A Note on the Distance to Monotonicity of Boolean Functions

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Abstract

Given a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), let \( \epsilon_M(f) \) denote the smallest distance between \( f \) and a monotone function on \( \{0,1\}^n \). Let \( \delta_M(f) \) denote the fraction of hypercube edges where \( f \) violates monotonicity. We give an alternative proof of the tight bound: \( \delta_M(f) \geq \frac{2}{n} \epsilon_M(f) \) for any boolean function \( f \). This was already shown by Raskhodnikova in [Ras99].

Let \( \mathcal{U} \) be a set of objects and let \( \mathcal{P} \subseteq \mathcal{U} \) be a property of the elements of \( \mathcal{U} \). For many natural definitions of \( \mathcal{U} \) and \( \mathcal{P} \), an object in \( \mathcal{U} \) that is “globally” far from being in \( \mathcal{P} \) also exhibits many “local” discrepancies. Thus, to test whether an object is globally far from being in \( \mathcal{P} \), one often only needs to make a few local checks for discrepancies. In this note, we characterize the relationship between global and local farness with respect to the property of monotonicity of boolean functions.

First, we fix some notation. For two elements \( x, y \in \{0,1\}^n \), \( x \) is said to be less than \( y \), or \( x \prec y \), if \( x \neq y \) and for all \( i \in [n] \), \( x_i \leq y_i \). We view the set \( \{0,1\}^n \) as vertices of the \( n \)-dimensional hypercube graph. An edge \((x,y)\) in this graph denotes a pair of strings \( x \) and \( y \) such that \( x \prec y \) and the Hamming distance between \( x \) and \( y \) is exactly 1. Note that the number of edges in \( \{0,1\}^n \) is exactly \( \frac{1}{2} n 2^n \). For a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), we say that an edge \((x,y)\) is violated by \( f \) if \( x \prec y \) but \( f(x) > f(y) \). The function \( f \) is monotone if and only if no edge in the hypercube is violated by \( f \).

Now, let us define the following two quantities:

**Definition 1.** For a function \( f : \{0,1\}^n \rightarrow \{0,1\} \),

- \( \epsilon_M(f) \) is the smallest distance between \( f \) and a monotone function on \( \{0,1\}^n \),

- \( \delta_M(f) = \frac{\Pr_{e \text{ edge in } \{0,1\}^n} [e \text{ violated by } f]}{\Pr_{x \in \{0,1\}^n} [f(x) \neq g(x)], \text{ where } g : \{0,1\}^n \rightarrow \{0,1\} \text{ ranges over all monotone functions}} \)

\( \epsilon_M(f) \) represents the global distance of \( f \) from the monotonicity property. \( \delta_M(f) \) is a local distance measure corresponding to the following natural test of monotonicity (analyzed, e.g., in [DGL+99, GGL+00]): choose some random edges in the hypercube and check whether they are violated by \( f \). The combinatorial question that now arises is the characterization of the relationship between the two distance measures. Goldreich et al. in [GGL+00] observed that this relationship is not simply determined; that is, \( \epsilon_M(\cdot) \) is not just a function of \( \delta_M(\cdot) \) or vice versa. In fact, they proved the following:

**Theorem 2** (Proposition 4 of [GGL+00]). For every \( c < 1 \), for any sufficiently large \( n \), and for any \( \alpha \) such that \( 2^{-c^n} \leq \alpha \leq \frac{1}{2} \):

1. There exists a function \( f : \{0,1\}^n \rightarrow \{0,1\} \) such that \( \alpha \leq \epsilon_M(f) \leq 2\alpha \) and

\[ \delta_M(f) = \frac{2}{n} \epsilon_M(f) \]

2. There exists a function \( f : \{0,1\}^n \rightarrow \{0,1\} \) such that \( (1 - o(1)) \alpha \leq \epsilon_M(f) \leq 2\alpha \) and

\[ \delta_M(f) = (1 - o(1)) \cdot (1 - c) \cdot \epsilon_M(f) \]

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[GGL+00] show that the function in the second part essentially achieves the maximum $\delta_M$ relative to $\epsilon_M$. They prove that, provided $\delta_M(f) > 2^{-o(n)}$ (or equivalently, $\epsilon_M(f) > 2^{-o(n)}$), $\delta_M(f) \leq (1 + o(1))\epsilon_M(f)$ which is approximately what is achieved in Theorem 2. The lower bound in $\delta_M(f)$ is also known to be tight. Sofya Raskhodnikova in her Masters thesis ([Ras99]) showed that $\delta_M(f) \geq \frac{2}{n}\epsilon_M(f)$. We reprove this claim with a different proof that might be of independent interest.

