1
1	1	1	0

$\#(s_{\hat{\nu}(i)},1)$	0	1	2	3
	0	1	2	0

Figure 1: NAND payoffs $p_i(s)$ for the symmetric and the anonymous case. Columns correspond to the different values of the commutative image of s w.r.t. $\nu(i)$ and $\hat{\nu}(i)$, respectively. In the symmetric case, rows correspond to the different actions of player i.

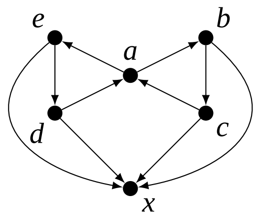


Figure 2: Output gadget. A directed edge between vertices i and j denotes that $j \in \nu(i)$. All players have payoffs as in Figure 1. Player x must play action 0 in every pure equilibrium of the game.

1. Let N be a set of players, $|N| = 3$, and for all $i \in N$, $\hat{\nu}(i) = N$. Then, an action profile s of $\Gamma(N)$ is a pure equilibrium if and only if $\#(s, 1) = 2$. In particular, for every $i \in N$, there exists a pure equilibrium where player i plays action 0 and a pure equilibrium where he plays action 1.

2. Let N and N' be two sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Then, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.

3. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium and consider two players $a, b \in N$. Further consider an additional player $c \notin N$ with $\nu(c) = \{a, b\}$. Then the game $\Gamma(N \cup \{c\})$ has a pure equilibrium, and in every pure equilibrium s of $\Gamma(N \cup \{c\})$, $s_c = 0$ if $s_a = s_b = 1$, and $s_c = 1$ otherwise. In other words, player c always plays the NAND of the actions played by players a and b.

4. Let N be a set of players and consider a particular player $x \in N$. Further consider five additional players $a, b, c, d, e \notin N$ with neighborhoods according to Figure 2, and denote $N' = N \cup \{a, b, c, d, e\}$. Then $\Gamma(N')$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s where $s_x = 0$. Assume that $\Gamma(N)$ has a pure equilibrium s where $s_x = 0$ and extend this to an action profile for $\Gamma(N')$ by letting $s_a = 0$ and $s_b = s_c = s_d = s_e = 1$. On the other hand, consider an action profile s for $\Gamma(N')$ where $s_x = 0$. If $s_a = 0$, then action 1 is the unique best response for players c and d, after which action 0 is the unique best response for players b and e. In this case, player a can change his action to 1 to get a higher payoff. If $s_a = 1$, then the unique best response for players c and d and for players b and e becomes action 0 and 1, respectively. Again, player a can change his action to get a higher payoff.

5. Let $N_1 = \{x, y, z\}$ be an instance of N in Property 1, and N_2 an instance of N' in Property 4 with $N = \{x\}$. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, $a \in N$, and denote $N' = N_1 \cup N_2 \cup N$. Further consider an additional player $c \notin N'$ with $\nu(c) = \{a, y\}$. Then, $\Gamma(N' \cup \{c\})$ has a pure equilibrium and in every pure equilibrium s of $\Gamma(N' \cup \{c\})$, $s_c = 1 - s_a$. To see this, observe that by Property 1 exactly two players in N_1 must play action 1, which, by Property 4, have to be players y and z. By Property 3, and since φ NAND *true* $= \neg\varphi$, the claim follows.

Now consider an instance C of CSAT, and assume w.l.o.g. that C consists exclusively of NAND gates and that no variable appears more than once as the input to the same gate. The latter assumption can be made by Property 5. We construct a game $\Gamma = \Gamma(N)$ as follows. For every input of C we augment N by three players (Property 1). We then inductively define Γ by adding, for a gate with inputs corresponding to players $a, b \in N$, a player c as described in Property 3. Finally, we construct a player according to Property 5 who plays the opposite action as the one corresponding to the output of C, and identify this player with x in a new instance of 4. It is now easily verified that a pure equilibrium of Γ corresponds to a computation of C which outputs *true*, and that such an equilibrium exists if and only if C has a satisfying assignment. □

3.2 Weak Anonymity and Two Different Payoffs

Theorem 1 allowed for a uniform proof, but its shortcomings will not have gone unnoticed. The result is not tight in that three different payoffs are required to show NP-hardness in the anonymous case. It is natural to enquire what happens for games with anonymity and only two different payoffs. In this section we will prove a tight result for most restricted case of *weak* anonymity, *i.e.*, the case with two different payoff functions.

