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Limits on the Social Welfare of Maximal-In-Range Auction Mechanisms

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Abstract

Many commonly-used auction mechanisms are "maximal-in-range". We show that any maximal-in-range mechanism for n bidders and m items cannot both approximate the social welfare with a ratio better than $\min(n, m^\eta)$ for any constant $\eta < 1/2$ and run in polynomial time, unless $NP \subseteq P/poly$. This significantly improves upon a previous bound on the achievable social welfare of polynomial time maximal-in-range mechanisms of $2n/(n+1)$ for constant n. Our bound is tight, as a $\min(n, 2m^{1/2})$ approximation of the social welfare is achievable.

1 Introduction

In this paper, we analyze n-bidder combinatorial auctions with budget constraints. In this model, we consider a group of n bidders bidding on m items at an auction. Each bidder i has a private valuation $v_{i,j}$ for each item j, and a budget b_i. Player i values receiving a set $S \subseteq [m]$ at

$$\min\left(\left(\sum_{j\in S} v_{i,j}\right), b_i\right).$$

The goal of the auction is to allocate items to bidders in order to maximize the total value summed over all of the bidders. This sum is called the *social welfare*. Even if we know the values $v_{i,j}$ and the budgets b_i for all i, it is NP-hard to maximize the social welfare, although an FPTAS does exist for a constant number of bidders [AM04].

The auction setting introduces the additional difficulty that the bidders will not reveal their true valuations unless it is to their advantage to do so. So in order to even learn the parameters of the problem, it is necessary to design an auction in such a way that each bidder's profit is maximized by revealing their true valuations. Such an auction mechanism is called *truthful*.

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A natural property shared by many truthful mechanisms is that they are maximal-in-range [MPSS09]. This means that the mechanism $\mathcal{M}$ has an associated range R of possible assignments of items to bidders, and $\mathcal{M}$ will always maximize the social welfare over R. All maximal-in-range allocation schemes can be implemented truthfully by a VCG mechanism.

Since a mechanism is only useful if it can be implemented, it is desirable that mechanisms do not require intractible computations. In this paper, we restrict our study to *polynomial-time mechanisms*, which only require computation time polynomial in n and m, the number of bidders and items being auctioned, respectively. We assume unless otherwise specified that $n = n(m) \leq poly(m)$, and that m is the main growing parameter. Super-polynomial n can be handled using stronger complexity assumptions; see Section 3.

The technique we use to demonstrate the limits of polynomial-time maximal-in-range mechanisms relies upon showing that the range has a large VC dimension[1] which permits the embedding of hard sub-problems. This general technique was originated in [PSS08] to show inapproximability for the Combinatorial Public Projects Problem. More recently, the same idea was used in [MPSS09] to show that a polynomial-time maximal-in-range mechanism cannot approximate the social welfare for n bidders with any constant ratio less than $\frac{2n}{n+1}$ unless $NP \subseteq P/poly$.

In this paper, we use the same general framework, together with several new ideas, to show that no polynomial-time maximal-in-range mechanism can approximate the social welfare with a ratio better than $\min(n, m^\eta)$ for any constant $\eta < 1/2$ unless $NP \subseteq P/poly$. There are three new ideas that lead to our improvements:

1. We devise a counting argument that shows that there must be a reasonably large subset of items that are *fully* allocated in exponentially many different ways by the range. This allows us to overcome one of the main difficulties in analyzing maximal-in-range mechanisms, which is dealing with unallocated items (indeed, [MPSS09] are able to achieve an *optimal* ratio for the 2 bidder case when the mechanism is *required* to allocate all items).

2. We use Sauer's Lemma in a non-direct way to argue that there exists a partition of the bidders into a single bidder and "everyone else," so that all possible splittings of the relevant items across this partition are realized by the range.

3. We show that this "splitting" structure, while weaker than the full n-ary shattering used in [MPSS09] (for $n > 2$), nevertheless admits an embedding of Subset Sum.