**Theorem 3.** For any $f : \{0, 1\}^n \rightarrow \{0, 1\}$,

$$\delta_M(f) \geq \frac{2}{n}\epsilon_M(f)$$

Our proof of Theorem 3 takes a slightly different approach than the corresponding proof of Theorem 2 in [Ras99]. Both proofs proceed by showing that if there are not too many violated edges, then $f$ can be made monotone by changing its value at only a few vertices, but the proof of [GGL+00] uses a switching/sorting operator while our proof shows more explicitly how to repair violated edges and is perhaps a little more intuitive in the sense that the claims in the proof require less “checking” and case analysis.

**Proof of Theorem 3.** Given a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ with at most $\epsilon 2^n$ violated edges, we show how to obtain a monotone function by modifying the value of $f$ at only $\epsilon 2^n$ many vertices of the hypercube. Then, by the contrapositive, we will have that $\delta_M(f) \geq \frac{\epsilon M(f) 2^n}{n} = \frac{2}{n}\epsilon M(f)$.

The modification of $f$ proceeds in $n$ stages, where at each stage $i$, we repair the violated edges that lie in direction $i$. That is, we define a sequence of boolean functions $g_0, g_1, \ldots, g_n$ such that $g_0$ equals $f$ and the violated edges of each $g_i$ lie in directions greater than $i$. So, the last function in the sequence, $g_n$, must be monotone.

Let us define $g_i$, given $g_{i-1}$, for some $i \in \{1, \ldots, n\}$. We assume inductively that $g_{i-1}$ does not have any violated edges in directions $< i$. Let $H^0_i = \{x : x_i = 0\}$ and let $H^1_i = \{x : x_i = 1\}$. The edges $\{(x, x \oplus e_i) : x \in H^0_i\}$ are a perfect matching between $H^0_i$ and $H^1_i$. Let $M_i$ be the subset of the edges of this matching which are violated by $g_{i-1}$. That is, if $(x, y) \in M_i$, $x_i = 0$ and $y_i = 1$ but $g_{i-1}(x) = 1$ and $g_{i-1}(y) = 0$. Additionally, we define the following sets of edges violated by $g_{i-1}$: $L_i = \{(x, y) : (x, y) \text{ violated by } g_{i-1}, x, y \in H^0_i\}$, $L^+_i = \{(x, y) \in L_i : (x, x \oplus e_i) \in M_i\}$, $L^-_i = L_i - L^+_i$, $R_i = \{(x, y) : (x, y) \text{ violated by } g_{i-1}, x, y \in H^1_i\}$, $R^+_i = \{(x, y) \in R_i : (y \oplus e_i, y) \in M_i\}$, and $R^-_i = R_i - R^+_i$. Now, $g_i$ is defined as follows. If $|L^+_i| \leq |R^-_i|$, then:

$$g_i(x) = \begin{cases} 1 & \text{for } (x \oplus e_i, x) \in M_i \\ g_{i-1}(x) & \text{otherwise} \end{cases}$$

(1)

Otherwise, if $|L^+_i| > |R^-_i|$, then:

$$g_i(x) = \begin{cases} 0 & \text{for } (x, x \oplus e_i) \in M_i \\ g_{i-1}(x) & \text{otherwise} \end{cases}$$

(2)

So, $g_i$ repairs $g_{i-1}$ in the $i$th dimension by changing $g_{i-1}$ so that either all the edges in $M_i$ have both endpoints labeled 1 or all the edges in $M_i$ have both endpoints labeled 0. Furthermore, we will show that $g_i$ does not introduce any new violated edges in directions less than $i$. Formally, we show the following:

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1If the proviso $\delta_M > 2^{-o(n)}$ is not true, then the second part of Theorem 2 is not tight. Consider $\delta_M = 2^{-c n}$ for $c = \Theta(1)$. In this case, we can better the second item of Theorem 2; there exists a function $f$ such that $\delta_M(f) = 2 \cdot (1 - H^1_2(1-c)) \cdot (1 + o(1)) \cdot \epsilon_M(f)$, where $H^1_2$ is the inverse of the binary entropy function, $H^1_2(p) = p \log_2 1/p + (1-p) \log_2 1/(1-p)$, when restricted to $p \in [0, \frac{1}{2}]$. This function $f$ has value 0 on all strings of weight 0, 1 on all strings of weight 1, 0 0 0 0 on all strings of weight 2, and so on, until there are $\delta_M \cdot n 2^{n-1}$ violated edges (and then value 1 for rest of the strings). The same analysis as in the proof of Proposition 4 in [GGL+00] combined with the fact that $\sum_{i=0}^{\log_2 n} (n) i \leq 2^{(H^1(2)+o(1))n}$ then yields the desired result. This construction can also be shown to be tight, that is, it maximizes $\delta_M$ relative to $\epsilon_M$.