The problem with anonymity and the construction used in the proof of Theorem 1 is that two different payoffs are not enough to make a player care about his own action no matter which actions are played by his neighbors. With four different values for $\#(s_{\hat{\nu}(i)}, 1)$, there will either be an equilibrium where all players play the same action, or a situation where a player is indifferent between both of his actions. When we want to use games to compute a function, such indifference is clearly undesirable. The key idea that will enable us to prove the following theorem is to isolate pure equilibria that are themselves *symmetric* in the actions of a subset of the players, *i.e.*, in which these players all play the same action. To enforce that two particular players play the same action in every equilibrium, we will add two additional players, each of which observes the other as well as one of the original players. Depending on the actions of the original players, the new players will either play a game with a unique pure equilibrium, or a game that is prototypical both for anonymous games and for games without pure equilibria, namely matching pennies. We proceed with the statement of the theorem.

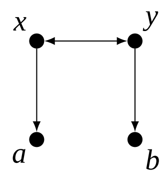
Theorem 2 *Deciding whether a graphical game has a pure Nash equilibrium is NP-complete, even if every player has only two neighbors, two actions, and two different payoffs, and when restricted to games with weak anonymity and two different payoff functions.*

Proof: Membership in NP is again obvious.

For *hardness*, we again provide a reduction from circuit satisfiability (CSAT). Let $\Gamma(N) = (N, \{0,1\}^N, (p_i)_{i\in N})$ denote a graphical game for a set N of players with neighborhood ν and payoff functions p_i satisfying weak anonymity. We observe the following:

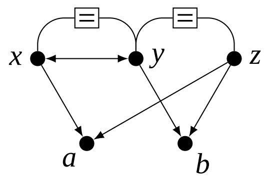
1. Let N be a set of players, $a, b \in N$, and consider two additional players $x, y \notin N$ with neighborhoods and payoffs according to Figure 3. We claim that $\Gamma(N \cup \{x, y\})$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s where $s_a = s_b$. Assume that $\Gamma(N)$ has a pure equilibrium s where $s_a = s_b$ and extend this to an action profile for $\Gamma(N')$ by letting $s_x = 0$ and $s_y = 1$. It is easily verified that under this action profile players x and y both receive the maximum payoff of 1, such that the equilibrium condition is trivially satisfied. On the other hand, assume that one of the players x and y observes action 0 being played by player a or b, while the other one observes action 1. Then players x and y effectively play the well-known matching pennies game. More precisely, the player observing 0 receives a payoff of 1 if and only if $\#(s_{\{x,y\}})$ is odd, while the same is true for the player observing 1 if and only if this number is even. Since both players can change between the two outcomes by changing their own action, there is no pure equilibrium.

2. Let $N = \{a, b, c\}$ with $\nu(i) = N$ for all $i \in N$, and payoffs according to Figure 3. It is then easily verified that s with $s_a = s_b = s_c = 1$ or with $s_a = s_b = 0$ and $s_c = 1$ is an equilibrium of $\Gamma(N)$. In particular, there exist equilibria where $s_a = 0$ and $s_a = 1$, respectively.



$\#(s_{\hat{\nu}(i)}, 1)$	0	1	2	3
$p_i(s)$	0	1	0	1

Figure 3: Equality gadget. A pure equilibrium exists if and only if players a and b play the same action.



$\#(s_{\hat{\nu}(i)}, 1)$	0	1	2	3
$p_i(s)$	0	1	1	0

Figure 4: NAND gadget. The construction of Figure 3 is used to ensure that players connected by "=" play the same action in every pure equilibrium.

3. Let N and N' be two sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Again, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.

4. Let $N = \{a, b, c\}$ with neighborhoods and payoffs as in Property 2, and assume by Property 1 that every pure equilibrium s of $\Gamma(N)$ is symmetric, *i.e.*, $s_a = s_b = s_c$. Then, s with $s_a = s_b = s_c = 1$ is the unique pure equilibrium of $\Gamma(N)$. Clearly, s is an equilibrium of $\Gamma(N)$, since all players receive the maximum payoff of 1. In the only other symmetric action profile, all players play action 0 and receive a payoff of 0. Either one of them can change his action to 1 to receive a higher payoff.

5. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, let $a, b \in N$, and consider three additional players $x, y, z \notin N$ with neighborhoods and payoffs according to Figure 4. Then, $\Gamma(N \cup \{x, y, z\})$ has a pure equilibrium, and for every pure equilibrium s of $\Gamma(N \cup \{x, y, z\})$, $s_x = 0$ if $s_a = s_b = 1$, and $s_x = 1$ otherwise. It is easily verified that players x, y, and z get the maximum payoff of 1, and thus will not deviate, under any action profile s where $s_x = s_y = s_z = 1$ and $\#(s_{\{a,b\},1}) \leq 1$ or where $s_x = s_y = s_z = 0$ and $s_a = s_b = 1$. On the other hand, let s be an arbitrary action profile of $\Gamma(N \cup \{x, y, z\})$. By Property 1, s cannot be an equilibrium unless $s_x = s_y = s_z$. If $s_a = s_b = s_z = 0$ or $s_a = s_b = s_z = 1$, then player z can change his action to receive a higher payoff. If otherwise $s_a \neq s_z$ and $s_x = s_y = 0$, then there exists $i \in \{x, y\}$ such that $\#(s_{\hat{\nu}(i)}, 1) = 0$, and player i will deviate.

6. Let N a set of players, $o \in N$. Let $N' = \{a, b, c\}$ with neighborhoods as in Property 4, $N'' = \{x, y\}$ with $\nu(x) = \{a, y\}$ and $\nu(y) = \{o, x\}$. Then, $\Gamma(N \cup N' \cup N'')$ has a pure equilibrium if and only if $\Gamma(N)$ has a pure equilibrium s with $s_o = 1$. Clearly, an action profile that is not an equilibrium of $\Gamma(N)$ cannot be extended to an equilibrium of $\Gamma(N \cup N' \cup N'')$. On the other hand, assume that s is an equilibrium of $\Gamma(N \cup N' \cup N'')$. Then, by Property 4, $s_a = 1$. Furthermore, by Property 1, $s_a = s_o$, and thus $s_o = 1$.

Now consider an instance C of CSAT. We assume w.l.o.g. that C consists exclusively of NAND gates. Since $\varphi \,\mathrm{NAND}\, \mathit{true} = \neg\varphi$, and using Property 4, we can further assume that no variable appears more than once as an input to the same gate. We construct a game $\Gamma = \Gamma(N)$ as follows: For every input of C, we add three players according to Property 2. For every gate of C with inputs corresponding to players $a, b \in N$, we add three players according to Property 5. Finally, we add five players according to Property 6, where o is the player corresponding to the output of C. It is now readily appreciated that Γ has a pure equilibrium if and only if C is satisfiable. $\square$

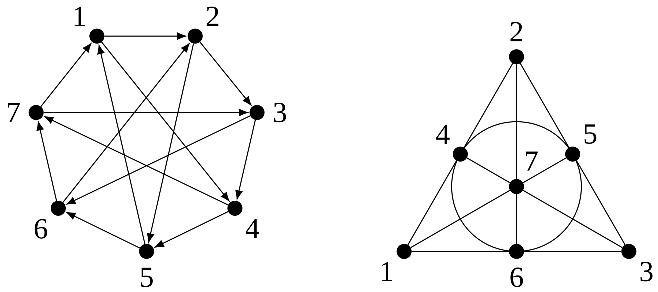


Figure 5: Neighborhood graph of a graphical game with seven players (left), corresponding to the three-uniform square hypergraph given by the lines of the Fano plane (right).

3.3 Strong Anonymity and Two Different Payoffs

Let us return to games with strong anonymity. Strongly anonymous games as studied by Brandt et al. (2007) always possess a pure Nash equilibrium due to the fact that they are common-payoff. This is not the case for graphical games with strong anonymity, even when there are only two different payoffs. In particular, there exists a graphical seven-player game with strong anonymity and without a pure equilibrium, such that each player has exactly two actions and two neighbors. It will be instructive to view a graphical game as a hypergraph, with each vertex corresponding to a player and each edge to the set of players in the neighborhood of one particular player *including* the player himself. Corresponding to the set of games with m-neighborhood is the set of $(m+1)$-uniform hypergraphs that possess a matching in the sense of Seymour (1974), *i.e.*, a bijective mapping from the set of vertices to the set of edges. Then, a game with strong anonymity and $\mathbf{p}_i = (0, 1, 1, 0)$ for all $i \in N$ has a pure Nash equilibrium if and only if the corresponding hypergraph is two-colorable. Given a two-coloring, every player observes either one or two players in his neighborhood, including himself, who play action 1. Every player thus obtains the maximum payoff of 1. On the other hand, if there is no two-coloring, then there is at least one player for every action profile who plays the same action as all of his neighbors and can deviate to obtain a higher payoff. Figure 5 shows the neighborhood of a graphical game with seven players and two neighbors for each player. This graph induces the 3-uniform square hypergraph corresponding to the lines of the Fano plane, which cannot be two-colored (see, *e.g.*, Seymour, 1974). We leave it to the reader to verify that there is no game with the above properties and less than seven players.

