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# Equivalence of Uniform Key Agreement and Composition Insecurity[*]

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**Abstract**

We prove that achieving adaptive security from composing two general non-adaptively secure pseudo-random functions is impossible if and only if a uniform-transcript key agreement protocol exists.

It is well known that proving the security of a key agreement protocol (even in a special case where the protocol transcript looks random to an outside observer) is at least as difficult as proving $P \neq NP$. Another (seemingly unrelated) statement in cryptography is the existence of two or more non-adaptively secure pseudo-random functions that do not become adaptively secure under sequential or parallel composition. In 2006, Pietrzak showed that *at least one* of these two seemingly unrelated statements is true. Pietrzak's result was significant since it showed a surprising connection between the worlds of public-key (i.e., "cryptomania") and private-key cryptography (i.e., "minicrypt"). In this paper we show that this duality is far stronger: we show that *at least one* of these two statements must also be false. In other words, we show their *equivalence*.

More specifically, Pietrzak's paper shows that if sequential composition of two non-adaptively secure pseudo-random functions is not adaptively secure, then there exists a key agreement protocol. However, Pietrzak's construction implies a slightly stronger fact: If sequential composition does not imply adaptive security (in the above sense), then a *uniform-transcript* key agreement protocol exists, where by uniform-transcript we mean a key agreement protocol where the transcript of the protocol execution is indistinguishable from uniform to eavesdroppers. In this paper, we complete the picture, and show the reverse direction as well as a strong equivalence between these two notions. More specifically, as our main result, we show that if there exists *any* uniform-transcript key agreement protocol, then composition does not imply adaptive security. Our result holds for both parallel and sequential composition. Our implication holds based on virtually all known key agreement protocols, and can also be based on general complexity assumptions of the existence of dense trapdoor permutations.

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# 1 Introduction

One of the central questions in cryptography is the question of *composition*, which very broadly is the study of various ways to compose several basic primitives in a way that amplifies the hardness of the composed object. Naturally, this central question has received a lot of attention in various settings and we continue the study of this question here. More specifically, we investigate a question of whether a composition of pseudo-random functions, to be defined shortly, constitutes stronger security by utilizing the security of the component functions. We consider two very natural types of conventional compositions: a parallel composition with respect to Exclusive-Or (XOR) operation denoted by $\oplus$ and a sequential composition. Briefly, on input $x$ in the domain of $\mathsf{F}$ and $\mathsf{G}$, the parallel XOR-composition of two functions $\mathsf{F}$ and $\mathsf{G}$ is defined as $\mathsf{F}(x) \oplus \mathsf{G}(x)$. The sequential composition of $\mathsf{F}$ and $\mathsf{G}$ is defined as $\mathsf{G}(\mathsf{F}(x))$ (or $\mathsf{F}(\mathsf{G}(x))$).

Seemingly unrelated to the notion of security amplification via composition, there is the question of designing Key Agreement protocol. Recall that Key Agreement (KA) is a protocol that enables two parties to generate a secret string (also called key) by communicating with each other over an insecure channel in the presence of a eavesdropping adversary. Uniform-transcript key agreement (UTKA) is a strengthened version of key agreement in which messages between two parties are indistinguishable from uniform distribution by all probabilistic polynomial-time (PPT) adversaries. The reason why key agreement seems unrelated to the security of composition is that key agreement belongs to the world of public-key cryptography (also known as "cryptomania") whereas the security of composed pseudo-random functions rather belongs to the world of private-key cryptography (also known as "minicrypt"). For further discussion on cryptomania and minicrypt, see [Imp95].

Now, let us recall briefly recall the definition of Pseudo-Random Functions (PRF) [GGM86]. There are two notions of security of PRF: adaptive security and non-adaptive security. Intuitively, a (pseudo-random) function is said to be non-adaptively secure if the function is indistinguishable from a random function against all PPT adversaries that evaluate the function on inputs chosen independently of the function outputs, that is, chosen prior to PPT adversary learning any of the outputs. Adaptive security is a far stronger notion of security than non-adaptive security: a PRF is said to be adaptively secure if the function remains indistinguishable from random function against all PPT adversaries preparing the current query based on the outputs of the function on all previous queries. Clearly, adaptive security implies non-adaptive security.

We show that the equivalence between the impossibility of achieving adaptive security by composing general non-adaptively secure pseudo-random functions and the existence of uniform transcript key-agreement protocol. We note that our impossibility result holds not only for the case in which the non-adaptively-secure component functions are drawn from the different function families (also known as the general composition) but also for the case where the component functions are drawn from the same function family (also known as self-composition).

## 1.1 Related Work

There has been extensive research on relationship between the security of component functions and the security of their parallel or sequential composition. In the information theoretic context, Vaudenay [Vau03] proved that if $\mathsf{F}$ is a pseudo-random permutation with security $\epsilon$ against any distinguisher making $q$ (non-)adaptive queries, then the sequential composition of $k$ $\mathsf{F}$'s has improved security $2^{k-1}\epsilon^k$ against a (non-)adaptive distinguisher. $\mathsf{F}$ only needs to be a function instead of a permutation for the same security in parallel composition. Luby and Rackoff [LR86] show the similar security amplification result in the computational context.

In the information theoretic setting, Maurer and Pietrzak [MP04] proved that composition of non-adaptive secure functions amplifies its security $\epsilon$ to security $2\epsilon(1+\ln(\epsilon^{-1}))$ against an adaptive distinguisher. In 2007, Maurer et al. improved this bound to $2\epsilon$ [MPR07].

Myers [Mye04] showed that the existence of oracles relative to which there are non-adaptively secure permutations, but where the composition of such permutations fails to achieve adaptive security. Recently, Pietrzak [Pie05] showed that the composition of non-adaptively secure functions does not imply adaptive security under the Decisional Diffie-Hellman (DDH) assumption. Pietrzak's more recent work [Pie06] showed that if sequential composition does not imply adaptive security, then there exists a key agreement protocol. Moreover, it turns out that Pietrzak's construction in [Pie06] implies a slightly stronger result: that his key agreement protocol satisfies the property of uniform-transcript. Thus, we can restate the Pietrazak's result as follows:

**Theorem 1.** [Pie06] *If sequential composition of pseudo-random functions is not adaptively secure, then there exists a UTKA.*

## 1.2 Our Results

Pietrzak's work left open the question of establishing the precise connection between the impossibility of adaptively secure composition and key agreement. Our main contribution is to establish sufficient and necessary conditions. In particular, we prove that the existence of UTKA implies the impossibility of obtaining an adaptively secure function from composing general non-adaptively secure functions. The main technique is the fully black-box construction of counter-example functions from UTKA. Therefore, our result holds with respect to any UTKA without relying on the actual code of the UTKA. We prove our result in both parallel and sequential compositions.

**Theorem 2.** *If there exists a UTKA, then parallel composition of non-adaptively secure pseudo-random functions does not imply a pseudo-random function with adaptive security.*

**Theorem 3.** *If there exists a UTKA, then sequential composition of non-adaptively secure pseudo-random functions does not imply a pseudo-random function with adaptive security.*

We also prove the analog of Pietrzak's Theorem 1 for parallel composition:

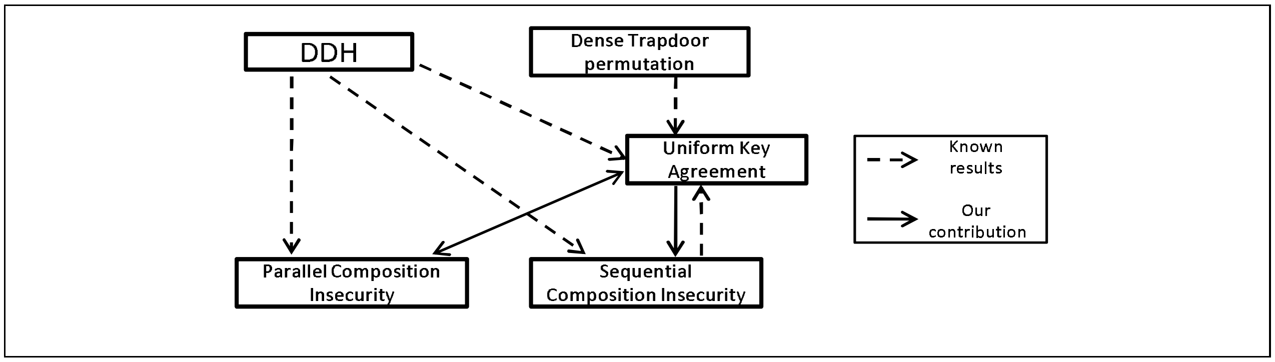
**Theorem 4.** *If a parallel composition of speudo-random functions is not adaptively secure, then there exists a UTKA.*

Putting all our results together with Theorem 1, we conclude the equivalence between the impossibility of adaptively secure composition and the existence of a uniform transcript key-agreement (both for parallel and sequential compositions). This is informally stated as follows.

**Theorem 5.** (MAIN) *The composition of two non-adaptively secure pseudo-random functions does not imply an adaptively secure pseudo-random function if and only if a UTKA exists.*

We emphasize that our main theorem holds regardless of whether PRFs being composed are taken from a single function family (called self-composition) or from two distinct function families (called general-composition). In particular, we show that the impossibility of secure general-compositions further implies the impossibility of secure self-compositions. The precise connection between the impossibility of adaptively secure composition and a UTKA protocol were not known prior to our work. We summarize these previously known results and our contributions in Figure 1.

**Organization of the rest of the paper:** In Section 2, we review all basic cryptographic notions and definitions. To build the intuition of our main construction, we first show in section 3 a high



DDH
Dense Trapdoor permutation
Uniform Key Agreement
Known results
Our contribution
Parallel Composition Insecurity
Sequential Composition Insecurity

**Figure 1:** Relationship between composition insecurity and other assumptions

level outline of somewhat weaker result. In particular, we outline the analogue of **Theorem 2** and **Theorem 3** not assuming UTKA, but rather assuming the existence of a family of enhanced trapdoor permutations. We note that even this weaker variant of our main result is a generalization from the result by [Pie05], which relies on a specific assumption (i.e., DDH assumption). In Section 4, we proceed to give the intuition of our main result assuming UTKA. In Section 5, we discuss the implications of out main results on the impossibility of adaptively secure self-composition.

