Electronic Colloquium on Computational Complexity, Report No. 47 (2008)

Continuity Properties of Equilibria in Some Fisher and Arrow-Debreu Market Models

Vijay V. Vazirani[*] *Lei Wang*[†]

Abstract

Following up on the work of Megiddo and Vazirani [7], who determined continuity properties of equilibrium prices and allocations for perhaps the simplest market model, Fisher's linear case, we do the same for:

- Fisher's model with piecewise-linear, concave utilities
- Fisher's model with spending constraint utilities
- Arrow-Debreu's model with linear utilities
- Arrow-Debreu's model with piecewise-linear, concave utilities

1 Introduction

Three basic properties that a desirable model of an economy should possess are existence, uniqueness, and continuity of equilibria (see [3], Chapter 15, "Smooth preferences"). These lead to parity between supply and demand, stability, and predictive value, respectively. In particular, without continuity, small errors in the observation of parameters of an economy may lead to entirely different predicted equilibria.

Although mathematical economists studied very extensively questions of existence and uniqueness for several concrete and realistic market models, the question of continuity was studied only in very abstract settings. For example, Debreu [3] (Chapter 19, "The application to economies of differential topology and global analysis: regular differentiable economies") assumed that demand functions of agents are continuously differentiable and, using differential topology, showed that the set of "bad" economies is of Lebesgue measure zero if the set of economies is finite-dimensional.

Megiddo and Vazirani [7] attempted to rectify this situation by starting with perhaps the simplest market model – the linear case of Fisher's model. An instance of this market is specified

*Address: College of Computing, Georgia Institute of Technology, Atlanta, GA 30332–0280, E-mail: vazirani@cc.gatech.edu.

†Address: Mathematics Department, Georgia Institute of Technology, Atlanta, GA 30332–0280, E-mail: wanglei@math.gatech.edu.

ISSN 1433-8092

by giving the initial amounts of money held by the agents, m and their utility functions, U. They showed that the mapping $p(m, U)$ giving the unique vector of equilibrium prices is continuous. They also showed that $X(m, U)$, the correspondence giving the set of equilibrium allocations, is upper hemicontinuous, but not lower hemicontinuous.

In this paper, we determine continuity properties of equilibrium prices and allocations for:

- Fisher's model with piecewise-linear, concave utilities
- Fisher's model with spending constraint utilities
- Arrow-Debreu's model with linear utilities
- Arrow-Debreu's model with piecewise-linear, concave utilities

Our results for these models are summarized in the following table. We note that among these cases, only Fisher's model with spending constraint utilities supports unique equilibrium prices. For the remaining cases, we need to consider the correspondence giving the set of equilibirum prices, and we need to establish its upper and lower hemicontinuity.

	Price	Allocation
Fisher+PL	not upper and not lower	not upper and not lower
Fisher+SC	continuous(unique)	upper but not lower
AD+L	not upper and not lower	upper but not lower
AD+PL	not upper and not lower	not upper and not lower

[7] crucially used the Eisenberg-Gale convex program [5] for proving their results. The optimal solution to this program gives equilibrium allocations for Fisher's linear model, and the optimal values of the Lagrange variables give equilibirum prices. Naturally, their proofs were steeped in polyhedral combinatorics. For the Arrow-Debreu model with linear utilities, we use a convex program due to [6].

For the remaining three cases, such convex programs are not known, and instead, we use the combinatorial structure of equilibria for proving our theorems. The groundwork for discovering such structure was laid in [4] in the context of obtaining a polynomial time algorithm for computing the equilibrium the linear case of Fisher's model. Each of the remaining three cases is a generalization of this case and the relevant structure is also a generalization of that for the Fisher's linear case. For Fisher's model with piecewise-linear, concave utilities, we use structure found by [9] (and used for proving that equilibrium prices and allocations are rational numbers). For Fisher's model with spending constraint utilities, we use structure found by [8] (and used for obtaining a polynomial time algorithm for computing the equilibrium).

The recent surge in interest on obtaining efficient algorithms for computing market equilibria is motivated in part by potential applications to electronic commerce on the Internet. In particular, the spending constraint model has been shown to be applicable to Google's AdWords market – it provides rich expressivity while at the same time maintaining simplicity and polynomial time solvability [8]. Clearly, a good understanding of continuity properties of various market models can go a long way in such applications.

Perhaps the most startling, and unsettling, finding of our work is that unlike the linear case of Fisher's model, the cases we have studied do not have good continuity properties. This raises the question of determining whether equilibria in these very basic models are robust in some other sense.

2 Market Models and Definitions

2.1 Fisher's model

Fisher's model [2] is the following: Consider a market consisting of a set B of *buyers* and a set A of divisible *goods*. Assume $|A| = n$ and $|B| = n'$. We are given for each buyer i the amount e_i of money she possesses and for each good j the amount b_j of this good. In addition, we are given the utility functions of the buyers, which are assumed to be additively separable. Let $u_{ij} : R \rightarrow R$ specify the utility derived by i as a function of the amount of good j she gets. If the latter is denoted by x_{ij}, the total utility derived by i is

$$U_i(x) = \sum_{j \in A} u_{ij}(x_{ij}).$$

Given prices $p_1, \ldots, p_n$ of the goods, one can compute baskets of goods (there could be many) that maximize i's utility, subject to her budget constraint of e_i. We will say that $p_1, \ldots, p_n$ are *market clearing* prices if after each buyer is assigned such a basket, there is no surplus or deficiency of any of the goods. The problem is to compute such prices.

