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On the complexity of cutting plane proofs using split cuts

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Abstract

We prove a monotone interpolation property for split cuts which, together with results from Pudlák (1997), implies that cutting-plane proofs which use split cuts have exponential length in the worst case. As split cuts are equivalent to mixed-integer rounding cuts and Gomory mixed-integer cuts, cutting-plane proofs using the above cuts also have exponential worst-case complexity.

Key words. cutting-plane proof, split cut, mixed-integer rounding, monotone interpolation

1 Introduction

The complexity of different types of cutting-plane proofs has been a much studied topic in recent years. Some well-known classes of cutting planes (linear inequalities satisfied by integral points in polyhedra) are Gomory-Chvátal cuts [12], split cuts [8], mixed-integer rounding (MIR) cuts [19], and lift-and-project cuts [1]. Let $Ax \leq b$ be a system of linear inequalities with rational coefficients in n variables. A *Gomory-Chvátal* (GC) cutting plane (or cut) for $Ax \leq b$ is a linear inequality $cx \leq \lfloor d \rfloor$ where c is an integral vector, and $cx \leq d$ is satisfied by solutions of $Ax \leq b$. A *Gomory-Chvátal cutting-plane proof* of $cx \leq d$ from $Ax \leq b$ with *length* L is a sequence of inequalities $a_i x \leq d_i$ $(i = 1, \dots, L)$ such that the last inequality in the sequence is $cx \leq d$, and for $i = 1, \dots, L$, the inequality $a_i x \leq d_i$ is a Gomory-Chvátal cut derived from the previous inequalities in the sequence and the inequalities in $Ax \leq b$. Cutting-plane proofs were introduced in [4]. Any inequality satisfied by integral solutions of $Ax \leq b$ has a GC cutting plane proof [12], [4].

Pudlák [20] proved, extending ideas in [3],[16], that GC cutting plane proofs of inequalities satisfied by 0-1 points in polyhedra have exponential length in the worst case. A similar

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result for lift-and-project cutting plane proofs can be found in [9]. An inequality $cx \le d$ is a *lift-and-project* cut for $P = \{x \in [0,1]^n \,|\, Ax \le b\}$ if for some index j, $cx \le d$ is satsfied by points in $P \cap \{x_j = 0\}$ and $P \cap \{x_j = 1\}$. A question left open in [9] is whether cutting-plane proofs using split cuts (or *split cut proofs*) have exponential worst-case complexity. An inequality $cx \le d$ is a *split cut* for $P = \{x \in \mathbb{R}^n \,|\, Ax \le b\}$ if $cx \le d$ is satisfied by points in $P \cap \{\alpha x \le \beta\}$ and $P \cap \{\alpha x \ge \beta + 1\}$, where α, β are integral. We say that $cx \le d$ is derived using the *disjunction* $\alpha x \le \beta \vee \alpha x \ge \beta + 1$. All integral points in P satisfy every split cut for P.

Lift-and-project cuts and GC cuts are special cases of split cuts. Split cuts are equivalent to MIR cuts and Gomory mixed-integer (GMI) cuts [13]; these are currently the most important classes of cutting planes for general integer programs. Further, split cuts are substantially stronger than GC cuts and lift-and-project cuts. For the family of instances (from J. A. Bondy) $P_k = \{x, y \,|\, x \le ky, x + ky \le k\}$, $k = 1, 2, \ldots$, the inequality $x \le 0$ is a split cut, but any GC cutting-plane proof has length at least $k/2$, which is exponential in the encoding size of P_k. The inequality $\sum_{i=1}^n x_i \ge 1$ is a split cut for $P = \{x \in [0,1]^n : \sum_{i=1}^n x_i \ge 1/2\}$, but any lift-and-project proof has length at least n [7].