2An edge $(x, y)$ is said to lie in direction $i$ if $x$ and $y$ differ only at the $i$th coordinate.
Lemma 4. For any \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) and any \( i \in [n] \), let \( \Delta_i(g) \) be the number of edges lying in the \( i \)’th direction that are violated by \( g \). Then:

(i) For all \( k \in [i] \), \( \Delta_k(g_i) = 0 \).

(ii) \( \sum_{k=i+1}^{n} \Delta_k(g_i) \leq \sum_{k=i+1}^{n} \Delta_k(g_{i-1}) \).

(iii) \( \sum_x |g_i(x) - g_{i-1}(x)| = \Delta_i(g_{i-1}) \).

Observe that Lemma 4 implies our claim. By property (i), \( g_0 \) is monotone, as desired. The number of modifications needed to transform \( f \) to \( g_n \) is \( \sum_x |g_n(x) - g_0(x)| \leq \sum_{i=1}^{n} \sum_x |g_i(x) - g_{i-1}(x)| \leq \sum_{i=1}^{n} \Delta_i(g_{i-1}) \) by property (iii). But by repeated application of property (ii), it follows that \( \sum_{i=1}^{n} \Delta_i(g_{i-1}) \leq \sum_{i=1}^{n} \Delta_i(g_0) \leq c2^n \). So, \( f \) can be converted to a monotone function by changing its value at \( c2^n \) vertices at most. All that remains now is to prove Lemma 4.

Proof of Lemma 4. Suppose \( |L_i^1| \leq |R_i^2| \) so that \( g_i \) is defined by equation (1); a symmetric argument works for the other case. Clearly, property (iii) is true, since \( |M_i| = \Delta_i(g_{i-1}) \) and \( g_i \) differs from \( g_{i-1} \) in only \( |M_i| \) many vertices. To see properties (i) and (ii), observe the following. Any edge between two vertices in \( H_0^i \) is violated by \( g_i \) if it was violated by \( g_{i-1} \); so the edges in \( L_i \) remain violated. Also, all the edges in \( R_i^2 \) are still violated by \( g_i \) because their endpoints are not incident to edges in \( M_i \). However, the edges in \( R_i^1 \) are now not violated by \( g_i \). This is so, because for any \( (x, y) \in R_i^1 \), it must have been that \( g_{i-1}(x) = 1 \) and \( g_{i-1}(y) = 0 \) but since \( g_i(y) = 1 \) and \( g_i(x) \) remains 1, the edge is now unviolated. Finally, there are some edges violated by \( g_i \) that were not violated by \( g_{i-1} \). Call this set of newly violated edges \( N_i \). We will show that any edge in \( N_i \) must lie in a direction greater than \( i \) and that \( |N_i| \leq |L_i^1| \). This proves properties (i) and (ii). Property (i) holds because clearly \( \Delta_i(g_i) = 0 \) and none of the newly violated \( N_i \) lie in directions \( k \leq i \).

Property (ii) holds because

\[
\sum_{k=i+1}^{n} \Delta_k(g_i) \leq |L_i| + |R_i^2| + |N_i|
\]

where the inequality in the last line is from the assumption that \( |L_i^1| \leq |R_i^2| \).

To see that any edge in \( N_i \) must lie in a direction greater than \( i \) and that \( |N_i| \leq |L_i^1| \), refer to Figure 1. Consider two edges \( (x, y) \in M_i \) and \( (y, z) \in N_i \). Let \( (y, z) \) lie in direction \( j \). It must have been the case that \( g_{i-1}(x) = 1 \) and \( g_{i-1}(y) = g_{i-1}(z) = 0 \). For \( (y, z) \) to become violated, it must also be the case that \( z \) is not incident to an edge in \( M_i \) so that \( g_i(z) \) is still 0. But this implies that \( g_{i-1}(w) = 0 \) where \( w = x \oplus e_j \).
Therefore, \((x, w) \in L_i^1\). Since \(g_{i-1}\) does not violate any edges in directions less than \(i\) and \((x, w)\) lies in direction \(j\), we see that \(j > i\). Also, we have shown a one-to-one mapping from edges in \(N_i\) to edges in \(L_i^1\), namely the one that takes \((y, z)\) to \((x, w)\); so, \(|N_i| \leq |L_i^1|\), completing our proof.

Note that our proof crucially exploits the fact that the edge expansion of the hypercube is 1. Very roughly speaking, it shows that if there are at least \(\epsilon 2^n\) many vertices that need to be changed in order to make the function monotone, then the at least \(\epsilon 2^n\) many edges incident to this set of vertices must be violated. We end this note with the outstanding open problem [DGL+99] in this area:

*Establish a tight characterization of the relationship between the global and local distance measures of monotonicity for functions of the form \(f : \{0, 1\}^n \rightarrow \mathbb{R}\) where \(R > 2\) is a positive integer.*

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**References**