2.2 The Arrow-Debreu model

The Arrow-Debreu model [1] generalizes Fisher's model in that there is no demarcation between buyers and sellers. Suppose there are n agents A and m goods G (money may be one of them). Each agent comes to the market with an initial endowment of these m goods. Once the prices of the goods are fixed, each agent can compute the worth of her initial endowment and baskets of goods she can obtain with it so as to maximize her utility. The problem again is to find prices so that the market clears.

2.3 Spending constraint utilities

We will consider 3 utility functions in this paper. The first two are well known: $u_{ij}(x_{ij})$ is either a (homogeneous) linear function or a piecewise-linear, concave function. Linear utilities have several deficiencies: typically, a buyer may end up spending all her money on a single item, and they fail to capture the important condition of buyers getting satiated with goods. Piecewise-linear, concave functions rectify both these deficiencies; however, unlike the linear case, no efficient algorithms are known for computing equilibria for such utilities.

The third utilities we consider are spending constraint utility functions, introduced in [8]. They rectify some of the deficiencies of linear utility functions and at the same time are amenable to efficient algorithms. We extend Fisher's model via spending constraint utilities as follows. For $i \in B$ and $j \in A$, let $f_j^i : [0, e_i] \to R$ be the *rate function* of buyer i for good j; it specifies the rate at which i derives utility per unit of j received, as a function of the amount of her budget spent on j. If the price of j is fixed at p_j per unit amount of j, then the function f_j^i/p_j gives the rate at which i derives utility per dollar spent, as a function of the amount of her budget spent on j. Define $g : [0, e_i] \to R$ as follows:

$$g(x) = \int_0^x \frac{f_j^i(y)}{p_j} dy$$

This function gives the utility derived by i on spending x dollars on good j at price p_j.

Observe that if f_j^i is a decreasing step function, g will be a piecewise-linear, concave function (the linear version of Fisher's problem is the special case in which each f_j^i is the constant function).

2.4 Continuity of set-valued functions

The concept of continuity is generalized to set valued functions in the following way:

Definition 1 *Suppose $f : A \to 2^S$ is a set value function. f is said to be upper hemicontinuous, if given any sequence $\{a^k\}_{k\geq 1}$ which has limit a^0 and for any sequence $\{x^k\}_{k\geq 1}$ such that $x^k \in f(a^k)$, there exists a convergent subsequence $\{x^{k_j}\}_{j\geq 1}$ such that $lim_{j\to\infty} x^{k_j} = x^0 \in f(a^0)$.*

Definition 2 *A set value function f is said to be lower hemicontinuous, if given any sequence $\{a^k\}_{k\geq 1}$ which has limit a^0 and for each element $x^0 \in f(a^0)$, there exists a sequence $\{x^k\}_{k\geq 1}$ such that $x^k \in f(a^k)$ and $x^k \to x^0$*

In this paper, we will study these properties of the set of equilibrium prices and allocations when they are not unique.

3 Piecewise-linear, Concave utilities for Fisher and Arrow-Debreu Models

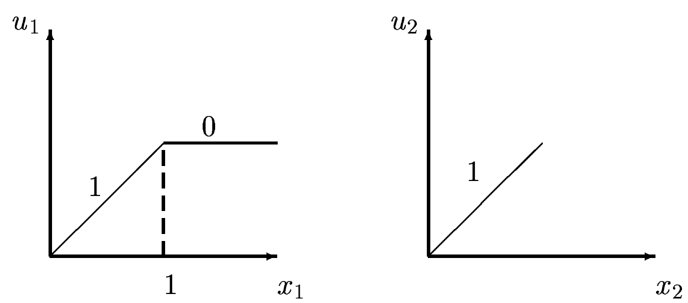
In this section, we provide examples to show:

Theorem 3 *In Fisher's model with piecewise-linear, concave utilities:*
(1) The set of equilibrium prices is not upper or lower hemicontinuous;
(2) The set of equilibrium allocations is not upper or lower hemicontinuous.

Similar results for the Arrow-Debreu model follow since the latter is a generalization of Fisher's model.

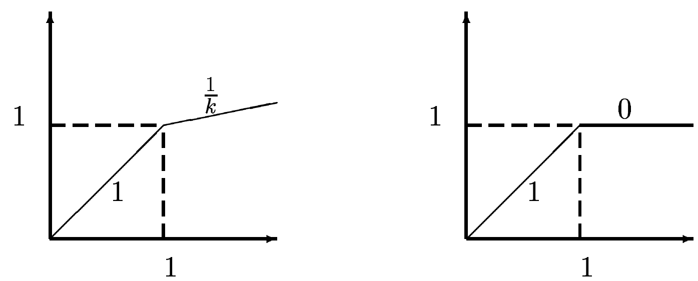
3.1 Equilibrium prices

Suppose in the market, there is only one person with money m and there are two goods. There is 1 unit of good 1 and 1 unit of good 2. Suppose u_1 is her utility function of good 1 and u_2 is her utility function of good 2 (see the picture). The slopes are indicated in the picture. Suppose the price for good 1 is p_1 and the price for good 2 is p_2. Then $p = (p_1, p_2)$ is an equilibrium price if and only if $p_1 > 0, p_2 > 0$, $\frac{1}{p_1} \geq \frac{1}{p_2}$ and $p_1 + p_2 = m$.