We prove that split-cut proofs have exponential worst-case complexity, thus generalizing Pudlák' result [20] and our result [9]. Pudlák showed how to map a GC cutting-plane proof with length L of a specific inequality to a *monotone real circuit* (see Section 2) with $O(poly(L))$ gates which tests for cliques of a certain size in an arbitrary graph. He separately proved that such monotone circuits have exponentially many gates. We show how to map a split cut proof of length L to a monotone circuit with $O(poly(L))$ gates for the clique testing problem mentioned above (this is called *monotone interpolation*). This mapping is very similar to the one in [9] for lift-and-project cuts, and is based on a property of split cuts which we prove: if $gx + hy \le d$ is a split cut for $\{x, y : Ax \le e,\ By \le f\}$, then there exist numbers r and s with $r + s \le d$, such that $gx \le r$ is implied by split cuts for $Ax \le e$, $hy \le s$ is implied by split cuts for $By \le f$, and r can be computed from $Ax \le e$ using monotone operations only. Our results imply that cutting-plane proofs which use GMI cuts, MIR cuts, two-step MIR cuts [10], or pairing inequalities [14] have exponential worst-case complexity.

Bonet, Pitassi and Raz [3] gave an exponential lower bound on the complexity of branch-and-cut proofs which use GC cuts, and branching on inequalities $\alpha x \le \beta$ and $\alpha x \ge \beta + 1$, where α and β are integral, subject to the restriction that the GC cuts and the branching inequalities have polynomially bounded coefficients (see also [16], [15]). The results in this paper, combined with results in [9], imply that branch-and-cut proofs which use split cuts but branch only on the inequalities $x_i \le 0$ and $x_i \ge 1$ for 0-1 variables x_i, have exponential worst-case complexity.

In the next section, we review well-known complexity results for monotone circuits. In Section 3, we present MIR cuts in the form given in [11], and discuss their equivalence with split cuts. Based on this equivalence, we prove a monotone interpolation result for split cuts in Section 4, and give our exponential lower bound result. The results in this paper are self-contained other than for Theorem 1 [20].

2 Monotone circuit complexity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *monotone* if for x, y in $\mathbb{R}^n$, $x \leq y$ implies $f(x) \leq f(y)$. *Monotone operations* are monotone functions with one or two inputs; some examples are

$$tx, \quad r + x, \quad x + y, \quad \lfloor x \rfloor, \quad thr(x, 0)$$

where t is a non-negative constant, x and y are real variables, and r is a real constant; $thr(x, 0)$ is a *threshold function* which returns 0, if $x < 0$, and 1 otherwise. The functions $\wedge$ and $\vee$ are monotone operations over the domain $\{0, 1\}$. A *monotone boolean circuit* is similar to a boolean circuit, except that it uses only $\wedge$ gates and $\vee$ gates. A *monotone real circuit* is one with arbitary monotone operations as gates.

Let $CLIQUE_{k,n}$ (say k is a fixed function of n) be the function which takes as input graphs on n nodes (given as incidence vectors of edges), and returns 1 if the input graph contains a clique of size k, and 0 if the graph contains a coloring of size $k-1$ (and is undefined for all other graphs). This function is monotone over the domain consisting of graphs containing a k-clique or a $k-1$ coloring as adding edges to a graph causes the maximum clique size to increase. Any monotone boolean circuit solving $CLIQUE_{k,n}$ (for appropriate k) has an exponential number of gates (see Razborov [22], and Alon and Boppana [2]). Pudlák, and independently Cook and Haken [5], proved a similar result for monotone real circuits.

Theorem 1 *[20] Let C_n be a monotone real circuit computing $CLIQUE_{k,n}$, with $k = \lfloor n^{2/3} \rfloor$. Then $|C_n| \geq 2^{\Omega((n/\log n)^{1/3})}$.*

Pudlák [20] defined a system of linear inequalities $\mathcal{I}$ such that if $\mathcal{I}$ has a 0-1 solution, then there is a graph on n nodes which has both a clique of size k and a coloring of size $k-1$. He proved that given a GC cutting plane proof $\mathcal{P}$ of $\mathbf{0}x \leq -1$ (which proves that $\mathcal{I}$ has no 0-1 solution) of length L, one can construct a monotone real circuit with $O(poly(L))$ gates solving $CLIQUE_{k,n}$. Theorem 1 then implies that L is exponential in n. Pudlák used the following properties of GC cuts in mapping short cutting-plane proofs to monotone real circuits with few gates:

(A) If $gx + hy \leq d$ is a GC cut for $Ax + By \leq c$, then for any 0-1 vector y', $gx \leq d - hy'$ is a GC cut for $Ax \leq c - By'$;

(B) if $gx + hy \leq d$ is a GC cut for $Ax \leq e,\ By \leq f$, then there are numbers r and s such that $gx \leq r$ is a GC cut for $Ax \leq e$, and $hy \leq s$ is a GC cut for $By \leq f$, and $r + s \leq d$;

(C) The number r (or s) can be computed from A, e (or B, f) with polynomially many monotone operations.

