

# Derandomizing the Isolation Lemma and Lower Bounds for Noncommutative Circuit Size

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## Abstract

We give a randomized polynomial-time identity test for noncommutative circuits of polynomial degree based on the isolation lemma. Using this result, we show that derandomizing the isolation lemma implies noncommutative circuit size lower bounds. More precisely, we consider two restricted versions of the isolation lemma and show that derandomizing each of them implies nontrivial circuit size lower bounds for noncommutative circuits. These restricted versions of the isolation lemma are natural and would suffice for the standard applications of the isolation lemma.

## 1 Introduction

We recall the Isolation Lemma [MVV87]. Let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ . Let  $U$  be a set of size  $n$  and  $\mathcal{F} \subseteq 2^U$  be any family of subsets of  $U$ . Let  $w : U \rightarrow \mathbb{Z}^+$  be a weight function that assigns positive integer weights to the elements of  $U$ . For  $T \subseteq U$ , define its weight  $w(T)$  as  $w(T) = \sum_{u \in T} w(u)$ . Then Isolation Lemma guarantees that for any family of subsets  $\mathcal{F}$  of  $U$  and for any random weight assignment  $w : U \rightarrow [2n]$ , with high probability there will be a unique minimum weight set in  $\mathcal{F}$ .

**Lemma 1.1 (Isolation Lemma)** [MVV87] Let  $U$  be an universe of size  $n$  and  $\mathcal{F}$  be any family of subsets of  $U$ . Let  $w : U \rightarrow [2n]$  denote a weight assignment function to elements of  $U$ . Then,

$$\text{Prob}_w[\text{There exists a unique minimum weight set in } \mathcal{F}] \geq \frac{1}{2},$$

where the weight function  $w$  is picked uniformly at random.

In the seminal paper [MVV87] Mulmuley et al apply the isolation lemma to give a randomized NC algorithm for computing maximum cardinality matchings for general graphs (also see [ARZ99]). Since then the isolation lemma has found several other applications. For example, it is crucially used in the proof of the result that  $\text{NL} \subseteq \text{UL}/\text{poly}$  [AR00] and in designing randomized NC algorithms for linear representable matroid problems [NSV94]. It is also known that the isolation lemma can be used to prove the Valiant-Vazirani lemma that SAT is many-one reducible via randomized reductions to USAT.

Whether the matching problem is in deterministic NC, and whether  $\text{NL} \subseteq \text{UL}$  are outstanding open problems. Thus, the question whether the isolation lemma can be derandomized is clearly important.

As noted in [Agr07], it is easy to see by a counting argument that the isolation lemma can not be derandomized, in general, because there are  $2^{2^n}$  set systems  $\mathcal{F}$ . More formally, the following is observed in [Agr07].

**Observation 1.2** [Agr07] *The Isolation Lemma can not be fully derandomized if we allow weight functions  $w : U \rightarrow [n^c]$  for a constant  $c$  (i.e. weight functions with a polynomial range). More precisely, for any polynomially bounded collection of weight assignments  $\{w_i\}_{i \in [n^{c1}]}$  with weight range  $[n^c]$ , there exists a family  $\mathcal{F}$  of  $[n]$  such that for all  $j \in [n^{c1}]$ , there exists two minimal weight subsets with respect to  $w_j$ .*

However that does not rule out the derandomization of any special usage of the isolation lemma. Indeed, for all applications of the isolation lemma (mentioned above, for instance) we are interested only in exponentially many set systems  $\mathcal{F} \subseteq 2^U$ .

We make the setting more precise by giving a general framework. Fix the universe  $U = [n]$  and consider an  $n$ -input boolean circuit  $C$  where  $\text{size}(C) = m$ . The set  $2^U$  of all subsets of  $U$  is in a natural 1-1 correspondence with the length  $n$ -binary strings  $\{0, 1\}^n$ : each subset  $S \subseteq U$  corresponds to its characteristic binary string  $\chi_S \in \{0, 1\}^n$  whose  $i^{\text{th}}$  bit is 1 iff  $i \in S$ . Thus the  $n$ -input boolean circuit  $C$  implicitly defines the set system

$$\mathcal{F}_C = \{S \subseteq [n] \mid C(\chi_S) = 1\}.$$

As an easy consequence of Lemma 1.1 we have the following.

