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On the Structure of Cubic and Quartic Polynomials

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Abstract

In this paper we study the structure of polynomials of degree three and four that have high bias or high Gowers norm, over arbitrary prime fields. In particular we obtain the following results.

1. Let f be a degree three polynomial with $\text{bias}(f) = \delta$ then there exist $r = O(\log(1/\delta))$ quadratic polynomials $\{q_i\}$, $c = O(\log^4(\frac{1}{\delta}))$ linear functions $\{\ell_i\}$ and a degree three polynomial g such that $f = \sum_{i=1}^{r} \ell_i \cdot q_i + g(\ell_1, \ldots, \ell_c)$. This result generalizes the corresponding result for quadratic polynomials.
2. Let $\deg(f) = 4$ and $\text{bias}(f) = \delta$. Then $f = \sum_{i=1}^{r} \ell_i \cdot g_i + \sum_{i=1}^{r} q_i \cdot q'_i$, where $r = \text{poly}(1/\delta)$, the ℓ_i-s are linear, the q_i-s are quadratics and the g_i-s are cubic.
3. Let $\deg(f) = 4$ and $\|f\|_{U^4} = \delta$. Then there exists a partition of a subspace $V \subseteq \mathbb{F}^n$, $\dim(V) \geq n - O(\log(1/\delta))$, to subspaces $\{V_\alpha\}$, such that $\forall \alpha$ $\dim(V_\alpha) \geq n/\exp(\log^2(1/\delta))$ and $\deg(f|_{V_\alpha}) = 3$.

Items 1,2 extend and improve previous results for degree three and four polynomials [KL08, GT07]. Item 3 gives a new result for the case of degree four polynomials with high U^4 norm. It is the first case where the inverse conjecture for the Gowers norm fails [LMS08, GT07], namely that such an f is not necessarily correlated with a cubic polynomial. Our result shows that instead f equals a cubic polynomial on a large subspace (in fact we show that a much stronger claim holds).

Our techniques are based on finding a structure in the space of partial derivatives of f. For example, when $\deg(f) = 4$ and f has high U^4 norm we show that there exist quadratic polynomials $\{q_i\}_{i\in[r]}$ and linear functions $\{\ell_i\}_{i\in[R]}$ such that (on a large enough subspace) every partial derivative of f can be written as $\Delta_y(f) = \sum_{i=1}^{R} \ell_i \cdot q_i^y + \sum_{i=1}^{r} q_i \cdot \ell_i^y + q_0^y$, where ℓ_i^y, q_i^y depend on y, the direction of the partial derivative, $r = O(\log^2(1/\|f\|_{U_4}))$ and $R = \exp(r)$.

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1 Introduction

Assume that we are given a degree d polynomial f that, in some sense, 'behaves' differently from a random degree d polynomial. Is there anything that we can deduce about the structure of f just by knowing this fact? Recently this question received a lot of attention, where the 'behavior' of f was examined with respect to its bias or the more general notion of the Gowers norm.

Definition 1.1. *Let $f : \mathbb{F}^n \to \mathbb{F}$ be a function. The bias of f is defined as*

$$\text{bias}(f) = \left|\mathbb{E}_{\bar{a}\in\mathbb{F}^n}[\omega^{f(\bar{a})}]\right| ,$$

where $\omega = e^{\frac{2\pi i}{|\mathbb{F}|}}$ is a complex primitive root of unity of order $|\mathbb{F}|$.

Intuitively, the bias of f measures how far is the distribution induced by f from the uniform distribution. We expect a random polynomial to have a vanishing small bias (as a function of the number of variables), so it is interesting to know what can be said when the bias is not too small. Indeed, Green and Tao [GT07] showed that if f is a degree d polynomial over $\mathbb{F}$, such that $d < |\mathbb{F}|$, and $\text{bias}(f) = \delta$ then f can be written as a function of a small number of lower degree polynomial. Formally, $f(x) = F(g_1, \ldots, g_{c_d})$ for some function F and $c_d = c_d(\text{bias}(f), |\mathbb{F}|)$ polynomials $\{g_i\}$ satisfying $\deg(g_i) < d$. Note that $c_d = c_d(\text{bias}(f), |\mathbb{F}|)$ does not depend on the number of variables, i.e. it is some constant. This result was later extended by Kaufman and Lovett [KL08] to arbitrary finite fields (i.e. without the restriction $d < |\mathbb{F}|$). Thus, if f has a noticeable bias, unlike a random degree d polynomial, then f is in fact very far from being random; simple counting arguments show that most degree d polynomials cannot be represented as functions of a few lower degree polynomials. This result is also interesting as it gives an average case - worst case reduction. Namely, if f has correlation δ with a lower degree polynomial then it is a function of a small number of lower degree polynomials. One drawback of the results of [GT07, KL08] is the dependance of the number of lower degree polynomials on the bias of f. In particular when $\deg(f) = 3$, [GT07, KL08] get the bound $c_3 = \exp(\text{poly}(1/\text{bias}(f)))$ and for $\deg(f) = 4$ they bound[1] c_4 by a tower of height c_3. On the other hand if $\deg(f) = 2$ and $\text{bias}(f) = \delta$ then it is known that f can be written as a function of at most $2\log(1/\delta) + 1$ linear functions. This can be immediately deduced from the following well known theorem.

Theorem 1.1 (Structure of quadratic polynomials)**.** *(Theorems 6.21 and 6.30 in [LN97]). For every quadratic polynomial $f : \mathbb{F}^n \to \mathbb{F}$ over a prime field $\mathbb{F}$ there exists an invertible linear transformation T, a linear polynomial ℓ, and field elements $\alpha_1, \ldots \alpha_n$ (some of which may be 0) such that:*

1. *If $\text{char}(\mathbb{F}) = 2$ then $(q \circ T)(x) = \sum_{1=i}^{\lfloor n/2 \rfloor} \alpha_i \cdot x_{2i-1} \cdot x_{2i} + \ell(x)$,*
2. *If $\text{char}(\mathbb{F})$ is odd then $(q \circ T)(x) = \sum_{1=i}^{n} \alpha_i \cdot x_i^2 + \ell(x)$.*

Moreover, the number of non zero α_i-s is invariant and depends only on f.

[1] These numbers are not explicitly computed there, but this is what the recursive arguments in the papers imply.

We thus see that there is a sharp contrast between the result for quadratic polynomials and the results for polynomials of degrees as low as three or four. We also note that the results of Kaufman and Lovett only guarantee that f can be represented as $f(x) = F(g_1, \ldots, g_c)$ but no nice structure like the one in Theorem 1.1 is known. It is thus an intriguing question whether a nice structural theorem exists for biased polynomials and what is the correct dependance of the number of lower degree polynomials on $\deg(f)$ and $\text{bias}(f)$.

As mentioned above, a more general measure of randomness that was considered is the so called Gowers norm of f. Intuitively, the U^d Gower norm tests whether f behaves like a degree $d-1$ polynomial on d dimensional subspaces. To define the Gowers norm we first define the notion of a discrete partial derivative.

Definition 1.2. *(Discrete partial derivative) For a function $f : \mathbb{F}^n \to \mathbb{F}$ and a direction $y \in \mathbb{F}^n$ we define $\Delta_y(f)(x) \triangleq f(x+y) - f(x)$ to be the discrete partial derivative of f in direction y at the point x.*

It is not difficult to see that if $\deg(f) = d$ then for every y, $\deg(\Delta_y(f)) \leq d-1$. We now define the d-th Gower norm of a function f.

Definition 1.3 (Gowers norm [AKK+05, Gow98, Gow01])**.** *The d-th Gower norm, U^d, of f is defined as*

$$\|f\|_{U^d} \triangleq |\mathbb{E}_{x,y_1,\ldots,y_d}[\omega^{\Delta_{y_d}\cdots\Delta_{y_1}(f)(x)}]|^{1/2^d} ,$$

where again $\omega = e^{\frac{2\pi i}{|\mathbb{F}|}}$.

Note that $\|f\|_{U^0} = \|f\|_{U^1} = \text{bias}(f)$. It is also clear that if $\deg(f) = d-1$ then $\|f\|_{U^d} = 1$. For more properties of the Gowers norm we refer the reader to [Gow98, Gow01, GT08, Sam07, VW07].

In [AKK+05] Alon et. al. showed that if $\|f\|_{U^d} > 1 - \delta$, for some small δ, then f can be well approximated by a degree $d-1$ polynomial. This raises the question whether any function that has a noticeable U^d norm is somewhat correlated with a lower degree polynomial and indeed in [Sam07, GT08] this was conjectured to be the case. This conjecture has become known as the inverse conjecture for the Gowers norm. Samorodnitsky [Sam07] proved that if $\|f\|_{U^3} = \delta$ where $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is an arbitrary function, then f has an exponentially high (in δ) correlation with a quadratic polynomial. Namely, there exists a quadratic polynomial q such that $\Pr_{x\in\mathbb{F}_2^n}[f(x) = q(X)] \geq 1/2 + \exp(-\text{poly}(1/\delta))$. Green and Tao [GT08] obtained a similar result for fields of odd characteristic. These results gave an affirmative answer for the case of the U^3 norm. More generally, Green and Tao proved that if $d < |\mathbb{F}|$ and f is a degree d polynomial with a high U^d norm then f is indeed correlated with a lower degree polynomial [GT07]. Recently, the case of large characteristic was solved by Tao and Ziegler [TZ08].[2] Using ideas from ergodic theory and the earlier [BTZ] they proved that if $f : \mathbb{F}^n \to \mathcal{D}$ (where $\mathcal{D}$ is the unit disk in $\mathbb{C}$) is a function with high U^d norm and $d \leq |\mathbb{F}|$ then f is correlated with a degree $d-1$ phase polynomial.[3] This completely settled the conjecture for the case $d \leq |\mathbb{F}|$. On the other hand, for the U^4 norm it was

[2] In fact,[TZ08] only get a qualitative result. No explicit connection is known between the Gowers norm and the correlation with polynomials.

[3] A degree $d-1$ phase polynomial is a function of the form $e^{2\pi i\theta}\omega^g$, for some degree $d-1-(p-1)t$ polynomial g where $\theta \in [0,1]$ and $\omega = e^{2\pi i/|\mathbb{F}|^t}$.

shown, independently, by Lovett, Meshulam and Samorodnitsky [LMS08] and by Green and Tao [GT07] that no such result is possible when $\mathbb{F} = \mathbb{F}_2$. Namely, [LMS08] proved that the symmetric polynomial $S_4(x_1, \ldots, x_n) \triangleq \sum_{T \subset [n], |T|=4} \prod_{i \in T} x_i$, which is of degree four, has a high U^4 norm but has an exponentially (in n) small correlation with any lower degree polynomial. Similar examples where given for other fields (when d is large enough compared to the size of the field). These examples show that for small fields the inverse conjecture for the Gowers norm is not true in its current form. In their work, Tao and Ziegler [TZ08] proved a variant of the conjecture for the case $d \leq |\mathbb{F}|$. Namely, that if a function f has high U^d norm then f is correlated with a phase polynomial of a certain constant degree (but not necessarily smaller than d). We note however, that if $\deg(f) = d$ then the results of [TZ08] do not give any information on f. In fact, even if $\deg(f) = 4$ and f has a high U^4 norm then nothing is known on the structure of f. It is thus a very interesting question to understand the structure of low degree polynomials having high Gowers norm over small fields.

Besides being natural questions on the own, results on the Gowers norms had many applications in mathematics and computer science. In his seminal work on finding arithmetic progressions in dense sets, Gowers first defined the U^d norm (for functions from $\mathbb{Z}_n$ to $\mathbb{Z}_n$) and proved an inverse theorem for them that was instrumental in his proofs [Gow98, Gow01]. Bogdanov and Viola [BV07] attempt for constructing a pseudo random generator for constant degree polynomials relied on the (erroneous) inverse conjecture for the Gowers norm, yet it paved the way for other papers solving the problem [Lov08, Vio08]. In [ST06] applications of an inverse theorem for the Gowers norm to PCP constructions was given. Samorodnitsky's proof of the inverse theorem for the U^3 norm [Sam07] implies a low degree test distinguishing quadratic functions from those that do not have a non trivial correlation with a quadratic function. This result also gives a test for checking the distance of a given word from the 2nd order Reed-Muller code, beyond the list decoding radius. For a more elaborate discussion of the connection between additive combinatorics and computer science see [Tre09].

1.1 Our results

In this work we are able to show analogs of Theorem 1.1 for polynomials of degree three and four. We also prove a structural result for the case that such a polynomial has a high Gowers norm. Our first main result is the following.

Theorem 1. *(biased cubic polynomials)Let $\mathbb{F}$ be a finite field and $f \in \mathbb{F}[x_1, \ldots, x_n]$ a cubic polynomial ($\deg(f) = 3$) such that $\mathrm{bias}(f) = \delta$. Then there exist $c_1 = O(\log(1/\delta))$ quadratic polynomials $q_1, \ldots, q_{c_1} \in \mathbb{F}[x_1, \ldots, x_n]$ and linear functions $\ell_1, \ldots, \ell_{c_1} \in \mathbb{F}[x_1, \ldots, x_n]$ and another $c_2 = O(\log^4(\frac{1}{\delta}))$ linear functions $\ell'_1, \ldots, \ell'_{c_2} \in \mathbb{F}[x_1, \ldots, x_n]$ such that $f = \sum_{j=1}^{c_1} \ell_j \cdot q_j + g(\ell'_1, \ldots, \ell'_{c_2})$, where g is cubic.*

We note that if it weren't for the $g(\ell'_1, \ldots, \ell'_{c_2})$ part then this result would be quantitatively the same as Theorem 1.1 (and tight of course). It is an interesting open question to decide whether we can do only with the $\sum_{j=1}^{O(\log_{|\mathbb{F}|} 1/\delta)} \ell_i \cdot q_i$ part. Using the same techniques we show a similar result for the case that $\|f\|_{U^3} > \delta$.

Theorem 2. *(cubic polynomials with high U^3 norm)Let $\mathbb{F}$ be a finite field and $f \in$*

$\mathbb{F}[x_1, \ldots, x_n]$ *a cubic polynomial such that* $\|f\|_{U^3} = \delta$. *Then there exist* $c + 1 = O(\log^2(\frac{1}{\delta}))$ *quadratic polynomials* $q_0, \ldots, q_c \in \mathbb{F}[x_1, \ldots, x_n]$ *and* c *linear functions* $\ell_1, \ldots, \ell_c \in \mathbb{F}[x_1, \ldots, x_n]$ *such that* $f = \sum_{j=1}^{c} \ell_j \cdot q_j + q_0$.

Note that the difference between the structure of f in Theorems 1 and 2 is the number of quadratic function required. Recall that in [Sam07] Samorodnitsky proved that if an $\mathbb{F}_2$ function f has a high U^3 norm then it has an exponentially (in $\|f\|_{U^3}$) high correlation with a quadratic polynomial. Thus, our theorem shows that when f is a cubic polynomial then a much stronger statement holds. Namely, f has correlation $\exp(\log^2(1/\delta))$ with a quadratic polynomial, and further, has a nice structure.

Our second main result is an analog of Theorem 1 for the case of quartic polynomials (i.e. $\deg(f) = 4$).