Now we take $p^k = (\frac{1}{k}, m - \frac{1}{k})$. Thus for each k, p^k is an equilibrium price. However, $p^k \to (0, m)$ which is not an equilibrium price. Thus the set of equilibrium prices is not closed, hence not upper hemicontinuous.

Next we provide an example to show that the set of equilibrium prices is not lower hemicontinuous. There is one buyer with money \$1 and there are two goods in the market. Suppose $\{U^k\}$ is a sequence of utilities and $U^k \to U^0$. The picture of U^k is the following:

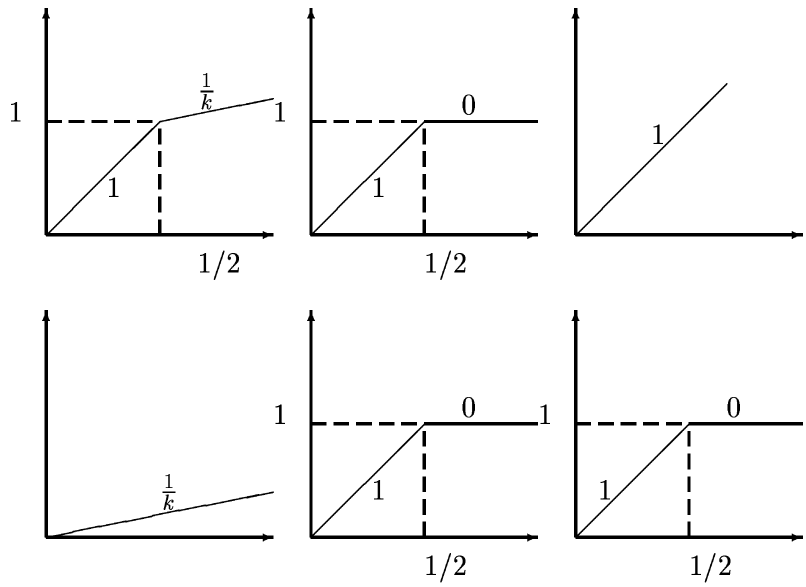


The slopes of the line segments are indicated in the picture. Now for each k, it is easy to see $p^k = (p_1^k, p_2^k)$ is an equilibrium price if and only if $p_2^k \geq p_1^k$ and $p_1^k + p_2^k = 1$. However, $p^0 = (p_1^0, p_2^0)$ is an equilibrium price for U^0 if and only if $p_1^0 + p_2^0 = 1$ (i.e. $p_1^0 > p_2^0$ is allowed). Thus we have some price vector, for instance (0.9,0.1), that can not be the limit of a sequence of equilibrium prices for U^k. Hence the set of equilibrium prices is not lower hemicontinuous.

3.2 Equilibrium allocations

In [7], the authors showed that in Fisher's model for linear utilities, the allocation is not lower hemicontinuous. Because the piecewise-linear case is a generalization of the linear case, the allocation is still not lower hemicontinuous. In this section, we provide an example to show the equilibrium allocation is not upper hemicontinuous.

Consider the following example. There are two buyers and three goods in the market. Suppose we have $\{(m^k, U^k)\}_{k\geq 0}$ where $m^k = (1, 1 + \frac{1}{k})$ and the picture of U^k is the following:



The slopes of the line segments are indicated in the picture. It is easy to see $p^k = (\frac{1}{k}, 1, 1)$ and $x^k = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{k} & \frac{1}{2} \\ 0 & \frac{1}{2} + \frac{1}{k} & \frac{1}{2} \end{pmatrix}$ are equilibrium price and allocation. However, $x^k \to x^0 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, which is not an equilibrium allocation for $(m^0, U^0) = \lim_{k\to\infty}(m^k, U^k)$, because if buyer 1 will prefer spending her money on good 3 to spending on more than 1/2 of good 1. under (m^0, U^0), $p = (\frac{1}{2}, \frac{1}{2}, 1)$ and $x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ are equilibrium price and allocation. Therefore the equilibrium allocation is not upper hemicontinuous.

4 Linear case of Arrow-Debreu Model

We start by giving a nonconvex and a convex program from [6] that capture equilibrium allocations for this model.

$$\forall j : \sum_i x_{i,j} = 1,$$
$$\forall i, j : x_{i,j} \geq 0,$$

$$\forall i,j: \frac{u_{i,j}}{p_j} \leq \frac{\sum_{1\leq t\leq n} u_{i,t}x_{i,t}}{p_i},$$
$$\forall i: p_i > 0$$

Theorem 4 *([6]) The feasible region of the above nonconvex program has all and only market equilibria.*

[6] also gives a convex program whose feasible solution gives all and only the equilibrium allocations. For any given utilities, construct a directed graph G with n vertices to represent the n people. Draw an edge from i to j if $u_{i,j} > 0$. Define $w(i,j) = \frac{\sum_{1\leq t\leq n} u_{i,t}x_{i,t}}{u_{i,j}}$. Then consider the following convex program:

$$\forall j: \sum_i x_{i,j} = 1,$$
$$\forall i,j: x_{i,j} \geq 0,$$
$$For\ every\ cycle\ C\ of\ G: \prod_{(i,j)\in C} w(i,j) \geq 1.$$

Theorem 5 *([6]) The feasible region of the above convex program has all and only equilibrium allocations.*

We will use these two theorems to prove the following:

Theorem 6 *In the Arrow-Debreu model with linear utilities:*
(1)The set of equilibrium allocations is upper but not lower hemicontinuous;
(2)The set of equilibrium prices is not upper or lower hemicontinuous.