**Lemma 1.3** *Let  $U$  be an universe of size  $n$  and  $C$  be an  $n$ -input boolean circuit of size  $m$ . Let  $\mathcal{F}_C \subseteq 2^U$  be the family of subsets of  $U$  defined by circuit  $C$ . Let  $w : U \rightarrow [2n]$  denote a weight assignment function to elements of  $U$ . Then,*

$$\text{Prob}_w[\text{There exists a unique minimum weight set in } \mathcal{F}_C] \geq \frac{1}{2},$$

where the weight function  $w$  is picked uniformly at random. Furthermore, there is a collection of weight functions  $\{w_i\}_{1 \leq i \leq p(m,n)}$ , where  $p(m,n)$  is a fixed polynomial, such that for each  $\mathcal{F}_C$  there is a weight function  $w_i$  w.r.t. which there is a unique minimum weight set in  $\mathcal{F}_C$ .

Lemma 1.3 allows us to formulate two natural and reasonable derandomization hypotheses for the isolation lemma.

**Hypothesis 1.** There is a deterministic algorithm  $\mathcal{A}_1$  that takes as input  $(C, n)$ , where  $C$  is an  $n$ -input boolean circuit, and outputs a collection of weight functions  $w_1, w_2, \dots, w_t$  such that  $w_i : [n] \rightarrow [2n]$ , with the property that for some  $w_i$  there is a unique minimum weight set in the set system  $\mathcal{F}_C$ . Furthermore,  $\mathcal{A}_1$  runs in time subexponential in  $\text{size}(C)$ .

**Hypothesis 2.** There is a deterministic algorithm  $\mathcal{A}_2$  that takes as input  $(m, n)$  in unary and outputs a collection of weight functions  $w_1, w_2, \dots, w_t$  such that  $w_i : [n] \rightarrow [2n]$ , with the property that for each size  $m$  boolean circuit  $C$  with  $n$  inputs there is some weight function  $w_i$  w.r.t. which  $\mathcal{F}_C$  has a unique minimum weight set. Furthermore,  $\mathcal{A}_2$  runs in time polynomial in  $m$ .

Clearly, Hypothesis 2 is stronger than Hypothesis 1. It demands a “black-box” derandomization in the sense that  $\mathcal{A}_2$  efficiently computes a collection of weight functions that will work for *any* set system in  $2^U$  specified by a boolean circuit of size  $m$ .

Notice that a random collection  $w_1, \dots, w_t$  of weight functions will fulfil the required property of either hypotheses with high probability. Thus, the derandomization hypotheses are plausible. Indeed, it is not hard to see that suitable standard hardness assumptions that yield pseudorandom generators for derandomizing BPP would imply these hypotheses. We do not elaborate on this here.

In this paper we show the following two results.

1. Hypothesis 1 implies that either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the noncommutative Permanent does not have polynomial size noncommutative circuits.
2. Hypothesis 2 implies that for each  $n$  there is an explicit polynomial  $f_n(x_1, x_2, \dots, x_n) \in \mathbb{F}\{x_1, x_2, \dots, x_n\}$  in noncommuting variables  $x_i$  (where by explicit we mean that the coefficients of the polynomial  $f_n$  are computable by a uniform algorithm in time exponential in  $n$ ) that does not have noncommutative arithmetic circuits of size  $2^{o(n \lg n)}$  (where the field  $\mathbb{F}$  is either the rationals or a finite field).

These two results are a consequence of an identity testing algorithm for noncommutative circuits that is based on the isolation lemma. This algorithm is based on ideas from [AMS08] where we used automata theory to pick matrices from a suitable matrix ring and evaluate the given arithmetic circuit on these matrices. In the next section, we describe the background and then give the identity test in the following section.

**Remark 1.4** *Notice that derandomizing the isolation lemma in specific applications like the RNC algorithm for matchings [MVB87] and the containment  $\text{NL} \subseteq \text{UL/poly}$  [AR00] might still be possible without implying such circuit size lower bounds.*

Noncommutative polynomial identity testing has been the focus of recent research [RS05, BW05, AMS08]. One reason to believe that it could be easier than the commutative case to derandomize is because lower bounds are somewhat easier to prove in the noncommutative setting as shown by Nisan [N91]. Using a rank argument Nisan has shown exponential size lower bounds for noncommutative formulas (and noncommutative algebraic branching programs) that compute the noncommutative permanent or determinant polynomials in the ring  $\mathbb{F}\{x_1, \dots, x_n\}$  where  $x_i$  are noncommuting variables. However, no superpolynomial lower bounds are known for the size of noncommutative circuits for explicit polynomials.