Theorem 3. *(biased quartic polynomials) Let* $\mathbb{F}$ *be a finite field and* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *a quartic polynomial (*$\deg(f) = 4$*) such that* $\mathrm{bias}(f) = \delta$. *Then there exist* $4c = \mathrm{poly}(|\mathbb{F}|/\delta)$ *polynomials* $\{\ell_i, q_i, q'_i, g_i\}_{i=1}^{c}$, *where the* ℓ_i*-s are linear, the* q_i*-s and* q'_i*-s are quadratic and the* g_i*-s are cubic such that* $f = \sum_{j=1}^{c} \ell_j \cdot g_j + \sum_{j=1}^{c} q_j \cdot q'_j$.

As mentioned above, prior to this result it was known that there exist C cubic polynomials $g_1, \ldots, g_C$ and a function F such that $f = F(g_1, \ldots, g_C)$, where C is a tower of height $\exp(\mathrm{poly}(1/\delta))$ [GT07, KL08]. Thus, our result greatly improves the dependance on δ and gives a nice structure for the polynomial. We note that in their work Green and Tao do show that such a nice structure exists when $d < |\mathbb{F}|$ [GT07], but no such result was known for smaller fields (in addition C needs to be even larger for such a nice representation to hold).

Our third main result is for the case where $\deg(f) = 4$ and $\|f\|_{U^4} = \delta$. In such a case it is known [LMS08, GT07] that we cannot hope to get a nice structure as in Theorem 2 as it may be the case that f has an exponentially small (in n) correlation with all lower degree polynomials. However, we do manage to show that there is some subspace $U \subset \mathbb{F}^n$ such that when restricted to V, $f|_U$ is equal to some degree three polynomial. Thus, f does not have a correlation with a cubic polynomial in the entire space but instead there is a large subspace on which it is of degree three. In fact we show a more general result. Namely, that there is a large subspace V, of dimension $n - O(\log(1/\delta))$, that can be partitioned to subspaces of dimension $n/\exp(\log^2(1/\delta))$ such that the restriction of f to any of the subspaces in the partition is of degree three.

Theorem 4. *(quartic polynomials with high* U^4 *norm) Let* $\mathbb{F}$ *be a finite field and* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *a degree four polynomial such that* $\|f\|_{U^4} = \delta$. *Then there exists a partition of a subspace* $V \subseteq \mathbb{F}^n$, *of dimension* $\dim(V) \geq n - O(\log(1|/\delta))$, *to subspaces* $\{V_\alpha\}_{\alpha \in I}$, *satisfying* $\dim(V_\alpha) = \Omega(n/|\mathbb{F}|^{\log^2(1/\delta)})$, *such that for every* $\alpha \in I$, $f|_{V_\alpha}$ *is a cubic polynomial.*

Remark 1.4. *Note that the structure guaranteed in Theorem 4 is shared by very few polynomials. Specifically, a random polynomial of degree four is unlikely to be equal to any degree three polynomial on any subspace of dimension larger than, say,* $n^{0.9}$. *To see this note that if* $|\mathbb{F}| = p$ *and* $\dim(V) = d$ *then there are roughly* p^{d^3} *cubic polynomials and* p^{d^4} *quartic polynomials over* V. *Furthermore, the map taking a quartic polynomial over* $\mathbb{F}^n$ *to its restriction is a linear map and so the fraction of quartic polynomials that equal a degree three polynomial on* V *is (roughly)* $p^{-d^4+d^3}$. *As the total number of subspaces can be bounded by* p^{n^2} *we get*

that the fraction of quartic polynomial that are equal to a degree three polynomial on some subspace of dimension greater than $n^{0.9}$ is at most $p^{n^2-n^{3.6}+n^3} = \exp(-n^{3.6})$.

This result has the same flavor as the inverse U^3 norm theorem of [GT08]. There it was shown that if $f : \mathbb{F}_5^n \to \mathbb{F}_5$ satisfies $\|f\|_{U^3} = \delta$ then there exists a subspace V of codimension $\mathrm{poly}(1/\delta)$, such that on an 'average' coset of V, f is correlated with a quadratic polynomial. Recently, Wolf [Wol09] proved a similar result for the case of characteristic two, thus extending Samorodnitsky's argument [Sam07]. The main difference between these results and our result is that ours only holds for polynomials of degree four whereas the results of [GT08, Sam07, Wol09] hold for arbitrary functions. On the other hand our result holds for the U^4 norm compared to the U^3 norm studied there. Moreover, when $\mathrm{char}(\mathbb{F}) > 4$, using the same techniques we can actually show that f must have a structure similar to the one guaranteed by Theorem 3.

Theorem 5. *Let $\mathbb{F}$ be a finite field with $\mathrm{char}(\mathbb{F}) > 4$ and $f \in \mathbb{F}[x_1, \ldots, x_n]$ a degree four polynomial such that $\|f\|_{U^4} = \delta$. Then*

$$f = \sum_{i=1}^{R} \ell_i \cdot g_i + \sum_{i=1}^{r} q_i \cdot q_i' \,,$$

for $r = O(\log^2(1/\delta))$ and $R = \exp(\log^2(1/\delta))$ where ℓ_i is linear, q_i, q_i' are quadratic and g_i cubic.

1.2 Proof Technique

The main approach in all the proofs is to consider the space of discrete partial derivatives of f and look for some structure there. We will explain the idea for the case of degree three polynomials and then its extension to degree four polynomials.

Let f be a degree three polynomials. Assume that f has high bias (alternatively, high U^3 norm). By a standard argument it follows that a constant fraction of its derivatives, which are degree 2 polynomials, have high bias (high U^2 norm). By Theorem 1.1 it follows that for a constant fraction of the directions, the partial derivatives depends on a small number of linear functions (same for the U^3 norm). Hence, in the space of partial derivatives, a constant fraction of the elements depend on a few linear functions. We now show that there must be a small number of linear functions that 'explain' this. More accurately, we show that there exist a subspace $V \subset \mathbb{F}^n$, of dimension $\dim(V) = n - O(1)$, and $O(1)$ linear functions $\ell_1, \ldots, \ell_c$, such that for every $y \in V$ it holds that $\Delta_y(f) = \sum_{i=1}^{c} \ell_i \cdot \ell_i^{(y)} + \ell_0^{(y)}$, where the $\ell_i^{(y)}$-s are linear functions determined by y.

We are now basically done. Consider the subspace $U = \{x : \ell_1(x) = \ldots = \ell_c(x) = 0\}$. Then, for every $y \in V$ it holds that $\Delta_y(f)|_U = \ell_0^{(y)}|_U$. This implies that $f|_V = \sum_{i=1}^{c} \ell_i \cdot q_i + q_0$, where the q_i-s are quadratic polynomials. As $\dim(V) = n - O(1)$ we obtain the same structure (with a different constant c) for f.

To prove the result for biased degree four polynomials we follow the footsteps of [KL08] with two notable differences. Let f be such a polynomial. First, we pass to a subspace on which all the partial derivatives of f have low rank as degree three polynomials. This steps

relies on our results for biased degree three polynomials. Then, as in [KL08], we show that f can be approximated by a function of a few of its derivatives. Because of the properties of the derivatives, this means that f can be approximated well by a function of a few quadratics and linear functions. We then show, again following [KL08], that in such a case f is actually a function of a few quadratics and linear functions. Here we heavily rely on properties of quadratic functions to avoid the blow up in the number of polynomials approximating f that occurs in [KL08, GT07]. Finally, we show that if a degree four polynomial is a function of several quadratic and linear functions then it actually have a nice structure.

The proof for the case of degree four polynomials with high U^4 norm is more delicate. Assume that f is such a polynomial. As before, a constant fraction of the partial derivatives of f are degree three polynomials with high U^3 norm. By the result for degree three polynomials we get that each of those partial derivatives is of the form $\Delta_y(f) = \sum_{i=1}^{c} \ell_i^{(y)} \cdot q_i^{(y)} + q_0^{(y)}$. Again we find a subspace V, of constant co-dimension, such that for every $y \in V$, $\Delta_y(f)$ has a nice structure. We now show that there exist a small number of linear and quadratic functions $\{\ell_i, q_i\}_{i=1}^{c}$ such that for every $y \in V$ it holds that $\Delta_y(f) = \sum_{i=1}^{c} \ell_i \cdot q_i^{(y)} + \sum_{i=1}^{c} \ell_i^{(y)} \cdot q_i + q_0^{(y)}$, where the polynomials $\{\ell_i^{(y)}, q_i^{(y)}\}$ depend on y. This is the technical heart of the proof. It now follows quite easily that there is a subspace $U \subseteq V$ of dimension $n/\exp(c)$ such that when restricted to U all the functions $\{\ell_i, q_i\}$ are fixed to constants. Thus, for every $y \in U$ it holds that $\deg(\Delta_y(f)) = 2$. So we get that $\deg(f|_U) = 3$. In fact, by closely examining the argument above we show an even stronger result. Namely, that we can partition a large subspace of $\mathbb{F}^n$ to (affine) subspaces of dimension $n/\exp(c)$ such that on each of the subspaces f is equal to some cubic polynomial (that may depend on the subspace).

1.3 Organization

In Section 2 we give some basic definitions and discuss properties of subadditive functions. In Section 3 we prove the theorems concerning degree three polynomials. In Section 4 we prove Theorem 3 and in Section 5 we prove Theorems 4 and 5.

2 Preliminaries

In this paper $\mathbb{F}$ will always be a prime field. We denote with $\mathbb{F}_p$ the field with p elements. As we will be considering functions over $\mathbb{F}_p$ we will work modulo the polynomials $x_i^p - x_i$. In particular, when we write $f = g$, for two polynomials, we mean that they are equal as functions and not just as formal expressions. This will be mainly relevant when we consider quadratic polynomials (or higher degree polynomials) over $\mathbb{F}_2$. More generally, we shall say that a function f has degree d if there is a degree d polynomial g such that $f = g$. Note that this does not have an affect on the bias and the Gowers norm. Namely, the bias and U^d norm of f do not change when adding multiplies of $x_i^p - x_i$. Finally we note that if all the partial derivative of f have degree at most $d-1$ then there is a polynomial g of degree at most d such that $f = g$ (this is easily proved by observing that a degree k polynomial, all of whose individual degrees are smaller than $|\mathbb{F}|$, always has a partial derivative whose degree is $k-1$). From this point on we shall use the notion of a function and a polynomial

arbitrarily without any real distinction.

The Fourier transform of a function $f : \mathbb{F}^n \to \mathbb{F}$ is defined as

$$\hat{f}(\alpha) = \mathbb{E}_{x \in \mathbb{F}^n}[f(x)\overline{\chi_\alpha(x)}] \,,$$

where for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\chi_\alpha(x) = \omega^{\sum_{i=1}^n \alpha_i x_i}$ where $\omega = e^{\frac{2\pi i}{|\mathbb{F}|}}$ is a complex primitive root of unity of order $|\mathbb{F}|$. For more on Fourier transform see [Ste03].

We say that a function h ϵ-approximates a function f if $\Pr_x[f(x) \neq h(x)] \leq \epsilon$.

Definition 2.1. *Following [KL08] we say that the distribution induced by a set of functions $\{h_i\}_{i=1}^m$ (all from $\mathbb{F}^n$ to $\mathbb{F}$) is γ close to the uniform distribution if for every $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ it holds that*

$$\left| \Pr_{x \in \mathbb{F}^n}[\forall 1 \leq i \leq m,\ h_i(x) = \alpha_i] - |\mathbb{F}|^{-m} \right| \leq \gamma |\mathbb{F}|^{-m} \,.$$

The following well known lemma bounds the distance between distributions using the Fourier transform.

Lemma 2.2. *For $i = 1 \ldots m$ let $h_i : \mathbb{F}^n \to \mathbb{F}$ be a function. Then, the distribution induced by the h_i-s is γ close to uniform if for every nontrivial linear combination $h_\alpha = \sum_{i=1}^m \alpha_i h_i$, we have that* $\mathrm{bias}(h_\alpha) \leq \gamma / |\mathbb{F}|^{3m/2}$.

Proof. Let $H : \mathbb{F}^n \to \mathbb{F}^m$ be defined as $H(x) = (h_1(x), \ldots, h_m(x))$. For $y \in \mathbb{F}^m$ let $f(y) = \Pr_{x \in \mathbb{F}^n}[H(x) = y]$. We have that

$$\begin{aligned} |\hat{f}(\alpha)| &= \left| \mathbb{E}_{y \in \mathbb{F}^m}\left[f(y)\overline{\chi_\alpha(y)} \right] \right| = \left| \mathbb{E}_{y \in \mathbb{F}^m}\left[\Pr_{x \in \mathbb{F}^n}[H(x) = y]\overline{\chi_\alpha(y)} \right] \right| \\ &= |\mathbb{F}|^{-n-m} \left| \sum_{x \in \mathbb{F}^n} \chi_\alpha[H(x)] \right| = |\mathbb{F}|^{-m} \mathrm{bias}\left(\sum_{i=1}^m \alpha_i h_i \right) \,. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\sum_{y \in \mathbb{F}^m} \left| f(y) - |\mathbb{F}|^{-m} \right| \right)^2 &\leq |\mathbb{F}|^m \sum_{y \in \mathbb{F}^m} \left| f(y) - |\mathbb{F}|^{-m} \right|^2 \\ &= |\mathbb{F}|^m \sum_{y \in \mathbb{F}^m} f(y)^2 - 2|\mathbb{F}|^{-m} f(y) + |\mathbb{F}|^{-2m} \\ &= \left(\sum_{\alpha \in \mathbb{F}^m} |\mathbb{F}|^{2m} \hat{f}(\alpha)^2 \right) - 1 = \left(\sum_{0 \neq \alpha \in \mathbb{F}^m} |\mathbb{F}|^{2m} \hat{f}(\alpha)^2 \right) \\ &= \sum_{0 \neq \alpha \in \mathbb{F}^m} \mathrm{bias}\left(\sum_{i=1}^m \alpha_i h_i \right)^2 < |\mathbb{F}|^{-2m} \gamma^2 \,. \end{aligned}$$

Hence, for every $y \in \mathbb{F}^m$ it holds that $|f(y) - |\mathbb{F}|^{-m}| < |\mathbb{F}|^{-m}\gamma$, which is what we wanted to prove. □

2.1 Subadditive functions

As described in Section 1.2 our proofs are based on finding a structure for the space of partial derivatives of the underlying polynomial f. For this end we need a special case of the Bogolyubov-Chang lemma (see e.g. [Gre]).

For a set $A \subseteq \mathbb{F}^n$ denote with $kA - kA$ the set

$$kA - kA = \{a_1 + \ldots + a_k - a_{k+1} - \ldots - a_{2k} \mid \forall i \; a_i \in A\} \; .$$

Lemma 2.3 (Bogolyubov-Chang)**.** *Let $A \subseteq U$ be a subset of a linear space U such that $|A| = \mu_0 \cdot |U|$. Then, for some $k \leq \max(1, \lceil \frac{1}{2}(\log_{\frac{|\mathbb{F}|}{|\mathbb{F}|-1/2}}(2/\mu_0) + 2)\rceil)$, $kA - kA$ contains a subspace W of co-dimension at most $\log_{\frac{|\mathbb{F}|-1/2}{|\mathbb{F}|-1}}(1/2\mu_0)$.*

For completeness we give the proof here.