4.1 Equilibrium allocations

In this section, we prove the set of equilibrium allocations is upper but not lower hemicontinuous.

Given the utility functions, we write the $u_{i,j}$'s in a matrix $U = (u_{i,j})$. From now on we view the price as a vector and the allocation as a matrix. The equilibrium prices form a set $P(U)$ and the equilibrium allocations form a set $X(U)$.

Suppose $\{U^k = (u^k_{i,j})\}_{k\geq 1}$ is a sequence of utilities and $U^k \to U^0$ when k approaches ∞. Suppose $\{x^k\}_{k\geq 1}$ is a sequence of allocations such that $x^k \in X(U^k)$ and $x^k \to x^0$. We want to show $x^0 \in X(U^0)$. By Theorem 5, we only need to prove x^0 is a feasible solution of the convex program.

Because $x^k \in X(U^k)$, by Theorem 5, we have for every k, x^k is a feasible solution to the convex program. Therefore, for each k, we have a graph G^k which is formed by all the edges (i,j) where $u^k_{i,j} > 0$. Thus we have:

$$\forall j: \sum_i x^k_{i,j} = 1, \tag{1}$$

$$\forall i,j : x^k_{i,j} \geq 0, \tag{2}$$

$$For\ every\ cycle\ C\ of\ G^k : \prod_{(i,j)\in C} w^k(i,j) \geq 1. \tag{3}$$

where $w^k(i,j) = \frac{\sum_{1\leq t\leq n} u^k_{i,t} x^k_{i,t}}{u^k_{i,j}}$.

Now for U^0, consider the graph G^0. In G^0, there is an edge between a pair of vertices (i,j) if and only if $U^0_{i,j} > 0$. Because $U^k \to U^0$, when k is large enough, for every (i,j) such that $u^0_{i,j} > 0$, we have $u^k_{i,j} > 0$. Therefore if G^0 has an edge (i,j), then G^k also has the edge (i,j). This implies if C is a cycle in G^0, then C is also a cycle in G^k. Therefore x^k satisfies a weaker condition than (3):

$$For\ every\ cycle\ C\ of\ G^0 : \prod_{(i,j)\in C} w^k(i,j) \geq 1. \tag{4}$$

Now in (1),(2) and (4), let k go to ∞, we have:

$$\forall j : \sum_i x^0_{i,j} = 1,$$
$$\forall i,j : x^0_{i,j} \geq 0,$$
$$For\ every\ cycle\ C\ of\ G^0 : \prod_{(i,j)\in C} w^0(i,j) \geq 1.$$

where $w^0(i,j) = \frac{\sum_{1\leq t\leq n} u^0_{i,t} x^0_{i,t}}{u^0_{i,j}}$. Thus by Theorem 5, we have $x^0 \in X(U^0)$. Thus $X(U)$ is upper hemicontinuous. □

The next example shows that the set of equilibrium allocations is not lower hemicontinuous. Suppose there are two people, each of them has one unit of distinct good. Suppose the utility matrix is $\begin{pmatrix} 1 & 1 \\ 1 & u \end{pmatrix}$ where $u \leq 1$. It is easy to see the equilibrium price is $(1,1)$ for each $u \leq 1$. When $u < 1$, there is only 1 equilibrium allocation: buyer 2 gets all the 1 unit of good 1 and buyer 1 gets all the 1 unit of good 2. However, when $u = 1$ there are infinitely many allocations. Thus the allocation is not lower hemicontinuous.

4.2 Equilibrium prices

If we scale an equilibrium price vector by a positive constant, we will still get an equilibrium price. In this sense, we are only interested in the equilibrium price whose l_1 norm is 1. Such an equilibrium price is called normalized. Now we provide examples to show that the set of normalized equilibrium prices in this model is not upper or lower hemicontinuous.

Consider the following example: There are two people, each of them has one good. Suppose $\{U^k\}_{k=1}^{\infty}$ is a sequence of utilities where $U^k = \begin{pmatrix} 1-\frac{1}{k} & 0 \\ 0 & 1-\frac{1}{k} \end{pmatrix}$. For each k, let $P(U^k)$ be the set of normalized equilibrium prices for utility U^k. Let $p^k = (\frac{1}{k}, 1 - \frac{1}{k})$. Then p^k is a normalized vector. Now we show the fact that $p^k \in P(U^k)$. Here we let $X^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to check that (p^k, X^k) satisfies all the restrictions in the nonconvex program. By Theorem 3, we have $p^k \in P(U^k)$. Since $U^k \to U^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but $p^k \to (0, 1) \notin P(U^0)$, we conclude that $P(U)$ is not upper hemicontinuous.