Property (A) is easy to prove. Consider property (B). As $gx + hy \leq d$ is a GC cut for $Ax \leq e,\ By \leq f$, we can assume that g, h, d are integral and there are multiplier vectors $\lambda, \mu \geq 0$ such that $g = \lambda A,\ h = \mu B,\ d = \lfloor \lambda e + \mu f \rfloor$. Clearly, $gx \leq \lfloor \lambda e \rfloor$ is a GC cut for $Ax \leq e$, and so is $hy \leq \lfloor \mu f \rfloor$, and $\lfloor \lambda e \rfloor + \lfloor \mu f \rfloor \leq \lfloor \lambda e + \mu f \rfloor$. Property (C) also follows from this; the number $\lfloor \lambda e \rfloor$ can be computed from e via polynomially many monotone operations (the coefficients of λ are treated as non-negative constants). Properties (A) and (B) hold for the matrix cuts of Lovász and Schrijver (cuts based on the N and N_+ operators); see [21], [9]. Property (C) is often hard to prove, and is not known to hold for matrix cuts. In Lemma 5, we prove (via the equivalence of split cuts and MIR cuts) that slight variants of properties (B) and (C) hold for split cuts. We use Lemma 5 and ideas from Pudlák's paper to show in Theorem 8 that any split cut proof of $\mathbf{0}x \leq -1$ from Pudlák's system $\mathcal{I}$ has exponential length. For completeness, we give the inequality system $\mathcal{I}$ below.

Let $k = \lfloor n^{2/3} \rfloor$. Let z be a vector of $n(n-1)/2$ 0-1 variables, such that every 0-1 assignment to z corresponds to the incidence vector of a graph on n nodes (assume nodes are numbered from $1, \ldots, n$). Let x be the 0-1 vector of variables $(x_i \,|\, i = 1, \ldots, n)$ and let y be the 0-1 vector of variables $(y_{ij} \,|\, i = 1, \ldots, n, j = 1, \ldots, k-1)$. Consider the inequalities

$$\sum_i x_i \geq k, \tag{1}$$

$$x_i + x_j \leq 1 + z_{ij}, \quad \forall\, i, j \in N, \text{ with } i < j, \tag{2}$$

$$\sum_{j=1}^{k-1} y_{ij} = 1, \quad \forall i \in N, \tag{3}$$

$$y_{is} + y_{js} \leq 2 - z_{ij}, \quad \forall\, i, j \in N \text{ with } i < j, \text{ and } \forall s \in \{1, \ldots, k-1\}. \tag{4}$$

In any 0-1 solution of the above inequalities, the set of nodes $\{i \,|\, x_i = 1\}$ forms a clique of size k or more, and for all $j \in \{1, \ldots, k-1\}$, the set $\{i \,|\, y_{ij} = 1\}$ is a stable set. Thus, the variables y_{ij} define a mapping of nodes in a graph to $k-1$ colors in a proper colouring. Therefore the inequalities (1) - (4) have no 0-1 solution. Let $Ax + Cz \leq e$ stand for the inequalities (1) and (2), along with the bounds $0 \leq x \leq 1$. Let $By + Dz \leq f$ stand for the inequalities (3) and (4), along with the bounds $0 \le y \le 1$ and $0 \le z \le 1$. Note that the above inequalities have $O(n^2)$ variables and $O(n^3)$ constraints; for technical reasons we need the fact that $C \le 0$. Theorem 1 implies that every monotone real circuit which takes graphs on n nodes as input (in the form of a 0-1 vector z') and outputs 0 or 1 such that output 0 implies $Ax \le e - Cz'$ has no 0-1 solution and output 1 implies $By \le f - Dz'$ has no 0-1 solution, has exponentially many gates.

