A Note on the Distance to Monotonicity of Boolean Functions

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Abstract

Given a function $f : \{0,1\}^n \to \{0,1\}$, let $\epsilon_M(f)$ denote the smallest distance between f and a monotone function on $\{0,1\}^n$. Let $\delta_M(f)$ denote the fraction of hypercube edges where f violates monotonicity. We give an alternative proof of the tight bound: $\delta_M(f) \geq \frac{2}{n}\epsilon_M(f)$ for any boolean function f. This was already shown by Raskhodnikova in [Ras99].

Let $\mathcal{U}$ be a set of objects and let $\mathcal{P} \subseteq \mathcal{U}$ be a property of the elements of $\mathcal{U}$. For many natural definitions of $\mathcal{U}$ and $\mathcal{P}$, an object in $\mathcal{U}$ that is "globally" far from being in $\mathcal{P}$ also exhibits many "local" discrepancies. Thus, to test whether an object is globally far from being in $\mathcal{P}$, one often only needs to make a few local checks for discrepancies. In this note, we characterize the relationship between global and local farness with respect to the property of monotonicity of boolean functions.

First, we fix some notation. For two elements $x, y \in \{0,1\}^n$, x is said to be less than y, or $x \prec y$, if $x \neq y$ and for all $i \in [n]$, $x_i \leq y_i$. We view the set $\{0,1\}^n$ as vertices of the n-dimensional hypercube graph. An edge (x,y) in this graph denotes a pair of strings x and y such that $x \prec y$ and the Hamming distance between x and y is exactly 1. Note that the number of edges in $\{0,1\}^n$ is exactly $\frac{1}{2}n2^n$. For a function $f : \{0,1\}^n \to \{0,1\}$, we say that an edge (x,y) is violated by f if $x \prec y$ but $f(x) > f(y)$. The function f is monotone if and only if no edge in the hypercube is violated by f.

Now, let us define the following two quantities:

Definition 1. *For a function $f : \{0,1\}^n \to \{0,1\}$,*

- $\epsilon_M(f) \stackrel{def}{=} \min_g \Pr_{x \in \{0,1\}^n}[f(x) \neq g(x)]$*, where $g : \{0,1\}^n \to \{0,1\}$ ranges over all monotone functions*
- $\delta_M(f) \stackrel{def}{=} \Pr_{e \text{ edge in } \{0,1\}^n}[e \text{ violated by } f]$

$\epsilon_M(f)$ represents the global distance of f from the monotonicity property. $\delta_M(f)$ is a local distance measure corresponding to the following natural test of monotonicity (analyzed, e.g., in [DGL+99, GGL+00]): choose some random edges in the hypercube and check whether they are violated by f . The combinatorial question that now arises is the characterization of the relationship between the two distance measures. Goldreich *et al.* in [GGL+00] observed that this relationship is not simply determined; that is, $\epsilon_M(\cdot)$ is not just a function of $\delta_M(\cdot)$ or vice versa. In fact, they proved the following:

Theorem 2 (Proposition 4 of [GGL+00])**.** *For every $c < 1$, for any sufficiently large n, and for any α such that $2^{-c \cdot n} \leq \alpha \leq \frac{1}{2}$:*

1. *There exists a function $f : \{0,1\}^n \to \{0,1\}$ such that $\alpha \leq \epsilon_M(f) \leq 2\alpha$ and*

$$\delta_M(f) = \frac{2}{n}\epsilon_M(f)$$

2. *There exists a function $f : \{0,1\}^n \to \{0,1\}$ such that $(1 - o(1))\alpha \leq \epsilon_M(f) \leq 2\alpha$ and*

$$\delta_M(f) = (1 \pm o(1)) \cdot (1 - c) \cdot \epsilon_M(f)$$

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[GGL+00] show that the function in the second part essentially achieves the maximum δ_M relative to ϵ_M. They prove that, provided $\delta_M(f) > 2^{-o(n)}$ (or equivalently, $\epsilon_M(f) > 2^{-o(n)}$), $\delta_M(f) \leq (1 + o(1))\epsilon_M(f)$ which is appproximately what is achieved [1] in Theorem 2. The lowerbound in $\delta_M(f)$ is also known to be tight. Sofya Raskhodnikova in her Masters thesis ([Ras99]) showed that $\delta_M(f) \geq \frac{2}{n}\epsilon_M(f)$. We reprove this claim with a different proof that might be of independent interest.

Theorem 3. *For any* $f : \{0,1\}^n \to \{0,1\}$,

$$\delta_M(f) \geq \frac{2}{n}\epsilon_M(f)$$

Our proof of Theorem 3 takes a slightly different approach than the corresponding proof of Theorem 2 in [Ras99]. Both proofs proceed by showing that if there are not too many violated edges, then f can be made monotone by changing its value at only a few vertices, but the proof of [GGL+00] uses a switching/sorting operator while our proof shows more explicitly how to repair violated edges and is perhaps a little more intuitive in the sense that the claims in the proof require less "checking" and case analysis.

