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On the Matching Problem for Special Graph Classes

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Abstract

An even cycle in a graph is called *nice* by Lovasz and Plummer in [LP86] if the graph obtained by deleting all vertices of the cycle has some perfect matching. In the present paper we prove some new complexity bounds for various versions of problems related to perfect matchings in graphs with a polynomially bounded number of nice cycles. We show that for graphs with a polynomially bounded number of nice cycles the perfect matching decision problem is in **SPL**, it is hard for **FewL**, and the perfect matching construction problem is in $\mathbf{L}^{\mathbf{C_=L}} \cap \oplus\mathbf{L}$. Furthermore, we significantly improve the best known upper bounds, proved by Agrawal, Hoang, and Thierauf in the STACS'07-paper [AHT07], for the polynomially bounded perfect matching problem by showing that the construction and the counting versions are in $\mathbf{C_=L} \cap \oplus\mathbf{L}$ and in $\mathbf{C_=L}$, respectively. Note that $\mathbf{SPL}, \oplus\mathbf{L}, \mathbf{C_=L}$, and $\mathbf{L}^{\mathbf{C_=L}}$ are contained in $\mathbf{NC}^2$.

Moreover, we show that the problem of computing a maximum matching for bipartite planar graphs is in $\mathbf{L}^{\mathbf{C_=L}}$. This solves Open Question 4.7 stated in the STACS'08-paper by Datta, Kulkarni, and Roy [DKR08] where it is asked whether computing a maximum matching even for bipartite planar graphs can be done in **NC**. We also show that the problem of computing a maximum matching for graphs with a polynomially bounded number of even cycles is in $\mathbf{L}^{\mathbf{C_=L}}$.

1 Introduction

A set M of edges in an undirected graph G such that no two edges of M share a vertex is called a matching in G. A matching with maximal cardinality is called maximum. A maximum matching is perfect if it covers all vertices in the graph. Graph matchings because of their fundamental properties are one of the most fundamental and well-studied objects in mathematics and in theoretical computer science (see e.g. [LP86, KR98]). In the wide research-topic on graph matchings, perfect matchings and maximum matchings w.r.t. parallel computations receive a great attention.

From the viewpoint of complexity theory it is well-known that a maximum matching can be constructed efficiently in polynomial time [Edm65]. Hence the problem of deciding whether a graph has a perfect matching (short: Decision-PM) and the problem of computing a perfect matching in a graph(short: Search-PM) are in **P**. Regarding parallel computations, computing a maximum matching is known to be in randomized **NC** [KUW86, MVV87], and particularly in nonuniform **SPL** [ARZ99] (see Section 2 for more detail on the complexity classes). Therefore, both problems Decision-PM and Search-PM are in nonuniform **SPL**. But it is a big open question whether even Decision-PM is in uniform **NC**. Note that if Search-PM would be in **NC** then also Decision-PM. Note further that there is a huge gap among the complexities of

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the search and the counting version of the perfect matching problem (short: COUNTING-PM) because computing the number of all perfect matching in a bipartite graph is known to be #**P**-complete [Val79].

By Tutte's Theorem [Tut47] (see next section for more detail), DECISION-PM can be reduced to the problem of testing if a symbolic determinant is zero. This algebraic setting puts DECISION-PM into a special case of the well-studied problem *Polynomial Identity Testing* (short: PIT), the problem of testing if a polynomial given in an implicit form, like an arithmetic circuit or a symbolic determinant, is zero. PIT can be solved by a randomized algorithm using Schwartz-Zippel Lemma [Sch80, Zip79], but whether the method can be derandomized is a prominent open question. Due to a result by Impagliazzo and Kabanets [KI04] stating that the problem of derandomizing PIT is computationally equivalent to the problem of proving lower bounds for arithmetic circuits, the matching problem attracts great attention.

In this paper we continue with the line of research that tries to characterize exactly the complexity of the matching problem. The motivation for the work comes directly from the crucial importance of the matching problem mentioned above. Since it is open whether the perfect matching problem is in **NC**, diverse special cases of the problem have been studied and solved before. For example, **NC** algorithms are known for DECISION-PM for: planar graphs [Kas67, Vaz89], regular bipartite graphs [LPV81], strongly chordal graphs [DK86], and dense graphs [DHK93]. SEARCH-PM is also known in **NC** for bipartite planar graphs [MN95, MV00, DKR08], and for graphs with a polynomially bounded number of perfect matchings [GK87, AHT07].

In the first part of the paper, in Section 3, we investigate the complexity of the perfect matching problem for graphs with a polynomially bounded number of so-called *nice cycles*. An even cycle C in a graph G is called nice [LP86] if the graph obtained from G by deleting all vertices of C has some perfect matching. The nice cycles play a crucial role for deterministic isolations of perfect matchings (see Lemma 3.1 on page 5), thereby a deterministic isolation in **NC** would bring both the decision and search versions of the perfect matching problem in **NC**. Thus, towards a derandomization of the perfect matching problem, our considered promise problems in Section 3 is not purposeless. Moreover, this promise problem is a generalization of the polynomially bounded perfect matching problem which has been studied in [GK87, AHT07], since on the one hand the number of all nice cycles in any graph is at most the square of the number of all perfect matchings in it and on the other hand the number of all perfect matchings in a graph with a polynomially bounded number of nice cycles might be exponentially big. The results in Section 3 can be summarized as follows:

- Following a general paradigm for derandomizing polynomial identity testing by Agrawal [Agr03] and introducing a method different from one in [AHT07] for solving the polynomially bounded perfect matching problem, in Section 3 we show that for graphs with a polynomially bounded number of nice cycles both the decision and search versions of the perfect matching problem are respectively in **SPL** and $\mathbf{L}^{\mathbf{C_=L}} \cap \oplus\mathbf{L}$, which are contained in $\mathbf{NC}^2$.