3 Mixed-integer rounding cuts

For a number v and an integer t, define $\hat{v} = v - \lfloor v \rfloor$. Define

$$Q = \{v \in R,\ z \in \mathbb{Z} : v + z \ \ge\ b,\ \ v \ge 0\}.$$

Wolsey [24] defined the *basic mixed-integer* inequality as

$$v + \hat{b}z \ge \hat{b}\lceil b \rceil, \tag{5}$$

and showed that it is valid and facet-defining for Q; also see Marchand and Wolsey [18]. In Figure 1(a), we depict the points in Q (when b is not integral) by horizontal lines. In Figure 1(b), the half-plane above the dashed line represents (5), and contains the shaded regions which are $Q \cap \{z \le \lfloor b \rfloor\}$ and $Q \cap \{z \ge \lceil b \rceil\}$. Therefore (5) is a split cut for Q. The

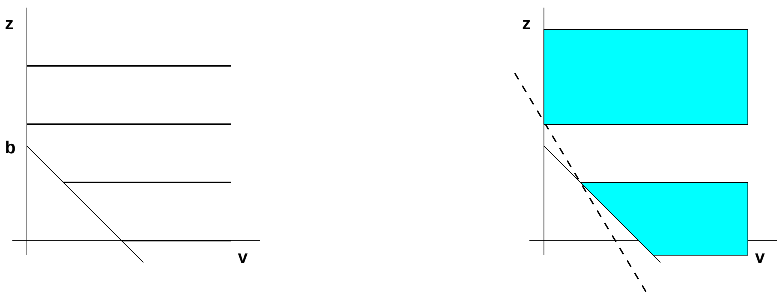

Figure 1: The basic mixed-integer inequality

following is a well-known property of the (5) and can easily be inferred from Figure 1.

Lemma 2 *conv*$(Q) = \{v, z \in R : v + z \ \ge\ b,\ \ v + \hat{b}z \ge \hat{b}\lceil b \rceil,\ \ v \ge 0\}$.

Therefore, any linear inequality satisfied by points in Q is implied by a non-negative linear combination of the inequalities $v \ge 0$, $v + z \ge b$ and (5).

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$, and assume A has m rows. Define $s = b - Ax$; clearly $s \ge 0$ for all $x \in P$. Let $\lambda \in \mathbb{R}^m$ be a row vector such that λA is integral. Define β as λb, and λ^+ by $\lambda_i^+ = \max\{\lambda_i, 0\}$. Then the equation $\lambda s + (\lambda A)x = \lambda b$ is valid for P and so is the inequality

$$\lambda^+ s + (\lambda A)x \geq \lambda b \tag{6}$$

For all points in P, $\lambda^+ s$ is non-negative and $(\lambda A)x$ is integral. The basic mixed-integer inequality implies that

$$\lambda^+ s + \hat{\beta}(\lambda A)x \geq \hat{\beta}(\lceil \beta \rceil), \tag{7}$$

or equivalently $\lambda^+(b - Ax) + \hat{\beta}(\lambda A)x \geq \hat{\beta}(\lceil \beta \rceil)$, is a valid inequality for integral points in P (recall that $\hat{\beta} = \beta - \lfloor \beta \rfloor$). This inequality is the *mixed-integer rounding (MIR)* cut for P. It follows from the discussion above that the MIR cut is a split cut for P derived using the disjunction $(\lambda A)x \leq \lfloor \beta \rfloor \vee (\lambda A)x \geq \lceil \beta \rceil$. Nemhauser and Wolsey defined [19] MIR cuts and proved that every split cut for P is also an MIR cut for P; see [11] for different ways of defining the MIR cut, and the equivalence of (7) with the definition in [19].

Definition 3 *The split closure of a polyhedron P, denoted by $sc(P)$, is the set of points in P satisfying all split cuts for P.*

The MIR closure (denoted by P^{MIR}) is defined similarly in terms of MIR cuts. The split closure of a polyhedron is therefore the same as its MIR closure.