## 2 Preliminaries

We let $n \in \mathbb{N}$ be a security parameter. An algorithm is considered efficient if its computation can be carried out by a PPT machine whose running time is expected polynomial in the input length. We use the notation $x \leftarrow_{\$} \{0,1\}^n$ when string $x$ is uniformly drawn from $\{0,1\}^n$. For further discussion on the following definitions and notions, see [Gol01].

**Definition 1** (Negligible and Overwhelming)**.** *A function $\epsilon : \mathbb{N} \to \mathbb{R}$ is* ***negligible*** *if for every $c > 0$ there exists an $N_c$ such that for all $n > N_c$, $\epsilon(n) \leq 1/n^c$. On the other hand, $1 - \epsilon$ is said to be* ***overwhelming*** *in $n$.*

**Definition 2** (Non-negligible)**.** *A function $\delta : \mathbb{N} \to \mathbb{R}$ is* ***non-negligible*** *if for every $c > 0$ there exists infinitely many $n$ such that $\delta(n) \geq 1/n^c$.*

**Definition 3** (Noticeable)**.** *A function $\mu : \mathbb{N} \to \mathbb{R}$ is* ***noticeable*** *if for every $c > 0$ there exists an $N_c$ such that for all $n > N_c$, $\mu(n) \geq 1/n^c$.*

**Definition 4** (Polynomial Indistinguishability)**.** *Two probability ensembles $X = \{X_n\}_{n \in \mathbb{N}}$ and $Y = \{Y_n\}_{n \in \mathbb{N}}$ are* ***polynomially indistinguishable*** *if for every Probabilistic Polynomial-Time (PPT) algorithm (distinguisher) $\mathcal{A}$, there exists a negligible function $\epsilon$ such that: for all random coin tosses $r$ and $r'$ of $\mathcal{A}$,*

$$|\Pr[\mathcal{A}_r(X_n) = 1] - \Pr[\mathcal{A}_{r'}(Y_n) = 1]| \leq \epsilon(n).$$

**Definition 5** (Pseudo-Random Function (Permutation))**.** *Given a randomly chosen key $k \in K$ for a key space $K$, an efficiently computable keyed function $F_k : K \times \{0,1\}^n \to \{0,1\}^n$ is called pseudo-random function (PRF), if for every probabilistic polynomial-time algorithm (distinguisher) $\mathcal{A}$, given access to the function $F_k$ and a uniform random function $U : \{0,1\}^n \to \{0,1\}^n$ and for all $n$, there exists a negligible function $\epsilon(n)$ such that: for all random coin tosses $r$ and $r'$ of $\mathcal{A}$,*

$$\left|\Pr[\mathcal{A}_r^{F_k}(1^n) = 1] - \Pr[\mathcal{A}_{r'}^{U}(1^n) = 1]\right| \leq \epsilon(n).$$

*In addition, if $F$ is one-to-one and onto for all $k$, then we call $F_k$ a pseudo-random permutation (PRP).*

**Definition 6** (Distinguishing Advantage)**.** *Given a polynomial-time distinguisher $\mathcal{A}(q,t)$, which runs in time $t$ and makes at most $q$ queries to the oracle $\mathsf{O}$ (i.e., a pseudo-random function $\mathsf{F}$) or random function $\mathsf{R}$, we define the advantage of $\mathcal{A}(q,t)$ as: for all random coin tosses $r$ and $r'$ of $\mathcal{A}$,*

$$\mathbf{Adv}^{\mathsf{O}}_{\mathcal{A}}(q,t) = \max_{\mathcal{A}} \left| \Pr[\mathcal{A}^{\mathsf{O}}_r = 1] - \Pr[\mathcal{A}^{\mathsf{R}}_{r'} = 1] \right| .$$

A distinguisher is a *non-adaptive* distinguisher if it makes all the queries before it receives the output. Otherwise, we call it an *adaptive* distinguisher.

**Definition 7** (Non-Adaptive Versus Adaptive Security)**.** *A pseudo-random function $f$ is non-adaptively secure if, for all polynomial-time non-adaptive distinguisher $\mathcal{A}(q,t)$, there exists a negligible function $\epsilon$ such that $\mathbf{Adv}^{\mathsf{O}}_{\mathcal{A}}(q,t) \leq \epsilon$. On the contrary, we say a pseudo-random function $f$ is adaptively secure if, for all polynomial-time adaptive distinguisher $\mathcal{A}(q,t)$, there exists a negligible function $\epsilon$ such that $\mathbf{Adv}^{\mathsf{O}}_{\mathcal{A}}(q,t) \leq \epsilon$.*

**Definition 8** (Parallel and Sequential Composition)**.** *The parallel XOR-composition of two functions $\mathsf{F}$ and $\mathsf{G}$, denoted as $\mathsf{F} \oplus \mathsf{G}$, is the operation that composes the output value of $\mathsf{F}$ and $\mathsf{G}$ over the bit-wise Exclusive-Or (XOR) operation. The sequential composition of $\mathsf{F}$ and $\mathsf{G}$, denoted as $\mathsf{F} \circ \mathsf{G}$, is the operation that applies two functions sequentially, i.e., $\mathsf{F} \circ \mathsf{G}(\cdot) = \mathsf{G}(\mathsf{F}(\cdot))$.*

Informally, a dense trapdoor permutation family is a trapdoor permutation family with a polynomially dense public description of permutation so that a public description is indistinguishable from uniform random [SP92, Hai04].

**Definition 9** (Dense Trapdoor One-Way Permutation (DTP))**.** *The algorithm triplet ($\mathsf{Gen}$, $f_k$, $f^{-1}_{t_k}$) is a family of dense trapdoor permutations if the following hold:*

- $\mathsf{Gen}(1^n)$ *is a probabilistic polynomial-time algorithm such that on input $1^n$, it outputs a pair of two strings $k \in \{0,1\}^n$ and $t_k$, where $|t_k| \leq p(n)$.*
- *Given $k$, algorithm $f_k : \{0,1\}^n \to \{0,1\}^n$ is a polynomial-time computable permutation.*
- *Given $t_k$, algorithm $f^{-1}_{t_k}$ is a polynomial-time computable inverse permutation of $f_k$. That is, $f^{-1}_{t_k}(f_k(x)) = x$ is efficiently computable for all $x \in \{0,1\}^n$.*
- *For all probabilistic polynomial-time algorithm $\mathcal{A}$, the following holds for any $(k, t_k)$, $x \in \{0,1\}^n$, for all random coin toss $r$ of $\mathcal{A}$,*

  $$\Pr[\mathcal{A}^{f_k}_r(k, f_k(x)) = f^{-1}_{t_k}(f_k(x))] \leq \epsilon(n)$$

  *where $\epsilon(n)$ is a negligible function in $n$.*
- *For all probabilistic polynomial-time algorithm $\mathcal{A}$, there exists a negligible function $\epsilon$ such that, given access to function $\mathsf{Gen}(1^n)$ or uniform function $U$ as oracle: for all random coin tosses $r$ and $r'$ of $\mathcal{A}$,*

  $$\left| \Pr[\mathcal{A}^{\mathsf{Gen}}_r(1^n) = 1] - \Pr[\mathcal{A}^{U}_{r'}(1^n) = 1] \right| \leq \epsilon(n).$$

**Definition 10** (Hard-Core Predicate)**.** *A polynomial-time computable function family* $\mathsf{B} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *is a hard-core predicate of one-way function* $f$ *if, for every probabilistic polynomial-time algorithm* $\mathcal{A}$ *and for all* $x \in \{0,1\}^n$*, there exists a negligible function* $\epsilon$ *such that*

$$\Pr[\mathcal{A}(f(x)) = \mathsf{B}(x,r))] \leq \frac{1}{2} + \epsilon(n).$$

Goldreich and Levin [GL89] presented the simple construction of hardcore predicate $\mathsf{B}(x,r)$ from any one-way function $f$ as $b = \bigoplus_i x_i r_i$, as usual, denoted as $b = < x, r >$, the inner product of two vectors.

**Definition 11** (Bit Agreement (BA))**.** *Informally, a bit agreement is a protocol in which two parties, Alice and Bob, secretly agree on a bit at the end. Formally, upon the security parameter* $n$ *as a common input, Alice and Bob output a bit* $b_A$ *and* $b_B$*, respectively. Then, the protocol is said to have the correlation* $\epsilon(n)$ *if for all* $n$*,*

$$\Pr[b_A = b_B] \geq \frac{1}{2} + \frac{\epsilon(n)}{2}.$$

*And the protocol is also said to be* $\delta(n)$*-secure if, for all* $n$ *and for any PPT adversary Eve, given the security parameter* $n$ *and the entire transcript (denoted by* $\mathsf{Trans}$*) between Alice and Bob,*

$$\Pr[b^{'} \leftarrow Eve(\mathsf{Trans}, 1^n) : b^{'} = b_A] \leq 1 - \frac{\delta(n)}{2}.$$

*If Alice and Bob exchange* $k$ *messages during the execution of the bit agreement protocol, it is called a* $k$*-pass bit agreement. Note that Alice and Bob output the same bit with overwhelming probability, and Eve can compute the bit* $b_A$ *with only negligible advantage over merely guessing it, as* $\epsilon$ *and* $\delta$ *are overwhelming.*

A key agreement protocol is one where two parties, Alice and Bob, given $n$ as a common input, secretly agree on a key in $\{0,1\}^n$ at the end of execution. The key agreement is known to be achieved by polynomially many parallel or sequential executions of the bit agreement protocol if the protocol has a noticeable $\epsilon$ and an overwhelming $\delta$ [Hol05]. Notice that the parallel repetitions of bit agreements achieve the $n$-bit key agreement without increasing the round complexity. See [Hol05] and [Hol06] for further details. By one round, we mean a unit process wherein Alice receives, computes and sends a message to Bob, and then Bob receives, computes and sends a message to Alice. Thus, a $\gamma$-round key agreement($\gamma$-KA) implies a $2\gamma$-pass key agreement. A $\gamma$-round uniform-transcript key agreement($\gamma$-UTKA) is a $\gamma$-KA whose transcripts are indistinguishable from uniform distribution.