Proof. For $\mu \in (0,1)$ define $\rho(\mu) = \frac{|\mathbb{F}|-1/2}{|\mathbb{F}|} \cdot \mu$. We shall think of A also as the characteristic function of the set A and denote with $\{\hat{A}(\alpha)\}$ its fourier coefficients. Assume that there is some $\alpha \neq 0$ such that $|\hat{A}(\alpha)| \geq \rho(\mu_0)$. This means that there is some (affine) subspace W of co-dimension at most one such that

$$|A \cap W|/|W| \geq \rho(\mu_0) \cdot |\mathbb{F}|/(|\mathbb{F}| - 1) = \frac{|\mathbb{F}| - 1/2}{|\mathbb{F}| - 1} \cdot \mu_0 = (1 + \epsilon)\mu_0 \; ,$$

where $\epsilon = \frac{1}{2|\mathbb{F}|-2}$. In other words, the density of A on W is $(1+\epsilon)$ larger than its density over the entire space. We continue restricting A to co-dim one subspaces (updating μ and considering $\rho(\mu)$ at each step) until after at most $t = \log_{\frac{|\mathbb{F}|-1/2}{|\mathbb{F}|-1}}(1/2\mu_0)$ steps we reach one of two possibilities. Either we get a subspace $V \subseteq U$ of co-dimension at most t such that $|A \cap V| > |V|/2$, or $\widehat{A \cap V}(\alpha) < \rho(\mu)$ for every $\alpha \neq 0$, where $\mu_0 < \mu = |A \cap V|/|V|$. In the first case it is clear that $(A \cap V) + (A \cap V) = V$ and so we found a subspace V of co-dimension at most t contained in $A + A$. In the second case where all the non-zero Fourier coefficients are smaller than $\rho(\mu)$ we show that for $k = \lceil \frac{1}{2}(\log_{\frac{|\mathbb{F}|}{|\mathbb{F}|-1/2}}(2/\mu) + 2)\rceil$ it holds that $k(A \cap V) - k(A \cap V) = V$. For this end we follow the proof of Lemma 4.4 in [Gre]. Let $B = A \cap V$. For $x \in V$ denote with $r_k(x)$ the number of representations of x as $a_1 + \ldots + a_k - a'_1 - \ldots - a'_k$ where the a_i-s and a'_i-s are from B. Clearly, $r_k(x)$ is equal to the sum, over all $(y_1, \ldots, y_k, z_1, \ldots, z_{k-1}) \in B^{2k-1}$, of $A(y_1) \cdot A(y_2) \cdot \ldots \cdot A(y_k) \cdot A(z_1) \cdot \ldots \cdot A(z_{k-1}) \cdot A(y_1 + \ldots + y_k - z_1 - \ldots - z_{k-1} - x)$. Writing the Fourier expansion A and using routine calculations we conclude that

$$r_k(x) = |\mathbb{F}|^{(2k-1)n} \cdot \sum_\alpha |\hat{B}(\alpha)|^{2k} \chi_\alpha(x) > |\mathbb{F}|^{(2k-1)n} \cdot \left(\hat{B}(0)^{2k} - \sum_{\alpha \neq 0} |\hat{B}(\alpha)|^{2k} \right) \geq$$

$$|\mathbb{F}|^{(2k-1)n} \cdot \left(\hat{B}(0)^{2k} - \rho(\mu)^{2k-2} \sum_\alpha |\hat{B}(\alpha)|^2 \right) = |\mathbb{F}|^{(2k-1)n} \cdot \left(\mu^{2k} - \rho(\mu)^{2k-2}\mu \right) > 0 \; ,$$

where the last inequality follows from the choice of k (we also used the fact that A is a 0/1 function). In particular, $V \subseteq kA - kA$ as needed. □

We will mainly apply the lemma on sets $A \subseteq \mathbb{F}^n$ containing all directions where the partial derivatives of our underlying polynomial f are either very biased or have a high Gowers norm. More generally we define the notion of a subadditive function below.

Definition 2.4. *Let $V \subset \mathbb{F}^n$ be a linear space. $\mathcal{F} : V \to \mathbb{R}^+$ is a subadditive function if for every $u, v \in V$ and $\alpha \in \mathbb{F}$ it holds that $\mathcal{F}(\alpha \cdot u + v) \leq \mathcal{F}(u) + \mathcal{F}(v)$.*

Lemma 2.5. *Let $\mathcal{F} : U \to \mathbb{R}^+$ be a subadditive function. Define, $A_r \triangleq \{x \in U \mid \mathcal{F}(x) \leq r\}$. If $|A_r| \geq \mu|U|$, then there exists a vector space V of co-dimension at most $\log_{\frac{|\mathbb{F}|-1/2}{|\mathbb{F}|-1}}(1/2\mu) = O(\log(1/\mu))$ such that for every $y \in V$ it holds that $\mathcal{F}(y) \leq 2r \cdot \lceil \frac{1}{2}(\log_{\frac{|\mathbb{F}|}{|\mathbb{F}|-1/2}}(2/\mu)+2)\rceil + 2r = O(r\log(1/\mu))$.*

Proof. The proof is immediate from Lemma 2.3. Let V be the subspace guaranteed by the lemma when applied on A_r. As $V \subseteq kA_r - kA_r$, for $k \leq \max(1, \lceil \frac{1}{2}(\log_{\frac{|\mathbb{F}|}{|\mathbb{F}|-1/2}}(2/\mu)+2)\rceil)$, we get that $\mathcal{F}(y) \leq 2kr$ for every $y \in V$. □

A typical example of a subadditive function will be the rank of a quadratic polynomial.

Definition 2.6. *Let q be a degree two function over a prime field $\mathbb{F}$. We define $\mathrm{rank}_2(q) = r$, where r is the number of α_i-s that are non zero when considering the canonical representation of q in Theorem 1.1.*

The following lemma is immediate.

Lemma 2.7. *For two quadratic polynomials q, q' and a constant $\alpha \in \mathbb{F}$ we have that $\mathrm{rank}_2(q+\alpha q') \leq \mathrm{rank}_2(q) + \mathrm{rank}_2(q')$.*

A more interesting example is given in the following lemma.

Lemma 2.8. *Let f be a cubic polynomial over a prime field $\mathbb{F}$. For every $y \in \mathbb{F}^n$ define $\mathcal{F}(y) = \mathrm{rank}_2(\Delta_y(f))$. Then $\mathcal{F}$ is a subadditive function.*

Proof. The proof follows from the following simple observation

$$\begin{aligned}
\Delta_y(f) + \Delta_z(f) &= f(x+y) - f(x) + f(x+z) - f(x) \\
&= f(x+y+z) - f(x) - (f(x+y+z) - f(x+y) - (f(x+z) - f(x))) \\
&= \Delta_{y+z}(f)(x) - (\Delta_z(f)(x+y) - \Delta_z(f)(x)) \\
&= \Delta_{y+z}(f)(x) - \Delta_y\Delta_z(f)(x)\ .
\end{aligned}$$

Indeed, we now get that $\mathcal{F}(y+z) = \mathrm{rank}_2(\Delta_{y+z}(f)) = \mathrm{rank}_2(\Delta_y(f)+\Delta_z(f)+\Delta_y\Delta_z(f)(x)) = \mathrm{rank}_2(\Delta_y(f) + \Delta_z(f)) \leq \mathrm{rank}_2(\Delta_y(f)) + \mathrm{rank}_2(\Delta_z(f)) = \mathcal{F}(y) + \mathcal{F}(z)$, where we used the fact that adding a linear function to a quadratic polynomial does not change its rank. □

3 The structure of cubic polynomials

In this section we prove Theorems 1 and 2. As described in Section 1.2 both proofs are based on finding a structure for the space of partial derivatives of f.

3.1 Restricting the polynomial to a 'good' subspace

In this section we show that if a cubic f ia biased or have a large U^3 norm then there is a subspace $V \subseteq \mathbb{F}^n$ such that for every $y \in V$ the rank of $\Delta_y(f)$ is relatively small. We start by showing that if f is biased or has a high Gowers norm then so do many of its partial derivatives. The following lemmas are well known and we prove them here for completeness.

Lemma 3.1. *Let $f : \mathbb{F}_p^n \to \mathbb{F}_p$ be such that $\mathrm{bias}(f) = \delta$. Then a fraction of at least $\frac{1}{2}\delta^2$ of the partial derivatives $\Delta_y(f)$ satisfy $\mathrm{bias}(\Delta_y(f)) \geq \frac{1}{2}\delta^2$.*

Proof. We first compute the expected bias of a partial derivative with respect to a random direction.

$$\begin{aligned}
\mathbb{E}_{y\in\mathbb{F}^n}\left[\mathrm{bias}(\Delta_y(f))\right] &= \mathbb{E}_{y\in\mathbb{F}^n}\left[\left|\mathbb{E}_{x\in\mathbb{F}^n}\left[\omega^{\Delta_y(f)(x)}\right]\right|\right] \geq \left|\mathbb{E}_{y\in\mathbb{F}^n}\left[\mathbb{E}_{x\in\mathbb{F}^n}\left[\omega^{f(x+y)-f(x)}\right]\right]\right| \\
&= \left|\mathbb{E}_{y\in\mathbb{F}^n,x\in\mathbb{F}^n}\left[\omega^{f(x+y)}\omega^{-f(x)}\right]\right| = \left|\mathbb{E}_{z\in\mathbb{F}^n,x\in\mathbb{F}^n}\left[\omega^{f(z)}\omega^{-f(x)}\right]\right| \\
&= \left|\mathbb{E}_{z\in\mathbb{F}^n}\left[\omega^{f(z)}\right]\right|\left|\overline{\mathbb{E}_{x\in\mathbb{F}^n}\left[\omega^{f(x)}\right]}\right| = \delta\cdot\delta = \delta^2 \ .
\end{aligned}$$

Therefore, by the fact that $\mathrm{bias}(f) \leq 1$, it follows that

$$\Pr_{y\in\mathbb{F}^n}\left[\mathrm{bias}(\Delta_y(f)) > \frac{1}{2}\delta^2\right] > \frac{1}{2}\delta^2 \ .$$

□

A similar result holds when f has a high U^d norm.

Lemma 3.2. *Let $f : \mathbb{F}_p^n \to \mathbb{F}_p$ be such that $\|f\|_{U^d} = \delta$. Then a fraction of at least $\frac{1}{2}\delta^{2^d}$ of the partial derivatives $\Delta_y(f)$ satisfy $\|\Delta_y(f)\|_{U^{d-1}} \geq \frac{1}{2}\delta^2$.*

Proof. The proof is again immediate from the definition.

$$\begin{aligned}
\delta^{2^d} = \|f\|_{U^d}^{2^d} &= \left|\mathbb{E}_{x,y_1,\ldots,y_d}\left[\omega^{\Delta_{y_1}\cdots\Delta_{y_d}(f)(x)}\right]\right| \\
&\leq \mathbb{E}_{y_d}\left|\mathbb{E}_{x,y_1,\ldots,y_{d-1}}\left[\omega^{\Delta_{y_1}\cdots\Delta_{y_{d-1}}(\Delta_{y_d}(f))(x)}\right]\right| \\
&= \mathbb{E}_y\left[\|\Delta_y(f)\|_{U^{d-1}}^{2^{d-1}}\right] \ .
\end{aligned}$$

As before we get that

$$\Pr_{y\in\mathbb{F}^n}\left[\|\Delta_y(f)\|_{U^{d-1}} > \frac{1}{2}\delta^2\right] > \frac{1}{2}\delta^{2^d} \ .$$

□

We thus see that in both cases a constant fraction of all partial derivatives of f have high bias or high U^2 norm. From Theorem 1.1 we get that if a partial derivative (which is a quadratic function) has a high bias then it depends on a few linear functions.

Lemma 3.3. *Let q be a quadratic polynomial over a prime field $\mathbb{F}$. Then q is a function of at most $\log_{|\mathbb{F}|}(\mathrm{bias}(q))+1$ linear functions. More accurately, in the notations of Theorem 1.1 the number of non zero α_i-s is at most $\log_{|\mathbb{F}|}(1/\mathrm{bias}(q))$.*

Proof. See e.g. Lemmas 15-17 of [BV07]. □

The next lemma of Bogdanov and Viola [BV07] shows that a similar result holds when a partial derivative has a high U^2 norm.

Lemma 3.4. *(Lemma 15 of [BV07]) Every quadratic polynomial q over a prime field $\mathbb{F}$ is a function of at most $\log_{|\mathbb{F}|}(1/\|q\|_{U^2}) + 1$ linear functions. Further, in the notations of Theorem 1.1 the number of non zero α_i-s is at most $\log_{|\mathbb{F}|}(1/\|q\|_{U^2})$.*

Concluding, we have proved the following lemma (recall Definition 2.6).

Lemma 3.5. *Let f be a cubic polynomial.*

1. *If $\mathrm{bias}(f) = \delta$, then for (at least) a $\frac{\delta^2}{2}$ fraction of $y \in \mathbb{F}^n$ it holds that $\mathrm{rank}_2(\Delta_y(f)) \leq \log_{|\mathbb{F}|}(\frac{2}{\delta^2})$.*
2. *If $\|f\|_{U^2} = \delta$, then for (at least) a $\frac{\delta^4}{2}$ fraction of $y \in \mathbb{F}^n$ it holds that $\mathrm{rank}_2(\Delta_y(f)) \leq \log_{|\mathbb{F}|}(\frac{2}{\delta^2})$.*

We now combine Lemma 2.8 with Lemma 3.5 and Lemma 2.5 and obtain the following corollary.

Corollary 3.6. *Let f be a cubic polynomial. If $\mathrm{bias}(f) = \delta$ or $\|f\|_{U^2} = \delta$, then there exists a subspace $V \subseteq \mathbb{F}^n$ such that $\dim(V) \geq n - O(\log(\frac{1}{\delta}))$ and such that for every $y \in V$ it holds that $\mathrm{rank}_2(\Delta_y(f)) = O(\log^2(\frac{1}{\delta}))$.*

3.2 The structure of low rank spaces

So far we have established the existence of a subspace $V \subseteq \mathbb{F}^n$ such that for every $y \in V$ it holds that $\mathrm{rank}_2(\Delta_y(f)) = O(\log^2(\frac{1}{\delta}))$. We now show that such spaces of low rank polynomials have a very restricted structure. Namely, there exist $r = O(\log^2(\frac{1}{\delta}))$ linear functions $\ell_1, \ldots, \ell_r$ such that every $\Delta_y(f)$ can be written as $\Delta_y(f) = \sum_{i=1}^r \ell_i \cdot \ell_i^{(y)} + \ell_0^{(y)}$, where the $\ell_i^{(y)}$-s are linear functions determined by y. The intuition behind this result is that $\mathrm{rank}_2(q + q')$ can be much smaller than $\mathrm{rank}_2(q) + \mathrm{rank}_2(q')$ only if there is some basis with respect to which q and q' share many linear functions when represented in the form of Theorem 1.1. From this observation we deduce that if we consider some function of maximal rank, $q = \sum_{i=1}^r \ell_i \cdot \ell_i'$, and set $\{\ell_i, \ell_i'\}_{i=1}^r$ to zero (namely, consider the subspace on which they all vanish), then on this subspace all the remaining quadratic functions become linear.

Lemma 3.7. *Let M be a linear space of quadratic functions satisfying $\mathrm{rank}_2(p) \leq r$ for all $p \in M$. Then there exists a subspace $V \subseteq \mathbb{F}^n$ of co-dimension $\leq 2r$ such that $p|_V$ is a linear function for all $p \in M$.*

We shall give the proof for the case that $\mathbb{F} = \mathbb{F}_2$. The proof for odd characteristic is very similar (except that in the odd characteristic case we have co-dimension of V is r whereas in the even characteristic case it $2r$).

Proof. Let g be a quadratic function such that $\text{rank}_2(g) = r$. By Theorem 1.1, g can be expressed as $g = \sum_{i=1}^{r} \ell_{2i-1} \cdot \ell_{2i} + \ell_0$. Denote $V \triangleq \{x \mid \ell_1(x) = \ell_2(x) = ... = \ell_{2r}(x) = 0\}$. We now show that V is the required subspace.