4 Complexity of split-cut proofs

We start off by proving the properties (A) and (B) given in Section 2 for split cuts. To do so, we invoke the equivalence of split cuts and MIR cuts, and use the lemma below. For $i = 1, 2$, let

$$P_i = \{v_i, z_i : v_i + z_i \geq b_i,\ v_i \geq 0,\ z_i \in \mathbb{Z}\}, \tag{8}$$

Lemma 4 *Let P_1, P_2 be defined as above. Let $b_3 = b_1 + b_2$. Then the inequality $(v_1 + v_2) + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil$ valid for $P_1 \cap P_2$ is implied by a non-negative combination of the inequalities defining P_1 and P_2 and the basic mixed-integer inequalities for P_1 and P_2.*

Proof By Lemma 2, for $i = 1, 2$,

$$conv(P_i) = \{v_i, z_i : v_i + z_i \geq b_i,\ v_i + \hat{b}_i z_i \geq \hat{b}_i \lceil b_i \rceil,\ v_i \geq 0\}.$$

Observe that all points $(v_1, v_2, z_1, z_2) \in P_1 \cap P_2$ satisfy $v_1 + v_2 + z_1 + z_2 \geq b_1 + b_2$ and $v_1 + v_2 \geq 0$ and $z_1 + z_2 \in \mathbb{Z}$. Using (5), we see that

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil. \tag{9}$$

is valid for $P_1 \cap P_2$. Therefore

$$\begin{aligned}\hat{b}_3 \lceil b_3 \rceil &\leq \min\{v_1 + v_2 + \hat{b}_3(z_1 + z_2) \,|\, (v_1, v_2, z_1, z_2) \in P_1 \cap P_2\} \\ &\leq \min\{v_1 + \hat{b}_3 z_1 \,|\, (v_1, z_1) \in P_1\} + \min\{v_2 + \hat{b}_3 z_2 \,|\, (v_2, z_2) \in P_2\}.\end{aligned}$$

as optimal solutions (v_1^*, z_1^*) and (v_2^*, z_2^*) of the last two minimization problems with optimal values μ_1 and μ_2, yield a feasible solution $(v_1^*, v_2^*, z_1^*, z_2^*)$ of the first minimization problem with objective value $\mu_1 + \mu_2$. As

$$v_1 + \hat{b}_3 z_1 \geq \mu_1 \text{ and } v_2 + \hat{b}_3 z_2 \geq \mu_2$$

are valid inequalities for $conv(P_1)$ and $conv(P_2)$, respectively, (9) is implied by a nonnegative combination of inequalities defining $conv(P_1)$ and $conv(P_2)$. ∎

Lemma 5 *Let x, y be vectors of integer variables, with no common components. Let*

$$Q_1 = \{x : Ax \leq e\}, \quad Q_2 = \{y : Cy \leq f\},$$

$$Q_3 = \{(x, y) : x \in Q_1, y \in Q_2\} = Q_1 \cap Q_2,$$

where A, C, e, f are matrices with appropriate dimensions. Let $ax + cy \leq d$ be a split cut for Q_3. Let $g = \max\{ax : x \in sc(Q_1)\}$ and $h = \max\{cy : y \in sc(Q_2)\}$. Then $g + h \leq d$.

Proof Let $ax + cy \leq d$ be a split cut for Q_3 derived as described above. It is also an MIR cut derived from the system

$$Ax + s = e, \;\; Cy + t = f, \;\; s, t \geq 0.$$

More precisely, there are vectors λ and μ such that the split cut above equals

$$\lambda^+ s + \mu^+ t + \hat{\beta}\big((\lambda A)x + (\mu C)y\big) \geq \hat{\beta} \lceil \beta \rceil, \tag{10}$$

$$\text{where } \beta = \lambda e + \mu f, \quad \lambda A \text{ and } \mu C \text{ are integral.} \tag{11}$$

Define

$$\begin{aligned} v_1 &= \lambda^+ s, \qquad z_1 = \lambda A x, \;\; b_1 = \lambda e, \\ v_2 &= \mu^+ t, \qquad z_2 = \mu C y, \;\; b_2 = \mu f. \end{aligned}$$