**Definition 12** ($\gamma$-round (Uniform-Transcript) Key Agreement $\Phi$ ($\gamma$-(UT)KA))**.** *For* $\gamma \geq 1$*, a* $\gamma$*-round key agreement (exchange) protocol* $\Phi$ *consists of two sub-protocols, Alice (*$\mathsf{A}$*) and Bob (*$\mathsf{B}$*), denoted as* $\Phi = (\mathsf{A}, \mathsf{B})$*. Let* $\alpha_i$ *and* $\beta_i$ *be the* $i$*th round messages of* $\mathsf{A}$ *and* $\mathsf{B}$ *respectively. Let* $\mathsf{Tran}_i^{\mathsf{A}}$ *be the transcript of all the messages up to the* $i$*th round from* $\mathsf{B}$ *as* $\mathsf{Tran}_i^{\mathsf{A}} = (\beta_1, \beta_2, \ldots, \beta_i)$ *and* $\mathsf{Tran}_i^{\mathsf{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_i)$*. Then,* $\mathsf{A}$ *consists of a family of message-generating algorithms* $\mathsf{A}_1, \mathsf{A}_2, \cdots, \mathsf{A}_{\gamma+1}$ *defined as follows. For* $\mathsf{A}$*'s random value* $r_{\mathsf{A}} \in \{0,1\}^n$*:*

$\mathsf{A}_1 : \{0,1\}^n \to \{0,1\}^n$ *is defined as* $\mathsf{A}_1(r_{\mathsf{A}}) \to \alpha_1$,

$\mathsf{A}_i : \{0,1\}^n \times (\{0,1\}^n)^{i-1} \to \{0,1\}^n$ *is defined as* $\mathsf{A}_i(r_{\mathsf{A}}, \mathsf{Tran}_{i-1}^{\mathsf{A}}) \to \alpha_i$ *for* $2 \leq i \leq \gamma$

$\mathsf{A}_{\gamma+1} : \{0,1\}^n \times (\{0,1\}^n)^{\gamma} \to \{0,1\}^n$ *is defined as* $\mathsf{A}_{\gamma+1}(r_{\mathsf{A}}, \mathsf{Tran}_{\gamma}^{\mathsf{A}}) \to s_{\mathsf{A}}$.

*The definition of* $\mathsf{B}$ *is identical to the definition of* $\mathsf{A}$ *except that all* $\mathsf{A}$*'s and* $\alpha$*'s are replaced with* $\mathsf{B}$*'s and* $\beta$*'s. Finally,* $\Phi = (\mathsf{A}, \mathsf{B})$ *satisfies the following conditions:*

1. *For any $\mathcal{A} \in PPT$, given* $\mathsf{Trans}$*, $\mathcal{A}$ cannot efficiently distinguish the exchanged $n$-bit key $s_\mathsf{A}$ from random. Formally, for all random coin tosses $r$ and $r'$ of $\mathcal{A}$ and random inputs $r_\mathsf{A}$ and $r_\mathsf{B}$ of A and B,*

$$\Pr[\mathsf{Trans} \leftarrow (\mathsf{A}_{r_\mathsf{A}}, \mathsf{B}_{r_\mathsf{B}}); s_\mathsf{A} \leftarrow \mathsf{A}_{\gamma+1}(r_\mathsf{A}, \mathsf{Trans}^\mathsf{A}_\gamma) : \mathcal{A}_r(\mathsf{Trans}, s_\mathsf{A}) = 1]$$

$$- \Pr[\mathsf{Trans} \leftarrow (\mathsf{A}_{r_\mathsf{A}}, \mathsf{B}_{r_\mathsf{B}}); \alpha \overset{rand}{\leftarrow} \{0,1\}^n : \mathcal{A}_{r'}(\mathsf{Trans}, \alpha) = 1] \leq \mu(n)$$

*for some negligible function $\mu(n)$.*

2. *At the end of execution of $\Phi$,* A *and* B *agree on a secret $s$. Formally, let $\tau(n)$ be a negligible function in $n$, then,*

$$\Pr[s_\mathsf{A} \leftarrow \mathsf{A}_{\gamma+1}(r_\mathsf{A}, \mathsf{Trans}^\mathsf{A}_\gamma); s_\mathsf{B} \leftarrow \mathsf{B}_{\gamma+1}(r_\mathsf{B}, \mathsf{Trans}^\mathsf{B}_\gamma) : s_\mathsf{A} = s_\mathsf{B}] \geq 1 - \tau(n).$$

3. *For the $\gamma$-round key agreement $\Phi$ to be a $\gamma$-round uniform-transcript key agreement, denoted as $\Phi_u$, the additional condition below is satisfied. For any PPT adversary $\mathcal{A}$,*

$$\left|\Pr[\mathcal{A}_r(\mathsf{Trans}) = 1] - \Pr[\mathsf{R}_\gamma \overset{rand}{\leftarrow} (\{0,1\}^n)^\gamma : \mathcal{A}_r(\mathsf{R}_\gamma) = 1]\right| \leq \epsilon(n),$$

*where $\epsilon(n)$ is a negligible function in $n$.* B *satisfies the same requirement. That is, no PPT adversary $\mathcal{A}$ distinguishes the messages of* A *and* B *from uniform distribution.*

## 3 Building intuition: Composition Insecurity vs. Dense Trapdoor Permutation

For gentle introduction to our main result, we first present a special case of our main result as an example − The existence of dense trapdoor permutation (DTP) implies the impossibility of achieving the adaptive security by composing (in a black-box way) non-adaptively secure pseudo-random functions. The main idea behind showing this, is that a family of DTPs is well-known to provide a 2-pass (uniform-transcript) key agreement.

A 2-pass key agreement can be achieved by $n$ parallel repetitions of an underlying 2-pass bit agreement without increasing its round complexity, which we describe as follows. Suppose that we are given a family of DTPs, $(\mathsf{Gen}(\cdot), f, f^{-1})$. Without loss of generality, Alice first chooses a pair of one-way permutation $f_k$ and its inverse permutation $f^{-1}_{t_k}$ by computing a public encryption information $k$ and its private corresponding trapdoor information $t_k$ using $\mathsf{Gen}(\cdot)$. Note that $k$ is computationally indistinguishable from a random string of the same length by the property of DTPs. Alice sends the public key $k$ to Bob. Upon $k$ from Alice, Bob chooses two strings $x$ and $r$. Bob encrypts $x$ with $f_k$, so let $y = f_k(x)$. Bob sends $y$ and $r$ to Alice and computes the secret bit $b = $ <x, r>. With $y$ and $r$, Alice obtains $x$ by inverting $y$ as $x = f^{-1}_{t_k}(y)$. Then, Alice achieves the bit agreement by computing $b = $ <x, r>. Notice that all the messages exchanged between Alice and Bob are either a uniformly random string of length $n$ (i.e., $y$ and $r$) or pseudo-random strings indistinguishable from uniform (i.e., $k$). Thus, the above bit agreement is a *uniform-transcript* bit agreement. Hence, the $n$ parallel repetitions of the 2-pass bit agreement achieve a 2-pass $n$-bit key agreement described in Protocol 1.[1]

[1]We remark that the same randomness $r$ can be used for all of $n$ parallel repetitions of bit agreement instead of using different $r's$ for each of bit agreements. However, this will complicate our exposition later on, so we will use different $r$'s. Clearly this does not affect the security of resulting key agreement protocol.

| Alice | Transcript | Bob |
|---|---|---|
| $(k, t_k) \leftarrow \mathsf{Gen}(n)$ | | |
| | $\xrightarrow{k}$ | $x_1, x_2, \cdots, x_n \leftarrow_{\$} \{0,1\}^n$ |
| | | $r_1, r_2, \cdots, r_n \leftarrow_{\$} \{0,1\}^n$ |
| | | $y_i \leftarrow f_k(x_i)$ for all $i \leq n$ |
| $x_i \leftarrow f_{t_k}^{-1}(y_i)$ for all $i \leq n$ | $\xleftarrow{y_1,\cdots,y_n,r_1,\cdots,r_n}$ | |
| $b_i \leftarrow < x_i, r_i >$ for all $i \leq n$ | | $b_i \leftarrow < x_i, r_i >$ for all $i \leq n$ |
| shared key $sk \leftarrow b_1, b_2, \cdots, b_n$ | | shared key $sk \leftarrow b_1, b_2, \cdots, b_n$ |

**Protocol 1:** 2-pass key agreement based on a DTP (Folklore)

## 3.1 Parallel Composition Insecurity from Dense Trapdoor Permutation

We construct two counter-example pseudo-random functions $\mathsf{F}$ and $\mathsf{G}$ which are secure against any PPT adversary non-adaptively. Then, we prove that their parallel composition is not secure against a particular sequence of four adaptive queries.

### 3.1.1 Intuitions of Parallel Composition of F and G

We provide the high-level overview and intuition of our construction of pseudo-random functions $\mathsf{F}$ and $\mathsf{G}$ based on DTP, and show how to break the adaptive security of their parallel composition. The main technique of our constructions of counter-example functions is to design the functions to detect the adaptive query throughout the input and output behavior. In particular, $\mathsf{F}$ and $\mathsf{G}$ emulate a 2-pass key agreement protocol (described in Protocol 1) via adaptive inputs and outputs. Once $\mathsf{F}$ and $\mathsf{G}$ internally obtain a shared key, they generate outputs which hide a special relation with respect to the shared key. As we input these specially generated outputs to the parallel composition again, $\mathsf{F}$ and $\mathsf{G}$ retrieve the previously shared key and verify the special relation with respect to the shared key. Hence, function $\mathsf{F}$ and $\mathsf{G}$ are convinced that the queries must be indeed adaptively generated, and reveal their private keys through their outputs, which break their security.

Our counter-example functions $\mathsf{F}$ and $\mathsf{G}$ are both defined over $(\{0,1\}^n)^{2n+3}$. $\mathsf{F}$ and $\mathsf{G}$ hide the secret keys $k_{\mathsf{F}}$ and $k_{\mathsf{G}}$ respectively. $\mathsf{P}$ denotes an adaptively secure pseudo-random permutation. Let $(\mathsf{Gen}(\cdot), f, f^{-1})$ be a family of DTPs. $r_{ij}$ and $s_{ij}$ denote the $i$th pseudo-random string generated by $\mathsf{F}$ and $\mathsf{G}$ using their secret keys on $j$th input respectively. In addition, $\mathsf{Enc}_k(x)$ is defined to be a pseudo-random private-key encryption of $x$ with respect to key $k$. Hence, we have $x = \mathsf{Dec}_k(\mathsf{Enc}_k(x))$.