The neighborhood graph on the left of Figure 5 does not have any cycles of even length. We will begin our investigation of the pure equilibrium problem in games with strong anonymity by generalizing this observation to games with arbitrary neighborhoods and $\mathbf{p}_i = (0, 1, 1, \dots, 1, 0)$ for all $i \in N$. The following lemma characterizes games with pure equilibria in the above subclass in terms of cycles in the neighborhood graph. Seymour (1974) provides a similar characterization of the minimal uniform square hypergraphs that do not have a two-coloring.

Lemma 1 *Let Γ be a graphical game with strong anonymity, two actions per player, and payoffs p_i such that for all $i \in N$, $\mathbf{p}_i = (0, 1, 1, \dots, 1, 0)$. Then, Γ has a pure Nash equilibrium if and only if for all $i \in N$, there exists $j \in N$ reachable from i that lies on a cycle of even length.*

Proof: For the implication from left to right, assume that there exists a pure equilibrium, *i.e.*, a two-coloring $c : N \to \{0, 1\}$ of the neighborhood graph such that the neighborhood of every player contains some player playing action 0 and some player playing action 1. Now consider an arbitrary player $v_1 \in N$. Using the above property of c, we can construct a path $v_1, v_2, \dots, v_{|N|+1}$, $v_i \in N$, such that for all i, $1 \leq i \leq |N|$,

$c(v_i) = 1 - c(v_{i+1})$. By the pigeonhole principle, there must exist i, j, $1 \leq i < j \leq |N| + 1$, such that $v_i = v_j$ and for all j', $i < j' < j$, $v_{j'} \neq v_i$. Then, $v_i, v_{i+1}, \ldots, v_j$ is a cycle of even length.

For the implication from right to left, let $N' \subseteq N$ be a set of players such that for every $i \in N$ there exists a directed path to some $j \in N'$ and such that N' induces a set of vertex-disjoint cycles of even length. We construct a two-coloring $c : N \to \{0, 1\}$, corresponding to an assignment of actions to players, as follows. First color the members of N' such that for all $i \in N'$ and $j \in \nu(i) \cap N'$, $c(i) = 1 - c(j)$. While there are uncolored vertices left, find $i, j \in N$ such that $j \in \nu(i)$, i is uncolored, and j is colored. Such a pair of vertices must always exist, since for every member N there is a directed path to some member of N', and thus to a vertex that has already been colored. Color i such that $c(i) = 1 - c(j)$. It is now easily verified that at any given time, and for all $i \in N$ that have already been colored, there exist $j, j' \in \hat{\nu}(i)$ with $c(j) = 0$ and $c(j') = 1$. If all vertices have been colored, then every neighborhood will contain at least one player playing action 0, and at least one player playing action 1. The corresponding action profile is a pure Nash equilibrium. $\square$

Thomassen (1985) has shown that for every k, there exists a directed graph without even cycles where every vertex has outdegree k. Together with Lemma 1, this means that the pure equilibrium problem for the considered class of games is nontrivial.

Corollary 2 *For every $m \in \mathbb{N}$, $m > 0$, there exist graphical games Γ, Γ' with strong anonymity where for all $i \in N$, $|\nu(i)| = m$ and $\mathbf{p}_i = (0, 1, 1, \ldots, 1, 0)$, such that Γ has a pure Nash equilibrium and Γ' does not.*

We are now ready to identify several classes of graphical games where the existence of a pure equilibrium can be decided in polynomial time.

Theorem 3 *Let Γ be a graphical game with with strong anonymity and payoffs p_i. The pure equilibrium problem for Γ can be decided in polynomial time if one of the following properties holds:*

(i) for all $i \in N$, $\mathbf{p}_i(0) \geq \mathbf{p}_i(1)$ or for all $i \in N$, $\mathbf{p}_i(|\hat{\nu}(i)|) \geq \mathbf{p}_i(|\hat{\nu}(i)| - 1)$;

(ii) for all $i \in N$ and all j, $1 \leq j \leq |\nu(i)|$, $\mathbf{p}_i(j-1) > \mathbf{p}_i(j)$ and $\mathbf{p}_i(j+1) > \mathbf{p}_i(j)$, or $\mathbf{p}_i(j-1) < \mathbf{p}_i(j)$ and $\mathbf{p}_i(j+1) < \mathbf{p}_i(j)$;

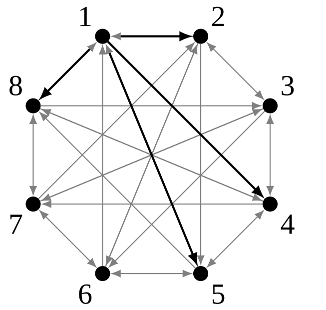
(iii) for all $i \in N$ and all j, $1 \leq j < |\nu(i)|$, $\mathbf{p}_i(j) = \mathbf{p}_i(j+1)$.