Clearly $\beta = b_1 + b_2$. Defining P_1, P_2 in Lemma 4 in terms of the variables v_1, v_2, z_1, z_2, and letting $b_3 = \beta$, we see that (10) can be written as

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil .$$

By the proof of Lemma 4, there are numbers μ_1 and μ_2 such that $\mu_1 + \mu_2 \geq \hat{b}_3 \lceil b_3 \rceil$ and for $i = 1, 2$, $v_i + \hat{b}_3 z_i \geq \mu_i$ is a non-negative combination of the inequalities defining P_i and the basic mixed-integer inequality for P_i. Consider the case $i = 1$. Substituting out the slacks s and t, we see that $v_1 \geq 0$ and $v_1 + z_1 \geq b_1$ are both implied by non-negative combinations of the inequalities defining Q_1, and $v_1 + \hat{b}_1 z_1 \geq \hat{b}_1 \lceil b_1 \rceil$ defines an MIR cut for Q_1. Finally, $v_1 + \hat{b}_3 z_1 \geq \mu_1$ becomes $ax \leq g'$ for some g'.

Therefore, substituting out the slacks s, t, we conclude that there are numbers g' and h' such that

(i) $g' + h' \leq d$,

(ii) $ax \leq g'$ is a non-negative linear combination of the inequalities defining Q_1 and some MIR cut for Q_1,

(iii) $cx \leq h'$ is a non-negative linear combination of the inequalities defining Q_2 and some MIR cut for Q_2.

If we define

$$g = \max\{ax : x \in Q_1^{MIR}\}, \quad h = \max\{cy : y \in Q_2^{MIR}\},$$

$$\text{then } g \leq g', \quad h \leq h' \text{ and } g + h \leq d.$$

But recall that $sc(Q_1) = Q_1^{MIR}$ and $sc(Q_2) = Q_2^{MIR}$. The lemma follows. ∎

It may be possible to obtain a direct proof of the previous result without using the equivalence of split cuts and MIR cuts, but we believe it is not trivial.

Definition 6 *A split cut proof of an inequality is a* simplified *split cut proof if every inequality is either a non-negative linear combination of previous inequalities or a split cut derived from two previous inequalities.*

Lemma 7 *Given a split cut proof of length L of some inequality, there is a* simplified *split cut proof of the same inequality with length $3L$.*

Proof Let $cx \leq d$ be a split cut for P. Then $cx \leq d$ is valid for $P_1 = P \cap \{\alpha x \leq \beta\}$ and $P_2 = P \cap \{\alpha x \geq \beta + 1\}$, for some integral row vector α and integer β. There are multipliers $\lambda_1, \lambda_2 \in \mathbb{R}^m$ and $\mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$ such that

$$c = \lambda_1 Ax + \mu_1 \alpha, \qquad c = \lambda_2 Ax - \mu_2 \alpha,$$
$$d \geq \lambda_1 b + \mu_1 \beta, \qquad d \geq \lambda_2 b - \mu_2 (\beta + 1).$$

Clearly $cx \leq d$ is a split cut for $\{x \mid \lambda_1 Ax \leq \lambda_1 b, \lambda_1 Ax \leq \lambda_1 b\}$. A simplified split cut proof can be obtained by replacing each split cut in the proof with three inequalities: the inequalities $\lambda_1 Ax \leq \lambda_1 b$ and $\lambda_2 Ax \leq \lambda_2 b$, followed by split cut itself. ∎

If $cx \leq d$ is a split cut for P as in the proof above, we say it is derived using the multipliers λ_1, λ_2 and μ_1, μ_2.

We now have all the tools required to show that any split cut proof of the 0-1 infeasibility of the system $\mathcal{I}$ described in Section 2 has exponential length. The next theorem is a straightforward application of Pudlák's proof technique for GC cutting plane proofs.