We first define $\mathsf{F}$ and $\mathsf{G}$ on the first *fixed* adaptive query $Q_1 = (0^n, 0^n, \cdots, 0^n)$:

- $\mathsf{F}$ generates $2n+3$ pseudo-random strings $r^*, r_{21}, r_{31}, \cdots, r_{(2n+3)1}$ computed by $\mathsf{P}_{k_{\mathsf{F}}}(Q_1)$.
- $\mathsf{G}$ on input $Q_1$ uses its secret key to first compute sufficiently long pseudo-random string which is then used to compute DTP pair $(k,\ t_k)$: a pair of a DTP key $k$ and its private trapdoor $t_k$ by $\mathsf{Gen}(1^n)$ of DTP. $\mathsf{G}$ generates $2n+2$ pseudo-random strings $s_{21}, s_{31}, \cdots, s_{(2n+3)1}$ by $\mathsf{P}_{k_{\mathsf{G}}}(Q_1)$, then it outputs $(k, s_{21}, \cdots, s_{(2n+3)1})$.

We describe the outputs of $\mathsf{F}$ and $\mathsf{G}$ and their parallel composition outputs below:

$$Q_1 \rightarrow \begin{bmatrix} \mathsf{F} \rightarrow (r^*, r_{21}, \cdots, r_{(2n+3)1}) \\ \mathsf{G} \rightarrow (k, s_{21}, \cdots, s_{(2n+3)1}) \end{bmatrix} \rightarrow (r^* \oplus k, r_{21} \oplus s_{21}, \cdots, r_{(2n+3)1} \oplus s_{(2n+3)1})$$

The second adaptive query is of the form $Q_2 = (u, 0^n, 0^n, \cdots, 0^n)$ where $u = r^* \oplus k$. We define $\mathsf{F}$ and $\mathsf{G}$ on $Q_2$ as follows.

- $\mathsf{F}$ first simulates the first-round of computation (by internally executing $\mathsf{P}_{k_\mathsf{F}}$ on the fixed query $Q_1$) to obtain $r^*$, then computes $u \oplus r^*$ which is equal to $k$; Now, $\mathsf{F}$ computes $2n+3$ pseudo-random strings $x_1, x_2, \cdots, x_n$ and $r_{(n+1)2}, r_{(n+2)2}, \cdots, r_{(2n+3)2}$ by $\mathsf{P}_{k_\mathsf{F}}(Q_2)$. $\mathsf{F}$ computes $y_i$ by $f_k(x_i)$ for $1 \leq i \leq n$, then outputs $(y_1, \cdots, y_n, r_{(n+1)2}, \cdots, r_{(2n+3)2})$.
- $\mathsf{G}$ generates fresh pseudo-random strings $(s_{12}, s_{22}, \cdots, s_{(2n+3)2})$ computed by $\mathsf{P}_{k_\mathsf{G}}(Q_2)$.

We describe what both $\mathsf{F}$ and $\mathsf{G}$ output individually and the output of their parallel composition:

$$Q_2 \to \begin{bmatrix} \mathsf{F} \to (y_1, \cdots, y_n, r_{(n+1)2}, \cdots, r_{(2n+3)2}) \\ \mathsf{G} \to (s_{12}, \cdots, s_{n2}, s_{(n+1)2} \cdots, s_{(2n+3)2}) \end{bmatrix}$$

$$\to (y_1 \oplus s_{12}, \cdots, y_n \oplus s_{n2}, r_{(n+1)2} \oplus s_{(n+1)2}, \cdots, r_{(2n+3)2} \oplus s_{(2n+3)2})$$

We define the third adaptive query $Q_3$ to consist of the selected coordinates in the previous outputs such that $Q_3 = (y_1 \oplus s_{12}, \cdots, y_n \oplus s_{n2}, r_{(n+1)2} \oplus s_{(n+1)2}, \cdots, r_{(2n)2} \oplus s_{(2n)2}, k \oplus r^*, 0^n, 0^n)$. On $Q_3$, we defined $\mathsf{F}$ and $\mathsf{G}$ as follows.

- $\mathsf{F}$ regenerates all the pseudo-random strings in the second round, $x_1, \cdots, x_n, r_{(n+1)2}, \cdots, r_{(2n+3)2}$ by $\mathsf{P}_{k_\mathsf{F}}(Q_2)$. Notice that $Q_2$ is $(k \oplus r^*, 0^n, \cdots, 0^n)$ where $\mathsf{F}$ can obtain $k \oplus r^*$ from $Q_3$. $\mathsf{F}$ can compute $b_i =< x_i, r_{(n+i)2} >$ for all $1 \leq i \leq n$ and retrieve a shared key $sk$ by letting $sk = b_1 b_2 \cdots b_n$. Now, $\mathsf{F}$ generates pseudo-random strings $r_{13}, r_{23}, \cdots, r_{(2n+3)3}$ by $\mathsf{P}_{k_\mathsf{F}}(Q_3)$ and encrypts $r_{13}$ with the shared key as $\mathsf{Enc}_{sk}(r_{13})$. Finally, $\mathsf{F}$ outputs $(\mathsf{Enc}_{sk}(r_{13}), r_{13}, r_{23}, \cdots, r_{(2n+2)3})$.
- $\mathsf{G}$ regenerates $s_{12}, s_{22}, \cdots, s_{(2n)2}$ by $\mathsf{P}_{k_\mathsf{G}}(Q_2)$. $\mathsf{G}$ can obtain $y_1, \cdots, y_n, r_{(n+1)2}, \cdots, r_{(2n)2}$ as it cancels $s_{12}, s_{22}, \cdots, s_{(2n)2}$ out of the first $2n$ coordinates in $Q_3$. By using the inverse permutation $f^{-1}_{t_k}$ with respect to the trapdoor $t_k$, $\mathsf{G}$ can obtain $x_i$ by computing $f^{-1}_{t_k}(y_i)$ for all i. Hence, $\mathsf{G}$ can compute $b_i =< x_i, r_i >$ for all i and retrieve the shared key $sk$ by letting $sk = b_1 b_2 \cdots b_n$. Then, $\mathsf{G}$ generates pseudo-random strings $s_{13}, s_{23}, \cdots, s_{(2n+3)3}$ by $\mathsf{P}_{k_\mathsf{G}}(Q_3)$ and creates an encryption $\mathsf{Enc}_{sk}(s_{13})$. Finally, $\mathsf{G}$ outputs $(\mathsf{Enc}_{sk}(s_{13}), s_{13}, s_{23}, \cdots, s_{(2n+2)3})$.

Below we depict the individual outputs of $\mathsf{F}$ and $\mathsf{G}$ and the output of their parallel composition:

$$Q_3 \to \begin{bmatrix} \mathsf{F} \to (\mathsf{Enc}_{sk}(r_{13}), r_{13}, r_{23}, \cdots, r_{(2n+2)3}) \\ \mathsf{G} \to (\mathsf{Enc}_{sk}(s_{13}), s_{13}, s_{23}, \cdots, s_{(2n+2)3}) \end{bmatrix}$$

$$\to (\mathsf{Enc}_{sk}(r_{13}) \oplus \mathsf{Enc}_{sk}(s_{13}), r_{13} \oplus s_{13}, r_{23} \oplus s_{23}, \cdots, r_{(2n+2)3} \oplus s_{(2n+2)3})$$

Our fourth query $Q_4$ is a selective collection of the outputs in the previous round such that $Q_4$ $= (y_1 \oplus s_{12}, \cdots, y_n \oplus s_{n2}, r_{(n+1)2} \oplus s_{(n+1)2}, \cdots, r_{(2n)2} \oplus s_{(2n)2}, k \oplus r^*, \mathsf{Enc}_{sk}(r) \oplus \mathsf{Enc}_{sk}(s), r \oplus s)$. Notice that $\mathsf{F}$ and $\mathsf{G}$ can simulate all the computations of previous rounds upon $Q_4$. Hence, $\mathsf{F}$ and $\mathsf{G}$ can retrieve shared key $sk$. $\mathsf{F}$ computes $\mathsf{Enc}_{sk}(r_{13})$ and $r_{13}$ by the simulation of computations on $Q_3$. Then, $\mathsf{F}$ checks to see if equality $\mathsf{Dec}_{sk}(\mathsf{Enc}_{sk}(r_{13}) \oplus (\mathsf{Enc}_{sk}(r_{13}) \oplus \mathsf{Enc}_{sk}(s_{13}))) = r_{13} \oplus (r_{13} \oplus s_{13})$ holds where $(\mathsf{Enc}_{sk}(r_{13}) \oplus \mathsf{Enc}_{sk}(s_{13}))$ and $(r_{13} \oplus s_{13})$ are obtained from $Q_4$. Since the equality holds, $\mathsf{F}$ deduces that the input query is indeed an adaptive query. Hence, $\mathsf{F}$ outputs $(k_\mathsf{F}, 0^n, 0^n, \cdots, 0^n)$ containing its secret key $k_\mathsf{F}$. $\mathsf{G}$ does the same and outputs $(0^n, k_\mathsf{G}, 0^n, \cdots, 0^n)$. The individual outputs of $\mathsf{F}$ and $\mathsf{G}$ and the output of the parallel composition are described below.

$$Q_4 \to \begin{bmatrix} \mathsf{F} \to (k_\mathsf{F}, 0^n, 0^n, \cdots, 0^n) \\ \mathsf{G} \to (0^n, k_\mathsf{G}, 0^n, \cdots, 0^n) \end{bmatrix} \to (k_\mathsf{F}, k_\mathsf{G}, 0^n, \cdots, 0^n)$$

### 3.1.2 Formal Construction of Non-Adaptively Secure Function F

We first provide the specifications of the underlying primitives used for the construction of $\mathsf{F}$. We have $\tilde{\pi} : K \times (\{0,1\}^n)^{2n+3} \to (\{0,1\}^n)^{2n+3}$ where $K$ is the key space of $\tilde{\pi}$. $\tilde{\pi}_k$ denotes a PRP with respect to private key $k$ and $\tilde{\pi}_k^{-1}$ is the inversion permutation to $\tilde{\pi}_k$. We are also given PRP $\pi : K \times \{0,1\}^n \to \{0,1\}^n$. Notice that both $\tilde{\pi}$ and $\pi$ have the same key space $K$. Without loss of generality, $K$ is $\{0,1\}^n$ throughout this paper. Finally, we are given a family of DTPs, $(\mathsf{Gen}(\cdot,\cdot), f, f^{-1})$. We denote $f_k$ and $f_{t_k}^{-1}$ as a permutation and its inverse permutation defined over $\{0,1\}^n$ where $(k, t_k)$ is generated by $\mathsf{Gen}(1^n, r)$ for randomness $r \in \{0,1\}^n$ and $|k| = n$ and $|t_k| = poly(n)$.