Our bound (of $\min(n, m^\eta)$ for any constant $\eta < 1/2$) is tight, as [DNS05] show a simple maximal-in-range algorithm which achieves a ratio of $\min(n, 2m^{1/2})$. For completeness, we include in the next subsection a brief description of their algorithm and a proof that it achieves a $\min(n, 2m^{1/2})$ ratio. That it is maximal-in-range and runs in polynomial time can be easily verified. This algorithm works for all *subadditive valuations with free disposal*. These are valuations in which $v_i(S) \leq v_i(S_1) + v_i(S_2)$ when S is the disjoint union of S_1 and S_2, and also $v_i(S) \leq v_i(S')$ when $S \subseteq S'$.

1.1 An algorithm achieving a $\min(n, 2m^{1/2})$ approximation ratio

Given valuation functions v_i for each bidder i, first form a bipartite graph with nodes on one side representing items and nodes on the other representing bidders. Form edges with weight $v_i(j)$ between the

[1]Actually we show that the VC dimension is large after an injective mapping, from which we infer that the range possesses a weaker, but still useful type of "diversity".

nodes representing bidder i and item j. Find a maximum weighted matching in this graph. Call the value of this matching $V_{matching}$. Now, consider $v_i([m])$, the value to player i of getting all the items. Let $V_{all} = \max_i v_i([m])$, and let i^* be the bidder that maximizes $v_i([m])$. If $V_{matching} \geq V_{all}$, assign items to bidders as in the maximum weighted matching. Otherwise, give every item to bidder i^*.

Theorem 1 ([DNS05], slightly rephrased)**.** *The above algorithm achieves a* $\min(n, 2m^{1/2})$ *approximation of the social welfare under subadditive valuations with free disposal.*

Proof. Consider an assignment A which maximizes the social welfare. There are at most $\sqrt{m}$ bidders which get $\sqrt{m}$ or more of the items each. Call this set of bidders B_{high}, and call the others B_{low}.

If the bidders in B_{high} get more than half of the social welfare, V_{all} will be at least as great as the maximum value received by any bidder in B_{high}. Thus, V_{all} is at least $1/\sqrt{m}$ times the social welfare from bidders in B_{high}. So the social welfare is at most $2\sqrt{m}$ times V_{all}. Similarly, since there are n bidders overall, the social welfare is at most n times V_{all} regardless of how the social welfare is distributed among B_{high} and B_{low}.

Otherwise, the bidders in B_{low} get at least half the social welfare. Consider the matching in the bidder-and-item graph in which every bidder in B_{low} receives the item maximizing $v_i(j)$ out of the items assigned to them in A. Since the valuations are subadditive and each bidder in B_{low} receives at most $\sqrt{m}$ items, the total value of B_{low} is at most $\sqrt{m}$ times the value of this matching. Since $V_{matching}$ is the maximal value over any matching, we see that the social welfare from B_{low} is at most $\sqrt{m}V_{matching}$. Thus, the social welfare of A is at most $2\sqrt{m}$ times $V_{matching}$.

Since V_{all} is always an n approximation and one of $V_{all}, V_{matching}$ is a $2\sqrt{m}$ approximation of the social welfare, assigning items to achieve the max of these two welfares yields a $\min(n, 2\sqrt{m})$ approximation. $\square$

1.2 History of this paper

The problem formulation and the general framework for attacking it appear in [MPSS09], which is as yet unpublished; our results would not have been possible without [MPSS09] and we are grateful to the authors of that paper for sharing their manuscript with us. Dughmi, Fu and Kleinberg [DFK09] have independently obtained the same results as in this paper, for all *constant* n, using different techniques. Both sets of authors are aware of the others' work, and we exchanged manuscripts at the beginning of June. Dughmi, Fu and Kleinberg have subsequently extended their results to a class of randomized mechanisms that they denote by MIWR, and a broader class of valuation functions (any valuation function for which the 2-player maximization problem is NP-hard).

2 Main Result

In this section, we prove the main theorem:

Theorem 2. *Let* $\mathcal{M}$ *be a polynomial-time maximal-in-range mechanism for auctions with* n *bidders and* m *items, with* $n = n(m) \leq m^\eta$ *for positive constant* $\eta < 1/2$*. If* $\mathcal{M}$ *approximates the social welfare with a ratio of* $n/(1+\epsilon)$ *for positive constant* ϵ*, then* $NP \subseteq P/poly$*.*

Theorem 2 is a direct consequence of Lemmas 5, 8 and 9 below. It also leads to the following theorem, which shows that it is not possible to find a polynomial-time maximal-in-range mechanism that achieves an approximation much better than the $\min(n, 2m^{1/2})$ in Theorem 1.