Our result in this paper is similar in flavor to the Impagliazzo-Kabanets result [KI03], where for *commutative* polynomial identity testing they show that derandomizing polynomial identity testing implies circuit lower bounds. Specifically, it implies that either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the integer Permanent does not have polynomial-size arithmetic circuits.

In [AMS08] we have observed that an analogous result also holds in the noncommutative setting. I.e., if noncommutative PIT has a deterministic polynomial-time algorithm then either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the *noncommutative* Permanent function does not have polynomial-size noncommutative circuits.

The connection that we show here between derandomizing the isolation lemma and noncommutative circuit size lower bounds is based on the above observation and our noncommutative polynomial identity test based on the isolation lemma.

## 2 Automata Theory background

We recall some standard automata theory [HU78]. Fix a finite automaton  $A = (Q, \delta, q_0, q_f)$  which takes inputs in  $\{0, 1\}^*$ ,  $Q$  is the set of states,  $\delta : Q \times \{0, 1\} \rightarrow Q$  is the transition function, and  $q_0$  and  $q_f$  are the initial and final states respectively (we only consider automata with unique accepting states). For each  $b \in \{0, 1\}$ , let  $\delta_b : Q \rightarrow Q$  be defined by:  $\delta_b(q) = \delta(q, b)$ . These functions generate a submonoid of the monoid of all functions from  $Q$  to  $Q$ . This is the transition monoid of the automaton  $A$  and is well-studied in automata theory [Str94, page 55]. We now define the 0-1 matrix  $M_b \in \mathbb{F}^{|Q| \times |Q|}$  as follows:

$$M_b(q, q') = \begin{cases} 1 & \text{if } \delta_b(q) = q', \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $M_b$  is the adjacency matrix of the graph of  $\delta_b$ . As  $M_b$  is a 0-1 matrix, we can consider it as a matrix over any field  $\mathbb{F}$ .

For a string  $w = w_1 w_2 \cdots w_k \in \{0, 1\}^*$  we define  $M_w$  to be the matrix product  $M_{w_1} M_{w_2} \cdots M_{w_k}$ . If  $w$  is the empty string, define  $M_w$  to be the identity matrix of dimension  $|Q| \times |Q|$ . Let  $\delta_w$  denote the natural extension of the transition function to  $w$ ; if  $w$  is the empty string,  $\delta_w$  is simply the identity function. We have

$$M_w(q, q') = \begin{cases} 1 & \text{if } \delta_w(q) = q', \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Thus,  $M_w$  is also a matrix of zeros and ones for any string  $w$ . Also,  $M_w(q_0, q_f) = 1$  if and only if  $w$  is accepted by the automaton  $A$ .

## 2.1 Noncommutative arithmetic circuits and automata

This subsection is reproduced from [AMS08] to make this paper self-contained.

Consider the ring  $\mathbb{F}\{x_1, \dots, x_n\}$  of polynomials with noncommuting variables  $x_1, \dots, x_n$  over a field  $\mathbb{F}$ . Let  $C$  be a noncommutative arithmetic circuit computing a polynomial  $f \in \mathbb{F}\{x_1, \dots, x_n\}$ . Let  $d$  be an upper bound on the degree of  $f$ . We can consider monomials over  $x_1, \dots, x_n$  as strings over an alphabet of size  $n$ . For our construction, it is more convenient to encode each  $x_i$  as a string over  $\{0, 1\}$ . We encode the variable  $x_i$  by the string  $v_i = 01^i 0$ . Clearly, each monomial over the  $x_i$ 's of degree at most  $d$  maps uniquely to a binary string of length at most  $d(n+2)$ .

Let  $A = (Q, \delta, q_0, q_f)$  be a finite automaton over the alphabet  $\{0, 1\}$ . We have matrices  $M_{v_i} \in \mathbb{F}^{|Q| \times |Q|}$  as defined in Section 2, where  $v_i$  is the binary string that encodes  $x_i$ . We are interested in the output matrix obtained when the inputs  $x_i$  to the circuit  $C$  are replaced by the matrices  $M_{v_i}$ . This output matrix is defined in the obvious way: the inputs are  $|Q| \times |Q|$  matrices and we do matrix addition and matrix multiplication at each addition gate (respectively, multiplication gate) of the circuit  $C$ . We define the *output of  $C$  on the automaton  $A$*  to be this output matrix  $M_{out}$ . Clearly, given circuit  $C$  and automaton  $A$ , the matrix  $M_{out}$  can be computed in time  $\text{poly}(|C|, |A|, n)$ .