Assume for contradiction that there is $h \in M$ such that $h|_V$ is not linear. I.e $\text{rank}_2(h|_V) = s \geq 1$. As before, h can be expressed as $h|_V = \sum_{i=1}^{s} m_{2i-1} \cdot m_{2i} + m_0$ (where the m_i-s are linear functions). We note that $\{\ell_i\}_{i=1}^{2r} \cup \{m_i\}_{i=1}^{2s}$ are linearly independent. Indeed, assume for contradiction that $\sum_{i=1}^{2s} \alpha_i m_i = m'$ for some $m' \in \text{span}(\{\ell_i\}_{i=1}^{2r})$ and some α_i-s (not all being zero). Assume w.l.o.g that $\alpha_{2s} = 1$. It holds that for $\beta = \left(\alpha_{2s-1} + \sum_{i=1}^{s-1} \alpha_{2i-1}\alpha_{2i}\right)$

$$\begin{aligned} h &= \sum_{i=1}^{s-1} m_{2i-1} \cdot m_{2i} + m_{2s-1} \cdot (m' - \sum_{i=1}^{2s-1} \alpha_i m_i) + m_0 \\ &=^{(*)} \sum_{i=1}^{s-1} (m_{2i-1} - \alpha_{2i} m_{2s-1}) \cdot (m_{2i} - \alpha_{2i-1} m_{2s-1}) + m_{2s-1} \cdot m' + (m_0 - \beta m_{2s-1}) \ , \end{aligned}$$

where equality $(*)$ holds as we think on all our polynomials as functions and look for representation of them modulo expressions of the form $x^2 - x$. As $m'|_V = 0$ we get that $\text{rank}_2(h|_V) \leq s - 1$, in contradiction. Hence, $\{\ell_i\}_{i=1}^{2r} \cup \{m_i\}_{i=1}^{2s}$ are linearly independent.

Consider $g + h = \sum_{i=1}^{r} \ell_{2i-1} \cdot \ell_{2i} + \sum_{i=1}^{s} m_{2i-1} \cdot m_{2i} + (\ell_0 + m_0)$. As the ℓ_i-s and m_i-s are linearly independent, we have that $\text{rank}_2(g + h) = r + s > r$ as opposed to the fact that every vector in M has rank at most r. □

3.3 Completing the proofs

We are no ready to complete the proofs of Theorems 1 and 2.

Proof of Theorem 2. By Corollary 3.6 we get that if $\|f\|_{U^2} = \delta$, then there exists a subspace $V \subseteq \mathbb{F}^n$ such that $\dim(V) \geq n - O(\log(1/\delta))$ and such that for every $y \in V$ it holds that $\text{rank}_2(\Delta_y(f)) = O(\log^2(\frac{1}{\delta}))$. Lemma 3.7 implies that there are at most $r = O(\log^2(\frac{1}{\delta}))$ linear functions $\ell_1, \ldots, \ell_r$ such that for every $y \in V$ we have that $\Delta_y(f) = \sum_{i=1}^{r} \ell_i \cdot \ell_i^{(y)} + \ell_0^{(y)}$. Let $U = \{x \in V \mid \ell_1(x) = \ldots \ell_r(x) = 0\}$. Then U is a linear space of dimension $\dim(U) \geq n - O(\log^2(\frac{1}{\delta}))$. For every $y \in U$ we have that $\Delta_y(f)|_U = \ell_0^{(y)}|_U$. Hence, for every $y \in U$, $\deg(\Delta_y(f)) \leq 1$. Therefore, $\deg(f|_U) \leq 2$. Let $\ell'_1, \ldots, \ell'_t$ be linearly independent linear functions such that $x \in U$ iff $\ell'_1(x) = \ldots = \ell'_t(x) = 0$. It follows that we can write $f = \sum_{i=1}^{t} \ell'_i \cdot q_i + q_0$ for some quadratic polynomials $\{q_i\}$. As $t = n - \dim(U) = O(\log^2(\frac{1}{\delta}))$ the result follows. □

The proof of Theorem 1 is essentially the same except that we make another small optimization that reduces the required number of quadratic functions.

Proof of Theorem 1. By the same argument as above we get that $f = \sum_{i=1}^{t} \ell_i \cdot q_i + q_0$ for some quadratic polynomials $\{q_i\}$ and linear functions $\{\ell_i\}$, where $t = O(\log^2(\frac{1}{\delta}))$. For convenience we shall assume w.l.o.g. that

$$f = \sum_{i=1}^{t} x_i \cdot q_i + q_0 \ . \tag{1}$$

The following lemma shows that by adding a few more linear functions we can assume that no nontrivial linear combination of the q_i-s has a low rank.

Lemma 3.8. *Let $q_1, \ldots, q_t$ be quadratic polynomials over $\mathbb{F}^n$. Then, for every r there exist a subspace $V \subset \mathbb{F}^n$ of dimension $\dim(V) \geq n - t(r+1)$, and $t' \leq t$ indices $i_1, \ldots, i_{t'}$ such that for every affine shift V' of V the following holds*

1. *For all i, $q_i|_{V'} \in \text{span}\{1, q_{i_1}|_{V'}, \ldots, q_{i_{t'}}|_{V'}\}$.*
2. *For any non trivial linear combination we have that $\text{rank}_2\left(\sum_{j=1}^{t'} \alpha_i q_{i_j}|_{V'}\right) > r$.*

Proof. The proof is by induction on t. For $t = 1$ the claim is clear: If $\text{rank}_2(q_1) > r$ then we are done. Otherwise we have $q_1 = \sum_{i=1}^{r} \ell_{2i-1}\ell_{2i} + \ell_0$. Letting $V = \{x \mid \ell_0(x) = \ell_2(x) = \ldots = \ell_{2r}(x) = 0\}$ the claim follows (indeed notice that passing to an affine shift of V simply means fixing the ℓ_i-s to arbitrary values). Assume now that we have $q_1, \ldots, q_t$ and that (w.l.o.g.) $\text{rank}_2\left(q_t + \sum_{i=1}^{t-1} \alpha_i q_i\right) \leq r$. Write $q_t + \sum_{i=1}^{t-1} \alpha_i q_i = \sum_{i=1}^{r} \ell_{2i-1}\ell_{2i} + \ell_0$. Set $V = \{x \mid \ell_0(x) = \ell_2(x) = \ldots = \ell_{2r}(x) = 0\}$. Then, $q_t|_V \in \text{span}\{q_1|_V \ldots, q_{t-1}|_V\}$. As $\dim(V) = n-(r+1)$ the claim follows by applying the induction argument to $q_1|_V \ldots, q_{t-1}|_V$ (again the claim about any affine shift follows easily). □

We continue with the proof of the theorem. Having Equation (1) in mind we set $U = \{(0, \ldots, 0, x_{t+1}, \ldots, x_n)\} \subset \mathbb{F}^n$. Applying Lemma 3.8 on $q_1|_U, \ldots, q_t|_U$ with $r = \log_{|\mathbb{F}|}(2/\delta)$ we get that there is a subspace $W \subset U$ and $t' \leq t$ such that: $\dim(W) \geq \dim(U) - (r+1)t \geq n - (r+2)t = n - O(\log^3(\frac{1}{\delta}))$; w.l.o.g. for every $i = 1 \ldots t$, $q_i|_W \in \text{span}\{q_1|_W, \ldots, q_{t'}|_W\}$; any nontrivial linear combination of $q_1|_W, \ldots, q_{t'}|_W$ has rank larger than r. By applying an invertible linear transformation[4] we can further assume that $W = \{x \in \mathbb{F}^m \mid x_1 = \ldots = x_m = 0\}$ for some $m \leq (r+2)t$. For $i = 1 \ldots t'$ let $q'_i = q_i|_W$. Note that q'_i does not contain any of the variables $x_1, \ldots, x_m$. We can rewrite Equation (1) as[5]

$$f = \sum_{i=1}^{t'} \ell'_i q'_i + \sum_{i=1}^{t} \sum_{j=1}^{m} x_i x_j \ell_{i,j} \ , \tag{2}$$

where the ℓ'_i-s are linearly independent linear functions in $x_1, \ldots, x_t$. We now show that $t' < \log_{|\mathbb{F}|}(2/\delta)$. Assume for contradiction that $t' \geq \log_{|\mathbb{F}|}(2/\delta)$. As

$$\text{bias}(f) = \mathbb{E}_{\alpha_1,\ldots,\alpha_{t'}}\left[\text{bias}(f(x_1, \ldots, x_n)|_{(\ell'_1,\ldots,\ell'_{t'})=(\alpha_1,\ldots,\alpha_{t'})}\right] \ ,$$

there exists an assignment $(x_1, \ldots, x_m) = (\beta_1, \ldots, \beta_m)$ satisfying $(\ell'_1, \ldots, \ell_{t'}) = (\alpha_1, \ldots, \alpha_{t'}) \neq 0$ such that

$$\text{bias}\left(\sum_{i=1}^{t'} \alpha_i q'_i + \sum_{i=1}^{t} \beta_i \sum_{j=1}^{m} \beta_j \ell_{i,j}\right) \geq \delta - \frac{1}{|\mathbb{F}|^{t'}} \geq \delta/2 \ .$$

[4]This step is not really required but we continue using it just to make the proofs easier to read.

[5]We will later explain why q_0 'disappeared' from this expression.

Therefore, for some constants $\alpha_1, \ldots, \alpha_{t'}$ (where not all $\alpha_1, \ldots, \alpha_{t'}$ are zero) we have that

$$\text{bias}\left(\sum_{i=1}^{t'} \alpha_i q'_i + \ell\right) \geq \delta/2 \ ,$$

for some linear function ℓ. By Lemma 3.3 we get that

$$\text{rank}_2\left(\sum_{i=1}^{t'} \alpha_i q'_i\right) = \text{rank}_2\left(\sum_{i=1}^{t'} \alpha_i q'_i + \ell\right) \leq \log_{|\mathbb{F}|}(1/(\delta/2)) = r \ ,$$

in contradiction to the choice of $q'_1, \ldots, q'_{t'}$.

To complete the proof we explain the reason for dropping q_0. Indeed, consider Equation (1). Let $U = \{x \mid x_1 = \ldots = x_t = 0\}$. Set $\tilde{q}_i = q_i|_U$. Then we can rewrite (1) as $\sum_{i=1}^t x_i \tilde{q}_i + \tilde{q}_0 + \sum_{i=1}^t x_i \sum_{j=1}^t x_j \ell_{i,j}$, for some linear functions $\ell_{i,j}$. Now, for some $\alpha_1, \ldots, \alpha_t$ we get that $\text{bias}(\sum_{i=1}^t \alpha_i \tilde{q}_i + \tilde{q}_0 + \sum_{i=1}^t \alpha_i \sum_{j=1}^t \alpha_j \ell_{i,j}) \geq \delta$. Lemma 3.3 implies that $\text{rank}_2(\sum_{i=1}^t \alpha_i \tilde{q}_i + \tilde{q}_0) \leq \log_{|\mathbb{F}|}(1/\delta)$ and so we can replace $\tilde{q}_0$ by a linear combination of the other $\tilde{q}_i$-s and a function depending on a few linear functions. By passing to a (possibly affine) subspace of dimension at least $n - \log_{|\mathbb{F}|}(1/\delta) - 1$ we get a representation for f without q_0. This operation increases t' in Equation (2) by no more than $\log_{|\mathbb{F}|}(1/\delta) + 1$ and so we are done. □

4 The structure of biased 4 degree polynomials

In this section we prove Theorem 3 on the structure of biased degree 4 polynomials. As in the case of cubic polynomials, we shall focus our attention on a subspace on which all of derivatives have a small rank (a cubic polynomial is of low rank if it depends on a small number of linear and quadratic functions). By a lemma of Bogdanov and Viola [BV07] (Lemma 4.3) we get that f can be well approximated by a function of a small number of its derivatives (which in our case, are all of low rank). Thus, f is well approximated by a function of a few linear and quadratic polynomials. By passing to a subspace we can assume that f is well approximated by a function of a small number of quadratic polynomials. Lemma 3.8 implies that (possibly on a slightly smaller subspace) f can be well approximated by a function of a small number of quadratics, that every nontrivial linear combination of them has a high rank. We then show that in this case those quadratic functions are in fact *strongly regular* (a notion that we later explain) and therefore by a theorem of Kaufman and Lovett [KL08], f in fact equals a function in those quadratic (on the subspace). We then finish the proof by showing that in this case f also have a nice structure.

4.1 Restricting the polynomial to a 'good' subspace

In this section we prove an analogous result to Corollary 3.6. We first define the rank of a cubic polynomial.

Definition 4.1. *Let g be a degree three polynomial. We define* $\mathrm{rank}_3(g)$ *to be the minimal integer r for which there are r linear functions $\ell_1, \ldots, \ell_r$ and $r+1$ quadratic functions $q_0, \ldots, q_r$ such that $g = \sum_{i=1}^{r} \ell_i q_i + q_0$.*

Lemma 4.2. *Let f be a degree four polynomial satisfying* $\mathrm{bias}(f) = \delta$. *Then there exist a linear subspace $V \subseteq \mathbb{F}^n$ of dimension* $\dim(V) \geq n - O(\log_{|\mathbb{F}|}(1/\delta))$, *such that for every $y \in V$* $\mathrm{rank}_3(\Delta_y(f)) = \log^{O(1)}(1/\delta)$.

Proof. As before, define $\mathcal{F}(y) \triangleq \mathrm{rank}_3(\Delta_y(f))$. It is again not difficult to see that $\mathcal{F}$ is a subadditive function. By Lemma 3.1 we get that there is a subset $S \subseteq \mathbb{F}^n$ of size $\frac{\delta^2}{2} \cdot \mathbb{F}^n$ such that for all $y \in S$, $\mathrm{bias}(\Delta_y(f)) \geq \frac{\delta^2}{2}$. Theorem 1 implies that for every $y \in S$ it holds that $\mathrm{rank}_3(\Delta_y(f)) = O(\log^4(\frac{1}{\delta}))$. From Lemma 2.5 it follows that there is a linear subspace $V \subseteq \mathbb{F}^n$ with $\dim(V) \geq n - O(\log_{|\mathbb{F}|}(1/\delta))$, such that for every $y \in V$ $\mathrm{rank}_3(\Delta_y(f)) = O(\log^5(\frac{1}{\delta}))$. □

By applying an invertible linear transformation we can assume that $V = \{x : x_1 = \ldots = x_m = 0\}$ for some $m = O(\log_{|\mathbb{F}|}(1/\delta))$. We now have

$$f = \sum_{i=1}^{m} x_i g_i + f' \,, \tag{3}$$

where $f' = f'(x_{m+1}, \ldots, x_n)$. Moreover, by Lemma 4.2 it follows that for every $y = (0, \ldots, 0, y_{m+1}, \ldots, y_n)$, $\mathrm{rank}_3(\Delta_y(f)) = O(\log^5(\frac{1}{\delta}))$. Notice that for every such y it holds that

$$\Delta_y(f) = \sum_{i=1}^{m} x_i \Delta_y(g_i) + \Delta_y(f') \,.$$

Hence, $\mathrm{rank}_3(\Delta_y(f')) \leq \mathrm{rank}_3(\Delta_y(f)) + m$. We now fix some value to $x_1, \ldots, x_m$, such that $\mathrm{bias}(f(\alpha_1, \ldots, \alpha_m, x_{m+1}, \ldots, x_n)) \geq \delta$. Let

$$\tilde{f}(x_{m+1}, \ldots, x_n) \triangleq f(\alpha_1, \ldots, \alpha_m, x_{m+1}, \ldots, x_n) \,. \tag{4}$$

It follows that $\mathrm{bias}(\tilde{f}) \geq \delta$ and that for every $y = (y_{m+1}, \ldots, y_n)$, $\mathrm{rank}_3(\Delta_y(\tilde{f})) = \mathrm{rank}_3(\Delta_y(f')) = O(\log^5(\frac{1}{\delta}))$ (note that $\deg\left(\Delta_y(\tilde{f}) - \Delta_y(f')\right) = 2$ so they have the same rank). From now on we will only consider $\tilde{f}$ and not f. Observe that if we prove Theorem 3 for $\tilde{f}$ then by considering Equations (3) and (4) we get the required result for f itself.