The next example shows the set of equilibrium prices is not lower hemicontinuous. Suppose the sequence of utilities is U^k where $U^k = \begin{pmatrix} 1 & \frac{1}{k} \\ \frac{1}{k} & 1 \end{pmatrix}$. It is easy to see p^k is an equilibrium price for U^k if and only if $p_1^k = p_2^k$. However, $U^k \to U^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For U^0, $p^0 = (p_1^0, p_2^0)$ is equilibrium if and only if $p_1^0 > 0$ and $p_2^0 > 0$. Hence the set of equilibrium prices is not lower hemicontinuous.

5 Fisher's Model with Spending Constraint Utilities

In this paper, we will deal with the case that f_j^i is a decreasing step function for each i, j. We first recall some definitions from [8]. We will call each step of f_j^i a *segment*. The set of segments defined in f_j^i is denoted by $seg(f_j^i)$. Suppose $s \in seg(f_j^i)$. If s ranges from a to b, then we define $good(s) = j$, $value(s) = b - a$, $rate(s)$ to be the value of f_j^i at segment s and $right(s)$ is the value of the right end of the segment s. Define $segments(i) = \cup_{j=1}^{n'} seg(f_j^i)$.

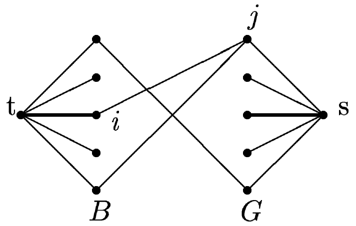
Given nonzero prices $p = (p_1, ..., p_{n'})$, we characterize optimal baskets for each buyer relative to p. Define the bang per buck relative to p for segment $s \in seg(f_j^i)$ to be $\frac{rate(s)}{p_j}$. Sort all segments $s \in segments(i)$ by decreasing bang per buck and partition by equality into classes: $Q_1, Q_2, ...$ For a class Q_l, define $value(Q_l)$ to be the sum of values of segments in it. Find k_i such that

$$\sum_{1 \leq l \leq k_i - 1} value(Q_l) < e_i \leq \sum_{1 \leq l \leq k_i} value(Q_l)$$

Thus buyer i will definitely want the goods corresponding to segments in $Q_1, ..., Q_{k_i - 1}$. We call these segments the *forced segments.* The buyer will buy some of the goods corresponding to segments in Q_{k_i}, we call these segments the *flexible segments.* The buyer will not buy any goods corresponding to segments in $Q_{k_i+1}, ...$, we call those segments the *undesirable segments.* By *forced allocations* we mean allocating all goods in the forced segments to the buyers.

Consider forced allocations made at price p. For each good j, define $\alpha(j)$ to be the sum over all the buyers of the money that they spend on good j. For each buyer i, we define $spend(i)$ to be the money she has spent on forced allocations. Define $m_i = e_i - spend(i)$ to be her left over money.

For a given money vector e and price vector p, let α be forced allocations. We can set up a network $N(e, p, \alpha)$ in the following way: There is a source s, a sink t, a set G of vertices representing the goods and a set B of vertices representing the buyers. For each $i \in B$ and segment $s \in Q_{k_i}$, suppose $good(s) = j$, then add edge (j, i) into the graph and set its capacity to be $value(s)$. For each $j \in G$, set the capacity of (s, j) to be $p_j - \alpha(j)$. For each $i \in B$, set the capacity of (i, t) to be m_i.



For any set of buyers $T \subseteq B$, we define $m(T) = \sum_{i \in T} m_i$. For any set $S \subseteq G$, we define $p(S) = \sum_{j \in S} p_j$ and $\alpha(S) = \sum_{j \in S} \alpha(j)$. Let $\Gamma(S) = \{i \in B : \exists j \in S, j \text{ is adjacent to } i\}$. Then we define the "best value" of the set S to be $best(S) = min\{m(T) + c(S, \Gamma(S) - T) : T \subseteq \Gamma(S)\}$ and let $bestT(S)$ be the optimal subset of $\Gamma(S)$. If $p(S) - \alpha(S) = best(S)$ then we say S is tight. If $p(S) - \alpha(S) > best(S)$ then we say S is overtight.

Given a price p, after each buyer is assigned an optimal basket of goods, it is easy to see that there is no good left if and only if in the network the cut $(t \cup B \cup G, s)$ is a min cut.

Lemma 7 *([8]) In $N(e, p, \alpha)$, $(t \cup B \cup G, s)$ is a min cut if and only if no subset of goods is overtight, i.e. $\forall S \subseteq G$, $p(S) - \alpha(S) \leq best(S)$.*

This lemma gives us a characterization of when $(t \cup B \cup G, s)$ is a min cut. Now by this network, we can tell when the price is an equilibrium price.

Lemma 8 *([8]) A price p is an equilibrium price if and only if $(t \cup B \cup G, s)$ and $(t, B \cup G \cup s)$ are both min cuts.*

Then in [8], the author gives a polynomial time algorithm to compute the equilibrium price and proves that the equilibrium price is unique

Theorem 9 *([8]) In Fisher's model with decreasing step spending constraint utility, the equilibrium price exists and is unique.*

5.1 Continuity of equilibrium prices

Let $p = p(m, U)$ be the equilibrium price as a function of money m and utility U in the spending constraint model. We prove:

Theorem 10 *p is a continuous function.*

In order to prove this, we only need to show that when the change of the money and the utility is small enough, the change of equilibrium price is also very small. We can view the change in the following point of view: We change the money of buyers one by one and after these changes, we change the utility of goods one by one. So we only need to consider two changes: (1) fix the utilities and change of buyer 1's money; (2) fix the money and change one of buyer 1's segments.