Proof: It is easy to see that a game Γ satisfying (i) possesses a pure equilibrium s in which $\#(s, 0) = 0$ or $\#(s, 1) = 1$.

For a game Γ satisfying (ii), we observe that in every equilibrium s, $p_i(s) = 1$ for all $i \in N$. The pure equilibrium problem for Γ thus corresponds to a variant of generalized satisfiability, with clauses induced by neighborhoods of Γ. The constraints associated with this particular variant require that the number of variables in each clause set to true is odd, and can be written as a system of linear equations over $GF(2)$. Tractability of the pure equilibrium problem for Γ then follows from Theorem 2.1 of Schaefer (1978).

Finally, a game satisfying (iii) but not (i) can be transformed into a best response equivalent one that satisfies the conditions of Lemma 1. We further claim that we can check in polynomial time whether for every $i \in N$, there exists $j \in N$ on a cycle of even length and reachable from i. For a particular $i \in N$, this problem is equivalent to checking whether the subgraph induced by the vertices reachable from i contains an even cycle. The latter problem has long been open, but was recently shown to be solvable in polynomial time (Robertson et al., 1999). $\square$

It is readily appreciated that every strongly anonymous game Γ with two different payoffs and neighborhoods of size two or three can be transformed into a game Γ' with the same set of players and the same neighborhoods, such that Γ and Γ' have the same set of pure equilibria and Γ' satisfies one of the conditions of Theorem 3. We thus have the following.



$\#(s_{\hat{\nu}(i)}, 1)$	0	1	2	3	4	5
$p_i(s)$	0	1	0	1	1	0

Figure 6: Neighborhood graph and payoffs of a graphical game with eight players and neighborhood of size four used in the proof of Theorem 4. The neighborhood graph satisfies rotational symmetry, the neighborhood of player 1 is highlighted.

Corollary 3 *The problem of deciding whether a graphical game with strong anonymity, two different payoffs, and three-bounded neighborhood has a pure equilibrium is in P.*

3.4 Strong Anonymity and Larger Neighborhoods

The remaining question is whether the pure equilibrium problem can be solved in polynomial time for all games with strong anonymity and two payoffs, or whether there is some bound on the neighborhood size where it again becomes hard. We will show in this section that the latter is true, and that the correct bound is indeed three, as suggested by Corollary 3.

To do so, we will essentially use the same tools as in Section 3.2 but will extract the necessary complexity from only a single payoff function. The additional insight necessary for this extraction will be that "constant" players, *i.e.*, players who play the same action in every pure equilibrium of a game, can be used to prune a larger payoff table and effectively obtain different payoff functions for smaller neighborhoods that can then be used to proceed with the original proof. Constructing the constant players will prove a rather difficult task in its own right. We are now ready to state the theorem.

Theorem 4 *Deciding whether a graphical game with strong anonymity and two different payoffs has a pure Nash equilibrium is NP-complete, even if every player has exactly four neighbors.*

Proof: Membership in NP is obvious. We can simply guess an action for each player and then verify that no player can increase his payoff by playing a different action instead.

For *hardness*, we again give a reduction from CSAT to the problem at hand. The central idea of this proof will be to guarantee that some players in a neighborhood only play certain well-defined actions in equilibrium. By this, the original payoff table is effectively "pruned" to a smaller one that can then be used, like in earlier proofs, to model the behavior of gates in a Boolean circuit.

As a first step, we will show how to construct "constants," *i.e.*, players who play action 0 or action 1, respectively, in *every* equilibrium of a game. To achieve this, we will construct a set of four players, such that in every equilibrium two of these players play action 0 and two of them play action 1. A player observing these four players can determine if the number of players in his neighborhood, including himself, who play action 1 is two or three. Clearly, such a player will play action 1 in every equilibrium. By a similar argument as above, a player who observes four players who play action 1 in every equilibrium will himself play action 0 in every equilibrium.