Built on the above underlying primitives, the counter-example function $\mathsf{F}$ is defined to be from $(\{0,1\}^n)^{2n+3}$ to $(\{0,1\}^n)^{2n+3}$ and internally hides a secret $k_{\mathsf{F}}$ in $K$, which is a private key applied to the underlying primitives in order to generate pseudo-random. Let $I = (u_1, u_2, \ldots, u_{2n+3})$ be an input vector to $\mathsf{F}$ where $u_j \in \{0,1\}^n$ for $i = 1, 2, \ldots, 2n+3$. Similarly, $(v_1, v_2, \ldots, v_{2n+3})$ denote an output vector. The formal construction of $\mathsf{F}$ is given in Algorithm 1.

**Claim 3.1.** *The function* $\mathsf{F}$ *is secure against any non-adaptive PPT adversary* $\mathcal{A}(q,t)$*, running in time* $t$ *and making at most* $q$ *non-adaptive queries, where* $t$ *and* $q$ *are any polynomials of security parameter* $n$*.*

*Proof.* For clarity in the following proof, we denote $r_{ij}$ as the $i$th randomness at the $j$th query. To prove the non-adaptive security of $\mathsf{F}$, we will show that

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{F}}(q,t) \le \mathbf{Adv}_{\mathcal{A}}^{f}(q,t') + \mathbf{Adv}_{\mathcal{A}}^{\tilde{\pi}}(q,t') + \mathbf{Adv}_{\mathcal{A}}^{\pi}(q,t') + \frac{q}{2^n} \tag{1}$$

where $t' = t + poly(n,q)$, accounting for the extra time costs resulting from our reduction.

Assume that non-adaptive adversary $\mathcal{A}$ chooses $q$ queries as follows. The first query is $Q_1 = (u^*, 0^n, \ldots, 0^n) \in (\{0,1\}^n)^{2n+3}$ and the rest of $q-1$ queries are $Q_i = (u_{i1}, u_{i2}, \ldots, u_{i(2n)}, u^*, 0^n, 0^n)$ for $2 \le i \le q$, in which $u_{i1}, u_{i2}, \ldots, u_{i(2n)}$ are arbitrarily chosen for all $i$. Notice that in cases 1 and 2 of Algorithm 1, once we fix the first coordinate of $Q_1$ and the $(2n+1)$th coordinate of $Q_i$'s to be equally $u^*$, we actually fix the shared key $k'$ through all the $q$ queries, so that the first coordinate is an encryption of the second coordinate by $\pi_{k'}$ in the last $q-1$ outputs. Hence, inverting the first $n$ coordinates of output on $Q_1$ will reveal the key $k'$, and consequently $\mathcal{A}$ distinguishes $\mathsf{F}$ from a uniform function $\mathsf{R}$. Since any PPT adversary can invert $f_k$ only with at most negligible probability $\epsilon_{f_k}$, the probability of retrieving $k'$ is at most $(\epsilon_{f_k})^n$, constituting the first term on the right-hand side (RHS) of inequality (1).

Assume that $\mathcal{A}$ makes $q$ queries in the form of $(u_1 \ne 0^n, u_2 \ne 0^n, \cdots, u_{2n+3} \ne 0^n)$, corresponding to case 3 and fixes the first $2n+1$ coordinates of all the queries. So, $\mathcal{A}$ fixes $k', r_1$, and $\alpha$ in the condition $\pi_{k'}^{-1}(\alpha \oplus u_{2n+2}) = r_1 \oplus u_{2n+3}$. Now, $\mathcal{A}$ only needs to find a pair of $u_{2n+2}$ and $u_{2n+3}$ satisfying the condition so that $\mathsf{F}$ reveals its secret key $k_{\mathsf{F}}$. Since $\pi$ is a permutation, there exists unique $u_{2n+3}$ to each $u_{2n+2}$, which satisfies the condition. Hence,

$$\Pr[\pi_{k'}^{-1}(\alpha \oplus u_{2n+2}) = r_1 \oplus u_{2n+3} : u_{2n+2}, u_{2n+3} \leftarrow_{\$} \{0,1\}^n] \le \frac{1}{2^n}.$$

With $q$ queries, $\mathcal{A}$ successfully guesses $k_{\mathsf{F}}$ with probability at most $q/2^n$; that is the second term on the RHS of the inequality.

Consider that $\mathcal{A}$ makes $q$ queries such that each input query falls into either case 3 or 4 of Algorithm 1. Since we already showed above that $\mathsf{F}$ in the case 3 outputs only pseudo-random strings computed by $\tilde{\pi}_{k_{\mathsf{F}}}$ with an overwhelming probability, we ignore the case that $\mathsf{F}$ outputs the

```
Construction of F

1. If $I = (u_1 \neq 0^n, u_2 = 0^n, \cdots, u_{2n+3} = 0^n)$, then
      Output $(v_1, v_2, \ldots, v_{2n+3})$ where
      $(a_1, a_2, \ldots, a_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(0^n, 0^n, \ldots, 0^n)$
      $(x_1, x_2, \ldots, x_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, \ldots, u_{2n+3})$
      Let $y_i = f_{u_1 \oplus a_1}(x_i)$ for $\forall i = 1, 2, \ldots, n$
      $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow (y_1, y_2, \cdots, y_n, x_{n+1}, \ldots, x_{2n+3})$

2. If $I = (u_1 \neq 0^n, u_2 \neq 0^n, \ldots, u_{2n+1} \neq 0^n, u_{2n+2} = 0^n, u_{2n+3} = 0^n)$, then
      Output $(v_1, v_2, \ldots, v_{2n+3})$ where
      $(x_1, x_2, \ldots, x_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_{2n+1}, 0^n, \ldots, 0^n)$
      Let $k_i =< x_i, x_{n+i} >$ for $\forall i = 1, 2, \ldots, n$
      Let $k' = k_1 k_2 \ldots k_n$
      $(r_1, r_2, \ldots, r_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, \ldots, u_{2n+3})$
      $(v_1, v_2, \ldots, v_{2n+3}) \leftarrow (\pi_{k'}(r_1), r_1, \ldots, r_{2n+2})$

3. If $I = (u_1 \neq 0^n, u_2 \neq 0^n, \cdots, u_{2n+3} \neq 0^n)$, then
      output $(v_1, v_2, \cdots, v_{2n+3})$ where
      $(x_1, x_2, \cdots, x_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_{2n+1}, 0^n, \cdots, 0^n)$
      Let $k_i =< x_i, x_{n+i} >$ for $\forall i = 1, 2, \cdots, n$
      Let $k' = k_1 k_2 \cdots k_n$
      $(r_1, r_2, \cdots, r_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, \cdots, u_{2n+1}, 0^n, 0^n)$
      $\alpha \leftarrow \pi_{k'}(r_1)$
   (a) If $\pi_{k'}^{-1}(\alpha \oplus u_{2n+2}) = r_1 \oplus u_{2n+3}$,
          then $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow (k_F, 0^n, 0^n, \cdots, 0^n)$
   (b)    else$(v_1, v_2, \cdots, v_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, \cdots, u_{2n+3})$

4. If $I$ is not of any previous cases, then
      output $(v_1, v_2, \cdots, v_{2n+3}) where$
      $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, \cdots, u_{2n+3})$
```

**Algorithm 1:** The algorithm of function F

secret key $k_{\mathsf{F}}$ on one of the $q$ queries. Then, $\mathsf{F}$ simply outputs a vector of pseudo-random strings generated by PRP $\tilde{\pi}_{k_{\mathsf{F}}}$ on each input query, which is indistinguishable from uniform randoms of $(\{0,1\}^n)^{2n+3}$. This constitutes $\mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t')$ in inequality (1).

Consider the case in which $\mathcal{A}$ makes $q$ non-adaptive queries in which one of the queries is $(0^n, 0^n, \cdots, 0^n)$, so it evokes case 4 and the rest of queries evoke case 1 of Algorithm 1. Towards a contradiction, assume that $\mathcal{A}$ distinguishes the outputs corresponding to the $q$ non-adaptive input queries described above. The output of case 4 is indistinguishable from uniform random by the security of PRP $\tilde{\pi}_{k_{\mathsf{F}}}$. This implies that $\mathcal{A}$ distinguishes the outputs of case 1 from uniform randoms. Notice that the first $n$ coordinates of an output of case 1 are strings generated in the following way. $\tilde{\pi}_{k_{\mathsf{F}}}$ on an input query first generates pseudo-random strings and then a trapdoor permutation $f_{u_1 \oplus a_1}$ re-encrypts these pseudo-random strings where $u_1 \oplus a_1$ is known since $\mathcal{A}$ can obtain $a_1$ from the output of $\mathsf{F}$ on $(0^n, 0^n, \cdots, 0^n)$. The rest of coordinates (all coordinates except for the first $n$ coordinates) are strings generated only by $\tilde{\pi}_{k_{\mathsf{F}}}$ on an input query. We recall the description of the output of case 1 on input query $(u_1, u_2, \cdots, u_{2n+3})$ as follows:

$$(\underbrace{f_{u_1 \oplus a_1}(r_1), \cdots, f_{u_1 \oplus a_1}(r_n)}_{\text{first n coordinates}},\ \underbrace{r_{n+1}, \cdots, r_{2n+3}}_{\text{the rest of coordinates}})$$