First of all, we fix the utility and consider raising the money of buyer 1 slightly. Now suppose $e_1' = e_1(1+\epsilon)$ and $e_i' = e_i$ for every $i = 2, 3, ..., n$. Suppose p is the original equilibrium price and α is the original forced allocation. Suppose $N(e, p, \alpha)$ is the original network.

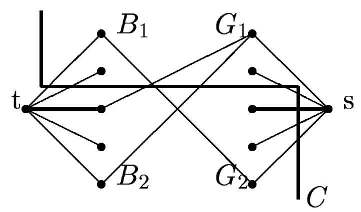
There are two cases: The first case is

$$\sum_{1\leq l\leq k_1-1} value(Q_l) < e_1 < \sum_{1\leq l\leq k_1} value(Q_l)$$

In this case, we can choose ϵ small enough such that

$$\sum_{1\leq l\leq k_1-1} value(Q_l) < e_1' < \sum_{1\leq l\leq k_1} value(Q_l)$$

Therefore the forced allocation does not change. Now consider the network: The only change is the capacity of the edge $(1, t)$ changes from m_1 to $m_1' = m_1 + \epsilon e_1$. Now $(t, B \cup G \cup s)$ is not a minimum cut, however, $(t \cup B \cup G, s)$ is still a minimum cut. We say that a cut $C = (t \cup B_2 \cup G_2, B_1 \cup G_1 \cup s)$ is a maximal min cut if the number of vertices in the s-part is maximal.



In the network, do the following:

- Step 1: Find a maximal min cut $C = (t \cup B_2 \cup G_2, B_1 \cup G_1 \cup s)$ in $N(e', p, \alpha)$ and goto Step 2.
- Step 2: First observe that any edge $e = (j, i)$ from G_2 to B_1 cannot carry any flow in a max-flow, since it goes from the t-side of the min-cut to the s-side. Next, suppose there is an edge $e = (j, i)$ from G_1 to B_2, corresponding to segment $s \in Q_{k_i}$ with $good(s) = j$. Because C is a min cut, the edge is saturated in a maximal flow, therefore buyer i must be able to afford the whole segment s of good j after she got all her forced goods. We update $\alpha(j)$ to be $\alpha(j) - value(s)$ and we update $m'(i)$ to $m'(i) - value(s)$. Then we delete this edge from the graph.

After doing the above two steps, we have: For $i = 1, 2$, each buyer in B_i is only adjacent to goods in G_i. We use $N(e', p, \alpha)$ to denote the network at this time.

Lemma 11 *No subset S of G_2 is tight in $N(e', p, \alpha)$.*

Proof: If $S \subseteq G_2$ and S is tight. By the property of $N(e', p, \alpha)$, we have $\Gamma(S) \subseteq B_2$. Suppose $bestT(S) = T$. Then $p(S) - \alpha(S) = m'(T) + c(S, \Gamma(S) - T)$. Now if we move S and T to the s-part, the change of the value of the cut is

$$-(p(S) - \alpha(S)) + m'(T) + c(S, \Gamma - T) = 0$$

Therefore we get a minimum cut of larger s-part, contradiction. □

Now let $\delta = \frac{\epsilon e_1}{p(G_2)}$. For each $g_j \in G_2$, let $p'_j = (1 + \delta)p_j$ and for each good which is not in G_2, keep its price unchange. In this way, we get a new price vector p'. If ϵ is small, then p' is close to p. We can choose ϵ small enough such that every subset of G_2 is still not tight under the price p'. Because p is our original equilibrium price, under the price p every good can be sold out, no subset of G is overtight. Now we didn't change the prices of goods in G_1, therefore no subset of G_1 is overtight under p'. Therefore $(t \cup B \cup G, s)$ is a minimum cut in the network $N(e', p', \alpha')$.

Lemma 12 *p' is the equilibrium price for the money vector e'.*

Proof: Because $\delta = \frac{\epsilon e_1}{p(G_2)}$, we can see the value of the cut $(t, B \cup G \cup s)$ equals the value of the cut $(t \cup B \cup G, s)$ in $N(e', p', \alpha')$. Therefore by the argument above, they are both minimum cuts. Therefore p' is the equilibrium price. □

The second case is

$$\sum_{1 \leq l \leq k_1 - 1} value(Q_l) < e_1 = \sum_{1 \leq l \leq k_1} value(Q_l)$$

Because p is the equilibrium price, buyer 1 must be able to use up all her money under the price p. Therefore all her flexible segments in Q_{k_1} are fully allocated. Allocate these goods to her and delete the corresponding edges from the network. Now if we raise e_1 to $e'_1 = e_1 + \epsilon$, then all the segments in Q_{k_1} become forced and some new segments become flexible. Suppose α' is the new forced allocation, then consider the network $N(e', p, \alpha')$. Because all the goods can be sold out under price p, now buyer 1 has even more money, all the goods can still be sold out under price p. Therefore $(t \cup B \cup G, s)$ is a minimum cut in $N(e', p, \alpha')$. Then just use the same argument in case 1, we can show that the change of the price is very small.