We observe the following property: the matrix output  $M_{out}$  of  $C$  on  $A$  is determined completely by the polynomial  $f$  computed by  $C$ ; the structure of the circuit  $C$  is otherwise irrelevant. This is important for us, since we are only interested in  $f$ . In particular, the output is always 0 when  $f \equiv 0$ .

More specifically, consider what happens when  $C$  computes a polynomial with a single term, say  $f(x_1, \dots, x_n) = cx_{j_1} \cdots x_{j_k}$ , with a non-zero coefficient  $c \in \mathbb{F}$ . In this case, the output matrix  $M_{out}$  is clearly the matrix  $cM_{v_{j_1}} \cdots M_{v_{j_k}} = cM_w$ , where  $w = v_{j_1} \cdots v_{j_k}$  is the binary string representing the monomial  $x_{j_1} \cdots x_{j_k}$ . Thus, by Equation 1 above, we see that the entry  $M_{out}(q_0, q_f)$  is 0 when  $A$  rejects  $w$ , and  $c$  when  $A$  accepts  $w$ . In general, suppose  $C$  computes a polynomial  $f = \sum_{i=1}^t c_i m_i$  with  $t$  nonzero terms, where  $c_i \in \mathbb{F} \setminus \{0\}$  and  $m_i = \prod_{j=1}^{d_i} x_{i_j}$ , where  $d_i \leq d$ . Let  $w_i = v_{i_1} \cdots v_{i_{d_i}}$  denote the binary string representing monomial  $m_i$ . Finally, let  $S_A^f = \{i \in \{1, \dots, t\} \mid A \text{ accepts } w_i\}$ .

**Theorem 2.1** [AMS08] *Given any arithmetic circuit  $C$  computing polynomial  $f \in \mathbb{F}\{x_1, \dots, x_n\}$  and any finite automaton  $A = (Q, \delta, q_0, q_f)$ , then the output  $M_{out}$  of  $C$  on  $A$  is such that  $M_{out}(q_0, q_f) = \sum_{i \in S_A^f} c_i$ .*

*Proof.* The proof is an easy consequence of the definitions and the properties of the matrices  $M_w$  stated in Section 2. Note that  $M_{out} = f(M_{v_1}, \dots, M_{v_n})$ . But  $f(M_{v_1}, \dots, M_{v_n}) = \sum_{i=1}^t c_i M_{w_i}$ , where  $w_i = v_{i_1} \cdots v_{i_{d_i}}$  is the binary string representing monomial  $m_i$ . By Equation 1, we know that  $M_{w_i}(q_0, q_f)$  is 1 if  $w_i$  is accepted by  $A$ , and 0 otherwise. Adding up, we obtain the result. ■

We now explain the role of the automaton  $A$  in testing if the polynomial  $f$  computed by  $C$  is identically zero. Our basic idea is to design an automaton  $A$  that accepts exactly one word from among all the words that correspond to the nonzero terms in  $f$ . This would ensure that  $M_{out}(q_0, q_f)$  is the nonzero coefficient of the monomial filtered out. More precisely, we will use the above theorem primarily in the following form, which we state as a corollary.

**Corollary 2.2** [AMS08] *Given any arithmetic circuit  $C$  computing polynomial  $f \in \mathbb{F}\{x_1, \dots, x_n\}$  and any finite automaton  $A = (Q, \delta, q_0, q_f)$ , then the output  $M_{out}$  of  $C$  on  $A$  satisfies:*

- (1) *If  $A$  rejects every string corresponding to a monomial in  $f$ , then  $M_{out}(q_0, q_f) = 0$ .*
- (2) *If  $A$  accepts exactly one string corresponding to a monomial in  $f$ , then  $M_{out}(q_0, q_f)$  is the nonzero coefficient of that monomial in  $f$ .*

Moreover,  $M_{out}$  can be computed in time  $\text{poly}(|C|, |A|, n)$ .

*Proof.* Both points (1) and (2) are immediate consequences of the above theorem. The complexity of computing  $M_{out}$  easily follows from its definition. ■ Another interesting corollary to the above theorem is the following.