- We improve significantly the best known upper bounds $\mathbf{L}^{\mathbf{C_=L}}$ and $\mathbf{NC}^1(\mathbf{GapL})$, proved in [AHT07] for the construction and the counting versions of the polynomially bounded perfect matching problem, to $\mathbf{C_=L} \cap \oplus\mathbf{L}$ and $\mathbf{C_=L}$, respectively.

Moreover, the results and techniques presented in Section 3 give evidence that in general the perfect matching problem might be solvable by a method we describe in Section 5.

In the second part of the paper, in Section 4, we show an algebraic method for constructing a maximum matching once some weight function for isolating a maximum matching is given. Thereby we solve Open Question 4.7 in [DKR08] which asked whether the problem of constructing a maximum matching even for bipartite planar graphs is in **NC**. In particular, we show that the maximum matching problem for bipartite planar graphs is in $\mathbf{L}^{\mathbf{C_=L}}$. Furthermore, using the results from Section 3 we show that the maximum matching problem for graphs with a polynomially bounded number of even cycles is also in $\mathbf{L}^{\mathbf{C_=L}}$. These results are significant because the considered problems were not known to be in **NC** previously.

2 Preliminaries

Algebraic Graph Theory. We describe some basic notions on graph matchings. For more detail we refer the readers to [LP86, MSV99], or to standard textbooks in linear algebra and graph theory.

Let $G = (V, E)$ be an undirected graph with n vertices, $V = \{1, 2, \ldots, n\}$, and m edges $E = \{e_1, \ldots, e_m\} \subseteq V \times V$. A *matching* in G is a set $M \subseteq E$, such that no two edges in M have a vertex in common. A matching M is called *perfect* if M covers all vertices of G, i.e. $|M| = \frac{1}{2}\, n$, M of maximal size is called *maximum*. The weight of a matching in a weighted graph is defined as the sum of all weights of the edges in the matching.

Graph G can be presented by its *adjacency matrix*. This is an $n \times n$ symmetric matrix $A \in \{0, 1\}^{n \times n}$ where $A_{i,j} = 1$ iff $(i, j) \in E$, for all $1 \leq i, j \leq n$. Assign weights $w(i, j)$ to edges (i, j) to get the *weighted* graph G. Assign orientations to the edges of weighted graph G, i.e. edge (i, j) gets one of two orientations, from i to j or from j to i, to obtain an orientation $\vec{G}$ for which we have a so-called *Tutte skew-symmetric matrix* T as follows:

$$T_{i,j} = \begin{cases} A_{i,j}\ w(i,j)\ , & \text{if an edge of } \vec{G} \text{ is directed from } i \text{ to } j, \\ -A_{i,j}\ w(i,j)\ , & \text{otherwise.} \end{cases}$$

In the case when all directed edges of $\vec{G}$ are oriented from smaller to larger vertices, the orientation $\vec{G}$ and the matrix T are called *canonical*. The Pfaffian of a skew-symmetric matrix T from an orientation $\vec{G}$, denoted by $\mathrm{pf}(T)$ or $\mathrm{pf}(\vec{G}, w)$, is defined as follows:

$$\mathrm{pf}(\vec{G}, w) \quad = \sum_{\text{perfect matching } M \text{ in } G} \mathrm{sign}(M)\ \mathrm{value}(M)$$

where $\mathrm{sign}(M) \in \{-1, +1\}$ is the sign of M that depends on the orientation $\vec{G}$, and $\mathrm{value}(M) = \prod_{(i,j) \in M} w(i, j)$ is the value of M that depends on the weighting scheme for G. It is known from linear algebra that $\det(S) = \mathrm{pf}^2(S)$ if S is a skew-symmetric matrix of even order, and $\mathrm{pf}(S) = 0$ for all skew-symmetric matrices of odd order. We refer the reader to [Kas67, MSV99] for more detail.

Assign indeterminates $x_{i,j}$ to the edges (i, j) of a graph G to get the graph $G(X)$. Let $T(X)$ be the canonical Tutte skew-symmetric matrix of $G(X)$. The perfect matching problem can be decided by a randomized algorithm using the following theorem and the Schwartz-Zippel Lemma [Sch80, Zip79].

Theorem 2.1 (Tutte, [Tut47]) *Graph G has no perfect matching iff* $\mathrm{pf}(T(X)) = 0$.

An orientation such that all perfect matchings in G have the same sign $+1$ (or -1) is called a *Pfaffian orientation* [Kas67]. Hence the number of perfect matchings in a graph G can be computed by finding a Pfaffian orientation in it and then by computing the Pfaffian. But there are graphs which do not admit any Pfaffian orientation, the complete bipartite graph $K_{3,3}$ is an example of them. However, planar graphs [Kas67] and $K_{3,3}$-free graphs [Vaz89] admit always Pfaffian orientations which are computable in **NC**, and thus the number of all perfect matchings in such a graph can be computed efficiently.