where $(r_1, r_2, \cdots, r_{2n+3}) \leftarrow \tilde{\pi}_{k_{\mathsf{F}}}(u_1, u_2, \cdots, u_{2n+3})$. Since $\mathcal{A}$ distinguishes these outputs of case 1 from uniform randoms, either of the following cases must be true. First, $\mathcal{A}$ distinguishes outputs of case 1 from uniform randoms by distinguishing the first $n$ coordinates of the outputs from uniform randoms. Then, $\mathcal{A}$ can also distinguish outputs of case 4 from uniform randoms as follows. Upon outputs of case 4, $\mathcal{A}$ applies $f_{u_1 \oplus a_1}$ to the first $n$ coordinates of each output and ignores the rest of coordinates of each output. This forces the distribution of the outputs of case 4 to be identical to the distributions of the outputs of case 1 that $\mathcal{A}$ distinguishes from uniform randoms. Therefore, $\mathcal{A}$ distinguishes outputs of case 4 from uniform randoms. This leads to a contradiction to the non-adaptive security of $\mathsf{F}$ in case 4 already proven above. Then, it must be true that $\mathcal{A}$ distinguishes outputs of case 1 from uniform randoms by distinguishing the rest of coordinates of outputs from uniform randoms. However, this also enables $\mathcal{A}$ to distinguish outputs of case 4 from uniform randoms by ignoring the first $n$ coordinates of each output. Hence, another contradiction arises to the non-adaptive security of $\mathsf{F}$ in case 4. Since we encounter contradictions in both cases, the advantage of $\mathcal{A}$ making $q$ non-adaptive queries of case 1 is also upper-bounded by $\mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t')$ in inequality (1).

Finally, consider that $\mathcal{A}$ makes $q$ queries of case 2 in Algorithm 1. Towards a contradiction, assume that $\mathcal{A}$ distinguishes theses outputs from uniform randoms. Notice that the distribution of the last $2n+1$ coordinates in output vectors (all the elements except for the first two elements) is equivalent to the distribution of the last $2n+1$ coordinates of an output of case 4. This is due to that both distributions are generated by $\tilde{\pi}_{k_{\mathsf{F}}}$ on input queries. We already showed above that if $\mathcal{A}$ distinguishes outputs of case 2 from uniform randoms by distinguishing the last $2n+1$ coordinates from uniform randoms, $\mathcal{A}$ can also distinguish outputs of case 4 from uniform randoms, which leads to a contradiction. This implies that $\mathcal{A}$ distinguishes outputs of case 2 from uniform random by distinguishing the first 2 coordinates from uniform randoms. Hence, distinguishing the outputs of case 2 from uniform randoms over $(\{0,1\}^n)^{2n+3}$ is equivalent to distinguishing

$$(\pi_k(r_1), r_1), (\pi_k(r_2), r_2), \cdots, (\pi_k(r_q), r_q) \tag{2}$$

where $k, r_1, r_2, \cdots r_q$ are uniformly random, from

$$(a_1, b_1), (a_2, b_2), \cdots, (a_q, b_q) \tag{3}$$

where $a_i, b_i$ for all $i = 1, \cdots, q$ are uniformly random.

Let $\mathcal{A}$ distinguish (2) from (3) with a non-negligible probability $\xi$. We define the $i$th hybrid distribution $H_i$ as

$$H_i = (\pi_k(r_1), r_1), \cdots, (\pi_k(r_i), r_i), (a_{i+1}, b_{i+1}), \cdots, (a_q, b_q)$$

where $k, r_i, a_i, b_i$ for all $i = 1, \cdots, q$ are uniformly random. Since $|H_0 - H_q| \geq \xi$, there exists $i$ such that $|H_i - H_{i+1}| \geq \xi/q$. Then, we can construct a distinguisher $\mathcal{D}'$ by using $\mathcal{A}$ such that $\mathcal{D}'$ distinguishes $\pi$ from a random function $\mathsf{R} : \{0,1\}^n \to \{0,1\}^n$ with non-negligible probability as follows. Upon an unknown distribution $(\alpha, \beta) \in (\{0,1\}^n)^2$, $\mathcal{D}'$ generates the $i$th hybrid distribution as

$$(\pi_k(t_1), t_1), \cdots, (\pi_k(t_{i-1}), t_{i-1}), (\alpha, \beta), (a_{i+1}, b_{i+1}), \cdots, (a_q, b_q)$$

where $t_1, \cdots, t_q, a_{i+1}, b_{i+1}, \cdots, a_q, b_q$ are uniformly random. Then, $\mathcal{D}'$ queries the $i$th hybrid distribution to $\mathcal{A}$. Since $\mathcal{A}$ distinguishes the $i$th hybrid distribution with non-negligible probability $\xi/q$, $\mathcal{D}'$ distinguishes $\pi$ from $\mathsf{R}$ with non-negligible probability $\xi/q$, which contradicts the indistinguishability of $\pi$. This contributes to $\mathbf{Adv}^{\pi}_{\mathcal{A}}(q, t')$ in inequality (1). □

### 3.1.3 Formal Construction of Non-Adaptively Secure Function G

The function $\mathsf{G}$ is also defined from $(\{0,1\}^n)^{2n+3}$ to $(\{0,1\}^n)^{2n+3}$ with a secret $k_{\mathsf{G}}$ in $K$. The notations and standard specifications of underlying primitives remain identical to those for the construction of $\mathsf{F}$ in the previous section. The formal construction of $\mathsf{G}$ is presented in Algorithm 2.

**Claim 3.2.** *The function $\mathsf{G}$ is secure against any non-adaptive PPT adversary $\mathcal{A}(q, t)$, running in time $t$ and making at most $q$ non-adaptive queries, where $t$ and $q$ are any polynomials of security parameter $n$.*

*Proof.* To prove the non-adaptive security of $\mathsf{G}$, we will also show that

$$\mathbf{Adv}^{\mathsf{G}}_{\mathcal{A}}(q, t) \leq \mathbf{Adv}^{f}_{\mathcal{A}}(q, t') + \mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q, t') + \mathbf{Adv}^{\pi}_{\mathcal{A}}(q, t') + \frac{q}{2^n} + \mathbf{Adv}^{\mathsf{Gen}}_{\mathcal{A}}(q, t') \tag{4}$$

where $t' = t + poly(n, q)$ as defined in Claim 3.1 and $\mathsf{Gen}$ is a key generation algorithm for a family of DTPs. Since all the cases of $\mathsf{G}$ are identical except that the output of $\mathsf{G}$ in case (1) is a public key $k$ for trapdoor permutation $f$, the first four terms on the RHS of (4) are identical to those in Lemma 3.1. Since the key $k$ is indistinguishable from uniform random over $\{0,1\}^n$ by the property of $\mathsf{Gen}$, any PPT adversary $\mathcal{A}$ distinguishes the key $k$ from uniform random over $\{1,0\}^n$ with only negligible advantage. Hence, the indistinguishability of dense keys constitutes $\mathbf{Adv}^{\mathsf{Gen}}_{\mathcal{A}}(q, t')$ in inequality (4). □

### 3.1.4 Adaptive Insecurity of Parallel Composition of F and G

In this paper, a pseudo-random function is said to be *breakable by q adaptive queries* if there is a PPT adversary $\mathcal{A}$ such that $\mathcal{A}$ distinguishes the pseudo-random function from a uniform random function by asking $q$ adaptive queries to the pseudo-random function.

**Claim 3.3.** *The parallel composition function $\mathsf{F} \oplus \mathsf{G}$ is breakable by four adaptive queries.*

```
Construction of G

1. If $I = (u_1 = 0^n, u_2 = 0^n, \cdots, u_{2n+3} = 0^n)$, then
     output $(v_1, v_2, \cdots, v_{2n+3})$ where
     $(s_1, s_2, \cdots, s_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(0^n, \cdots, 0^n)$
     $(k, t_k) \leftarrow \mathsf{Gen}(1^n, s_1)$
     $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow (k, s_2, \cdots s_{2n+3})$

2. If $I = (u_1 \neq 0^n, u_2 \neq 0^n, \cdots, u_{2n+1} \neq 0^n, u_{2n+2} = 0^n, u_{2n+3} = 0^n)$, then
     output $(v_1, v_2, \cdots, v_{2n+3})$ where
     $(a_1, a_2, \cdots, a_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(0^n, 0^n, \cdots, 0^n)$
     $(k, t_k) \leftarrow \mathsf{Gen}(1^n, a_1)$
     $(b_1, b_2, \cdots, b_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_{2n+1}, 0^n, \cdots, 0^n)$
     Let $x_i = f_{t_k}^{-1}(u_i \oplus b_i)$ for $i = 1, 2, \cdots, n$
     Let $k_j =< x_i, (u_{n+j} \oplus b_{n+j}) >$ for $j = 1, 2, \cdots, n$
     Let $k' = k_1 k_2 \cdots k_n$
     $(s_1, s_2, \cdots, s_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+3})$
     $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow (\pi_{k'}(s_1), s_1, \cdots, s_{2n+2})$

3. If $I = (u_1 \neq 0^n, u_2 \neq 0^n, \cdots, u_{2n+3} \neq 0^n)$, then
     output $(v_1, v_2, \cdots, v_{2n+3})$ where
     $(a_1, a_2, \cdots, a_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(0^n, 0^n, \cdots, 0^n)$
     $(k, t_k) \leftarrow \mathsf{Gen}(1^n, a_1)$
     $(b_1, b_2, \cdots, b_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_{2n+1}, 0^n, \cdots, 0^n)$
     Let $x_i = f_{t_k}^{-1}(u_i \oplus b_i)$ for $i = 1, 2, \cdots, n$
     Let $k_j =< x_j, (u_{n+j} \oplus b_{n+j}) >$ for $j = 1, 2, \cdots, n$
     Let $k' = k_1 k_2 \cdots k_n$
     $(c_1, c_2, \cdots, c_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+1}, 0^n, 0^n)$
     $\beta \leftarrow \pi_{k'}(c_1)$

   (a) If $\pi_{k'}^{-1}(\beta \oplus u_{2n+2}) = c_2 \oplus u_{2n+3}$,
         then $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow (0^n, k_G, 0^n, \cdots, 0^n)$
   (b)   else $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+3})$

4. If $I$ is not of any previous cases, then
     output $(v_1, v_2, \cdots, v_{2n+3})$ where
     $(v_1, v_2, \cdots, v_{2n+3}) \leftarrow \tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+3})$
```