**Corollary 2.3** [AMS08] *Given any arithmetic circuit  $C$  over  $\mathbb{F}\{x_1, \dots, x_n\}$ , and any monomial  $m$  of degree  $d_m$ , we can compute the coefficient of  $m$  in  $C$  in time  $\text{poly}(|C|, d_m, n)$ .*

*Proof.* Apply Corollary 2.2 with  $A$  being any standard automaton that accepts the string corresponding to monomial  $m$  and rejects every other string. Clearly,  $A$  can be chosen so that  $A$  has a unique accepting state and  $|A| = O(nd_m)$ . ■

**Remark 2.4** *Corollary 2.3 is very unlikely to hold in the commutative ring  $\mathbb{F}[x_1, \dots, x_n]$ . For, it is easy to see that in the commutative case computing the coefficient of the monomial  $\prod_{i=1}^n x_i$  in even a product of linear forms  $\prod_i \ell_i$  is at least as hard as computing the permanent over  $\mathbb{F}$ , which is #P-complete when  $\mathbb{F} = \mathbb{Q}$ .*

### 3 Noncommutative identity test based on isolation lemma

We now describe a new identity test for noncommutative circuits based on the isolation lemma. It is directly based on the results from [AMS08]. This is conceptually quite different from the randomized identity test of Bogdanov and Wee [BW05].

**Theorem 3.1** *Let  $f \in \mathbb{F}\{x_1, x_2, \dots, x_n\}$  be a polynomial given by an arithmetic circuit  $C$  of size  $m$ . Let  $d$  be an upper bound on the degree of  $f$ . Then there is a randomized algorithm which runs in time  $\text{poly}(n, m, d)$  and can test whether  $f \equiv 0$ .*

*Proof.* Let  $[d] = \{1, 2, \dots, d\}$  and  $[n] = \{1, 2, \dots, n\}$ . Consider the set of tuples  $U = [d] \times [n]$ . Let  $v = x_{i_1} x_{i_2} \dots x_{i_t}$  be a nonzero monomial of  $f$ . Then the monomial can be identified with the following subset  $S_v$  of  $U$ :

$$S_v = \{(1, i_1), (2, i_2), \dots, (t, i_t)\}$$

Let  $\mathcal{F}$  denotes the family of subsets of  $U$  corresponding to the nonzero monomials of  $f$  i.e,

$$\mathcal{F} = \{S_v \mid v \text{ is a nonzero monomial in } f\}$$

By the Isolation Lemma we know that if we assign random weights from  $[2dn]$  to the elements of  $U$ , with probability at least  $1/2$ , there is a unique minimum weight set in  $\mathcal{F}$ . Our aim will be to construct a family of small size automaton which are indexed by weights  $w \in [2nd^2]$  and  $t \in [d]$ , such that the automata  $A_{w,t}$  will precisely accept all the strings (corresponding to the monomials)  $v$  of length  $t$ , such that the weight of  $S_v$  is  $w$ . Then from the isolation lemma we will argue that the automata corresponding to the minimum weight will precisely accept only one string (monomial). Now for  $w \in [2nd^2]$ , and  $t \in [d]$ , we describe the construction of the automaton  $A_{w,t} = (Q, \Sigma, \delta, q_0, F)$  as follows:  $Q = [d] \times [2nd^2] \cup \{(0, 0)\}$ ,  $\Sigma = \{x_1, x_2, \dots, x_n\}$ ,  $q_0 = \{(0, 0)\}$  and  $F = \{(t, w)\}$ . We define the transition function  $\delta : Q \times \Sigma \rightarrow Q$ ,

$$\delta((i, V), x_j) = (i + 1, V + W),$$

where  $W$  is the random weight assign to  $(i + 1, j)$ . Our automata family  $\mathcal{A}$  is simply,

$$\mathcal{A} = \{A_{w,t} \mid w \in [2nd^2], t \in [d]\}.$$

Now for each of the automaton  $A_{w,t} \in \mathcal{A}$ , we mimic the run of the automaton  $A_{w,t}$  on the circuit  $C$  as described in Section 2. If the output matrix corresponding to any of the automaton is nonzero, our algorithm declares  $f \neq 0$ , otherwise declares  $f \equiv 0$ .

The correctness of the algorithm follows easily from the Isolation Lemma. By the Isolation Lemma we know, on random assignment, a unique set  $S$  in  $\mathcal{F}$  gets the minimum weight  $w_{\min}$  with probability at least  $1/2$ . Let  $S$  corresponds to the monomial  $x_{i_1}x_{i_2} \dots x_{i_\ell}$ . Then the automaton  $A_{\ell, w_{\min}}$  accepts the string (monomial)  $x_{i_1}x_{i_2} \dots x_{i_\ell}$ . Furthermore, as no other set in  $\mathcal{F}$  get the same minimum weight,  $A_{\ell, w_{\min}}$  rejects all the other monomials. So the  $(q_0, q_f)$  entry of the output matrix  $M_o$ , that we get in running  $A_{\ell, w_{\min}}$  on  $C$  is nonzero. Hence with probability at least  $1/2$ , our algorithm correctly decide that  $f$  is nonzero. The success probability can be boosted to any constant by standard independent repetition of the same algorithm. Finally, it is trivial to see that the algorithm always decides correctly if  $f \equiv 0$ .  $\blacksquare$

## 4 Noncommutative identity testing and circuit lower bounds

For commutative circuits, Impagliazzo and Kabanets [KI03] have shown that derandomizing PIT implies circuit lower bounds. It implies that either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the integer Permanent does not have polynomial-size arithmetic circuits.