Complexity Classes. The complexity classes **P**, **L**, **NP**, and **NL** are well known. We mention briefly some other classes we work with. We refer the reader to [AO96, ABO99, ARZ99] for more detail.

The classes $\mathbf{NC}^k$, for fixed k, consists of families of Boolean circuit with $\wedge$-, $\vee$-gates of fan-in 2, and $\neg$ -gates, of depth $O(\log^k n)$ and of polynomial size. $\mathbf{NC} = \cup_{k\geq 0}\mathbf{NC}^k$. The class $\mathbf{AC}^0$ is defined as the set of families of Boolean circuit with (unbounded fan-in) $\wedge$-, $\vee$-gates, and $\neg$ -gates, of constant-depth and of polynomial-size. It is know that

$$\mathbf{AC}^0 \subseteq \mathbf{NC}^1 \subseteq \mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{NC}^2 \subseteq \mathbf{NC} \subseteq \mathbf{P}.$$

For an **NL** machine M, we denote the number of accepting and rejecting computation paths on input x by $\#acc_M(x)$ and $\#rej_M(x)$, respectively. **FewL** is the class of languages accepted by **NL** machines with at most a polynomial number of accepting computations [BDHM91]. The class **GapL** consists of all functions gap_M, where M is an **NL**-machine, and for all x, $gap_M(x) = \#acc_M(x) - \#rej_M(x)$. This class is characterized by the determinant of integer matrices [Dam91, Tod91, Vin91, Val92]. Note that the problem of computing the determinant of an integer matrix is in $\mathbf{NC}^2$ [Ber84]. **GapL** is closed under addition, subtraction, multiplication, and restricted composition [AO96, AAM03]. The following classes are related to **GapL**.

- $\oplus\mathbf{L}$ is the class of sets A for which there exists a function $f \in \mathbf{GapL}$ such that $\forall\, x:\ x \in A \iff f(x) \not\equiv 0 (\mathrm{mod}\ 2)$. Obviously, we have $\mathbf{L}^{\oplus\mathbf{L}} = \oplus\mathbf{L}$.

- $\mathbf{C_{=}L}$ (*Exact Counting in Logspace*) consists of all problems of verifying a **GapL**-function, i.e. it is the class of sets A for which there exists a function $f \in \mathbf{GapL}$ such that $\forall\, x:\ x \in A \iff f(x) = 0$.

- The Hierarchy over $\mathbf{C_{=}L}$ collapses to $\mathbf{L}^{\mathbf{C_{=}L}}$ [ABO99] which is equal to $\mathbf{AC}^0(\mathbf{C_{=}L})$, the class of all problems $\mathbf{AC}^0$-reducible to $\mathbf{C_{=}L}$. The problem of computing the rank of an integer matrix is complete for $\mathbf{L}^{\mathbf{C_{=}L}} = \mathbf{AC}^0(\mathbf{C_{=}L})$ [ABO99].

- **SPL** [ARZ99] is the class of all languages for which their characteristic functions are in **GapL**, i.e. $\mathbf{SPL} = \{L \in \Sigma^* | \chi_L \in \mathbf{GapL}\}$. It is known that **SPL** is closed under complement, moreover $\mathbf{L}^{\mathbf{SPL}} = \mathbf{SPL}$. Note that the inclusion $\mathbf{NL} \subseteq \mathbf{SPL}$ remains open.

We list some known inclusions among the mentioned classes:

$$\begin{gathered}
\mathbf{L} \subseteq \mathbf{FewL} \subseteq \mathbf{SPL} \subseteq \mathbf{C_{=}L} \subseteq \mathbf{L}^{\mathbf{C_{=}L}} \subseteq \mathbf{NC}^2,\\
\mathbf{SPL} \subseteq \oplus\mathbf{L} \subseteq \mathbf{NC}^2,\\
\mathbf{L} \subseteq \mathbf{FewL} \subseteq \mathbf{NL} \subseteq \mathbf{C_{=}L},\\
\mathbf{L} \subseteq \mathbf{GapL} \subseteq \mathbf{NC}^2.
\end{gathered}$$

The Pfaffian of an integer skew-symmetric matrix is known to be in **GapL** [MSV99]. Given a univariate polynomial matrix $A(x)$, i.e. the elements of $A(x)$ are polynomials in x of logarithmic bit length in the degree, the problem of computing $\det(A(x))$ is known to be in **GapL** [AAM03]: all the coefficients of $\det(A(x))$ are computable in **GapL**. By following the latter and the combinatorial setting for Pfaffians in [MSV99], it is not hard to show that in the case when $A(x)$ is skew-symmetric, all the coefficients of $\mathrm{pf}(A(x))$ are **GapL**-computable.

By Decision-PM, Search-PM, and Counting-PM we denote the decision, the search, and the counting version of the perfect matching problem, respectively. By Search-MM we denote the problem of computing a maximum matching in a graph.