Theorem 8 *Let $\mathcal{R}$ be a simplified split cut proof of $0^T x + 0^T y + 0^T z \leq -1$ from $\mathcal{I} = \{x, y, z \mid Ax + Cz \leq e, By + Dz \leq f\}$ of length L. Then there exists a monotone real circuit of size $O(Ln^2)$ solving $CLIQUE_{k,n}$.*

Proof Let $a_i x + b_i y + c_i z \leq d_i$ be the ith inequality in $\mathcal{R}$ and call this $\mathcal{R}_i$. Let $\mathcal{R}_1, \ldots, \mathcal{R}_m$ be just the inequalities in $Ax + Cz \leq e$ and $By + Dz \leq f$. Then $\mathcal{R}_L$ equals $\mathbf{0}x + \mathbf{0}y + \mathbf{0}z \leq -1$. Let I_i stand for $\{1, \ldots, i-1\}$. By definition, for $i > m$ $\mathcal{R}_i$ is either a non-negative linear combination of the inequalities $\mathcal{R}_1, \ldots, \mathcal{R}_{i-1}$ with the multipliers $\lambda_{ij} \geq 0\,(j \in I_i)$, or a split cut derived from $\mathcal{R}_k$ and $\mathcal{R}_l$ for some $k, l \leq i-1$.

Let z' stand for a 0-1 assignment to z. The inequality sequence $\mathcal{R}'$, where

$$\mathcal{R}'_i \text{ is } a_i x + b_i y \leq d_i - c_i z',$$

is a simplified split cut proof of $\mathbf{0}x + \mathbf{0}y \leq -1$ from $Ax \leq e - Cz'$ and $By \leq f - Dz'$ with the same length as $\mathcal{R}$: $\mathcal{R}'_i$ can be derived with the same multipliers as $\mathcal{R}_i$. Define $d'_i = d_i - c_i z'$. We construct a sequence of inequalities $\mathcal{S}$ involving only x, and another sequence $\mathcal{T}$, involving only y, such that

$$\mathcal{S}_i \equiv a_i x \leq g_i, \quad \mathcal{T}_i \equiv c_i y \leq h_i, \text{ and } g_i + h_i \leq d'_i$$

$$\mathcal{S}_i, \mathcal{T}_i \text{ are valid for integral solutions of } Ax \leq e - Cz',\ By \leq f - Dz'. \tag{12}$$

Thus $\mathcal{S}_i + \mathcal{T}_i$ has the same left hand side as $\mathcal{R}'_i$, but an equal or smaller right-hand side.

For $i = 1, \ldots, m$, if $\mathcal{R}'_i$ involves only x, then set $\mathcal{S}_i$ to $\mathcal{R}'_i$ and $\mathcal{T}_i$ to $\mathbf{0}y \leq 0$, otherwise set $\mathcal{S}_i$ to $\mathbf{0}x \leq 0$ and $\mathcal{T}_i$ to $\mathcal{R}'_i$. Define subsequent terms of $\mathcal{S}$ and $\mathcal{T}$ as follows. For $i = m+1, \ldots, k$, if $\mathcal{R}'_i$ is a non-negative linear combination of inequalities $\mathcal{R}'_j\,(j \in I_i)$ with the multipliers $\lambda_{ij} > 0\,(j \in I_i)$, then let $\mathcal{S}_i$ and $\mathcal{T}_i$ be non-negative linear combinations of $\mathcal{S}_j\,(j \in I_i)$ and $\mathcal{T}_j\,(j \in I_i)$, respectively, with the same multipliers $\lambda_{ij}(j \in I_i)$. If $\mathcal{R}'_i$ is a split cut derived from $\mathcal{R}'_k$ and $\mathcal{R}'_l$ for some $k, l \leq i-1$, then define

$$Q_1 = \{x : a_k x \leq g_k, a_l x \leq g_l\},\ Q_2 = \{y : c_k y \leq h_k, c_l y \leq h_l\}.$$

It follows from Lemma 5 that $\mathcal{R}'_i$ is implied by the inequalities defining Q_1 and Q_2 and some split cuts for these sets. More precisely, if we define

$$g_i = \max\{a_i x : x \in sc(Q_1)\}, \;\; h_i = \max\{c_i y : y \in sc(Q_2)\}, \;\; \text{then } g_i + h_i \leq d'_i. \tag{13}$$

We then define $\mathcal{S}_i$ to be $a_i x \leq g_i$, and $\mathcal{T}_i$ to be $c_i y \leq h_i$.