In [AMS08] we have observed that this also holds in the noncommutative setting. I.e., if noncommutative PIT has a deterministic polynomial-time algorithm then either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the *noncommutative* Permanent function does not have polynomial-size noncommutative circuits. We note here that noncommutative circuit lower bounds are sometimes easier to prove than for commutative circuits. E.g. Nisan [N91] has shown exponential-size lower bounds for noncommutative formula size and further results are known for pure noncommutative circuits [N91, RS05]. However, proving superpolynomial size lower bounds for general noncommutative circuits computing the Permanent has remained an open problem.

To keep this paper self contained, we briefly recall the discussion from [AMS08].

The noncommutative Permanent function  $Perm(x_1, \dots, x_n) \in R\{x_1, \dots, x_n\}$  is defined as

$$Perm(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \prod_{i=1}^n x_{i, \sigma(i)},$$

where the coefficient ring  $R$  is any commutative ring with unity. Specifically, for the next theorem we choose  $R = \mathbb{Q}$ .

Let SUBEXP denote  $\bigcap_{\epsilon > 0} \text{DTIME}(2^{n^\epsilon})$  and NSUBEXP denote  $\bigcap_{\epsilon > 0} \text{NTIME}(2^{n^\epsilon})$ .

**Theorem 4.1** [AMS08] *If PIT for noncommutative circuits of polynomial degree  $C(x_1, \dots, x_n) \in \mathbb{Q}\{x_1, \dots, x_n\}$  is in SUBEXP, then either  $\text{NEXP} \not\subseteq \text{P/poly}$  or the noncommutative Permanent function does not have polynomial-size noncommutative circuits.*

*Proof.* Suppose  $\text{NEXP} \subseteq \text{P/poly}$ . Then, by the main result of [IKW02] we have  $\text{NEXP} = \text{MA}$ . Furthermore, by Toda's theorem  $\text{MA} \subseteq \text{P}^{Perm_z}$ , where the oracle computes the integer permanent. Now, assuming PIT for noncommutative circuits of polynomial degree is in deterministic polynomial-time we will show that the (noncommutative) Permanent function does not have polynomial-size noncommutative circuits. Suppose to the contrary that it does have polynomial-size noncommutative circuits. Clearly, we can use it to compute the integer permanent as well. Furthermore, as in [KI03] we notice that the noncommutative  $n \times n$  Permanent is also uniquely characterized by the identities  $p_1(x) \equiv x$  and  $p_i(X) = \sum_{j=1}^i x_{1j} p_{i-1}(X_j)$  for  $1 < i \leq n$ , where  $X$  is a matrix of  $i^2$  noncommuting variables and  $X_j$  is its  $j$ -th minor w.r.t. the first row. I.e. if arbitrary polynomials  $p_i, 1 \leq i \leq n$  satisfies these  $n$  identities over *noncommuting* variables  $x_{ij}, 1 \leq i, j \leq n$  if and only if  $p_i$  computes the  $i \times i$  permanent of noncommuting variables. The rest of the proof is exactly as in Impagliazzo-Kabanets [KI03]. We can easily describe an NP machine to simulate a  $\text{P}^{Perm_z}$  computation. The NP machine guesses a polynomial-size noncommutative circuit for  $Perm$  on  $m \times m$  matrices, where  $m$  is a polynomial bound on the matrix size of the queries made. Then the NP verifies that the circuit computes the permanent by checking the  $m$  *noncommutative* identities it must satisfy. This can be done in SUBEXP by assumption. Finally, the NP machines uses the circuit to answer all the integer permanent queries. Putting it together, we get  $\text{NEXP} = \text{NSUBEXP}$  which contradicts the nondeterministic time hierarchy theorem. ■

## 5 The Results

We are now ready to prove our first result. Suppose the derandomization Hypothesis 1 holds (as stated in the introduction): i.e. suppose there is a deterministic algorithm  $\mathcal{A}_1$  that takes as input  $(C, n)$  where  $C$  is an  $n$ -input boolean circuit and in subexponential time computes a set of weight functions  $w_1, w_2, \dots, w_t, w_i : [n] \rightarrow [2n]$  such that the set system  $\mathcal{F}_C$  defined by the circuit  $C$  has a unique minimum weight set w.r.t. at least one of the weight functions  $w_i$ .