4.2 Computing $\tilde{f}$ using a few quadratics

We now show that there is a large subspace on which f can be approximated by a function of a few quadratic polynomials. The following lemma of Bogdanov and Viola shows that if f is biased then it can be well approximated by a small set of partial derivatives.

Lemma 4.3. *(Lemma 24 from [BV07]) Let $f : \mathbb{F}^n \to \mathbb{F}$ be a function over a finite field $\mathbb{F}$ with* $\mathrm{bias}(f) = \delta$. *Then there are t directions $a_1, ..., a_t$ and a function H such that $H(\Delta_{a_1}(f), \ldots, \Delta_{a_t}(f))$ ϵ-approximates f, where $t \leq (1 + \log\frac{1}{\epsilon})\,(|\mathbb{F}|/\delta)^{O(1)}$.*

By the construction of $\tilde{f}$ we know that each of its partial derivatives is of rank $\log^{O(1)}(1/\delta)$ and that $\text{bias}(\tilde{f}) \geq \delta$. Thus, Lemma 4.3 guarantees that $\tilde{f}$ can be well approximated using a few quadratics.

Corollary 4.4. *For every $\epsilon > 0$ there are $c = (1 + \log \frac{1}{\epsilon}) (|\mathbb{F}|/\delta)^{O(1)}$ quadratic polynomials $Q_1, \ldots, Q_c$ and a function H such that $\tilde{f}$ is ϵ-approximated by $H(Q_1, \ldots, Q_c)$.*

The next lemma, which is the main lemma of [KL08] shows that if the approximation is good enough (i.e. ϵ is small), and if the quadratics satisfy the *strong regularity* property then $\tilde{f}$ can in fact be computed by a small number of quadratics.

Definition 4.5. *(strongly regular quadratic functions) We say that a family of quadratic functions $\{Q_i\}_{i=1}^m$ is γ-* strongly regular *if the following holds for every $x_0 \in \mathbb{F}^n$: for independent uniform random variables $Y_1, ..., Y_5$ the joint distribution of*

$$\left\{ Q_j \left(x_0 + \sum_{i \in I} Y_i \right) \mid j \in [m], I \subseteq [5], 1 \leq |I| \leq 2 \right\}$$

is γ close to the uniform distribution (recall Definition 2.1).

This definition is a restricted version of Definition 8 of [KL08] for quadratic polynomials. The interested reader is referred to that paper for the general definition for higher degree polynomials.

Lemma 4.6. *(Lemma 13 from [KL08]) Let $f(x)$ be a degree d polynomials, $h_1, ..., h_m$ polynomials of degree less than d and $H : \mathbb{F}^m \to \mathbb{F}$ a function such that*

- *$H(h_1, ..., h_m)$ ϵ-approximates f where $\epsilon \leq 2^{-2(d+1)}$.*
- *$\{h_i\}_{i=1}^m$ is a γ-strongly regular family where $\gamma \leq \min\left\{2^{-d}, 2^{-m}\right\}$.*

Then there exists a function $F : \mathbb{F}^m \to \mathbb{F}$ such that $f = F(h_1, ..., h_m)$.

In other words, the lemma says that if f is well approximated by a family of strongly regular functions then it can actually be computed everywhere by the functions in the family. We shall now show that if $q_1, \ldots, q_c$ are quadratic polynomials such that the rank of every nontrivial linear combination of them is high, then they are strongly regular. This will imply (by Corollary 4.4) that $\tilde{f}$ is a function of a few quadratics and therefore so is f.

Lemma 4.7. *Let $\{Q_i\}_{i=1}^m$ be a family of quadratic functions such that for every nontrivial linear combination, $\text{rank}_2(\sum_{i=1}^m \alpha_i Q_i) \geq R$. Then $\{Q_i\}_{i=1}^m$ is a γ-strongly regular family for $\gamma = |\mathbb{F}|^{3m/2 - R/4}$.*

Proof. The proof is based on the analogy between quadratic functions and matrices.

Definition 4.8. *Let $Q : \mathbb{F}^n \to \mathbb{F}$ be a quadratic polynomial and $A \in \mathbb{F}^{n \times n}$ an $n \times n$ matrix. We say that A represents Q if there exists a linear function ℓ such that $Q(x) = x^t A x + \ell(x)$.*

Notice that there may be many different matrices representing the same polynomial Q. For example, every antisymmetric matrix represents the zero function. More generally, if S is antisymmetric then A and $A + S$ represent the same polynomial.

Lemma 4.9. *Let q be a quadratic polynomial. Then* $\mathrm{rank}_2(q)$ *(recall Definition 2.6) is equal to the minimal rank of a matrix representing q. Moreover, for every matrix A representing q we have that* $\mathrm{rank}(A+A^t)/2 \leq \mathrm{rank}_2(q) \leq \mathrm{rank}(A+A^t)$.

We shall prove the lemma for $\mathbb{F} = \mathbb{F}_2$. The proof for other fields is similar.

Proof. Let $\mathrm{rank}_2(q) = r$. Then q can be expressed as $\sum_{i=1}^{r}\left(\sum_{j=1}^{n} a_{i,j}x_j\right)\left(\sum_{j=1}^{n} b_{i,j}x_j\right) + \ell(x)$. Set $A = (a_{i,j}), B = (b_{i,j}) \in \mathbb{F}^{r\times n}$. It is clear that A^tB represents q and that $\mathrm{rank}(A^tB) \leq r$. On the other hand, if q can be represented by a rank r matrix A, then let $\ell_1,\ldots,\ell_r$ be a basis for the rows of A, when interpreted as linear functions.[6] Let A_i be the i-th row of A and denote $A_i = \sum_{j=1}^{r}\alpha_{i,j}\ell_j$. We have that for some linear function ℓ,

$$q-\ell = x^tAx = \sum_{i=1}^{n} x_iA_i(x) = \sum_{i=1}^{n} x_i\sum_{j=1}^{r}\alpha_{i,j}\ell_j = \sum_{j=1}^{r}\ell_j\left(\sum_{i=1}^{n}\alpha_{i,j}x_i\right) = \sum_{j=1}^{r}\ell_j\ell'_j\ ,$$

where $\ell'_1,\ldots,\ell'_r$ are linear functions. This implies that $\mathrm{rank}_2(q) \leq r$. Thus, $\mathrm{rank}_2(q) = \min\{\mathrm{rank}(A) \mid q(x) = x^tAx+\ell(x)\}$.

To prove the second claim, let A be any matrix representing q. We first change the basis of the space so that with respect to the new basis q will have the form of Theorem 1.1. Let T be an invertible matrix representing the change of basis. Clearly, T^tAT represents $q\circ T = \sum_{i=1}^{r} x_{2i-1}x_{2i} + \ell$, where $r = \mathrm{rank}_2(q)$. Thus, the matrix T^tAT can be written as $D+S$ where D is a block diagonal matrix consisting of r nonzero blocks of size 2×2 and S is a symmetric matrix. We also note that for each 2×2 diagonal block C of D it holds that $C+C^t \neq 0$. We thus get that

$$\mathrm{rank}(A+A^t) = \mathrm{rank}(T^t(A+A^t)T) = \mathrm{rank}(D+S+D^t+S^t) = \mathrm{rank}(D+D^t)\ .$$

Now, for every 2×2 diagonal block C of D we have that $1 \leq \mathrm{rank}(C+C^t) \leq 2$ and so

$$\mathrm{rank}_2(q) = r \leq \mathrm{rank}(D+D^t) \leq 2r = 2\mathrm{rank}_2(q)\ .$$

This completes the proof of the Lemma.[7] □

We continue the proof of Lemma 4.7. Using the above observation we now prove that any nontrivial linear combination $\sum_{k\in[m],I\subseteq[5],1\leq|I|\leq 2}\alpha_{k,I}Q_j(x+\sum_{i\in I}Y_i)$ has high rank (as a quadratic polynomial in the variables $Y_1\cup\ldots\cup Y_5$).

Fix $x = x_0$ and let A_k be a matrix representing Q_k. Notice that the quadratic polynomial $Q_k(x_0+\sum_{i\in I}Y_i)$ (in the variables $\cup_{i=1}^{5}Y_i$) can be represented by a block matrix $B^{k,I} \in \mathbb{F}^{5n\times 5n}$. Indeed, consider a 5×5 matrix that has 1 in the (i,j)-position iff $i,j\in I$, and zeros otherwise. Now, replace any 1 by the matrix A_k and every 0 by the $n\times n$ zero matrix. It is an easy calculation to see that this matrix represents $Q_k(x_0+\sum_{i\in I}Y_i)$. We shall abuse notations and for $i,j\in I$ say that $(B^{k,I})_{i,j} = A_k$, and that otherwise $(B^{k,I})_{i,j} = 0$.

[6] I.e. $(a_1,\ldots,a_n) \leftrightarrow \sum_{i=1}^{n} a_i\cdot x_i$.

[7] From the proof it actually follows that over $\mathbb{F}_2$, $\mathrm{rank}_2(q) = \mathrm{rank}(A+A^t)/2$ but this is not the case for other prime fields.

Clearly, the linear combination

$$Q' \triangleq \sum \{\alpha_{k,I} Q_k(x + \sum_{i \in I} Y_i) \mid k \in [m], I \subseteq [5], 1 \leq |I| \leq 2\}$$

is represented by the matrix

$$C \triangleq \sum \{\alpha_{k,I} B^{k,I} \mid k \in [m], I \subseteq [5], 1 \leq |I| \leq 2\} .$$

Observe that for $i \neq j \in [5]$, $C_{i,j} = \sum_{k \in [m]} \alpha_{k,\{i,j\}} A_k$. We now show that if for some $i \neq j \in [5]$ and $k \in [m]$ it holds that $\alpha_{k,\{i,j\}} \neq 0$ then the rank of $C^t + C$ (and hence of Q') is high.

$$\begin{aligned}
\text{rank}_2(Q') &= \text{rank}_2 \left(\sum \left\{ \alpha_{k,I} Q_k(x + \sum_{i \in I} Y_i) \mid k \in [m], I \subseteq [5], 1 \leq |I| \leq 2 \right\} \right) \\
&\geq \frac{1}{2} \text{rank}(C + C^t) \geq \frac{1}{2} \text{rank}(C_{i,j} + C^t_{j,i}) \\
&= \frac{1}{2} \text{rank} \left(\sum_{k \in [m]} \alpha_{k,\{i,j\}} \left(A_k + A_k^t \right) \right) \\
&\geq \frac{1}{4} \text{rank}_2 \left(\sum_{k \in [m]} \alpha_{k,\{i,j\}} Q_k \right) > \frac{1}{4} R .
\end{aligned}$$

If it is not the case, namely, for all $i \neq j \in [5], k \in [m]$ $\alpha_{k,\{i,j\}} = 0$, then there is some $i \in [5]$ and $k \in [m]$ such that $\alpha_{k,\{i\}} \neq 0$ and we get that same result by considering $C_{i,i}$ instead.

To conclude, every nontrivial linear combination of $\left\{ Q_j(x + \sum_{i \in I} Y_i) \right\}_{k \in [m], I \subseteq [5], 1 \leq |I| \leq 2}$ has rank grater than $\frac{1}{4} R$. Lemma 3.3 implies that the bias of every such linear combination is bounded by $|\mathbb{F}|^{-R/4}$. It now follows by Lemma 2.2 that the distribution is $|\mathbb{F}|^{3m/2 - R/4}$ close to the uniform distribution as needed. □

We thus get the following corollary.

Corollary 4.10. *Let $g(x)$ be a degree d polynomials, $q_1, ..., q_m$ quadratic polynomials and $H : \mathbb{F}^m \to \mathbb{F}$ a function such that*

- *$H(h_1, ..., h_m)$ ϵ-approximates g where $\epsilon \leq 2^{-2(d+1)}$.*
- *The bias of every non trivial combination of $h_1, \ldots, h_m$ is $|\mathbb{F}|^{-\Omega(m+d)}$.*

Then there exists a function $G : \mathbb{F}^m \to \mathbb{F}$ such that $g = G(h_1, ..., h_m)$.

We now show that $\tilde{f}$ can be computed by a few quadratics.

Lemma 4.11. *Let $g : \mathbb{F}^n \to \mathbb{F}$ be a quartic polynomial such that for every y, $\text{rank}_3(\Delta_y(f)) \leq \text{poly}(1/\delta)$. Then there exist a subspace W, $c = \text{poly}(|\mathbb{F}|/\delta)$ quadratics $q'_1, \ldots, q'_c$ and a function G such that, $\dim(W) = n - \text{poly}(|\mathbb{F}|/\delta)$ and $g|_W = G(q'_1, \ldots, q'_c)$.*

Proof. Applying Lemma 4.3, and using the fact that every partial derivative of g has a low rank, we conclude that for $\epsilon = 2^{-20}$ there exist $c = \text{poly}(|\mathbb{F}|/\delta)$ linear and quadratic functions, and a function H, such that $H(\ell_1, \ldots, \ell_c, q_1, \ldots, q_c)$ ϵ-approximates g. Let $r = \text{poly}(|\mathbb{F}|/\delta)$ and $U = \{x : \ell_1(x) = \alpha_1, \ldots, \ell_c(x) = \alpha_c\}$ be some subspace such that $H(\alpha_1, \ldots, \alpha_c, q_1|_U, \ldots, q_c|_U)$ ϵ-approximates $g|_U$. Applying Lemma 3.8 on $q_1|_U, \ldots, q_c|_U$ and r we get that there exists a (possible affine) subspace $W \subseteq U$ and $c' \leq c$ such that: $\dim(W) \geq \dim(U) - (r+1)c \geq n - (r+2)c = n - \text{poly}(|\mathbb{F}|/\delta)$; w.l.o.g. for every $i = 1 \ldots c$, $q_i|_W \in \text{span}\{q_1|_W, \ldots, q_{c'}|_W\}$; any nontrivial linear combination of $q_1|_W, \ldots, q_{c'}|_W$ has rank larger than r; $g|_W$ is ϵ-approximated by $H(\ell_1|_W, \ldots, \ell_c|_W, q_1|_W, \ldots, q_c|_W)$ (this follows by picking an adequate shift of the linear space in the lemma). Hence, $g|_W$ is ϵ-approximated by $H(\ell_1|_W, \ldots, \ell_c|_W, q_1|_W, \ldots, q_c|_W) = H'(q_1|_W, \ldots, q_{c'}|_W)$ for some H'. The reason for passing to W is that now any nontrivial linear combination of $q_1|_W, \ldots, q_{c'}|_W$ has rank larger than r. We thus get by Corollary 4.10 that there is some function G such that $g|_W = G(q_1|_W, \ldots, q_{c'}|_W)$. □

Recall that we assume w.l.o.g. that for every $y \in \mathbb{F}^{n-m}$, $\text{rank}_3(\Delta_y(\tilde{f})) \leq \text{poly}(1/\delta)$. Thus, the lemma above implies the following corollary.