Theorem 3. *For any positive constant ϵ and $n = n(m) \leq poly(m)$, no polynomial-time maximal-in-range auction mechanism can approximate the social welfare with a ratio better than* $\min(n, m^{1/2-\epsilon})$ *by a constant factor unless $NP \subseteq P/poly$.*

Proof. This follows from Theorem 2 by simply noting that any mechanism $\mathcal{M}$ which performs well on $n = n(m) \leq m^{1/2-\epsilon}$ bidders will perform well on $n = n(m) \leq poly(m)$ bidders when all but $m^{1/2-\epsilon}$ of the bidders have valuation functions which are identically zero. Thus, by setting all but $m^{1/2-\epsilon}$ of the valuation functions to 0, and simulating $\mathcal{M}$, we are effectively simulating $\mathcal{M}$ on an auction with $n = m^{1/2-\epsilon}$, as assigning items to bidders with valuations functions equal to zero has the same effect as not assigning them at all. Thus, setting $n' = \min(n, m^{1/2-\epsilon})$, we see by Theorem 2 that achieving an approximation ratio better than n' implies $NP \subseteq P/poly$. $\square$

We begin the proof of Theorem 2 by examining the structure of the range. Below we omit floors and ceilings when dealing with them would be routine.

2.1 The Counting Argument

Let $\mathcal{M}$ be a maximal-in-range mechanism with range $R \subseteq ([n] \cup \{\star\})^m$. For a vector $x \in R$, $x_i = j$ means that item i is given to bidder j, while $x_i = \star$ indicates that no bidder is given item i. For $S \subseteq [m]$, we define R_S to be the subset of the range where all of the items in S are assigned to bidders,

$$R_S = \{x \in R \ : \ x_i \in [n] \text{ for all } i \in S\}.$$

When considering R_S we wish to focus on the bidders that the items in S are assigned to, so we define T_S to be the projection of R_S to the indices in S. So $T_S \subseteq [n]^{|S|}$.

In order to show that $\mathcal{M}$ can solve a hard problem, we will show that there is some T_S with sufficiently many elements so that subset sum can be embedded in the valuations of S by the various bidders in such a way that $\mathcal{M}$ will solve it. This differs from the approach in [MPSS09] in that by focusing on a portion of the range such that there are no unassigned items within a fixed subset S, we can ignore the difficulties associated with unassigned items. This idea allows for simpler and more powerful arguments. First, we show that there must be some exponentially large T_S. We begin with a helpful lemma.

Lemma 4. *For any positive constant ϵ and any m, n for which the binomial coefficients below are positive,*

$$\frac{\binom{m}{\epsilon m/n}}{\binom{((1+2\epsilon)/n)m}{\epsilon m/n}} < \left(\frac{n}{1+\epsilon}\right)^{\epsilon m/n}.$$

Proof. First, note that

$$\frac{\binom{m}{\epsilon m/n}}{\binom{((1+2\epsilon)/n)m}{\epsilon m/n}} = \prod_{i=0}^{\epsilon m/n-1} \frac{m-i}{((1+2\epsilon)/n)m-i}.$$

Now,

$$\begin{aligned}
\frac{m-i}{((1+2\epsilon)/n)m-i} &= \frac{m-i}{((1+2\epsilon)/n)m-i} \\
&< \frac{m}{(1+2\epsilon)m/n - \epsilon m/n} \\
&= \frac{n}{1+\epsilon}
\end{aligned}$$