3 Isolating and computing perfect matchings

In this section we show that the perfect matching problem for graphs with a polynomially bounded number of nice cycles is in $\mathbf{NC}^2$. It is well known that for any computational problem the decision version is reducible to the search version. Thus, in order to obtain an upper bound for the perfect matching problem we can concentrate on Search-PM. Our method for searching a perfect matching consists of two standard steps: a) isolating a perfect matching by a weight function, and b) computing the isolated perfect matching.

Isolating a perfect matching. Given a graph $G = (V, E)$ with n vertices $V = \{1, 2, \ldots, n\}$, m edges $E = \{e_1, e_2, \ldots, e_m\}$, and with at most n^k nice cycles, where k is a fixed positive integer, (recall that an even cycle C in G is called nice if the graph obtained by deleting from G all vertices of C has some perfect matching or it is empty), we show how to deterministically isolate a perfect matching in G.

Let w be a weight function for the edges of G, i.e. edge e gets the weight $w(e)$, for every e. Observe that a simple cycle C (in G) with $2l$ edges, $l > 0$, has exactly two perfect matchings N_1 and N_2, each of them is of size l. By $W(N_1)$ and $W(N_2)$ we denote the weights of N_1 and N_2, respectively. Recall that the weight of a matching is the sum of the weights on its edges. The difference of the weights of the two perfect matchings in an even cycle is called the *circulation* of the cycle [DKR08]:

$$\text{circulation}(C) = |W(N_1) - W(N_2)|.$$

This function has been used in Lemma 3.2 in [DKR08] as follows: *if all the cycles of a bipartite graph have non-zero circulations, then the minimum weight perfect matching in it is unique.* In general, Lemma 3.2 in [DKR08] holds also for non-bipartite graphs by considering circulations of only nice cycles. We omit the proof of the following lemma because it is in analogy to the proof of Lemma 3.2 in [DKR08].

Lemma 3.1 ([DKR08]) *If all nice cycles in a weighted graph have non-zero circulations, then there is a unique minimum weight perfect matching in it.*

Thus the circulations of nice cycles play a central role for isolating a perfect matching in graphs. It is easy to see that the converse of Lemma 3.1 is not true. For example: we can easily assign integer weights to 6 edges of K_4, the complete graph with 4 vertices, so that the minimum weight perfect matching is unique but there is a nice cycle of zero circulation.

We call a weight function *admissible* for G if it assigns positive integers with a logarithmically bounded number of bits to the edges in G so that a minimum weight perfect matching becomes unique. By Lemma 3.1, in order to isolate deterministically a perfect matching we can determine an admissible weight function such that all nice cycles in the graph get non-zero circulations. We show the following lemma for isolating a perfect matching in graphs having a polynomially bounded number of nice cycles.

Lemma 3.2 *Let $G = (V, E)$ be an undirected graph with $|V| = n$ vertices and m edges $E = \{e_1, e_2, \ldots, e_m\}$, and let the number of nice cycles in G be at most n^k, for some positive constant k. Then there exists a prime number $p < 2n^k(m+1)$ such that the weight function $w_p : E \mapsto \mathbb{Z}_p$ where $w_p(e_i) = 2^i \bmod p$ is admissible for G.*

Proof. Assign 2^i to every edge e_i in G. Then each nice cycle C in G has a non-zero circulation because two perfect matchings defined in C have different weights. Consider the product of all the circulations:

$$Q = \prod_{C \text{ is a nice cycle}} \text{circulation}(C).$$

Since the number of nice cycles in G is at most n^k and since $0 < \text{circulation}(C) < 2^{m+1}$ holds for every nice cycle C, we get $0 < Q < 2^{n^k(m+1)}$. It is well-known from Number Theory that

$$\prod_{\text{all primes } p_i \leq 2N} p_i > 2^N, \text{ for all } N > 2.$$

Therefore, there exists a prime $p < 2n^k(m+1)$ such that p is not a factor of Q, i.e. we have $Q \bmod p \neq 0$, or equivalently: $\text{circulation}(C) \bmod p \neq 0$ for all nice cycles C in G. Hence by Lemma 3.1 a minimum weight perfect matching becomes unique under the weight function $w_p : E \to \mathbb{Z}_p$ where

$$w_p(e_i) = 2^i \bmod p, \text{ for } i = 1, 2, \ldots, m.$$

Note that all the prime numbers $q < 2n^k(m+1)$ and the weight functions w_q are computable in logspace. This completes the proof of the lemma. □

Observe that the set of all nice cycles in any graph is the union of all symmetric differences between two different perfect matchings. Hence it is easy to see that the number of all nice cycles in a graph is at most the square of the number of all perfect matchings in the graph. Therefore, Lemma 3.2 can be used also for isolating a perfect matching in graphs with polynomially bounded number of perfect matchings.

Note that it is still open if there is an **NC**-computable admissible weight function for an arbitrary graph (without any restriction of the number of nice cycles). This open question is similar to the open question of whether Isolating Lemma [MVV87] for randomly isolating a perfect matching can be derandomized. We believe that there is an affirmative answer to this open question. In Section 5 we give a discussion on this topic. Isolating Lemma can be stated in general as follows:

Lemma 3.3 (Isolating Lemma [MVV87]) *Let U be a universe of size m and S be a considered family of subsets of U. Let $w : U \to \{1, \ldots, 2m\}$ be a random weight function. Then with probability at least $\frac{1}{2}$ there exists a unique minimum weight subset in S.*

Computing perfect matchings.