Proof of Theorem 3. Given a function $f : \{0,1\}^n \to \{0,1\}$ with at most $\epsilon 2^n$ violated edges, we show how to obtain a monotone function by modifying the value of f at only $\epsilon 2^n$ many vertices of the hypercube. Then, by the contrapositive, we will have that $\delta_M(f) \geq \frac{\epsilon_M(f)2^n}{n2^{n-1}} = \frac{2}{n}\epsilon_M(f)$.

The modification of f proceeds in n stages, where at each stage i, we repair the violated edges that lie[2] in direction i. That is, we define a sequence of boolean functions $g_0, g_1, \ldots, g_n$ such that g_0 equals f and the violated edges of each g_i lie in directions greater than i. So, the last function in the sequence, g_n, must be monotone.

Let us define g_i, given g_{i-1}, for some $i \in \{1, \ldots, n\}$. We assume inductively that g_{i-1} does not have any violated edges in directions $< i$. Let $H_i^0 = \{x : x_i = 0\}$ and let $H_i^1 = \{x : x_i = 1\}$. The edges $\{(x, x \oplus e_i) : x \in H_i^0\}$ are a perfect matching between H_i^0 and H_i^1. Let M_i be the subset of the edges of this matching which are violated by g_{i-1}. That is, if $(x, y) \in M_i$, $x_i = 0$ and $y_i = 1$ but $g_{i-1}(x) = 1$ and $g_{i-1}(y) = 0$. Additionally, we define the following sets of edges violated by g_{i-1}: $L_i = \{(x, y) : (x, y) \text{ violated by } g_{i-1}, x, y \in H_i^0\}$, $L_i^1 = \{(x, y) \in L_i : (x, x \oplus e_i) \in M_i\}$, $L_i^2 = L_i - L_i^1$, $R_i = \{(x, y) : (x, y) \text{ violated by } g_{i-1}, x, y \in H_i^1\}$, $R_i^1 = \{(x, y) \in R_i : (y \oplus e_i, y) \in M_i\}$, and $R_i^2 = R_i - R_i^1$. Now, g_i is defined as follows. If $|L_i^1| \leq |R_i^1|$, then:

$$g_i(x) = \begin{cases} 1 & \text{for } (x \oplus e_i, x) \in M_i \\ g_{i-1}(x) & \text{otherwise} \end{cases} \tag{1}$$

Otherwise, if $|L_i^1| > |R_i^1|$, then:

$$g_i(x) = \begin{cases} 0 & \text{for } (x, x \oplus e_i) \in M_i \\ g_{i-1}(x) & \text{otherwise} \end{cases} \tag{2}$$

So, g_i repairs g_{i-1} in the i'th dimension by changing g_{i-1} so that either all the edges in M_i have both endpoints labeled 1 or all the edges in M_i have both endpoints labeled 0. Furthermore, we will show that g_i does not introduce any new violated edges in directions less than i. Formally, we show the following:

[1] If the proviso $\delta_M > 2^{-o(n)}$ is not true, then the second part of Theorem 2 is not tight. Consider $\delta_M = 2^{-c \cdot n}$ for $c = \Theta(1)$. In this case, we can better the second item of Theorem 2; there exists a function f such that $\delta_M(f) = 2 \cdot (1 - H_2^{-1}(1 - c)) \cdot (1 \pm o(1)) \cdot \epsilon_M(f)$, where $H_2^{-1}(\cdot)$ is the inverse of the binary entropy function, $H_2(p) = p \log_2 1/p - (1-p) \log_2 1/(1-p)$, when restricted to $p \in [0, \frac{1}{2}]$. This function f has value 0 on all strings of weight 0, value 1 on all strings of weight 1, value 0 on all strings of weight 2, and so on, until there are $\delta_M \cdot n2^{n-1}$ violated edges (and then value 1 for rest of the strings). The same analysis as in the proof of Proposition 4 in [GGL+00] combined with the fact that $\sum_{i=0}^{kn} \binom{n}{i} \leq 2^{(H_2(k)+o(1))n}$ then yields the desired result. This construction can also be shown to be tight, that is, it maximizes δ_M relative to ϵ_M.

[2] An edge (x, y) is said to lie in direction i if x and y differ only at the i'th coordinate.

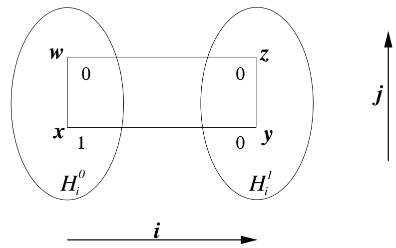

Figure 1: The values of g_{i-1} at the four vertices are labelled.