**Algorithm 2:** The algorithm of function $\mathsf{G}$

*Proof.* To show the claim, we present a particular sequence of four adaptive queries in which the parallel composition $\mathsf{F} \oplus \mathsf{G}$ reveals all the secret keys of $\mathsf{F}$ and $\mathsf{G}$, such as $k_F$ and $k_G$. Let $r_{ij}$ (or $s_{ij}$) denote the $i$th randomness of $\mathsf{F}$ (or $\mathsf{G}$) upon the $j$th adaptive query $Q_j$. Then, our first query $Q_1$ to $\mathsf{F} \oplus \mathsf{G}(\cdot)$ is $Q_1 = (0^n, 0^n, \cdots, 0^n) \in (\{0,1\}^n)^{2n+3}$. Since the input always (with probability 1) evokes case 4 of Algorithm 1, we have

$$\mathsf{F}(0^n, 0^n, \cdots, 0^n) = \tilde{\pi}_{k_F}(0^n, 0^n, \cdots, 0^n) = (r_{11}, r_{21}, \cdots, r_{(2n+3)1}).$$

In $\mathsf{G}$, since the input falls into case 1 of Algorithm 2, $\mathsf{G}$ computes $(s_{11}, s_{21}, \cdots, s_{(2n+3)1})$ by $\tilde{\pi}_{k_G}(0^n, 0^n, \cdots, 0^n)$ and then obtains $(k, t_k)$ by executing $\mathsf{Gen}(1^n, s_{11})$. Finally, $\mathsf{G}$ outputs $(k, s_{21}, \cdots, s_{(2n+3)1})$. Thus, the output of $\mathsf{F} \oplus \mathsf{G}(0^n, 0^n, \cdots, 0^n)$ is

$$(r_{11}, r_{21}, \cdots, r_{(2n+3)1}) \oplus (k, s_{21}, \cdots, s_{(2n+3)1}) = (r_{11} \oplus k, r_{21} \oplus s_{21}, \cdots, r_{(2n+3)1} \oplus s_{(2n+3)1}).$$

Obtaining the above output, we define our second adaptive query $Q_2$ to be:

$$Q_2 = (r_{11} \oplus k, 0^n, 0^n, \cdots, 0^n) \in (\{0,1\}^n)^{2n+3}.$$

Note that $Q_2$ fails to evoke case 1 of Algorithm 1 when $r_{11} \oplus k = 0^n$. That is, if $r_{11} = k$, then the second adaptive query cannot succeed to lead $\mathsf{F}$ and $\mathsf{G}$ into the proper case. The probability that $r_{11} = k$ is $1/2^n$ which is negligible in $n$. Therefore, $Q_2$ successfully evokes case 1 of Algorithm 1 with probability $1 - 1/2^n$.

Upon $Q_2$ which evokes case 1 of Algorithm 1, $\mathsf{F}$ internally simulates the computations of itself on $Q_1$ to obtain $r_{11}$ by executing $\tilde{\pi}_{k_F}(0^n, 0^n, \cdots, 0^n) = \tilde{\pi}_{k_F}(Q_1) = (r_{11}, r_{22}, \cdots, r_{(2n+3)1})$. Now, $\mathsf{F}$ can retrieve the public key $k$ from $Q_2$ by computing $r_{11} \oplus k \oplus r_{11} = k$. Obtaining $k$, $\mathsf{F}$ computes $y_{i2} = f_k(x_{i2})$ for $i = 1, 2, \cdots, n$ where $\mathsf{F}$ computes fresh pseudo-random strings as $\tilde{\pi}_{k_\mathsf{F}}(Q_2) = (x_{12}, x_{22}, \cdots, x_{(2n+3)2})$. Finally, $\mathsf{F}$ outputs $(y_{12}, \cdots, y_{n2}, x_{(n+1)2}, \cdots, x_{(2n+3)2})$.

In $\mathsf{G}$, the input $Q_2$ is of case 4 of Algorithm 2. Hence, $\mathsf{G}$ outputs fresh pseudo-randoms as follows.

$$\mathsf{G}(r_{11} \oplus k, 0^n, \cdots, 0^n) = \tilde{\pi}_{k_G}(r_{11} \oplus k, 0^n, \cdots, 0^n) = (s_{12}, s_{22}, \cdots, s_{(2n+3)2}) \tag{5}$$

Thus, we have

$$\begin{aligned}(\mathsf{F} \oplus \mathsf{G})(Q_2) &= (y_{12}, \cdots, y_{n2}, x_{(n+1)2}, \cdots, x_{(2n+3)2}) \oplus (s_{12}, s_{22}, \cdots, s_{(2n+3)2}) \\ &= (y_{12} \oplus s_{12}, \cdots, y_{n2} \oplus s_{n2}, x_{(n+1)2} \oplus s_{(n+1)2}, \cdots, x_{(2n+3)2} \oplus s_{(2n+3)2}).\end{aligned}$$

We define our third adaptive query $Q_3 \in (\{0,1\}^n)^{2n+3}$ to be

$$Q_3 = (y_{12} \oplus s_{12}, \cdots, x_{(2n)2} \oplus s_{(2n)2}, r_{11} \oplus k, 0^n, 0^n).$$

Assuming that $Q_2$ succeeds to evoke case 1 of Algorithm 1 and case 4 of Algorithm 2 (Recall that $Q_1$ always succeeds.), the third adaptive query $Q_3$ succeeds to evoke case 2 of Algorithm 1 and Algorithm 2 only when none of the first $2n + 1$ coordinates of $Q_3$ is $0^n$. Since the $(2n + 1)$th coordinate (i.e., $r_{11} \oplus k$) is taken from $Q_2$, the $(2n + 1)$th coordinate is guaranteed not to be $0^n$. Hence, for $Q_3$ to be a valid adaptive query, it must be the case that $y_{12} \neq s_{12}$, $\cdots$, $x_{(2n)2} \neq s_{(2n)2}$. In other words, $Q_3$ fails if at least one of equalities $y_{12} = s_{12}$, $\cdots$, $x_{(2n)2} = s_{(2n)2}$ occurs. Each of the equalities occurs with probability $1/2^n$ so that the total failing probability of $Q_3$ is $2n/2^n$ which is negligible in $n$. Thus, $Q_3$ succeeds to evoke case 2 of Algorithm 1 and Algorithm 2 with probability $1 - 2n/2^n$.

Then $\mathsf{F}$ upon $Q_3$ computes,

$$\tilde{\pi}_{k_F}(u_{2n+1}, 0^n, \cdots, 0^n) = \tilde{\pi}_{k_F}(r_{11} \oplus k, 0^n, \cdots, 0^n) = \tilde{\pi}_{k_F}(Q_2) = (x_{12}, x_{22}, \cdots, x_{(2n+3)2}).$$

Then, $\mathsf{F}$ computes $n$ hard-core bits by computing

$$k_i =< x_{i2}, x_{(n+i)2} > \text{ for } \forall i = 1, 2, \cdots, n. \tag{6}$$

So, $\mathsf{F}$ obtains a shared key $k' = k_1 k_2 \cdots k_n$. Finally, $\mathsf{F}$ computes fresh pseudo-random as

$$\tilde{\pi}_{k_F}(u_1, u_2, \cdots, u_{2n+3}) = (r_{13}, r_{23}, \cdots, r_{(2n+3)3}),$$

and outputs $(\pi_{k'}(r_{13}), r_{13}, \cdots, r_{(2n+2)3})$.

As $Q_3$ evokes case 2 of Algorithm 2, $\mathsf{G}$ retrieves $(k, t_k)$ by computing $\mathsf{Gen}(1^n, s_{11})$ where $(s_{11}, s_{21}, \cdots, s_{(2n+3)1}) = \tilde{\pi}_{k_{\mathsf{G}}}(Q_1)$. Then, $\mathsf{G}$ proceeds to compute the followings:

$$\begin{aligned} \mathsf{G}(u_{2n+1}, 0^n, \cdots, 0^n) &= \mathsf{G}(r_{11} \oplus k, 0^n, \cdots, 0^n) \\ &= (a_1, a_2, \cdots, a_{2n+3}) \\ &= (s_{12}, s_{22}, \cdots, s_{(2n+3)2}) \text{ by (5).} \end{aligned}$$

Using the trapdoor $t_k$, $\mathsf{G}$ computes

$$x_i = f_{t_k}^{-1}(u_i \oplus a_1) = f_{t_k}^{-1}(y_{i2} \oplus s_{i2} \oplus s_{i2}) = f_{t_k}^{-1}(y_{i2}) = x_{i2} \text{ for } \forall i = 1, 2, \cdots, n.$$

Then, $\mathsf{G}$ can compute for $\forall j = 1, 2, \cdots, n$

$$\begin{aligned} k_j &=< x_j, (u_{n+j} \oplus a_{n+j}) > \\ &=< x_{j2}, (x_{(n+j)2} \oplus s_{(n+j)2} \oplus s_{(n+j)2}) > \\ &=< x_{j2}, x_{(n+j)2} > . \end{aligned}$$

$\mathsf{G}$ constructs a shared key $k'$ by letting $k' = k_1 k_2 \cdots k_n$. Notice that $x_{j2}$ and $x_{(n+j)2}$ are respectively equivalent to $x_{i2}$ and $x_{(n+i)2}$ in $\mathsf{F}$'s computation at (6). Thus, $\mathsf{G}$'s $k' = \mathsf{F}$'s $k'$. Finally, $\mathsf{G}$ computes fresh pseudo-randoms as

$$\tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+3}) = (s_{13}, s_{23}, \cdots, s_{(2n+3)3}),$$

and outputs $(\pi_{k'}(s_{13}), s_{13}, \cdots, s_{(2n+2)3})$. Therefore, the output of $\mathsf{F} \oplus \mathsf{G}$ on the third input is

$$(\pi_{k'}(r_{13}) \oplus \pi_{k'}(s_{13}), r_{13} \oplus s_{13}, \cdots, r_{(2n+3)3} \oplus s_{(2n+3)3}).$$