So multiplying the $\epsilon m/n$ terms together, we have

$$\begin{aligned}
\frac{\binom{m}{\alpha m}}{\binom{((1+2\epsilon)/n)m}{\epsilon m/n}} &= \prod_{i=0}^{\epsilon m/n-1} \frac{m-i}{((1+2\epsilon)/n)m-i} \\
&< \prod_{i=0}^{\epsilon m/n-1} \frac{n}{1+\epsilon} \\
&= \left(\frac{n}{1+\epsilon}\right)^{\epsilon m/n},
\end{aligned}$$

which proves the lemma. □

Lemma 5. *Let $\mathcal{M}$ be a maximal-in-range mechanism for auctions with n bidders and m items that approximates the social welfare with a ratio of $n/(1+2\epsilon)$, for positive constant ϵ. Then there exists a set $S \subseteq [m]$ with $|S| = \epsilon m/n$ where T_S has size $|T_S| \geq (1+\epsilon)^{\epsilon m/n}$.*

Proof. To begin, we associate with each $x \in [n]^m$ a set of valuation functions. The valuation functions are such that

$$\begin{aligned}
v_{i,j} &= \begin{cases} 1 & x_j = i \\ 0 & \text{otherwise} \end{cases} \\
b_i &= m.
\end{aligned}$$

Let $x \in [n]^m$. Because $\mathcal{M}$ approximates the social welfare with a ratio of $(1+2\epsilon)/n$ and the maximum social welfare is m, there must be a member $r \in R$ of the range such that $r_i = x_i$ for at least $((1+2\epsilon)/n)m$ different indices i. Let S_x be the set of these indices,

$$S_x = \{i \,:\, r_i = x_i\}.$$

There are at least $\binom{|S_x|}{\epsilon m/n} \geq \binom{((1+2\epsilon)/n)m}{\epsilon m/n}$ subsets $S' \subseteq S_x$ of size $\epsilon m/n$. For each such set S', $T_{S'}$ contains the projection of x to S'. If $T_{S'}$ contains the projection of x to S', we say that x is *covered* by $T_{S'}$. If $t \in T_{S'}$ is the projection of x to S', we say that t *covers* x.

For a subset $S \subseteq [m]$, define $C(S)$ to be the number of vectors $x \in [n]^m$ which are covered by T_S. Since each $x \in [n]^m$ is covered by at least $\binom{((1+2\epsilon)/n)m}{\epsilon m/n}$ sets T_S with $|S| = \epsilon m/n$,

$$\sum_{S\subseteq[m],|S|=\epsilon m/n} C(S) \geq n^m \binom{((1+2\epsilon)/n)m}{\epsilon m/n}. \tag{1}$$

We now bound the sum $\sum_{S\subseteq[m],|S|=\epsilon m/n} C(S)$. Suppose by way of contradiction that for every subset $S \subseteq [m]$ of size $\epsilon m/n$, $|T_S| < (1+\epsilon)^{\epsilon m/n}$. Consider a subset $S \subset [m]$ such that $|S| = \epsilon m/n$. Each $t \in T_S$ covers $n^{m-\epsilon m/n}$ elements of $[n]^m$. So $C(S) < (1+\epsilon)^{\epsilon m/n} n^{m-\epsilon m/n}$, which gives the bound

$$\sum_{S\subseteq[m],|S|=\epsilon m/n} C(S) < \binom{m}{\epsilon m/n} (1+\epsilon)^{\epsilon m/n} n^{m-\epsilon m/n}. \tag{2}$$

So by Equations 1 and 2, we have

$$\binom{m}{\epsilon m/n} (1+\epsilon)^{\epsilon m/n} n^{m-\epsilon m/n} > n^m \binom{((1+2\epsilon)/n)m}{\epsilon m/n},$$

which we simplify to

$$\frac{\binom{m}{\epsilon m/n}}{\binom{((1+2\epsilon)/n)m}{\epsilon m/n}}(1+\epsilon)^{\epsilon m/n} > n^{\epsilon m/n}. \tag{3}$$

By Lemma 4, we get

$$\frac{\binom{m}{\epsilon m/n}}{\binom{((1+2\epsilon)/n)m}{\epsilon m/n}}(1+\epsilon)^{\epsilon m/n} < \left(\frac{n}{1+\epsilon}\right)^{\epsilon m/n}(1+\epsilon)^{\epsilon m/n} = n^{\epsilon m/n},$$

which contradicts Equation 3, proving that there exists some $S \subseteq [m]$ with $|S| = \epsilon m/n$ such that $|T_S| \geq (1+\epsilon)^{\epsilon m/n}$. □