Theorem 3.4 *There exists a zero-one-valued* **GapL***-function h that on input a graph G with a polynomially bounded number of nice cycles $h(G) = 1$ if G has some perfect matching, and $h(G) = 0$ if G has no perfect matching. A perfect matching in G, if one exists, can be constructed in $\mathbf{L}^{\mathbf{C_{=}L}} \cap \oplus\mathbf{L}$.*

Proof. Let $G = (V, E)$ be a graph with n vertices, m edges $E = \{e_1, \ldots, e_m\}$, and with at most n^k nice cycles, for some positive constant k. Let U be the set of all prime numbers at most $2n^k(m+1)$. Define the weight functions $w_p : E \to \mathbb{Z}_p$, for each $p \in U$, where $w_p(e_i) = 2^i \bmod p$ for every edge e_i.

Let x be an indeterminate. Assign $x^{w_p(e)}$ to each edge e in G to get the graphs $G_p(x)$, for every $p \in U$. By $G_p^{(-e)}(x)$ we denote the result of deleting edge e from $G_p(x)$. The canonical Tutte skew-symmetric matrices of $G_p(x)$ and $G_p^{(-e)}$ we denote respectively by $T_p(x)$ and by $T_p^{(-e)}(x)$. Considering the Pfaffian polynomials of these matrices we observe that the value of a perfect matching M becomes $x^{W(M)}$ where $W(M)$ is the weight of M, the coefficient of $x^{W(M)}$ in the polynomial is the sum of all signs of all perfect matchings having the same weight $W(M)$. Define $K = n^{k+1}(m+1)$. Then we can write:

$$\begin{aligned} \text{pf}(T_p(x)) &= c_{p,0} + c_{p,1}x^1 + \cdots + c_{p,K}x^K, \\ \text{pf}(T_p^{(-e)}(x)) &= c_{p,0}^{(-e)} + c_{p,1}^{(-e)}x^1 + \cdots + c_{p,K}^{(-e)}x^K. \end{aligned}$$

It is clear that all $\mathrm{pf}(T_p(x))$ and $\mathrm{pf}(T_p^{(-e)}(x))$ vanish if G has no perfect matching.

Consider the case when G has some perfect matching. By Lemma 3.2 there exists some $p \in U$ such that the graph G under w_p has a unique minimum weight perfect matching. Let M_0 be this unique matching and let I be its weight under w_p. Observe that the coefficient of x^I in $\mathrm{pf}(T_p(x))$, occurred as the lowest non-zero coefficient in the polynomial, should be $c_{p,I} = \mathrm{sign}(M_0) \in \{+1, -1\}$, or equivalently $c_{p,I}^2 = 1$. Recall from Section 2 that all the coefficients of the polynomials we consider are computable in **GapL**. Therefore

$$h(G) = 1 - \prod_{0 \le i \le K,\ p \in U} (1 - c_{p,i}^2)$$

is a *zero-one-valued* **GapL**-function that can be seen as the characteristic function for the problem of testing if G has a perfect matching, i.e. $h(G) = 1$ iff G has some perfect matching.

It remains to show that we can construct a perfect matching of G in $\mathbf{L}^{\mathbf{C_=L}} \cap \oplus\mathbf{L}$. Observe that if w_p is admissible for G, then G has the unique minimum weight perfect matching M_0 with weight $0 \le I \le K$. Thus we have

$$c_{p,I}^2 = 1 \text{ and } c_{p,I}^{(-e)} = \begin{cases} 0\ , & \text{if } e \in M_0 \\ c_{p,I}\ , & \text{otherwise.} \end{cases}$$

Therefore, in $\mathbf{C_=L}$ we can construct all edge-sets $M_{p,i}$ as follows:

$$e \in M_{p,i} \text{ iff } c_{p,i}^2 = 1 \textbf{ and } c_{p,i}^{(-e)} = 0, \text{ for each edge } e, \text{ for all } p \in U \text{ and } 0 \le i \le K.$$

It is easy to see that the same edge-sets will be constructed by the same procedure in $\mathbb{Z}_2$, i.e. in $\oplus\mathbf{L}$ we can construct all the sets $M_{p,i}$. After that we can easily determine and output in logspace all perfect matchings from the constructed edge-sets $M_{p,i}$. Note that there is at least one edge-set, namely $M_{p,I}$ from our construction, is indeed a perfect matching in G. Our construction is in $\mathbf{L}^{\mathbf{C_=L}} \cap \oplus\mathbf{L}$ because $\mathbf{L}^{\oplus\mathbf{L}} = \oplus\mathbf{L}$. This completes the proof of the theorem.

□

Note that the formulation "DECISION-PM *for graphs with a polynomially bounded number of nice cycles is in* **SPL**", used sometimes in the paper, is not completely formal. It means that there is a **GapL**-function h that on input a graph with a polynomially bounded number of nice cycles h is zero-one valued and it tests the existence of perfect matching in G, but this **GapL**-function h might be not zero-one-valued for graphs outside the considered promise. We further note that it is easily to see in the proof of Theorem 3.4 that DECISION-PM for graphs with a polynomially bounded number of nice cycles is in $\mathbf{C_=L} \cap \mathbf{coC_=L}$. (This avoids any possibility of confusion.)