Now we drop the money of buyer 1. Suppose $e'_1 = (1 - \epsilon)e_1$. We can choose ϵ small enough such that $\sum_{1 \leq l \leq k_1 - 1} value(Q_l) < e'_1 < \sum_{1 \leq l \leq k_1} value(Q_l)$. Therefore the forced and flexible segments will not change.

Let $p_0 = min\{p_j : 1 \leq j \leq n'\}$ and $\delta' = \frac{\epsilon e_1}{p_0}$. For each good j, let $p'_j = (1 - \delta')p_j$.

Lemma 13 *In the network $N(e', p', \alpha)$, $(t \cup B \cup G, s)$ is a minimum cut.*

Proof: We only need to prove that no subset of G is overtight in the network. For any $S \subseteq G$, let $best'(S)$ be the new best value of S under e'. Therefore $best'(S) \geq best(S) - \epsilon e_1$. Because p is the equilibrium price, we have $p(S) - \alpha(S) \leq best(S)$. Therefore we have:

$$
\begin{aligned}
& best'(S) \\
& \geq best(S) - \epsilon e_1 \\
& \geq p(S) - \alpha(S) - \epsilon e_1 \\
& \geq p(S) - \alpha(S) - \delta' p(S) \\
& = p'(S) - \alpha(S) \quad \square
\end{aligned}
$$

Now we can play the same trick to show the change of the price is small.

Next we fix the money and change buyer 1's utility. Suppose s is one of the segments. If s is her forced or undesirable segment, then if we change $rate(s)$ or $right(s)$ slightly, the network will not change so the equilibrium price will not change. Now we suppose s is her flexible segment and without loss of generality we may assume $good(s) = 1$. Let $S = \{j \in G : j \neq 1, j \text{ is adjacent to buyer } 1\}$. Now we increase $rate(s)$ to be $(1+\epsilon)rate(s)$. In the network, all the edges from S to buyer 1 are deleted.

Suppose in the original network N, f is a maximal flow. Now we delete the edges between S and buyer 1, we just decrease the flow values of f to 0 at these edges and keep the balance condition to get a flow f'. So we can view f' as a flow in our new network N'. Because f is a maximal flow in N, there is no path from s to t in N's residue graph N_f. Therefore there is no path from s to t in $N'_{f'}$. Thus f' is a maximal flow.

Now in N', find a maximal min cut $C = (t \cup B_2 \cup G_2, B_1 \cup G_1 \cup s)$. We may assume buyer 1 is in B_2. Because f' is a maximal flow, edges from G_2 to B_1 and from G_1 to B_2 are saturated by f' in N', hence they are also saturated by f is N. Therefore we can arrange the corresponding segments to forced allocation, then delete the edges from the network. Therefore we may assume in N' the edges always go from G_i to B_i, $i = 1, 2$. We may assume good 1 is in G_2.

By lemma 8, every subset of G_2 is not tight at this time. Now for each $j \in G_2$, let $p'(j) = (1+\delta_2)p(j)$; for each $j \in G_1$, let $p'(j) = p(j)(1-\delta_1)$. Here δ_1 and δ_2 satisfy the following equations:

$$\delta_1 P(G_1) = \delta_2 P(G_2) \tag{5}$$

$$\frac{1}{1-\delta_1} = \frac{1+\epsilon}{1+\delta_2} \tag{6}$$

If δ_1 and δ_2 satisfy (6) then every edge from $S \cap G_1$ to buyer 1 will come back. If δ_1 and δ_2 satisfy (5) then the cut value of $(t \cap B \cap G, s)$ will be equal to the cut value of $(t, B \cap G \cap s)$. If we choose ϵ small enough, every subset of G_2 is still not tight. Because all the edges from $S \cap G_1$ to buyer 1 come back, no subset of G_1 will be overtight, therefore $(t \cap B \cap G, s)$ and $(t, B \cap G \cap s)$ will be min cuts hence the price p' is the equilibrium price. Since ϵ is small, the change of the price is also small.

Now if we decrease $rate(s)$, we can use a totally analogous argument to prove that the change of the price will be small.

At last, we fix the money, fix the rates of segments but change the right end $right(s)$ of one of buyer 1's segments s. Again, if the segment is her forced segment or undesirable segment, then we can make the change small enough such that everything keeps the same so the equilibrium price keeps the same. Now we suppose s is her flexible segment. If in the original equilibrium, the segment s is not fully allocated, then we can also make the change small such that the equilibrium price keeps unchange. Now we suppose s is fully allocated. We may assume $good(s) = 1$. If $right(s)$ increases slightly, we don't need to make any change, the equilibrium price will not change. So the only case remaining is to decrease $right(s)$ slightly.

In the original network before we change $right(s)$, allocate all the fully allocated segments in the maximal flow and delete the corresponding edges in the network. Suppose now we decrease the $right(s)$ by ϵ, therefore good 1 will have surplus value ϵ and buyer 1 will spend her money to some other goods. In the new network, suppose edges $(j_1, 1), (j_2, 1), ..., (j_k, 1)$ are added into the graph. Pick a j among $\{j_1, ..., j_k\}$. Now we get a new price p' by decreasing the price of p_1 by ϵ and increasing the price of j by ϵ. If there is an edge (j, i) in the graph, since we have already allocated all the fully allocated edges, the edge (j, i) is not saturated in the original maximal flow, which means the corresponding segment is not fully allocated in the original equilibrium. Therefore we can choose ϵ small enough such that all these segments are not fully allocated in the new equilibrium. Therefore p' is the equilibrium price. Therefore the change of the equilibrium price is also very small.