Lemma 4. *For any $g : \{0,1\}^n \to \{0,1\}$ and any $i \in [n]$, let $\Delta_i(g)$ be the number of edges lying in the i'th direction that are violated by g. Then:*

(i) *For all $k \in [i]$, $\Delta_k(g_i) = 0$.*

(ii) $\sum_{k=i+1}^n \Delta_k(g_i) \le \sum_{k=i+1}^n \Delta_k(g_{i-1})$

(iii) $\sum_x |g_i(x) - g_{i-1}(x)| = \Delta_i(g_{i-1})$

Observe that Lemma 4 implies our claim. By property (i), g_n is monotone, as desired. The number of modifications needed to transform f to g_n is $\sum_x |g_n(x) - g_0(x)| \le \sum_{i=1}^n \sum_x |g_i(x) - g_{i-1}(x)| \le \sum_{i=1}^n \Delta_i(g_{i-1})$ by property (iii). But by repeated application of property (ii), it follows that $\sum_{i=1}^n \Delta_i(g_{i-1}) \le \sum_{i=1}^n \Delta_i(g_0) \le \epsilon 2^n$. So, f can be converted to a monotone function by changing its value at $\epsilon 2^n$ vertices at most. All that remains now is to prove Lemma 4.

Proof of Lemma 4. Suppose $|L_i^1| \le |R_i^1|$ so that g_i is defined by equation (1); a symmetric argument works for the other case. Clearly, property (iii) is true, since $|M_i| = \Delta_i(g_{i-1})$ and g_i differs from g_{i-1} in only $|M_i|$ many vertices. To see properties (i) and (ii), observe the following. Any edge between two vertices in H_i^0 is violated by g_i iff it was violated by g_{i-1}; so the edges in L_i remain violated. Also, all the edges in R_i^2 are still violated by g_i because their endpoints are not incident to edges in M_i. However, the edges in R_i^1 are now *not* violated by g_i. This is so, because for any $(x, y) \in R_i^1$, it must have been that $g_{i-1}(x) = 1$ and $g_{i-1}(y) = 0$ but since $g_i(y) = 1$ and $g_i(x)$ remains 1, the edge is now unviolated. Finally, there are some edges violated by g_i that were not violated by g_{i-1}. Call this set of newly violated edges N_i. We will show that any edge in N_i must lie in a direction greater than i and that $|N_i| \le |L_i^1|$. This proves properties (i) and (ii). Property (i) holds because clearly $\Delta_i(g_i) = 0$ and none of the newly violated N_i lie in directions $k \le i$. Property (ii) holds because

$$\begin{aligned}\sum_{k=i+1}^n \Delta_k(g_i) &= |L_i| + |R_i^2| + |N_i| \\ &\le |L_i| + |R_i^2| + |L_i^1| \\ &\le |L_i| + |R_i^2| + |R_i^1| = |L_i| + |R_i| = \sum_{k=i+1}^n \Delta_k(g_{i-1})\end{aligned}$$

where the inequality in the last line is from the assumption that $|L_i^1| \le |R_i^1|$.

To see that any edge in N_i must lie in a direction greater than i and that $|N_i| \le |L_i^1|$, refer to Figure 1. Consider two edges $(x, y) \in M_i$ and $(y, z) \in N_i$. Let (y, z) lie in direction j. It must have been the case that $g_{i-1}(x) = 1$ and $g_{i-1}(y) = g_{i-1}(z) = 0$. For (y, z) to become violated, it must also be the case that z is not incident to an edge in M_i so that $g_i(z)$ is still 0. But this implies that $g_{i-1}(w) = 0$ where $w = x \oplus e_j$.

Therefore, $(x, w) \in L_i^1$. Since g_{i-1} does not violate any edges in directions less than i and (x, w) lies in direction j, we see that $j > i$. Also, we have shown a one-to-one mapping from edges in N_i to edges in L_i^1, namely the one that takes (y, z) to (x, w); so, $|N_i| \leq |L_i^1|$, completing our proof. □

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Note that our proof crucially exploits the fact that the edge expansion of the hypercube is 1. Very roughly speaking, it shows that if there are at least $\epsilon 2^n$ many vertices that need to be changed in order to make the function monotone, then the at least $\epsilon 2^n$ many edges incident to this set of vertices must be violated. We end this note with the outstanding open problem [DGL+99] in this area:

> *Establish a tight characterization of the relationship between the global and local distance measures of monotonicity for functions of the form* $f : \{0, 1\}^n \rightarrow [R]$ *where* $R > 2$ *is a positive integer.*

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References

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