Allender et. al. [ARZ99] show that in general a perfect matching can be constructed in nonuniform **SPL**. Unfortunately, in the proof of Theorem 3.4 we do not know how to perform in (uniform) **SPL** the decision of which prime p from U is "right" for isolating a minimum weight perfect matching.

The polynomially bounded perfect matching problem. The best known upper bounds for the polynomially bounded perfect matching problem, taken from [GK87, AHT07], are given in the following theorem.

Theorem 3.5 ([AHT07]) *For graphs with a polynomially bounded number of perfect matchings, the perfect matching decision problem is in* $\mathbf{coC_=L}$*, the counting problem is in* $\mathbf{AC}^0(\mathbf{C_=L})$*, and all the perfect matchings can be constructed in* $\mathbf{NC}^1(\mathbf{GapL})$*.*

Note that $\mathbf{coC_=L} \subseteq \mathbf{AC^0(C_=L)} = \mathbf{L^{C_=L}} \subseteq \mathbf{NC^1(GapL)} \subseteq \mathbf{NC^2}$ where $\mathbf{NC^1(GapL)}$ is the class of all problems $\mathbf{NC^1}$-reducible to the determinant.

Obviously, upper bounds for the decision version of the polynomially bounded perfect matching problem come directly from the preceded paragraph. We show the following:

Theorem 3.6 *For graphs with a polynomially bounded number of perfect matchings*

(1) Decision-PM *is hard for* **FewL***,*

(2) Search-PM *is in* $\mathbf{C_=L} \cap \oplus\mathbf{L}$*, and* Counting-PM *is in* $\mathbf{C_=L}$*.*

Proof. (1) We omit the proof that Decision-PM is hard for **FewL** since it is straightforward by modifying the reduction from the directed connectivity problem, which is **NL**-complete, to the bipartite unique perfect matching problem [HMT06], or to the bipartite perfect matching problem [CSV84].

(2) Let $G = (V, E)$ be an undirected graph with n vertices, m edges $|E| = \{e_1, \ldots, e_m\}$, and with at most n^k perfect matchings. We show how to construct all perfect matchings in G. Our construction consists of two steps as follows:

a) compute a prime p such that $w_p : w_p(e_i) = 2^i \bmod p$ isolates all perfect matchings,

b) construct all perfect matchings from the Pfaffians $\mathrm{pf}(T_p(x))$ and $\mathrm{pf}(T_p^{(-e)}(x))$.

Consider Step a). Let's call a prime p from Step a) *"right"* if w_p isolates all perfect matchings in G. Observe that any perfect matchings M and N have different weight under $w: \ w(e_i) = 2^i$, i.e. $0 < |W(M) - W(N)| < 2^{m+1}$ where $W(M)$ and $W(N)$ are the weights of M and N, respectively.

$$0 < Q = \prod_{M \neq N} |W(M) - W(N)| \quad < \quad 2^{(m+1)\binom{n^k}{2}} \quad < \quad 2^{\frac{1}{2}(m+1)n^{2k}}.$$

Therefore, in analogy to Lemma 3.2 there exists a prime $p < (m+1)n^{2k}$ such that $Q \bmod p \neq 0$.

Define U as the set of all prime numbers at most $(m+1)n^{2k}$. Observe that a prime $p \in U$ is "right" iff in $\mathrm{pf}(T_p(x))$ *all coefficients are from* $\{-1, 0, +1\}$ **and** *the number of non-zero coefficients is maximum.* The latter is the number of all perfect matchings in G when p is "right".

Define $K = (m+1)n^{2k}$, and for every $q, q' \in U$ define the following **GapL**-functions:

$$h_q \;=\; \sum_{i=0}^{K} (c_{q,i}^2 - 1)\; c_{q,i}^2 \;, \qquad g_q \;=\; \sum_{i=0}^{K} c_{q,i}^2,$$

$$H_{q,q'} \;=\; \prod_{a=1}^{Kn^{4k}} (h_{q'} - a) \prod_{a=0}^{Kn^{2k}} (g_q - g_{q'} - a).$$

We see that $h_q = 0$ iff all $c_{q,i} \in \{-1, 0, 1\}$. For a "right" prime p, g_p is the number of all non-zero coefficients. Moreover, observe that $H_{q,q'} = 0$ iff $h_{q'} \neq 0$ or $g_q = g_{q'} + a$ for some non-negative integer a. Hence we get $g_q > g_{q'}$ as long as $h_{q'} = 0$. Thus, in $\mathbf{C_=L}$ we can select a "right" prime p from U as follows:

$$p \text{ is "right" iff} \quad h_p = 0 \textbf{ and } H_{p,q} = 0 \text{ for all } p \neq q \in U.$$

Consider Step b). In $\mathbf{C_=L}$ we can construct the edge-sets $M_{p,i}$ corresponding to $c_{p,i} \in \{-1, +1\}$ in $\mathrm{pf}(T_p(x))$ as stated in the proof of Theorem 3.4. Note that after Step b) we do not check again whether the constructed edge-sets are perfect matchings. This shows that all perfect matchings in G can be constructed in $\mathbf{C_=L}$. The problem is in $\oplus\mathbf{L}$ by following the proof of Theorem 3.4.