Let  $C'(x_1, x_2, \dots, x_n)$  be a noncommutative arithmetic circuit of degree  $d$  bounded by a polynomial in  $size(C')$ . By Corollary 2.3, there is a deterministic polynomial-time algorithm that takes as input  $C'$  and a monomial  $m$  of degree at most  $d$  and accepts if and only if the monomial  $m$  has nonzero coefficient in the polynomial computed by  $C'$ . Thus, we have a boolean circuit  $C$  of size polynomial in  $size(C')$  that accepts only the (binary encodings of) monomials  $x_{i_1} x_{i_2} \dots x_{i_k}, k \leq d$  that have nonzero coefficients in the polynomial computed by  $C'$ . Now, as a consequence of Theorem 3.1 and its proof we have a *deterministic* subexponential algorithm for checking if  $C' \equiv 0$ , assuming algorithm  $\mathcal{A}_1$  exists. Namely, we compute the boolean circuit  $C$  from  $C'$  in polynomial time. Then, invoking algorithm  $\mathcal{A}_1$  with  $C$  as input we compute

at most subexponentially many weight functions  $w_1, \dots, w_t$ . Then, following the proof of Theorem 3.1 we construct the automata corresponding to these weight functions and evaluate  $C'$  on the matrices that each of these automata define in the prescribed manner. By assumption about algorithm  $\mathcal{A}_1$ , if  $C' \neq 0$  then one of these  $w_i$  will give matrix inputs for the variables  $x_j, 1 \leq j \leq n$  on which  $C'$  evaluates to a nonzero matrix. We can now show the following theorem.

**Theorem 5.1** *If the subexponential time algorithm  $\mathcal{A}_1$  satisfying Hypothesis 1 exists then noncommutative identity testing is in SUBEXP which implies that either NEXP  $\not\subseteq$  P/poly or the Permanent does not have polynomial size noncommutative circuits.*

*Proof.* The result is a direct consequence of the discussion preceding the theorem statement and Theorem 4.1. ■

We now turn to the second consequence result under the *stronger* derandomization Hypothesis 2 (stated in the introduction). More precisely, suppose there is a deterministic algorithm  $\mathcal{A}_2$  that takes as input  $(m, n)$  and in time polynomial in  $m$  computes a set of weight functions  $w_1, w_2, \dots, w_t, w_i : [n] \rightarrow [2n]$  such that for *each*  $n$ -input boolean circuit  $C$  of size  $m$ , the set system  $\mathcal{F}_C$  defined by the circuit  $C$  has a unique minimum weight set w.r.t. at least one of the weight functions  $w_i$ . We show that there is an *explicit* polynomial<sup>1</sup>  $f(x_1, \dots, x_n)$  in noncommuting variables  $x_i$  that does not have subexponential size noncommutative circuits.

**Theorem 5.2** *Suppose there is a polynomial-time algorithm  $\mathcal{A}_2$  satisfying Hypothesis 2. Then for all but finitely many  $n$  there is an explicit polynomial  $f(x_1, \dots, x_n) \in \mathbb{F}\{x_1, x_2, \dots, x_n\}$  (where the field  $\mathbb{F}$  is either rationals or any finite field) in noncommuting variables  $x_i$  that is computable in  $2^{n^{O(1)}}$  time (by a uniform algorithm) and does not have noncommutative arithmetic circuits of size  $2^{o(n \lg n)}$ .*