Observe that the inequality $g_i + h_i \leq d'_i$ in (12) is by definition true for $i = 1, \ldots, m$; either $g_i = d'_i$ and $h_i = 0$, or $h_i = d'_i$ and $g_i = 0$. Let $i > m$, and assume by induction that (12) is true for smaller values of i. If $\mathcal{R}_i$ is a non-negative combination of inequalities, then (12) is clearly true. If $\mathcal{R}_i$ is a split cut derived from two previous inequalities, then (12) holds as(13) holds (by Lemma 5). Therefore the last inequalities in $\mathcal{S}$ and $\mathcal{T}$ are, respectively, $\mathbf{0}x \leq g_L$ and $\mathbf{0}y \leq h_L$. As $d'_L = d_L = -1$, one of g_k and h_k is less than 0, and we have a proof of integer infeasibility of either $Ax \leq e - Cz'$ or $By \leq f - Dz'$.

Define a monotone circuit $\mathcal{C}$ as follows. It first computes $e - Cz'$ by monotone operations (recall $C \leq 0$) from the input vector z'. It then computes $g_1, g_2, \ldots, g_L$ by monotone operations as follows. First, $g_1, \ldots, g_m$ are trivially obtained from $d'_1, \ldots, d'_m$ as discussed above. For $i > m$, if $\mathcal{R}_i = \sum_{j \in I_i} \lambda_{ij} R_j$, then $g_i = \sum_{j \in I_i} \lambda_{ij} g_j$. We can assume that $O(n^2)$ multipliers are non-zero, by Caratheódory's theorem. Therefore we can assume $\mathcal{C}$ computes g_i using at most $O(n^2)$ monotone operations from (2) (the λ_{ij}s are fixed as $\mathcal{R}$ is fixed; they are also non-negative). If $\mathcal{R}_i$ is a split cut derived from two previous inequalities, then $\mathcal{C}$ computes g_i as in (13). Note that only g_k and g_l are variable in this computation, and thus the computation of g_i is a monotone operation. Finally, the circuit returns $thr(g_L, 0)$, which is a monotone operation. Therefore, if the circuit returns 0, then $g_k < 0$ and $Ax \leq e - Cz'$ has no integral solutions. If the output is 1, then h_L must be negative (if it were computed) and $By \leq f - Dz'$ has no integral solutions. ∎

Corollary 9 *Every split cut proof of* $\mathbf{0}x + \mathbf{0}y + \mathbf{0}z \leq -1$ *from* $\mathcal{I}$ *has exponential length.*

A natural question is: why do we prove the above result for split cuts, and not directly for MIR cuts, given that we use MIR cuts to prove Lemma 5? One can avoid introducing split cuts, and work entirely with MIR cuts, but one has to carefully deal with the additional slack variables in the definition of the MIR cut (7) to ensure that only monotone operations are used to generate $\mathcal{S}$.

Dash [9, Lemma 5.7] proved that a branch-and-cut proof of (integer) infeasibility $\mathcal{R}$ using lift-and-project cuts and GC cuts and branching on 0-1 variables can be transformed into a cutting plane proof of infeasibility $\mathcal{S}$ with length $s + t$, where s and t are the number of cuts and branching decisions in $\mathcal{R}$, respectively. In this proof, every branching decision is replaced by a lift-and-project cut. One can easily obtain the following result using the proof technique for the result above.

Theorem 10 *If there exists a branch-and-cut proof of the fact that a polyhedron P has no integral solutions using s split cuts, and branching on 0-1 variables with t branching decisions, there exists a split-cut proof of the same fact with length $s + t$. Therefore, any branch-and-cut proof of integer infeasibility of $\mathcal{I}$ has exponentially many cuts plus nodes.*

Many cutting planes discussed in the literature – besides GC cuts and lift-and-project cuts – are either special cases of split cuts (see [18]) or can be derived via a short (polynomial in the number of variables) split cut proof. For example, the pairing inequalities [14] form a subclass of split cuts. The two-step MIR inequalities in [10] can be obtained by a split cut proof of length 2. Any cutting plane proof of the integer infeasibility of $\mathcal{I}$ using the above cuts has exponential length.

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