The number of all perfect matchings in G can be computed in $\mathbf{C_=L}$ by verifying $g_p = a$, for some $a \leq n^k$ and by testing if p from U is "right".

This completes the proof of the theorem. $\square$

4 Isolating and computing a maximum matching

In this section we investigate the maximum matching problem. W.l.o.g., assume that the considered graphs in this section have no perfect matching. We show the following lemma.

Lemma 4.1 *Given a weight function w that assigns logarithmic bit long positive integers to the edges of a graph G such that the weight of a maximum matching in G becomes unique, the problem of computing a maximum matching in G is $\mathbf{L}^{\mathbf{C_=L}}$-reducible to the problem of computing a perfect matching in a subgraph of G.*

Proof. Let $G = (V, E)$ be a graph with n vertices and m edges. Let M be a maximum matching of G, and let $|M| = l$ for some positive integer l. Suppose the weight of M is unique under the weight function w. By G_M we denote the subgraph of G, obtained by deleting $n - 2l$ vertices which are not covered by M.

Observe that the maximum matching M in G becomes perfect and unique in G_M under the weight function w. Therefore, the computation of M can be done by computing G_M and then by extracting a perfect matching in G_M.

Let x be an indeterminate. By $G(x)$ we denote the graph G by assigning $x^{w(e)}$ to every edge e of G. By this weighting scheme we obtain $G_M(x)$ from G_M. Let $T_G(x)$ and $T_{G_M}(x)$ be the canonical Tutte skew-symmetric matrix of $G(x)$ and of $G_M(x)$, respectively.

Since in G_M the weight of the perfect matching M is unique under w, the Pfaffian polynomial $\mathrm{pf}(T_{G_M}(x))$ should be non-zero and the order of $T_{G_M}(x)$ should be $2l$. Hence we have $\det(T_{G_M}(x)) = \mathrm{pf}^2(T_{G_M}(x)) \neq 0$, and $\mathrm{rank}(T_{G_M}(x)) = 2l$. Moreover, since l is maximum, $T_{G_M}(x)$ is a maximal non-singular polynomial skew-symmetric sub-matrix of $T_G(x)$. As a consequence we have $\mathrm{rank}(T_G(x)) = \mathrm{rank}(T_{G_M}(x)) = 2l$.

Conversely, consider an n-bit vector $\vec{b}$ associated to a maximal set of linearly independent columns of $T_G(x)$. We call vector $\vec{b}$ a *column-basis* of $T_G(x)$. Observe that the subgraph $G_{\vec{b}}$ of G that contains all vertices i of G such that $\vec{b}_i = 1$ has always perfect matchings of the size l, and these matchings are maximum in G. Thus, in order to compute a subgraph having a perfect matching which is a maximum matching of G we can compute a column-basis of $T_G(x)$.

The problem of computing a column-basis of an integer matrix [zG93] is reducible to the rank of an integer matrix. The latter is known to be in $\mathbf{L}^{\mathbf{C_=L}}$ [ABO99]. For polynomial matrices, we show that a) the problem of computing a column-basis is reduced to the problem of computing the rank and b) the rank can be computed in $\mathbf{L}^{\mathbf{C_=L}}$.

a) Let $A(x)$ be an $n \times n$ univariate polynomial matrix where the degrees of matrix elements are at most n^c, for some positive constant c. Let $\vec{a_1}(x), \ldots, \vec{a_n}(x)$ be its columns. One has to compute a column-basis of $A(x)$.

   Let $A_i(x)$ be the matrix formed by the first i columns $\vec{a_1}(x), \ldots, \vec{a_i}(x)$ of $A(x)$, for all $1 \leq i \leq n$. It is well known from linear algebra that a column-basis can be selected as the collection of all $\vec{a_i}(x)$ where $\mathrm{rank}(A_{i-1}(x)) + 1 = \mathrm{rank}(A_i(x))$, for every $1 \leq i \leq n$. Therefore, the computation of a column-basis is reduced to the problem of computing the rank of a polynomial matrix.

b) Let $B(x)$ be an $n \times m$ univariate polynomial matrix, where the degrees of the matrix-elements are at most n^c, for some positive constant c. One has to compute $\mathrm{rank}(B(x))$.

It is known that $2\mathrm{rank}(B(x)) = \mathrm{rank}(C(x))$ where $C(x) = \begin{pmatrix} \mathbf{0} & B(x) \\ B^t(x) & \mathbf{0} \end{pmatrix}$ and $B^t(x)$ is the transpose of $B(x)$. Since $C(x)$ is an $N \times N$ symmetric matrix, where $N = m+n$, we can compute $\mathrm{rank}(C(x))$ by the characteristic polynomial $\chi_C(x) = \det(yI - C(x))$, where y is an indeterminate, as follows: Let $\chi_C(x) = y^N + p_{N-1}(x)y^{N-1} + \cdots + p_1(x)y + p_0(x)$, where $p_i(x)$ is a polynomial in x. Then for some $0 \leq j \leq N$ we have

$$\mathrm{rank}(C(x)) = j \text{ iff } p_0(x) = \cdots = p_{N-j-1}(x) = 0 \textbf{ and } p_{N-j}(x) \neq 0.$$