2.2 Using the VC Dimension

In previous work [MPSS09], showing a size $2^{\Omega(m)}$-sized subset was used to show large VC-dimension in the $n = 2$ case. Unfortunately, this does not generalize well to auctions with three or more bidders because for $n > 2$ there exist sets of size $(n-1)^m > 2^m$ with n-ary VC dimension equal to 0. To get around this difficulty, we map T_S injectively from $[n]^{\epsilon m/n}$ into $[2]^{\epsilon m}$, and show that the image of this map has a large VC dimension. The large VC dimension then permits the embedding of an NP-hard problem (see Section 2.3. In order to show a lower-bound on the VC dimension, we use Sauer's Lemma:

Lemma 6 (Sauer's Lemma)**.** *Let S be a subset of $[2]^\ell$ with $|S| > \sum_{i=0}^{k-1}\binom{\ell}{i}$. The VC dimension of S is at least k.*

We will make use of the following corollary:

Corollary 7. *Let T be a subset of $[2]^\ell$. For any constant $\delta > 1/2$ and any $\epsilon > 0$, the following holds for all sufficiently large ℓ: if $|T| > (1+\epsilon)^{\epsilon\ell^\delta}$ then T has VC dimension at least $\ell^{1/2}$.*

Proof. Since for sufficiently large ℓ, $\ell^{1/2} < \ell/2$,

$$\begin{aligned}
\sum_{i=0}^{\ell^{1/2}-1}\binom{\ell}{i} &\leq \sum_{i=0}^{\ell^{1/2}-1}\binom{\ell}{\ell^{1/2}} \\
&\leq \ell^{1/2}\left(\frac{e\ell}{\ell^{1/2}}\right)^{\ell^{1/2}} \\
&= \ell^{1/2}\left(e\ell^{1/2}\right)^{\ell^{1/2}} \\
&= (1+\epsilon)^{1/2\log_{1+\epsilon}\ell+\ell^{1/2}\log_{1+\epsilon}(e\ell^{1/2})} \\
&= (1+\epsilon)^{\ell^{1/2}((1/2)\log_{1+\epsilon}\ell+\log_{1+\epsilon}e+o(1))} \\
&= (1+\epsilon)^{\ell^{1/2+o(1)}}
\end{aligned}$$

which is less than $|T| = (1+\epsilon)^{\epsilon\ell^\delta}$ for sufficiently large ℓ, since $\delta > 1/2$. □

Let ϕ_i be the map

$$\phi_i(j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

The next lemma is the main lemma in this section; it refers to the range R and the subsets T_S defined in Section 2.1.

Lemma 8. *Let $\mathcal{M}$ be a maximal-in-range mechanism for auctions with n bidders and m items, with $n = n(m) \leq m^\eta$ for positive constant $\eta < 1/2$. For all sufficiently large m, if there exists a subset $S \subseteq [m]$ with $|S| = \epsilon m/n$ such that $|T_S| \geq (1+\epsilon)^{\epsilon m/n}$, then there exists a bidder i^* such that $\phi_{i^*}(R)$ has VC dimension at least $\sqrt{\epsilon} \cdot m^{1/2-\eta}$.*

Proof. Define vectors $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the single 1 is in position j, and the number of coordinates of e_j is n. We define $f : [n]^{\epsilon m/n} \to [2]^{n\epsilon m/n} = [2]^{\epsilon m}$ by $f(x) = e_{x_1} e_{x_2} \cdots e_{x_{\epsilon m/n}}$. We write $f(T)$ for a subset T to mean the set $\{f(t) : t \in T\}$.

The function f is injective, so

$$|f(T_S)| = |T_S| \geq (1+\epsilon)^{\epsilon m/n}.$$

Note that $(1+\epsilon)^{\epsilon m/n} \geq (1+\epsilon)^{\epsilon m^{1-\eta}} \geq (1+\epsilon)^{\epsilon(\epsilon m)^{1-\eta}}$. We are assuming that m is sufficiently large, so we can apply Corollary 7 (with $\delta = 1 - \eta > 1/2$ and $\ell = \epsilon m$) to conclude that $f(T_S)$ has VC dimension at least $(\epsilon m)^{1/2}$.