Consider the graphical game Γ with eight players and neighborhood of size four given by Figure 6. We will argue that in every pure equilibrium of this game, *exactly* two players $i, j \in N$ play action 0 and $i - j = 2$ (mod 8). We exploit the following properties of the neighborhood graph:

1. For any $N' \subseteq N$, $|N'| = 3$, there exists a player $i \in N$ such that $N' \subseteq \hat{\nu}(i)$. Due to the rotational symmetry of the neighborhood graph, we can assume w.l.o.g. that $1 \in N'$. The property then follows by a straightforward if somewhat tedious case analysis.

2. For any $N' \subseteq N$, $|N'| = 3$, there exists a player $i \in N$ such that $|N' \cap (\hat{\nu}(i))| = 2$. Showing this property is again straightforward by assuming w.l.o.g. that $1 \in N'$ and showing that for any pair of distinct players, there exists a player $i \in N$ such that either $\hat{\nu}(i)$ contains player 1 and exactly one element of the pair, or both elements of pair but not player 1.

3. For any $N' \subseteq N$, $|N'| = 4$, there exists a player $i \in N$ such that $|N' \cap (\hat{\nu}(i))| = 3$. To show this property, we can again assuming w.l.o.g. that $1 \in N'$, and distinguishing neighborhoods that contain player 1 from neighborhoods that do not. The analysis is again straightforward.

Now consider an equilibrium s of Γ, and observe that, due to the structure of the payoffs, it must be the case that $p_i(s) = 1$ for all $i \in N$. If $\#(s, 0) < 2$ or $\#(s, 1) < 2$, then there exists a player $i \in N$ such that $\#(s_{\hat{\nu}(i)}, 0) = 0$ or $\#(s_{\hat{\nu}(i)}, 1) = 0$. Now consider the case $\#(s, 0) = 2$, and assume for contradiction that $s_i = 0$ for $i \in N \setminus \{1, 3, 7\}$. Inspection of the neighborhood graph reveals that in this case there exists a player $j \in N$ such that $\#(s_{\nu(j)}, 0) = 0$. If $\#(s, 0) = 3$, then by Property 1 there must exist a player $i \in N$ such that $\#(s_{\hat{\nu}(i)}, 0) = 3$ and thus $\#(s_{\hat{\nu}(i)}, 1) = 2$, contradicting the assumption that s is an equilibrium. By Property 3, the same holds if $\#(s, 0) = 4$. If $\#(s, 0) = 5$ and thus $\#(s, 1) = 3$, then by Property 2 there must yet again exists a player $i \in N$ such that $\#(s, 1) = 2$, a contradiction. The same trivially holds if $\#(s, 1) = 2$.

Now we augment Γ by a set $\{9, \dots, 13\}$ of five additional players such that

$$\nu(i) = \begin{cases} \{1, 3, 5, 7\} & \text{if } i \in \{9, 10\} \\ \{2, 4, 6, 8\} & \text{if } i \in \{11, 12\} \\ \{9, 10, 11, 12\} & \text{if } i = 13. \end{cases}$$

By construction of the original game with eight players, every pure equilibrium has either two or four players in the common neighborhood of players 9 and 10 play action 1. Furthermore, if players 9 and 10 observe two players who play action 1, then players 11 and 12 will observe four players who play action 1, and *vice versa*. As a consequence, either players 9 and 10 will play action 0, and players 11 and 12 will play action 1, or the other way round. In any case, exactly two players in the neighborhood of player 13 will play action 1 in every equilibrium of the augmented game, and player 13 himself will therefore play action 1.

In the following, we will denote by $\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3 \in N$ three players who play action 0 in every equilibrium, and by $\mathbf{1}_1, \mathbf{1}_2 \in N$ two players that constantly play action 1. Using these players to prune the payoff table, we will proceed to design games that simulate Boolean circuits. These games will satisfy strong anonymity, and the payoff of all players will therefore be determined by the table already used above and shown in Figure 6. As for the inputs of the circuit, it is easily verified that a game with players N, $|N| = 5$, such that for all $i \in N$, $\hat{\nu}(i) = N$, has pure equilibria s and s' such that for an arbitrary $i \in N$, $s_i = 0$ and $s'_i = 1$.