Corollary 4.12. *In the notations of the proof, there exist a subspace $Z \subset \mathbb{F}^{n-m}$ of dimension $\dim(Z) \geq n - \text{poly}(|\mathbb{F}|/\delta)$ such that $\tilde{f}|_Z = F(q_1, \ldots, q_c)$, for $c = \text{poly}(|\mathbb{F}|/\delta)$ quadratic polynomials and some function F.*

4.3 The structure of f

We now show that we can represent $\tilde{f}$ as $\tilde{f} = \sum_{i=1}^{k} \ell_i \cdot g_i + \sum_{i=1}^{k} q_i' \cdot q_i''$ where $k = \text{poly}(|\mathbb{F}|/\delta)$, the ℓ_i-s are linear, the q_i'-s and q_i''-s are quadratic and the g_i-s are cubic polynomials. For this we will transform the quadratic polynomials to be what we denote as *disjoint polynomials*.

Definition 4.13. *We say that the quadratic polynomials $\{Q_i\}_{i=1}^m$ are disjoint if there is a linear transformation T, $2m$ variables $\{x_i\}_{i=1}^m \cup \{y_i\}_{i=1}^m$, where possibly for several i-s $x_i = y_i$, and quadratic functions $\{Q_i'\}_{i=1}^m$ such that for every $k \in [m]$, $Q_k \circ T = x_k y_k + Q_k'$ where no degree two monomial in Q_k' contains a variable from $\{x_i\}_{i=1}^m \cup \{y_i\}_{i=1}^m$.*

Lemma 4.14. *Let $q_1, \ldots, q_c$ be quadratic polynomials from $\mathbb{F}^n$ to $\mathbb{F}$. Assume that the rank of every nontrivial linear combination of them is at least r. Then there exist a subspace $V \subseteq \mathbb{F}^n$ of dimension $\geq n - 2c^2$ and $c' \leq c$ quadratic polynomials $q_1', \ldots, q_{c'}' : V \to \mathbb{F}$ satisfying: the q_i'-s are disjoint; every nontrivial linear combination of the q_i'-s has rank at least $r - 2c^2$; $\text{span}(q_1', \ldots, q_{c'}') = \text{span}(q_1|_V, \ldots, q_c|_V)$.*

Proof. We prove the lemma by iteratively changing each q_i to a 'disjoint' form. We shall give the proof over $\mathbb{F}_2$ but almost the same proof holds for odd characteristics as well. We start with q_1. Assume w.l.o.g. that $x_1 \cdot x_2$ appears in q_1. Now, from every other q_i subtract an appropriate multiple of q_1 such that at the end $x_1 \cdot x_2$ only appears in q_1. For simplicity we call the new polynomial q_i as well. Now, for $2 \leq i$ and $j \in \{1, 2\}$ let $x_j \cdot \ell_{i,j}$ be the degree two monomials involving x_j in q_i. For q_1 let $x_j \cdot \ell_{1,j}$ be the degree 2 monomials involving x_j in $q_1 - x_1 \cdot x_2$. Let $V_1 = \{x \mid \ell_{1,1}(x) = \ldots = \ell_{2,c}(x) = 0\}$. Notice that none of the $\ell_{i,j}$-s contain x_1 or x_2. After restricting the polynomials to V_1 we have that $x_1 \cdot x_2$ appears in q_1

and every other appearance of either x_1 or x_2 is in degree one monomials. We now move to (the 'new') q_2 and continue this process. At the end we obtain a subspace V and quadratics $q'_1, \ldots, q'_{c'}$ (c' may be smaller than c if some polynomials vanished in the process). As at each step we set at most $2c$ linear functions to zero, for a total of at mots $2c^2$ linear functions, the claims about the dimension of V and the rank of every linear combination of the q_i-s follow. It is clear that the $q_i|_v$-s span the q'_i-s and so the lemma is proved.

When dealing with odd characteristics instead of looking for $x_1 \cdot x_2$ we search for x_1^2. By applying an invertible linear transformation such a monomial always exists and we continue with the same argument. □

The usefulness of the definition is demonstrated in the following lemma.

Lemma 4.15. *Let $q_1, \ldots, q_c$ be disjoint quadratic polynomials. Assume that $\deg(f) = 2d$ and $f = F(q_1, \ldots, q_c)$ for some function $F(z_1, \ldots, z_c)$. Then as a polynomial over $\mathbb{F}$, $\deg(F) \leq d$.*

Proof. We shall give the proof over $\mathbb{F}_2$ but it is again similar over odd characteristic fields. Let $z_1^{e_1} \cdots z_c^{e_c}$ be a monomial of maximal degree in F. When composing it with $q_1, \ldots, q_c$ we get that $q_1^{e_1} \cdots q_c^{e_c}$ contains the monomial $\prod_{i=1}^{c}(x_i \cdot y_i)^{e_i}$. As $z_1^{e_1} \cdots z_c^{e_c}$ is of maximal degree and each x_i and y_i appear only as linear terms in all the q_j-s (except the monomial $x_i \cdot y_i$ in q_i) we see that this monomial cannot be cancelled by any other monomial created in $F(q_1, \ldots, q_c)$. Therefore the monomial $\prod_{i=1}^{c}(x_i \cdot y_i)^{e_i}$ belongs to f as well. Since $\deg(f) = 2d$ it must be the case that $2e_1 + \ldots + 2e_c \leq 2d$. Hence, $\deg(F) = \sum_{i=1}^{c} e_i \leq d$. □

We are now ready to complete the proof of Theorem 3.

Proof of Theorem 3. Combining Corollary 4.12, Lemma 4.15 and Lemma 4.14 we get that for the subspace Z of Corollary 4.12, there exist a subspace $Z' \subseteq Z$, of dimension $\dim(Z') \geq \dim(Z) - \text{poly}(|\mathbb{F}|/\delta)$, $b = \text{poly}(|\mathbb{F}|/\delta)$ quadratic polynomials $Q_1, \ldots, Q_b$ and a quadratic polynomial H such that $\tilde{f}|_{Z'} = H(Q_1, \ldots, Q_b)$. In other words $\tilde{f}|_{Z'} = \sum_{i \leq j} \alpha_{i,j} Q_i Q_j + Q_0$.

As $f|_{Z'} = \tilde{f}|_{Z'}$ it follows that $f|_{Z'} = \sum_{i \leq j} \alpha_{i,j} Q_i Q_j + Q_0$. Assume w.l.o.g.[8] that Z' is defined as $Z' = \{x \mid x_1 = \beta_1, \ldots, x_k = \beta_k\}$ for some $k = \text{poly}(|\mathbb{F}|/\delta)$. Then it is clear that we can write $f = \sum_{i=1}^{k} x_i \cdot g_i + \sum_{i \leq j} \alpha_{i,j} Q_i Q_j + g_0$ for cubic polynomials $g_0, \ldots, g_k$.

□

5 Quartic polynomials with high U^4 norm

In this section we prove Theorems 4 and 5. Intuitively, the notion of $d+1$ Gowers norm indicates how close a given function is to a degree d polynomial. In fact, it was conjectured that if the U^{d+1} norm is bounded away from zero then the function has a noticeable correlation with a degree d polynomial. This conjecture turned to be false even when the function is a degree four polynomial and $d = 3$ [LMS08, GT07]. Here we will show that for this special case a weaker conclusion holds. Namely, that for any degree four polynomial f there exists a subspace of dimension $n/\exp(\text{poly}(1/\|f\|_{U^4}))$ on which $f|_V$ is equal to some

[8] This is true up to an invertible linear transformation and an affine shift and has no real effect on the result, but rather simplifies the notations.

cubic polynomial. In fact an even stronger conclusion holds - there exists a partition of (a subspace of small co dimension of) $\mathbb{F}^n$ to such subspaces on which f equals a cubic. To ease the reading we restate Theorem 4 here.

Theorem (Theorem 4)**.** *Let $\mathbb{F}$ be a finite field and $f \in \mathbb{F}[x_1, \ldots, x_n]$ a degree four polynomial such that $\|f\|_{U^4} = \delta$. Then there exists a partition of a subspace $V \subseteq \mathbb{F}^n$, of dimension $\dim(V) = n - \mathrm{poly}(|\mathbb{F}|/\delta)$, to subspaces $\{V_\alpha\}_{\alpha \in I}$, satisfying $\dim(V_\alpha) = \Omega(n/|\mathbb{F}|^{\mathrm{poly}(1/\delta)})$, such that for every $\alpha \in I$, $f|_{V_\alpha}$ is a cubic polynomial.*

In other words, the theorem says that for $r = \mathrm{poly}(1/\delta)$ any such f (possibly after a change of basis of $\mathbb{F}^n$) can be written as $f = \sum_{i=1}^{r} x_{n-r+i} g_i(x_1, \ldots, x_n) + f'(x_1, \ldots, x_{n-r}) + g_0$, where the g_i-s are degree three polynomials and f' is a polynomial for which there exists a partition of $\mathbb{F}^{n-r}$ to subspaces $\{V_\alpha\}_{\alpha \in I}$, satisfying $\dim(V_\alpha) = \Omega(n/\exp(\mathrm{poly}(1/\delta)))$, such that for every $\alpha \in I$, $f'|_{V_\alpha}$ is a cubic polynomial.

As in the proof of Theorem 2 we start by passing to a subspace of a constant codimension on which every derivative has low rank, i.e $\Delta_y(f) = \sum_{i=1}^{r} \ell_i Q_i + Q_0$. Then we shall deduce that there is some common 'basis' $\{\ell_i\}_{i=1}^{t_2}, \{Q_i\}_{i=1}^{t_1}$ to all the derivatives. Namely, every derivative $\Delta_y(F)$ can be expressed as $\sum_{i=1}^{t_1} \ell_i^y Q_i + \sum_{i=1}^{t_2} \ell_i Q_i^y + Q_0^y$ (where y in the exponent means that the polynomial may depend on y). This is the main technical difficulty of the proof and it is based on an extension of Lemma 3.7 to the case of low rank cubic polynomials. Then, we conclude that for every setting α of $\{\ell_i\}_{i=1}^{t_2}, \{Q_i\}_{i=1}^{t_1}$ we obtain a subspace V_α on which all the derivative are quadratic polynomials, i.e $f|_{V_\alpha}$ is cubic.

5.1 The case of the symmetric polynomial

Let $S_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \ldots < i_k \le n} x_{i_1} \cdot x_{i_2} \cdots x_{i_k}$. In [GT07, LMS08] it was shown that over $\mathbb{F}_2$, it holds that $\|S_4\|_{U^4} \ge \delta$, for some absolute constant $0 < \delta$, but for every degree three polynomial g, $\Pr[S_4 = g] \le 1/2 + \exp(-n)$. To make the claim of Theorem 4 clearer we shall work out the case of S_4 as an example.

Consider a partial derivative $\Delta_y(S_4)$. For simplicity assume that $n = 4m$. Computing we get that

$$\Delta_y(S_4) = S_2 \cdot \sum_{i \ne j}^{n} x_i y_j + S_1 \cdot \sum_{i \ne j}^{n} x_i y_j + \sum_{i \ne j}^{n} x_i y_j \ . \tag{5}$$

In particular, S_2 is a 'basis' for the set of partial derivatives of S_4. Continuing, we have that

$$S_2(x_1, \ldots, x_n) = \sum_{k=1}^{2m} \left(\sum_{i=1}^{2k-1} x_i \right) \cdot \left(x_{2k} + \sum_{i=1}^{2k-2} x_i \right) + \sum_{i=1}^{m} (x_{4i-3} + x_{4i-2}) \ . \tag{6}$$

For $k = 1, \ldots, 2m$ let $\ell_k = \sum_{i=1}^{2k-1} x_i$. Notice that fixing $\ell_1, \ldots, \ell_{2m}$ reduces the degree of S_2 to one and so every partial derivative of S_4 will have degree two. For example, consider the space $V_0 = \{x \mid \ell_1(x) = \ldots = \ell_{2m}(x) = 0\}$. Rewriting we get $V_0 = \{(0, y_1, y_1, y_2, y_2, \ldots, y_{2m-1} y_{2m-1}, y_{2m})\}$. Computing we get that

$$S_4|_{V_0} = S_2(y_1, \ldots, y_{2m-1}) \ .$$

A closer inspection shows that no matter how we set $\ell_1, \ldots, \ell_{2m}$ we will get that the degree of S_4 becomes two.

5.2 Finding a 'basis' for a space of low rank cubic polynomials

In this section we prove the main technical result showing that a subspace of degree 3 polynomials with low rank has a small 'basis'.

Lemma 5.1 (Main Lemma)**.** *Let M be a vector space of cubic polynomials satisfying $\mathrm{rank}_3(f) \leq r$ for all $f \in M$. Then there exists a set of linear and quadratic functions $\{Q_i\}_{i=1}^{t_1} \cup \{\ell_i\}_{i=1}^{t_2}$, for $t_1 \leq r$ and $t_2 = 2^{O(r)}$, such that every $f \in M$ can be represented as $f = \sum_{i=1}^{t_1} \ell_i^f Q_i + \sum_{i=1}^{t_2} \ell_i Q_i^f + Q_0^f$ for some linear and quadratic functions $\{\ell_i^f\}_{i=1}^{t_1} \cup \{Q_i^f\}_{i=0}^{t_2}$.*

The rest of this section is devoted to proving this lemma. Similarly to the proof of Lemma 3.7 we will work modulo a collection of linear and quadratic polynomials. For this we shall need the following definition.

Definition 5.2. *For a cubic polynomial f we say that $\mathrm{rank}_3^c(f) = r$ if r is the minimal integer such that f can be written as*

$$f = \sum_{i=1}^{r} \ell_i Q_i + \sum_{i=1}^{c} \ell_i^{(1)} \ell_i^{(2)} \ell_i^{(3)} + Q_0 \ , \tag{7}$$

where the ℓ-s are linear functions and the Q-s are quadratics.

To see that difference from the previous notion of rank_3 (Definition 4.1) we observe that if f is a degree three polynomial with $\mathrm{rank}_3(f) = r$ then $f = \sum_{i=1}^{r} \ell_i Q_i + Q_0$. If we also know that some nontrivial linear combination of $Q_1, \ldots, Q_r$ has rank (as a quadratic polynomial) less than c then $\mathrm{rank}_3^c(f) < r$. I.e. $\mathrm{rank}_3^c(f)$ ignores, in some sense, low rank quadratic functions in the representation of f.

Definition 5.3. *Let $A = \{Q_i\}_{i=1}^{t_1} \cup \{\ell_i\}_{i=1}^{t_2}$ be a set of linear and quadratic functions and let $f : \mathbb{F}^n \to \mathbb{F}$ be a degree three polynomial. Denote*

$$[f]_A \triangleq \left\{ f + \sum_{i=1}^{t_1} \ell_i' Q_i + \sum_{i=1}^{t_2} \ell_i Q_i' + Q_0' \mid \text{ for linear and quadratic functions } \{\ell_i'\}_{i=1}^{t_1},\ \{Q_i'\}_{i=0}^{t_2} \right\}$$

For a linear space M of degree three functions, we define the subspace $[M]_A$ to be

$$[M]_A \triangleq \{[f]_A \mid f \in M\} \ .$$

As before we define $\mathrm{rank}_3^c([f]_A)$ to be the lowest rank of functions in $[f]_A$.

$$\mathrm{rank}_3^c([f]_A) \triangleq \min \{\mathrm{rank}_3^c(g) \mid g \in [f]_A\} \ .$$

The definition of $[f]_A$ resembles, in some sense, the notion of working modulo an ideal. However, we note that as opposed to the usual definition, where for every f, $\{Q_i'\}_{i=1}^{t_1} \cup \{\ell_i'\}_{i=1}^{t_2}$ can be arbitrary functions, in our definition they are restricted to being quadratic and linear functions, respectively.