Consider on of the polynomials $p_i(x)$. If $p_i(x) = 0$ then it is clear that $p_i(a) = 0$ for all a's. Otherwise, if $p_i(x) \neq 0$ then there exists an integer a from $\{0, 1, \ldots, \deg(p_i(x))\}$ such that $p_i(a) \neq 0$. Since $\deg(p_i(x)) \leq Nn^c = (m+n)n^c$, for all $0 \leq i \leq N-1$, where c is a constant, we define $S = \{0, 1, \ldots, (m+n)n^c\}$. Then the rank of $B(x)$ is equal to the maximum of the ranks $\mathrm{rank}(B(a))$ over all $a \in S$. The rank of an integer matrix is known to be in $\mathbf{L}^{\mathbf{C_=L}}$ [ABO99]. Therefore, $\mathrm{rank}(B(x))$ is in $\mathbf{L}^{\mathbf{C_=L}}$.

The proof of the lemma is complete. □

Now we solve the maximum matching problem for bipartite planar graphs. A deterministic isolation of maximum matchings is due to Datta, Kulkarni, and Roy [DKR08]:

Lemma 4.2 ([DKR08]) *In logspace one can assign polynomially bounded weights to the edges of a bipartite planar graph so that the circulation of any cycle is non-zero.*

By Lemma 4.1, a subgraph G_M of a given bipartite planar graph G can be computed in $\mathbf{L}^{\mathbf{C_=L}}$ so that perfect matchings in G_M are maximum in G. Computing a perfect matching for bipartite planar graphs is in **SPL** [DKR08]. Since $\mathbf{SPL} \subseteq \mathbf{C_=L} \subseteq \mathbf{L}^{\mathbf{C_=L}}$, the maximum matching problem for bipartite planar graphs is in $\mathbf{L}^{\mathbf{C_=L}}$. We obtain the following theorem which is a positive answer to Open Question 4.7 stated in [DKR08] whether a maximum matching in bipartite planar graphs can be computed in **NC**.

Theorem 4.3 *The maximum matching problem for bipartite planar graphs is in* $\mathbf{L}^{\mathbf{C_=L}}$.

A method within **NC** for computing a maximum matching under the promise that the input graphs have a polynomial number of even cycles is given by the following theorem.

Theorem 4.4 *The maximum matching problem for graphs with a polynomially bounded number of even cycles is in* $\mathbf{L}^{\mathbf{C_=L}}$.

Proof. Let G be a graph with a polynomially bounded number of even cycles. In analogy to Lemma 3.2 we can show that there exists a small prime p such that all even cycles in G have non-zero circulations under $w_p : E \mapsto \mathbb{Z}_p$ where $w_p(e_i) = 2^i \bmod p$, for every edge e_i. Thus, all nice cycles in any subgraph H of G such that perfect matchings in H are maximum matchings in G have non-zero circulations under w_p. Hence by Lemma 3.1 H has a unique minimum weight perfect matching. By Lemma 4.1 such a graph H can be computed in $\mathbf{L}^{\mathbf{C_=L}}$. By Theorem 3.4 a perfect matching in H can be computed in $\mathbf{L}^{\mathbf{C_=L}}$. Therefore a maximum matching in G can be computed in $\mathbf{L}^{\mathbf{C_=L}}$. The proof of the theorem is complete. □

5 Discussion

As seen in the paper, isolations of graph matchings play a crucial role for a potential **NC** algorithm for both the decision and the search versions of the matching problem. Deterministic isolations of perfect matchings have been shown only for bipartite planar graphs [DKR08] and for graphs with a polynomially bounded number of nice cycles (the present paper). We conjecture that the method stated bellow can be used in general for isolating a perfect matching in graphs (without any promise).

Assign to each edge e_i of the graph G a polynomial $g_i(x)$ in x such that the circulation polynomial $p_C(x)$ of each even cycle C is non-zero in the ring $\mathbb{Z}[x]$. For example: $g_i(x) = a_i x^i$ for arbitrary small integers a_i. Consider $p_C(x)$ in the field $\mathbb{F} = \mathbb{Z}_P[x]/(h(x))$ where P is a small prime number and $h(x)$ is an irreducible polynomial in the polynomial ring $\mathbb{Z}_P[x]$. Since $\mathbb{F}$ has $P^{\deg(h)}$ elements, we have to choose $h(x)$ of constant degree, say $\deg(h(x)) \leq l$ for a constant l. If all the polynomials $p_C(x)$ are non-zero in $\mathbb{F}$, then there exists $a \in \mathbb{Z}_Q$, where Q is a small prime number of size at least $P^l \leq n^{kl}$, such that all the circulation-polynomials do not vanish at point a. Formally, we have $(p_C(x) \bmod P, h(x)) \bmod Q, x - a \neq 0$ for all C under the weight function $w: \; w(e_i) = (a_i x^i \bmod P, h(x)) \bmod Q, x - a$, for every edge e_i. The main problem we have to solve is how to define $g_i(x), h(x)$, and P such that $p_C(x)$ is in $\mathbb{F} \setminus 0$ for every nice cycle C. A positive answer to this question would give a deterministic isolation as described. Note that the described isolation works for bipartite planar graphs.

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