As before, we will now construct a subgame that simulates a functionally complete Boolean gate, in this case NOR, and a subgame that has a pure equilibrium if and only if a particular player plays action 1. For a set N of players with appropriately defined neighborhoods, let $\Gamma(N) = (N, \{0, 1\}^N, (p_i)_{i \in N})$ be a graphical game with neighborhood ν and payoff functions p_i satisfying strong anonymity as in Figure 6. We observe the following:

1. Let N and N' be two sets of players with neighborhoods such that for all $i \in N$, $\nu(i) \subseteq N$, and for all $i \in N'$, $\nu(i) \subseteq N'$. Again, s is a pure equilibrium of $\Gamma(N \cup N')$ if and only if s_N and $s_{N'}$ are pure equilibria of $\Gamma(N)$ and $\Gamma(N')$, respectively.

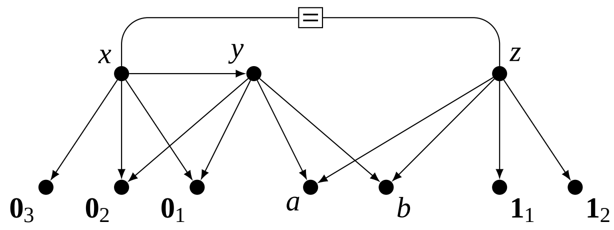


Figure 7: NOR gadget. Payoffs are identical to those in Figure 6. A construction analogous to Figure 3 is used to ensure that players x and z play the same action in every pure equilibrium.

2. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, let $a, b \in N$, and consider two additional players $x, y \notin N$ with $\nu(x) = \{\mathbf{0}_1, \mathbf{0}_2, a, y\}$, and $\nu(y) = \{\mathbf{0}_1, \mathbf{0}_2, b, x\}$. Then every pure equilibrium of $\Gamma(N \cup \{x, y\})$ satisfies $s_a = s_b$.

3. Letting $b = \mathbf{1}_1$ in the previous construction, we have that $\Gamma(N \cup \{x, y\})$ has a pure equilibrium if and only if $s_a = 1$ in some pure equilibrium of Γ.

4. Let N be a set of players such that $\Gamma(N)$ has a pure equilibrium, let $a, b \in N$, and consider two additional players $x, y \notin N$ with neighborhoods given by $\nu(x) = \{\mathbf{0}_1, \mathbf{0}_2, \mathbf{0}_3, y\}$ and $\nu(y) = \{\mathbf{0}_1, \mathbf{0}_2, a, b\}$. Then $\Gamma(N \cup \{x, y\})$ has a pure equilibrium, and every pure equilibrium s of $\Gamma(N \cup \{x, y\})$ satisfies $s_x = 1$ whenever $s_a = s_b = 0$, and $s_x = 0$ whenever $s_a \neq s_b$. For every pure equilibrium s with $s_a = s_b = 1$, there exists a pure equilibrium s' such that $s_x \neq s'_x$, and $s_i = s'_i$ for all $i \in N$.

5. Consider an additional player $z \notin N \cup \{x, y\}$, and let $\nu(z) = \{\mathbf{1}_1, \mathbf{1}_2, a, b\}$. Then $\Gamma(N \cup \{x, y, z\})$ has a pure equilibrium, and every pure equilibrium s of $\Gamma(N \cup \{x, y, z\})$ satisfies $s_z = 1$ whenever $s_a = s_b = 0$, and $s_z = 0$ whenever $s_a = s_b = 1$. For every pure equilibrium s with $s_a \neq s_b$, there exists a pure equilibrium s' such that $s_z \neq s'_z$, and $s_i = s'_i$ for all $i \in N$.

6. By Property 2, we can assume that every equilibrium s of $\Gamma(N \cup \{x, y, z\})$ satisfies $s_x = s_z$, and thus that $s_x = 1$ if and only if $s_a = s_b = 0$.

Steps 4 through 6 are illustrated in Figure 7.

Now consider an instance C of CSAT. We assume w.l.o.g. that C consist exclusively of NOR gates and that no variable appears more than once as an input to the same gate. The latter assumption can be made since $\varphi\,\mathrm{NOR}\,\mathit{false} = \neg\varphi$, and since there exists a game with strong anonymity and a player in this game who plays action 0 in every pure equilibrium. As before, we construct a game Γ by simulating every gate of C according to Property 6 and identifying the player that corresponds to the output of the circuit with a in Property 3. It is now readily appreciated that Γ has a pure equilibrium if and only if C is satisfiable. □

Observing that in the constructions used in the proofs of Theorems 1, 2, and 4 there is a one-to-one correspondence between satisfying assignments of a Boolean circuit and pure equilibria of a game, we have that counting the number of pure equilibria in the respective games is as hard as computing the permanent of a matrix.