Let Q be a size $(\epsilon m)^{1/2}$ subset of $[\epsilon m]$ that is shattered by $f(T_S)$. Recall that each member of $f(T_S)$ is the concatenation of vectors of length n, where a 1 in the ith position of the jth such vector corresponds to the ith bidder getting item j. In this way each element in Q corresponds to one of the n bidders. Partition Q into sets Q_i, where Q_i contains those coordinates that correspond to bidder i. There are n parts in the partition, so there is some $i^* \in [n]$ for which Q_{i^*} has size at least $(\epsilon m)^{1/2}/n$.

Since Q is shattered by $f(T_S)$, so is the subset Q_{i^*}. This means exactly that $\phi_{i^*}(T_S)$ has VC dimension at least $|Q_{i^*}| \geq (\epsilon m)^{1/2}/n$. Since the members of T_S are projections of members of R onto the coordinates in S, this implies that $\phi_{i^*}(R)$ also has VC dimension at least $(\epsilon m)^{1/2}/n \geq \sqrt{\epsilon} \cdot m^{1/2-\eta}$. □

2.3 Embedding Subset Sum

We now show that if $\phi_{i^*}(R)$ has VC dimension at least m^γ for constant $\gamma > 0$, we can embed a subset sum instance into the auction in such a way that it is solved by $\mathcal{M}$. We use a reduction similar to one used in [LLN06] to show that exactly maximizing the social welfare of these auctions is NP-hard.

Lemma 9. *Let $\mathcal{M}$ be a polynomial-time maximal-in-range mechanism for auctions with n bidders and m items. Suppose there exists a constant $\gamma > 0$ such that for all sufficiently large m, there exists a bidder i^* such that $\phi_{i^*}(R)$ has VC dimension at least m^γ (where R is the range). Then $NP \subseteq P/poly$.*

Proof. We take as advice the set $L \subseteq [m]$ of size m^γ that is shattered by $\phi_{i^*}(R)$. For ease of exposition we re-order the items so that L is the set of the first m^γ items. Let $a_1, \ldots, a_{m^\gamma}$ be a subset sum instance with target sum K. For all bidders $i \neq i^*$, we set

$$\begin{aligned} v_{i,j} &= \begin{cases} a_j, & j \leq \gamma m \\ 0, & j > \gamma m \end{cases} \\ b_i &= \sum_j a_j \end{aligned}$$

and for bidder i^*, we set

$$\begin{aligned} v_{i^*,j} &= \begin{cases} 2a_j, & j \leq \gamma m \\ 0, & j > \gamma m \end{cases} \\ b_{i^*} &= 2K. \end{aligned}$$

If there is a subset V of $\{a_1, \ldots, a_{m^\gamma}\}$ summing to K, there is an assignment in R with social welfare of $\sum_j a_j + K$. This can be any assignment where bidder i^* gets the items in V, and the other items are distributed among the other bidders. R must contain such an assignment because $\phi_{i^*}(R)$ shatters L. Since $\mathcal{M}$ is maximal-in-range with range R, it will output an assignment with at least this social welfare.

If there is no subset of $\{a_1, \ldots, a_{m^\gamma}\}$ summing to K, $\mathcal{M}$ will assign bidder i^* a subset $V \subseteq M$ such that $\sum_{j \in V} a_j \neq K$. If $\sum_{j \in V} a_j < K$, the total value is at most

$$\begin{aligned} \sum_{j \notin V} a_j + \sum_{j \in V} 2a_j &= \sum_j a_j + \sum_{j \in V} a_j \\ &< \sum_j a_j + K. \end{aligned}$$

If $\sum_{j \in V} a_j > K$, bidder i^* gets value $2K$. Let $\ell = \sum_{j \in V} 2a_j - 2K$. The total value is at most

$$\begin{aligned} \sum_{j \notin V} a_j + \sum_{j \in V} 2a_j - \ell &= \sum_j a_j + \sum_{j \in V} a_j - \ell \\ &= \sum_j a_j + 2K - \sum_{j \in V} a_j \\ &< \sum_j a_j + 2K - K \\ &= \sum_j a_j + K. \end{aligned}$$