We are now ready to prove the main lemma of this section that shows the existence of a small 'basis' for any linear space of cubic polynomials of low rank.

Lemma 5.4. *Let $A = \{Q_i\}_{i=1}^{t_1} \cup \{\ell_i\}_{i=1}^{t_2}$ be a set of linear and quadratic polynomials. Let M be a linear space of cubic polynomials such that for every $[f]_A \in [M]_A$, $\text{rank}_3^c([f]_A) \leq r$. Then, there are r linear functions $\{\ell_i\}_{i=1}^r$ and a quadratic polynomial Q such that for $A' \triangleq A \cup \{\ell_i\}_{i=1}^r \cup \{Q\}$ it holds that every $[f]_{A'} \in [M]_{A'}$ satisfies $\text{rank}_3^{c'}([f]_{A'}) \leq r-1$, for $c' = 11c + 3r + t_1$.*

In other words, the lemma says that we can find a small set of linear functions and one quadratic polynomial such that by adding them to A and increasing c by a constant factor, we can decrease the $\text{rank}_3^{c'}$ of every polynomial in $[M]_{A'}$.

Proof. Assume that there is some $[g]_A \in [M]_A$ such that $\text{rank}_3^{c'}(g) = \text{rank}_3^{c'}([g]_A) = r$. If no such g exists then for every $[f]_A \in [M]_A$, $\text{rank}_3^{c'}([f]_A) \leq r-1$ and there is nothing to prove. As $c < c'$ it also holds that $\text{rank}_3^c([g]_A) = r$. Hence, g can be represented as $\sum_{i=1}^r \ell_i^g Q_i^g + \sum_{i=1}^c \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)}$.[9] Note that $\text{rank}_2([Q_1^g]_A) > c' - c$ as otherwise we could replace Q_1^g with a function of the form $\sum_{i=1}^{t_1} \alpha_i Q_i + \sum_{i=1}^{t_2} \ell_i \ell_i' + \sum_{j=1}^{c'-c} m_i m_i'$, where the m-s are linear functions, and get that $\text{rank}_3^{c'}([g]_A) \leq r-1$.

Set $A' \triangleq A \cup \{\ell_i^g\}_{i=1}^r \cup \{Q_1^g\}$. Assume for contradiction that there is some $h \in M$ satisfying $\text{rank}_3^{c'}([h]_{A'}) = r$. This implies that $\text{rank}_3^c([h]_A) = r$ and that $\text{rank}_3^{c'-c}([h+g]_{A'}) = r$ as well. Indeed, if the latter equation was not true then by expressing $h+g$ as a low $\text{rank}_3^{c'-c}$ polynomial and moving g to the other side we would get that $\text{rank}_3^{c'}([h]_{A'}) < r$ in contradiction (recall that $\{\ell_i^g\} \subset A'$). From this we get that $\text{rank}_3^c([h+g]_A) = r$ as well. Let $f \in [h+g]_A$ be such that $\text{rank}_3^c(f) = r$. Express h and f as $h = \sum_{i=1}^r \ell_i^h Q_i^h + \sum_{i=1}^c \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)}$ and $f = \sum_{i=1}^r \ell_i^f Q_i^f + \sum_{i=1}^c \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)}$. Note that we can assume that w.l.o.g $\{\ell_i\}_{i=1}^{t_2}$ are linearly independent as otherwise we can just replace them with a linearly independent subset. Similarly, we can also assume that $\{\ell_i\}_{i=1}^{t_2} \cup \{\ell_i^g\}_{i=1}^r$ are linearly independent as otherwise we can find a representation for a function in $[g]_A$ with a smaller rank. Using the same argument again we conclude that $\{\ell_i\}_{i=1}^{t_2} \cup \{\ell_i^g\}_{i=1}^r \cup \{\ell_i^h\}_{i=1}^r$ are linearly independent as well (by considering $[h]_{A'}$).

Since $g + h - f \in [0]_A$, we can express this polynomial as $g + h - f = \sum_{i=1}^{t_1} \ell_i' Q_i + \sum_{i=1}^{t_2} \ell_i Q_i' + Q_0'$. In other words:

$$\sum_{i=1}^r \ell_i^g Q_i^g + \sum_{i=1}^c \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)} + \sum_{i=1}^r \ell_i^h Q_i^h + \sum_{i=1}^c \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)} \quad - \\ \left(\sum_{i=1}^r \ell_i^f Q_i^f + \sum_{i=1}^c \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)} + \sum_{i=1}^{t_1} \ell_i' Q_i + \sum_{i=1}^{t_2} \ell_i Q_i' + Q_0' \right) \quad = \quad 0\,. \tag{8}$$

To ease notations, using the fact that $\{\ell_i\}_{i=1}^{t_2} \cup \{\ell_i^g\}_{i=1}^r \cup \{\ell_i^h\}_{i=1}^r$ are linearly independent, let

[9] By definition of $[g]_A$ we can add any quadratic polynomial to g so we can assume that there is no extra Q_0^g term in the representation of g.

us assume w.l.o.g. that $\forall i$, $\ell_i^g = x_i$, $\ell_i^h = x_{r+i}$ and $\ell_i = x_{2r+i}$. Thus, Equation (8) becomes

$$\sum_{i=1}^{r} x_i Q_i^g + \sum_{i=1}^{c} \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)} + \sum_{i=1}^{r} x_{r+i} Q_i^h + \sum_{i=1}^{c} \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)} \quad - \\ \left(\sum_{i=1}^{r} \ell_i^f Q_i^f + \sum_{i=1}^{c} \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)} + \sum_{i=1}^{t_1} \ell_i' Q_i + \sum_{i=1}^{t_2} x_{2r+i} Q_i' + \tilde{Q}_0 \right) \quad = \quad 0 \,, \tag{9}$$

where we remember that variables from $\{x_i\}_{i=1}^{2r+t_2}$ may appear in the linear and quadratic functions in the expression. Consider all terms involving x_1 (recall that $\ell_1^g = x_1$) in Equation (9). Clearly they sum to zero, but they can also be written as

$$0 = Q_1^g + \sum_{i=1}^{r} x_i m_i^g + \sum_{i=1}^{3c} \alpha_i^g m_i^{g,(1)} m_i^{g,(2)} + \sum_{i=1}^{r} x_{r+i} m_i^h + \sum_{i=1}^{3c} \alpha_i^h m_i^{h,(1)} m_i^{h,(2)} - \\ \left(\sum_{i=1}^{r} \beta_i^f Q_i^f + \sum_{i=1}^{r} \ell_i^f m_i^f + \sum_{i=1}^{3c} \beta'^f_i m_i^{f,(1)} m_i^{f,(2)} + \sum_{i=1}^{t_1} \beta_i' Q_i + \sum_{i=1}^{t_1} \ell_i' m_i + \sum_{i=1}^{t_2} x_{2r+i} m_i' + m_0 \right) , \tag{10}$$

where the m-s are linear functions and the α-s and β-s are field elements. Rearranging terms we conclude that

$$\mathrm{rank}_2 \left(Q_1^g - \sum_{i=1}^{r} \beta_i^f Q_i^f - \sum_{i=1}^{t_1} \beta_i' Q_i - \sum_{i=1}^{t_2} x_{2r+i} m_i' \right) \leq 3r + 9c + t_1 = c' - 2c \,. \tag{11}$$

This implies that

$$\mathrm{rank}_2 \left(\left[\sum_{i=1}^{r} \beta_i^f Q_i^f \right]_{A'} \right) \leq c' - 2c \,.$$

We now have two cases to consider. If $(\beta_1^f, \ldots, \beta_r^f)$ are not all zero then, by arguments described above, this implies that $\mathrm{rank}_3^{c'-c}([f]_{A'}) \leq r - 1$. Recalling that $[h+g]_{A'} = [f]_{A'}$ we get a contradiction. If, on the other hand, $(\beta_1^f, \ldots, \beta_r^f) = 0$ then Equation (11) implies that $\mathrm{rank}_2([Q_1^g]_A) \leq c' - 2c$ and so $\mathrm{rank}_3^{c'-c}([g]_A) \leq r - 1$ in contradiction to the choice of g. Concluding, we have that for every $f \in M$, $\mathrm{rank}_3^{c'}([f]_{A'}) \leq r - 1$ as required. □

By applying Lemma 5.4 r times we obtain the following corollary.

Corollary 5.5. *Let M be a vector space of cubic polynomials satisfying $\mathrm{rank}_3(f) \leq r$ for every $f \in M$. Then there exists a set of quadratic and linear functions $A = \{Q_i\}_{i=1}^{r} \cup \{\ell_i\}_{i=1}^{r(r-1)/2}$, such that for $c = \exp(r)$, $\mathrm{rank}_3^c([f]_A) = 0$ for every $f \in M$.*

We now have that every function in M, modulo some set A of linear and quadratic functions, can be expressed as $\sum_{i=1}^{c} \ell_i^{(1)}, \ell_i^{(2)}, \ell_i^{(3)}$, for some c. Next we show that we can add $3c$ additional linear functions to A such that modulo the new set every function becomes zero. We again give an iterative procedure for finding those linear functions.

Before proving this result we define the notion of $\dim_3^c([f]_A)$ that will serve as a potential function in our argument (in a similar way to the role played by rank_3^c).

Definition 5.6. *Let A be a set of quadratic and linear functions and $[f]_A$ a class of cubic functions such that* $\text{rank}_3^c([f]_A) = 0$. *We define the dimension of the class as follows:*

$$\dim_3^c([f]_A) = \min \left\{ \dim \left(\text{span} \left\{ \ell_i^{(1)}, \ell_i^{(2)}, \ell_i^{(3)} \right\}_{i=1}^{c} \right) \mid \sum_{i=1}^{c} \ell_i^{(1)} \ell_i^{(2)} \ell_i^{(3)} \in [f]_A \right\} .$$

To better understand the reason for the definition we note that if $\text{rank}_3^c([f]_A) = 0$ then $\sum_{i=1}^{c} \ell_i^{(1)} \ell_i^{(2)} \ell_i^{(3)} + Q \in [f]_A$ for some linear functions and quadratic Q. Thus, our goal will be to find a small set of linear functions that, simultaneously, form a basis to all those linear functions for all $f \in M$. The next lemma shows that by joining $\left\{ \ell_i^{(1)}, \ell_i^{(2)}, \ell_i^{(3)} \right\}_{i=1}^{c}$ from some polynomial f, of maximal dimension in $[M]_A$, to A, the dimension of every other element in $[M]_A$ decreases.

Lemma 5.7. *Let $A = \{Q_i\}_{i=1}^{t_1} \cup \{\ell_i\}_{i=1}^{t_2}$ be a set of linear and quadratic functions. Assume that the rank of any nontrivial linear combination of $\{Q_i\}_{i=1}^{t_1}$ is greater than $9c + t_1 + t_2$. Let M be a linear space of cubic polynomials such that for every $[f]_A \in [M]_A$,* $\text{rank}_3^c([f]_A) = 0$ *and* $\dim_3^c([f]_A) \leq d$. *Then, there are d linear functions $\{\ell_i'\}_{i=1}^{d}$ such that for $A' \triangleq A \cup \{\ell_i'\}_{i=1}^{d}$,* $\dim_3^c([f]_{A'}) \leq d - 1$ *for all $[f]_{A'} \in [M]_{A'}$.*

The proof is very similar in nature to the proof of Lemma 5.4.

Proof. We start by passing to the subspace $V = \{x \mid \ell_1(x) = \ldots = \ell_{t_2}(x) = 0\}$. When restricting the Q_i-s to V the rank of every linear combination can drop by at most t_2 so it is still at least $9c + t_1$. From now on we shall work over V. Note that if we prove the theorem over V then it clearly holds over $\mathbb{F}^n$ as well.

Let $[g]_A \in [M]_A$ be a class satisfying $\dim_3^c([g]_A) = d$. By definition we can assume that g is such that $g = \sum_{i=1}^{c} \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)}$, and that for some d linearly independent linear functions $\{\ell_i^g\}_{i=1}^{d}$ it holds that $\left\{ \ell_i^{g,(1)}, \ell_i^{g,(2)}, \ell_i^{g,(3)} \right\}_{i=1}^{c} \subseteq \text{span}\{\ell_i^g\}_{i=1}^{d}$. Set $A' = A \cup \{\ell_i^g\}_{i=1}^{d}$. We will show that for every $f \in M$ it holds that $\dim_3^c([f]_{A'}) \leq d - 1$.

Assume for contradiction that there is some $[h]_{A'} \in [M]_{A'}$ such that $\dim_3^c([h]_{A'}) = d$. Clearly, $\dim_3^c([h]_A) = d$ as well. W.l.o.g. let $h = \sum_{i=1}^{c} \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)}$. We also denote with $\left\{ \ell_i^h \right\}_{i=1}^{d}$ a basis for $\left\{ \ell_i^{h,(1)}, \ell_i^{h,(2)}, \ell_i^{h,(3)} \right\}_{i=1}^{c}$. As $\dim_3^c([h]_A)$ does not decreases modulo $\{\ell_i^g\}_{i=1}^{d}$, it follows that $\{\ell_i^g\}_{i=1}^{d} \cup \left\{ \ell_i^h \right\}_{i=1}^{d}$ are linearly independent. By definition of A' we have that $\dim_3^c([g+h]_{A'}) = \dim_3^c([h]_{A'}) = d$. Let $f \in [g+h]_A$ be such that $f = \sum_{i=1}^{c} \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)}$ and $\dim(\text{span}\{\ell_i^{f,(j)}\}) = d$. Since $g + h - f \in [0]_A$ we have that $g + h - f = \sum_{i=1}^{t_1} Q_i \ell_i' + Q'$. We now show that all the ℓ_i'-s are zero. Assume for contradiction that this is not the case. Namely, $\{\ell_i'\}_{i=1}^{t_1}$ are not all zero. In particular, some ℓ_i' depends on some variable x. Write $g + h - f = xF + H$ where H does not depend on x. We now estimate $\text{rank}_2(F)$. On the one hand F can be expressed as $\sum_{i=1}^{t_1} \alpha_i Q_i + \sum_{i=1}^{t_1} m_i \ell_i' + m_0$ for some coefficients $\{\alpha_i\}_{i=1}^{t_1}$ (not all of them are zero) and some linear functions $\{m_i\}_{i=0}^{t_1}$. Hence, $\text{rank}_2(F)$ is larger than $9c$ (remember that $\text{rank}_2(\sum_{i=1}^{t_1} \alpha_i Q_i) > 9c + t_1$ on V). On the other hand, $g + h - f$ is equal to

$$g + h - f = \sum_{i=1}^{c} \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)} + \sum_{i=1}^{c} \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)} - \sum_{i=1}^{c} \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)} ,$$

so F can be expressed as $\sum_{i=1}^{9c} \hat{m}_i \tilde{m}_i + \ell$, i.e it's rank is at most $9c$, in contradiction. It follows that $g + h - f = Q$, for some quadratic Q. Thus,

$$\sum_{i=1}^{c} \ell_i^{g,(1)} \ell_i^{g,(2)} \ell_i^{g,(3)} + \sum_{i=1}^{c} \ell_i^{h,(1)} \ell_i^{h,(2)} \ell_i^{h,(3)} = \sum_{i=1}^{c} \ell_i^{f,(1)} \ell_i^{f,(2)} \ell_i^{f,(3)} + Q \,. \tag{12}$$

For simplicity, assume w.l.o.g. that for $i = 1 \dots d$, $\ell_i^g = y_i$, $\ell_i^h = z_i$. We would like to show that if Equation (12) holds then $\deg(h) = 2$ in contradiction to the choice of h. To further simplify notations we assume w.l.o.g. that the ℓ_i^f-s are linear functions in the variables $y_1 \dots, y_d, z_1, \dots, z_d$ (as we can set all other variables to zero and still obtain a similar equality). In particular, every ℓ_i^f can be expressed as $\ell_i^f = \ell_i^{f,g}(y) + \ell_i^{f,h}(z)$. Hence, Equation (12) can be rewritten as $Q(y,z) + f(y,z) = g(y) + h(z)$. Therefore, it holds that $g(y) = f(y,0) + Q(y,0)$ and $h(y) = f(0,z) + Q(0,z)$.[10] In particular, there is some representation of g and h as sums of products of linear functions such that $\left\{\ell_i^{f,g}(y)\right\}$ and $\left\{\ell_i^{f,h}(z)\right\}$ are their basis, respectively. By applying an invertible linear transformation we can further assume that $\ell_i^{f,g}(y) = y_i$ and $\ell_i^{f,h}(z) = z_i$. Thus, the basis for $\{\ell_i^{f,(j)}\}$ is $\ell_1^f = y_1 + z_1, \dots, \ell_d^f = y_d + z_d$. As a consequence we have that $f = \sum_{i=1}^{c} \ell_i^{f,(1)}(y+z)\ell_i^{f,(2)}(y+z)\ell_i^{f,(3)}(y+z)$.