Corollary 4 *For graphical games with neighborhoods of size two, counting the number of pure Nash equilibria is #P-hard, even when restricted to games with strong symmetry and two different payoffs, to games with weak anonymity with two different payoffs and two different payoff functions, or to games with strong anonymity and three different payoffs. The same holds for graphical games with neighborhoods of size four, strong anonymity, and two different payoffs*

4 Interlude: Generalized Satisfiability in the Presence of Matchings

The analysis at the end of the previous section allows us to derive a corollary that may be of independent interest. Schaefer (1978) completely characterizes which variants of the generalized satisfiability problem are in P and which are NP-complete. Some of the variants, like not-all-equal three-satisfiability, become tractable if there exists a matching, *i.e.*, a bijective mapping from variables to clauses. This follows from the equivalence of this problem with two-colorability of three-uniform hypergraphs and from the work of Robertson et al. (1999). On the other hand, the proof of Theorem 4 identifies a variant that is NP-complete and remains so in the presence of matchings. We thus have the following.

Corollary 5 *Generalized satisfiability is NP-complete, even if there exists a matching and all clauses have size five.*

We leave a complete characterization for future work. While the proof techniques developed in this paper will certainly be useful in this respect, it should be noted that the equivalence between generalized satisfiability and the pure equilibrium problem covered by Theorem 4 may fail to hold for instances of the latter where $p_i(s) = p_i(s') = 0$ for s, s' such that $\#(s_{\hat{\nu}(i)}, 1) = \#(s'_{\hat{\nu}(i)}, 1) + 1$.

5 Mixed Equilibria

Let us now briefly look at the problem of finding a mixed equilibrium. The following theorem states that this problem is tractable in graphical games with strong symmetry if the number of actions grows slowly in the neighborhood size.

Theorem 5 *Let* $\Gamma = (N, A^N, (p_i)_{i\in N})$ *be a graphical game with strong symmetry such that for all* $i \in N$, $A = O(\log|\nu(i)|/\log\log|\nu(i)|)$*. Then, a Nash equilibrium of* Γ *can be computed in polynomial time.*

Proof: We show that Γ possesses a symmetric equilibrium, *i.e.*, one where all players play the same (mixed) strategy, and that this equilibrium can be computed efficiently. For this, choose an arbitrary player $i \in N$ and construct a game $\Gamma' = (N', A^{N'}, (p'_i)_{i\in N})$ with players $N' = \hat{\nu}(i)$, and for all $j \in N'$, $\nu(j) = N'$, and $p'_j(s') = p_i(s)$ if for all $a \in A$, $\#(s', a) = \#(s_{\nu(i)}, a)$. It is easily verified that Γ' is strongly symmetric game, and must therefore possess a symmetric equilibrium s', which can be computed in polynomial time if $|A| = O(\log|N'|/\log\log|N'|)$ (Papadimitriou and Roughgarden, 2005).

Now define a strategy profile s of Γ by letting, for each $j \in N$, $s_j = s'_i$, and assume for contradiction that s is *not* an equilibrium. Then there exists a player $j \in N$ and some strategy $t \in \Delta(A)$ for this player such that $p_j(s_{N\setminus\{j\}}, t) > p_j(s)$. Then, by definition of p', $p'_i(s'_{N'\setminus\{i\}}, t) > p'_i(s)$, contradicting the assumption that s' is an equilibrium of Γ'. □

Observe that the above theorem applies in particular to the case where both the number of actions and the neighborhood size are bounded, and recall that the pure equilibrium problem in graphical games with strong symmetry is NP-complete even if $k = 2$. In other words, we have identified a class of games where computing a mixed equilibrium is computationally easier than deciding the existence of a pure one, unless P=NP. A different class of games with the same property is implicit in Theorem 3.4 of Daskalakis and Papadimitriou (2005).

6 Open Problems

In this paper we have mainly considered neighborhoods of constant size. The construction used in the proof of Theorem 4 can be generalized to arbitrary neighborhoods of even size. It is unclear what happens for odd-sized neighborhoods. The extreme case when the neighborhood of every player consists of all other players yields ordinary symmetric games, and it is known from the work of Brandt et al. (2007) that the pure equilibrium problem is in P in these games when the number of actions is bounded. It is an open problem at what neighborhood size the transition between membership in P and NP-hardness occurs.

Another interesting question concerns the complexity of the mixed equilibrium problem games with weak symmetry or weak anonymity. A promising direction for proving hardness would be to make the construction of Goldberg and Papadimitriou (2006) symmetric.

Finally, it would be interesting to study the complexity of generalized satisfiability problems in the presence of matchings.

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