So every assignment has social welfare less than $\sum_j a_j + K$. So taking L as advice, we can solve a subset sum instance with k integers in polynomial time (in $m = k^{1/\gamma}$ and the size of the binary representations of the integers). Therefore, subset sum is in $P/poly$, so $NP \subseteq P/poly$. $\square$

2.4 Proof of the Main Result

We can now prove Theorem 2. We have a polynomial-time maximal-in-range mechanism $\mathcal{M}$ for auctions with n bidders and m items, with $n = n(m) \leq m^\eta$ for positive constant $\eta < 1/2$. By Lemma 5, for each m there exists a subset $S \subseteq [m]$ of size $(\epsilon/2)m/n$ such that $|T_S| \geq (1 + \epsilon/2)^{(\epsilon/2)m/n}$. By Lemma 8, this implies that for sufficiently large m, the range of $\mathcal{M}$ has VC dimension at least $\sqrt{\epsilon/2} \cdot m^{1/2-\eta}$. Since $\eta < 1/2$, we have $\sqrt{\epsilon/2} \cdot m^{1/2-\eta} \geq m^\gamma$ for some fixed positive constant γ and sufficiently large m. By Lemma 9, we thus have that $NP \subseteq P/poly$.

3 Super-polynomially many bidders

In this section, we observe that our results can be extended to handle the case of n super-polynomial in m, at the expense of a stronger complexity assumption. For n larger than m, our technique shows a limit of $m^{1/2-\epsilon}$ on the approximation ratio of any mechanism which runs in time polynomial in m. However, by our definition an efficient mechanism need only run in time polynomial in n and m, which is greater than $poly(m)$ for super-polynomial n. By strengthening the complexity assumption, we can still prove limits on the achievable social welfare.

For instance, if n is sub-exponential in m, we can begin by assuming that NP does not have sub-exponential size circuits. Then applying the same reduction leads to a circuit family of size $poly(n, m)$ (or sub-exponential in m), which solves subset sum instances of size m^γ for constant $\gamma > 0$, and this implies that NP has subexponential size circuits.

If n is sufficiently large as a function of m, it can even become possible to maximize the social welfare exactly in polynomial time.

Theorem 10. *There exists a maximal-in-range mechanism $\mathcal{M}$ for auctions with n bidders and m items, which maximizes the social welfare and runs in polynomial time when $B_m \in O(poly(n))$, where B_m is the mth Bell number, the number of partitions of $[m]$ into any number of disjoint subsets with union $[m]$.*

Proof. If $B_m \in O(poly(n))$, it is possible to enumerate all of the partitions of $[m]$ into any number of disjoint subsets in polynomial time. For each such partition into sets $S_1, \ldots, S_k$, form a bipartite graph where one side has nodes representing the sets $S_1, \ldots, S_k$ and the other has nodes representing the bidders. The edge between bidder i and partition S_j has weight $v_i(S_j)$.

After finding a maximum weighted matching on each such bipartite graph, we can choose the maximum matching over all partitions. This matching represents the assignment which maximizes the social welfare. This can be easily seen because every assignment corresponds to a matching in the bipartite graph for some partition. □

4 Conclusions

We have shown that no polynomial-time maximal-in-range auction mechanism can approximate the social welfare to a ratio better than $\min(n, m^{1/2-\epsilon})$ by a constant factor. This essentially resolves the maximum social welfare achievable by efficient maximal-in-range auction mechanisms for any class of valuations including the valuation functions we considered, as a $\min(n, 2m^{1/2})$ ratio is achievable.

There is an asymmetry as to the strength of the n and $m^{1/2-\epsilon}$ bounds, however, as the n bound eliminates the possibility of a ratio of $n/(1+\epsilon)$ being achieved, but the $m^{1/2-\epsilon}$ bound leaves open the possibility of achieving a $m^{1/2-o(1)}$ approximation.

For super-polynomial n, we have demonstrated similar limits under stronger complexity assumptions, up to n being sub-exponential in m. We also showed that for sufficiently large n, a polynomial-time maximal-in-range auction mechanism exists.

While this largely resolves the performance of maximal-in-range mechanisms, it leaves open the larger question of how well truthful mechanisms perform.

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