By the above argument, the equilibrium price is continuous.

5.2 Equilibrium allocations

In this section, we prove the following theorem:

Theorem 14 *The set of equilibrium allocations $X(m, U)$ is upper but not lower hemicontinuous.*

Proof: First of all, we prove that the set of equilibrium allocations is upper hemicontinuous. Given a sequence of money vectors $\{e^k\}$ utilities $\{U^k\}$, let $\{p^k\}$ be the corresponding equilibrium prices and $\{x^k\}$ be a corresponding equilibrium allocations. Suppose $e^k \to e^0$, $U^k \to U^0$, $p^k \to p^0$ and $x^k \to x^0$. Here $U^k \to U^0$ means for each segment s^k in U^k, there is a segment s^0 in U^0 such that $rate(s^k) \to rate(s^0)$ and $right(s^k) \to right(s^0)$. We want to show $x^0 \in X(e^0, U^0)$. Let N^k be the corresponding networks for (e^k, U^k) and N^0 be the network for (e^0, U^0). We have the following two lemmas:

Lemma 15 *If s^0 is a forced segment of buyer i under (e^0, U^0, p^0), then for large enough k, s^k is a forced segment under (e^k, U^k, p^k).*

Lemma 16 *If (j,i) is not in the edge set of N^0, then when k is large enough, (j,i) is not in the edge set of N^k.*

Now for fixed i and j, suppose $s_1^0, ..., s_r^0$ are all the segments for good j of buyer i. Suppose s_f^0 is the last forced segment.

Lemma 17 $x_{ij}^0 \geq right(s_f^0)$.

Proof: If $x_{ij}^0 < right(s_f^0)$, then for k large enough, $x_{ij}^k < right(s_f^k)$. Because s_f^0 is forced, by lemma 15, s_f^k is forced. However, x^k is equilibrium, so $x_{ij}^k \geq right(s_f^k)$, contradiction.

Definition 18 *We use $force^k(j)$ to denote the right end of buyer i's last forced segment for good j under (e^k, U^k, p^k), $k = 0, 1, 2, ...$*

Now we have two cases:

Case 1: (j,i) is not in the edge set of N^0. Therefore buyer i does not have any flexible segment for good j. By Lemma 16, when k is large, (j,i) is not in the edge set of N^k. Therefore $x_{ij}^k = force^k(j)$ and s_f^k is the last forced segment for each k. Thus $right(s_f^k) = force^k(j) = x_{ij}^k$. Therefore $force^0(j) = right(s_f^0)$ =lim $right(s_f^k)$ =lim $x_{ij}^k = x_{ij}^0$. So in this case, we just allocate value x_{ij}^0 of good j to buyer i.

Case 2: (j,i) is in the edge set of N^0. In this case, s_{f+1}^0 is buyer i's flexible segment for good j. If $x_{ij}^0 > right(s_{f+1}^0)$, then for k large enough, we have $x_{ij}^k > right(s_{f+1}^k)$. Therefore s_{f+1}^k is forced. Since s_{f+1}^0 is flexible, the only possibility is s_{f+1}^k is forced for each k and $e_i^0 = \sum_{1 \leq l \leq k_i} Q_l^0$ where $Q_1^0, ..., Q_{k_i-1}^0$ are sets of forced segments and $Q_{k_i}^0$ is the set of flexible segment. Thus we have $x_{ij}^k - right(s_{f+1}^k) \to 0$, hence $x_{ij}^0 = right(s_{f+1}^0)$, contradiction. Therefore we have $x_{ij}^0 \leq right(s_{f+1}^0)$, then we can allocate value x_{ij}^0 of good j to buyer i. The above implies that $x^0 \in X(e^0, U^0)$.

Because the linear utility is a special case of the spending constraint utility(see [8]), the set of equilibrium allocations is not lower hemicontinuous.

References

[1] K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," *Econometrica*, **22** (1954) 265–290.

[2] W. C. Brainard and H. E. Scarf, "How to compute equilibrium prices in 1891," Cowles Foundation Discussion Paper 1270, 2000.

[3] G. Debreu, *Mathematical Economics: Twenty papers of Gerard Debreu*, Cambridge University Press, 1986.

[4] N. Devanur, C. H. Papadimitriou, A. Saberi, and V. V. Vazirani. Market equilibrium via a primal-dual-type algorithm. In *Proceedings of IEEE Annual Symposium on Foundations of Computer Science*, 2002. To appear in *JACM*.

[5] E. Eisenberg and D. Gale, "Consensus of subjective probabilities: the Pari-Mutuel method," *The Annals of Mathematical Statistics*, **30** (1959) 165–168.

[6] K. Jain. A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities. In *Proceedings of FOCS*, 2004.

[7] N. Megiddo and V. V. Vazirani. Continuity properties of equilibrium prices and allocations in linear fisher markets. In *Proceedings of The 3rd Workshop on Internet and Network Economics*, 2007.

[8] V. V. Vazirani. Spending constraint utilities, with applications to the Adwords market. Submitted to *Math of O.R.*, 2006.

[9] V. V. Vazirani and M. Yannakakis. In preparation, 2008.

ECCC ISSN 1433-8092
http://eccc.hpi-web.de/