Define $F : \mathbb{F}^d \to \mathbb{F}$ as $F(u) = \sum_{i=1}^{c} \ell_i^{f,(1)}(u)\ell_i^{f,(2)}(u)\ell_i^{f,(3)}(u)$. Hence, $f = F(y+z)$, $g = f(y,0) + Q(y,0) = F(y) + Q'(y)$ and $h = F(z) + Q''(z)$. Thus, for every $\alpha, \beta \in \mathbb{F}^d$ $F(\alpha+\beta) = F(\alpha) + F(\beta) + \tilde{Q}(\alpha,\beta)$. It is not difficult to check that if F is a polynomial such that $\deg\left(F(\alpha+\beta) - F(\alpha) - F(\beta)\right) \leq 2$ then $\deg(F) \leq 2$. Therefore, $[h]_A = [F(z)]_A = [0]_A$ (because F is quadratic), in contrary to the fact that $\dim_3^c([h]_A) = d$. We thus deduce that for every $[h]_{A'} \in [M]_{A'}$, $\dim_3^c([h]_{A'}) < d$ as required. □

Combining Lemma 5.4 and Lemma 5.7 we are now able to prove Lemma 5.1.

Proof of Lemma 5.1. Corollary 5.5 implies that there exists a set of quadratic and linear functions $A = \{Q_i\}_{i=1}^{r} \cup \{\ell_i\}_{i=1}^{r(r-1)/2}$, such that for $c = \exp(r)$, $\operatorname{rank}_3^c([f]_A) = 0$ for every $f \in M$. By Lemma 3.8 we can assume w.l.o.g. that every nontrivial linear combination of the Q_i's have rank larger than $10c$ (possibly after passing to a subspace V of dimension at least $n - \operatorname{poly}(c) = n - \exp(r)$ and throwing some of the Q_i-s (without changing the property of $[M]_A$)). By applying Lemma 5.7 $d = 3c$ times we get a set $A' = \{Q_i'\}_{i=1}^{t_1} \cup \{\ell_i'\}_{i=1}^{t_2}$, for $t_1 \leq r$ and $t_2 = \exp(r)$, such that $\dim_3^c([f]_{A'}) = 0$ for every $[f]_{A'} \in [M]_{A'}$. In particular, every $f \in M$ can be represented as $f = \sum_{i=1}^{t_1} \ell_i^f Q_i' + \sum_{i=1}^{t_2} \ell_i' Q_i^f + Q_0^f$ for some linear and quadratic functions $\left\{\ell_i^f\right\}_{i=1}^{t_1} \cup \left\{Q_i^f\right\}_{i=0}^{t_2}$ depending on f. □

5.3 Completing the proof

We can now complete the proof of Theorem 4. We first give a lemma summarizing what we have achieved so far.

[10] We can assume w.l.o.g. that ℓ_i^g and ℓ_i^h do not have a constant term.

Lemma 5.8. *Let f be a degree four polynomial with $\|f\|_{U^4} = \delta$. Then for $r = O(\log^2(1/\delta))$ there exist a subspace V, satisfying $\dim(V) \geq n - O(\log(1/\delta))$, r quadratic polynomials $Q_1, \ldots, Q_r$ and $R = \exp(r)$ linear functions $\ell_1, \ldots, \ell_R$ such that for every $y \in V$ we have that $\Delta_y(f|_V) = \sum_{i=1}^{r} Q_i \cdot \ell_i^y + \sum_{i=1}^{R} \ell_i \cdot Q_i^y + Q_0^y$.*

Proof. Let f be a quartic function such that $||f||_{U^4} > \delta$. By Lemma 3.2, Theorem 2 and Lemma 2.5 there is a subspace V, satisfying $\dim(V) \geq n - O(\log(1/\delta))$, such that every partial derivative of $f|_V$ is a cubic polynomial of rank at most $r = O(\log^2(1/\delta))$. Let $f' = f|_V$. Lemma 5.1 gives a set $A = \{Q_i\}_{i=1}^{r} \cup \{\ell_i\}_{i=1}^{\exp(r)}$ such that every $\Delta_y(f')$ can be written as $\Delta_y(f') = \sum_{i=1}^{r} Q_i \cdot \ell_i^y + \sum_{i=1}^{\exp(r)} \ell_i \cdot Q_i^y + Q_0^y$. Notice that the lemma concerns a linear space of cubic polynomials. In our case the linear space will be the span of all the partial derivatives of f'. As for every $y, z \in V$ it holds that $\deg\left(\Delta_y(f') + \Delta_z(f') - \Delta_{y+z}(f')\right) = 2$, we see that in order to 'close' the space we only need to add quadratic polynomials and so the assumption about the rank of the cubic polynomials in the space does not change. □

Proof of Theorem 4. By Lemma 5.8 for $r = O(\log^2(1/\delta))$ there exist r quadratics $Q_1, \ldots, Q_r$ and $R = \exp(r)$ linear functions $\ell_1, \ldots, \ell_R$ such that for every $y \in V$ we have that $\Delta_y(f|_V) = \sum_{i=1}^{r} Q_i \cdot \ell_i^y + \sum_{i=1}^{R} \ell_i \cdot Q_i^y + Q_0^y$.

We now wish to express each Q_i in the form of Theorem 1.1. We have two cases. Assume first that $\mathbb{F} = \mathbb{F}_2$. Then for every $1 \leq i \leq r$ we have that $Q_i = \sum_{i=1}^{n/2} \ell_{i,j} \cdot \ell'_{i,j} + \ell_{i,0}$. For $\alpha \in \mathbb{F}^R$ let $V_\alpha = \{x \in V \mid \forall 1 \leq i \leq R,\ \ell_i(x) = \alpha_i\}$. Clearly, $\dim(V_\alpha) \geq \dim(V) - R$. Let $f_\alpha = f|_{V_\alpha}$. Then for every $y \in V_\alpha$, $\Delta_y(f_\alpha) = \sum_{i=1}^{r} Q_i|_V \cdot \ell_i'^y + Q'^y_0$. We now repeat the following process for each $1 \leq i \leq r$. Assume that we are working over a subspace $V_{\alpha,\beta^1,\ldots,\beta^{i-1}}$, of dimension $d_{i-1} = \dim\left(V_{\alpha,\beta^1,\ldots,\beta^{i-1}}\right)$. Consider $Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^{i-1}}}$. By Theorem 1.1 we can write $Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^{i-1}}} = \sum_{i=1}^{d_{i-1}/2} \ell_{i,j} \cdot \ell'_{i,j} + \ell_{i,0}$. For $\beta^i \in \mathbb{F}^{d_{i-1}/2}$ define $V_{\alpha,\beta^1,\ldots,\beta^i} = \left\{x \in V_{\alpha,\beta^1,\ldots,\beta^{i-1}} \mid \forall 1 \leq j \leq d_{i-1}/2,\ \ell_{i,j} = (\beta^i)_j\right\}$. Note that $\cup_{\beta^i \in \mathbb{F}^{d_{i-1}/2}} V_{\alpha,\beta^1,\ldots,\beta^i} = V_{\alpha,\beta^1,\ldots,\beta^{i-1}}$. Thus, the set $\{V_{\alpha,\beta^1,\ldots,\beta^r}\}$ forms a partition of V. Moreover, observe that for every $\alpha, \beta^1, \ldots, \beta^i$, $\deg\left(Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^i}}\right) \leq 1$. Thus, for every $\alpha, \beta^1, \ldots, \beta^r$, all the partial derivatives of $f|_{V_{\alpha,\beta^1,\ldots,\beta^r}}$ are of degree two and so $\deg\left(f|_{V_{\alpha,\beta_1,\ldots,\beta_r}}\right) \leq 3$ as claimed. To finish the proof we note that $\dim\left(V_{\alpha,\beta^1,\ldots,\beta^i}\right) \geq \dim\left(V_{\alpha,\beta^1,\ldots,\beta^{i-1}}\right)/2$. Therefore, $\dim\left(V_{\alpha,\beta^1,\ldots,\beta^r}\right) \geq (n-R))/2^r = n/\exp(\log^2(1/\delta))$.

When $\mathrm{char}(\mathbb{F}) = p > 2$ we have the representation $Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^{i-1}}} = \sum_{i=1}^{d_{i-1}} \ell_{i,j}^2 + \ell_{i,0}$. Rewriting we obtain

$$\begin{aligned}
Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^{i-1}}} &= \sum_{i=1}^{r} \ell_{i,j}^2 + \ell_0 \\
&= \sum_{i=1}^{d_{i-1}/p} \sum_{j=0}^{p-1} \ell_{pi+j}^2 + \ell_0 \\
&= \sum_{i=1}^{d_{i-1}/p} \left(\sum_{j=1}^{p-1} (\ell_{pi+j} - \ell_{pi})^2 + 2\ell_{pi} \sum_{j=1}^{p-1} (\ell_{pi+j} - \ell_{pi}) \right) + \ell_0
\end{aligned}$$

Observe that after fixing $\forall 1 \leq j \leq p-1$, $\ell_{pi+j} - \ell_{pi} = (\beta^i)_j$, $Q_i|_{V_{\alpha,\beta^1,\ldots,\beta^{i-1}}}$ becomes linear. Thus, the same argument as before gives the required result here as well. □

Combining the idea of the above proof with the notion of disjoint polynomials we prove Theorem 5.

Proof Sketch of Theorem 5. As in the proof of Theorem 4 we obtain linear $\{\ell_i\}_{i=1\ldots R}$ and quadratic $\{q_i\}_{i=1\ldots r}$, where $r = O(\log^2(1/\delta))$ and $R = \exp(r)$, that form a 'basis' to the set of partial derivatives. By passing to a subspace of codimension R and using Lemma 4.14 we can assume w.l.o.g. that the q_i-s are disjoint and that every partial derivative has the form $\Delta_y(f) = \sum_{i=1}^r q_i \cdot \ell_i^{(y)} + q_0^{(y)}$. As $\mathrm{char}(\mathbb{F}) > 4$ we can assume w.l.o.g. that $q_i = x_i^2 + q_i'$ and that x_j can appear in q_i' only as a linear term. We now subtract from f terms of the form $\alpha q_i q_j$ such that in the resulting polynomial f' there will be no monomial of the form $x_i^2 x_j^2$ for $i \leq j$. Note that f' also has the property that for every y, $\Delta_y(f') = \sum_{i=1}^r q_i \cdot \ell_i'^{(y)} + q_0'^{(y)}$. We now show that degree four monomials in f' may only contain x_i or x_i^3 but not x_i^2, for $i \in [r]$. Indeed, assume for a contrary that x_i^2 appears in a degree four monomial. Then, x_i appears in $\Delta_{x_i}(f')$ in a degree three monomial. This monomial comes from some $\ell_j^{(x_i)} q_j$ for $j \neq i$. Therefore, we also have the term $x_i x_j^2$ in $\Delta_{x_i}(f')$ (it is not difficult to see that this term cannot be cancelled by any other $\ell_k^{(x_i)} q_k$). As $\mathrm{char}(\mathbb{F}) > 4$, integration w.r.t. x_i gives that the term $x_i^2 x_j^2$ appears in f' in contradiction. We can thus write $f' = \sum_{i=1}^r x_i^3 \tilde{\ell}_i + f''$, where in f'' each x_i has degree at most one. Consider any y 'orthogonal' to $\{x_1, \ldots, x_r, \tilde{\ell}_1, \ldots, \tilde{\ell}_r\}$ (namely, substituting y in any of those linear functions gives zero). Then for each i, $\Delta_y(x_i^3 \tilde{\ell}_i) = 0$. Hence, x_i is the highest power of x_i appearing in $\Delta_y(f')$. As the q_i-s are disjoint and $\Delta_y(f') = \sum_{i=1}^r q_i \cdot \ell_i'^{(y)} + q_0'^{(y)}$ we obtain that it must be the case that $\deg(\Delta_y(f')) \leq 2$. Thus, f' can be rewritten as a polynomial in at most $2r$ variables plus a degree three polynomial. Therefore, possibly after a change of basis we can write $f = \sum_{i \leq j} \alpha_{i,j} q_i \cdot q_j + \sum_{i=1}^{2r+R} y_i \cdot g_i + g_0$ as needed. □

6 Conclusions

In this paper we gave strong structural results for degree three and four polynomials that have a high bias. It is a very interesting question whether such a structure exists for higher degree biased polynomials. Green and Tao [GT07] proved such a result when $\deg(f) < |\mathbb{F}|$ (with much worse parameters for degrees three and four), so this question is mainly open for small fields. Another interesting question is improving the parameters in the results of [GT07, KL08]. There it was shown that when $\deg(f) = d$ and f is biased then $f = F(g_1, \ldots, g_{c_d})$, where $\deg(g_i) < \deg(f)$. However, the dependence of c_d on the degree d and the bias δ is terrible. Basically, $c_3 = \exp(\mathrm{poly}(1/\delta)$ and c_d is a tower of height c_{d-1}. In contrast, our results give that $c_3 = \log^2(1/\delta)$ and $c_4 = \mathrm{poly}(1/\delta)$. Thus, it is an intriguing question to find the true dependence of c_d on δ. In particular, as far as we know, it may be the case that c_d is polynomial in $1/\delta$ (where the exponent may depend on d), or even $\mathrm{poly}(\log(1/\delta))$.

For the case of degree four polynomials with high U^4 norm we proved an inverse theorem showing that on many subspaces, of dimension $\Omega(n)$, f equals to a degree three polynomial

(a different polynomial for each subspace). Such a result seems unlikely to be true for higher degrees. However, it may be the case that if $\deg(f) = d$ and f has a high U^d norm then f is correlated with a lower degree polynomial on a high dimensional subspace.

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