*Proof.* Let  $T_n$  denote the set of all sequences  $(i_1, i_2, \dots, i_n)$ , for  $i_j \in [n], 1 \leq j \leq n$ . For each such sequence  $\alpha = (i_1, i_2, \dots, i_n) \in T_n$  let  $m_\alpha$  denote the monomial  $x_{i_1} x_{i_2} \dots x_{i_n}$ . Now, we write

$$f(x_1, x_2, \dots, x_n) = \sum_{\alpha \in T_n} c_\alpha m_\alpha,$$

where we will pick the scalars  $c_\alpha$  appropriately so that the polynomial  $f$  has the claimed property. Suppose  $\mathcal{A}_2$  runs in time  $m^c$  for constant  $c > 0$ , where  $m$  denotes the size bound of the boolean circuit  $C$  defining set system  $\mathcal{F}_C$ . Notice that the number  $t$  of weight functions is bounded by  $m^c$ . As explained in Theorem 3.1, each weight function will give rise to a collection of  $2n^4$  automata  $\mathcal{A}_k$ , each of which will prescribe matrices of dimension at most  $r = \text{poly}(n)$  to be assigned for the input variables  $x_j, 1 \leq j \leq n$ . Call these matrices  $M_{i,j}^{(k)}$ . For each weight function  $w_i$  write down linear equations for each  $k \in [2n^4]$ .

$$f(M_{i,1}^{(k)}, M_{i,2}^{(k)}, \dots, M_{i,n}^{(k)}) = 0.$$

This will actually give us a system of at most  $2n^4 r^2$  linear equations in the unknown scalars  $c_\alpha$ . Since there are  $t \leq m^c$  weight functions in all, all the linear constraints put together give us a system of at most  $2n^4 r^2 m^c$  linear equations. Now, the number of distinct (noncommuting) monomials  $m_\alpha$  is  $n^n = 2^{n \lg n}$  which asymptotically exceeds  $2n^4 r^2 m^c$  for  $m = 2^{o(n \lg n)}$ , since  $r$  is polynomially bounded. Thus,

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<sup>1</sup>By explicit we mean that the coefficients of  $f$  are computable in time exponential in  $n$ .



the system of linear equations has a nontrivial solution in the  $c_\alpha$ 's that can be computed using Gaussian elimination in time exponential in  $n$ .

Notice that the polynomial  $f(x_1, \dots, x_n)$ , defined by the solution to the  $c_\alpha$ 's, is a nonzero polynomial. We claim that  $f$  cannot have a noncommutative circuit of size  $2^{o(n \lg n)}$ . Assume to the contrary that  $C'(x_1, \dots, x_n)$  is a noncommutative circuit of size  $s = 2^{o(n \lg n)}$  for  $f$ . Then, by Corollary 2.3 there is an  $n'$ -input boolean circuit  $C$  of size  $m = s^{O(1)} = 2^{o(n \lg n)}$  that accepts precisely the (binary encodings) of those monomials that are nonzero in  $C'$ . Let  $w_1, \dots, w_t$  be the weight functions output by  $\mathcal{A}_2$  for input  $(m, n')$ . By Hypothesis 2, for some weight function  $w_i$  and some  $k \in [2n^4]$  the circuit  $C'$  must be nonzero on matrices  $M_{i,j}^{(k)}$ . However,  $f$  evaluates to zero, by construction, on the matrix inputs prescribed by all the weight functions  $w_1, \dots, w_t$ . This is a contradiction to the assumption and it completes the proof. ■

**Remark 5.3** We can formulate both Hypothesis 1 and Hypothesis 2 more generally by letting the running time of algorithms  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be a function  $t(m, n)$ . We will then obtain suitably quantified circuit lower bound results as consequence.

## 6 Discussion

An interesting open question is whether derandomizing similar restricted versions of the Valiant-Vazirani lemma also implies circuit lower bounds. We recall the Valiant-Vazirani lemma as stated in the original paper [VV86].

**Lemma 6.1** Let  $S \subseteq \{0, 1\}^t$ . Suppose  $w_i, 1 \leq i \leq t$  are picked uniformly at random from  $\{0, 1\}^t$ . For each  $i$ , let  $S_i = \{v \in S \mid v \cdot w_j = 0, 1 \leq j \leq i\}$  and let  $p_t(S)$  be the probability that  $|S_i| = 1$  for some  $i$ . Then  $p_t(S) \geq 1/4$ .

Analogous to our discussion in Section 1, here too we can consider the restricted version where we consider  $S_C \subseteq \{0, 1\}^n$  to be the set of  $n$ -bit vectors accepted by a boolean circuit  $C$  of size  $m$ . We can similarly formulate derandomization hypotheses similar to Hypotheses 1 and 2.

We do not know if there is another randomized polynomial identity test for noncommutative arithmetic circuits based on the Valiant-Vazirani lemma. The automata-theoretic technique of Section 3 does not appear to work. Specifically, given a matrix  $h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^k$ , there is no deterministic finite automaton of size  $\text{poly}(n, k)$  that accepts  $x \in \mathbb{F}_2^n$  if and only if  $h(x) = 0$ .

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