Our final adaptive input $Q_4$ to $\mathsf{F} \oplus \mathsf{G}$ is

$$Q_4 = (y_{12} \oplus s_{12}, \cdots, y_{n2} \oplus s_{n2}, x_{(n+1)2} \oplus s_{(n+1)2}, \cdots, x_{(2n)2} \oplus s_{(2n)2}, r_{11} \oplus s_{11}, \pi_{k'}(r_{13}) \oplus \pi_{k'}(s_{13}), r_{13} \oplus s_{13}).$$

Conditioned that all the previous adaptive queries are successful, consider the probability that $Q_4$ succeeds to evoke case 3 of Algorithm 1 and Algorithm 2. That is, it is the probability that none of the coordinates of $Q_4$ is $0^n$. Notice that the first $2n + 1$ coordinates are taken from $Q_3$ (i.e., the first $2n$ coordinates) and $Q_2$ (i.e., the $(2n + 1)$th coordinate). Hence, none of the first $2n + 1$ coordinates are guaranteed to be $0^n$ as we conditioned that $Q_2$ and $Q_3$ are successful adaptive queries. $Q_4$ fails if $\pi_{k'}(r_{13}) = \pi_{k'}(s_{13})$ or $r_{13} = s_{13}$ in which each of the cases occurs with probability $1/2^n$. Therefore, $Q_4$ succeeds to evoke case 3 of Algorithm 1 and Algorithm 2 with probability $1 - 2/2^n$.

On $Q_4$, which evokes case 3 of Algorithm 1, $\mathsf{F}$ retrieves the same shared key $k'$ as in (6) since $\mathsf{F}$ only requires $u_{2n+1}$ to be $r_{11} \oplus k$ as in the previous round. Obtaining $k'$, $\mathsf{F}$ simulate the computations of the third round. In particular, $\mathsf{F}$ computes

$$\tilde{\pi}_{k_F}(u_1, u_2, \cdots, u_{2n+1}, 0^n, 0^n) = \tilde{\pi}_{k_F}(Q_3) = (r_{13}, r_{23}, \cdots, r_{(2n+3)3}).$$

Since $\alpha = \pi_{k'}(r_{13})$,

$$\begin{aligned}\pi_{k'}^{-1}(\alpha \oplus u_{2n+2}) &= \pi_{k'}^{-1}(\pi_{k'}(r_{13}) \oplus \pi_{k'}(r_{13}) \oplus \pi_{k'}(s_{13})) \\ &= \pi_{k'}^{-1}(\pi_{k'}(s_{13})) \\ &= s_{13} \\ &= r_{13} \oplus r_{13} \oplus s_{13} \\ &= r_{13} \oplus u_{2n+3}.\end{aligned}$$

Therefore, $\mathsf{F}$ outputs $(k_{\mathsf{F}}, 0^n, \cdots, 0^n)$.

$Q_4$ evokes in case 3 of Algorithm 2. Again, $\mathsf{G}$ starts the fourth round computation by retrieving $(k, t_k)$ as it computes $\mathsf{Gen}(1^n, s_{11})$ where $(s_{11}, s_{21}, \cdots, s_{(2n+3)1}) = \tilde{\pi}_{k_{\mathsf{G}}}(Q_1)$. Then, similarly to $\mathsf{F}$, $\mathsf{G}$ also computes the shared key $k'$ by using the first $2n$ elements of $Q_4$, which are equivalent to the first $2n$ coordinates of $Q_3$. Thus, $\mathsf{G}$ retrieves the shared key $k'$. Then, $\mathsf{G}$ proceeds to check if the equality in case 3.(a) holds as follows:

$$\tilde{\pi}_{k_G}(u_1, u_2, \cdots, u_{2n+1}, 0^n, 0^n) = \tilde{\pi}_{k_G}(Q_3) = (b_1, b_2, \cdots, b_{2n+3}) = (s_{13}, s_{23}, \cdots, s_{(2n+3)3}).$$

Since $\beta = \pi_{k'}(s_{13}) = \pi_{k'}(b_1)$,

$$\begin{aligned}\pi_{k'}^{-1}(\beta \oplus u_{2n+2}) &= \pi_{k'}^{-1}(\pi_{k'}(s_{13}) \oplus \pi_{k'}(r_{13}) \oplus \pi_{k'}(s_{13})) \\ &= \pi_{k'}^{-1}(\pi_{k'}(r_{13})) \\ &= r_{13} \\ &= s_{13} \oplus s_{13} \oplus r_{13} \\ &= b_{13} \oplus u_{2n+3}.\end{aligned}$$

Consequently, $\mathsf{G}$ outputs $(0^n, k_{\mathsf{G}}, 0^n, \cdots, 0^n)$. Therefore, the output of $\mathsf{F} \oplus \mathsf{G}$ on $Q_4$ is

$$(k_{\mathsf{F}}, 0^n, \cdots, 0^n) \oplus (0^n, k_{\mathsf{G}}, 0^n, \cdots, 0^n) = (k_{\mathsf{F}}, k_{\mathsf{G}}, 0^n, \cdots, 0^n),$$

which reveals all of the secret keys of $\mathsf{F}$ and $\mathsf{G}$. □

By Claim 3.1, 3.2, and 3.3, we immediately obtains the following lemma:

**Lemma 1.** *Suppose that a dense trapdoor permutation exists. Then, there exist non-adaptively secure functions* $\mathsf{F}$ *and* $\mathsf{G}$ *whose parallel composition* $\mathsf{F} \oplus \mathsf{G}$ *is breakable by four adaptive queries.*

## 3.2 Sequential Composition Insecurity from Dense Trapdoor Permutation

We now present that a sequential composition of two non-adaptively secure functions does not imply the adaptive security in the presence of dense trapdoor permutation. We specifically construct non-adaptively secure pseudo-random functions $\mathsf{F}$ and $\mathsf{G}$ whose sequential composition reveals the secret keys of both component functions upon only three adaptive queries which is less by one adaptive query than the number of adaptive queries required for breaking the parallel composition in the previous section.

### 3.2.1 Intuitions of Sequential Composition of F and G

We provide the intuitive description of our counter-example functions $\mathsf{F}$ and $\mathsf{G}$ for which we provide the high-level overview of their formal constructions. The standard notions and specifications of the underlying primitives are identical to the ones in the previous section. $\mathsf{F}$ (resp. $\mathsf{G}$) contains two secret keys $k_{\mathsf{F}}$ and $k_{\mathsf{F}}^{'}$ (resp. $k_{\mathsf{G}}$ and $k_{\mathsf{G}}^{'}$).

In fact, in the following, the generation of secret key to be shared is simplified for the precise construction and clearer presentation. The secret is merely a pseudo-random string of leghith $n$ while in the previous section, the secret key was generated as the concatenation of $n$ hard-core prediates. We remark that the non-adaptive security of counter-example functions is not affected by this simplification.

We define the first adaptive query $Q_1$ to be a query, $(u^*, 0^n, 0^n, 0^n, 0^n)$ where $u^*$ can be any arbitrary binary string of length $n$. Then, $\mathsf{F}$ and $\mathsf{G}$ are defined on $Q_1$ as follows.

- $\mathsf{F}$ computes $(k, t_k)$ by $\mathsf{Gen}(1^n, \mathsf{P}_{k_\mathsf{F}}(u^*))$, a pair of a public key defining a one-way permutation and its corresponding trapdoor for the inverse permutation. $\mathsf{F}$ also computes pseudo-random strings $(r_{11}, r_{21}, r_{31}, r_{41}, r_{51})$ by $\mathsf{P}_{k_\mathsf{F}}(Q_1)$. $\mathsf{F}$ retrieves a alledged secret key $sk_\mathsf{F}$ (from $\mathsf{G}$) by computing $f_{t_k}^{-1}(0^n)$, which is the output of inverting the second coordinate of $Q_1$ with respect to the trapdoor $t_k$. Then, $\mathsf{F}$ outputs $(k, \mathsf{P}_{sk_\mathsf{F}}(r_{31}), r_{31}, r_{41}, r_{51})$.

- On $(k, \mathsf{P}_{sk_\mathsf{F}}(r_{31}), r_{31}, r_{41}, r_{51})$, function $\mathsf{G}$ is defined to generate a pseudo-random string $sk_\mathsf{G}$ by $\mathsf{P}_{k_\mathsf{G}}(k)$ of length $n$. First, $\mathsf{G}$ checks if a condition holds that the second coordinate of the input is the encryption of the third coordinate with respect to $sk_\mathsf{G}$. The condition does not hold since $sk_\mathsf{G}$ equals $sk_\mathsf{F}$ with only negligible probability. Hence, $\mathsf{G}$ outputs $(f_k(sk_\mathsf{G})$, $s_{21}$, $s_{31}$, $s_{41}$, $s_{51})$ where $\mathsf{G}$ generates pseudo-random strings $(s_{11}, s_{21}, s_{31}, s_{41}, s_{51})$ by computing $\mathsf{P}_{k_\mathsf{G}}(k, \mathsf{P}_{sk_\mathsf{F}}(r_{31}), r_{31}, r_{41}, r_{51})$.

The computation of the sequential composition of $\mathsf{F}$ and $\mathsf{G}$ on $Q_1$ is described below:

$$Q_1 \xrightarrow{\mathsf{F}} (k, \mathsf{P}_{sk_\mathsf{F}}(r_{31}), r_{31}, r_{41}, r_{51}) \xrightarrow{\mathsf{G}} (f_k(sk_\mathsf{G}), s_{21}, s_{31}, s_{41}, s_{51}).$$

We define our second adaptive query $Q_2$ to be the combination of the first adaptive input $Q_1$ and the output of the sequential composition on $Q_1$ such that $Q_2 = (u^*, f_k(sk_\mathsf{G}), s_{21}, s_{31})$. On $Q_2$, we define $\mathsf{F}$ and $\mathsf{G}$ as follows.

- $\mathsf{F}$ obtains the same pair of key and trapdoor $(k, t_k)$ by $\mathsf{Gen}(1^n, \mathsf{P}_{k_\mathsf{F}}(u^*))$ as it did on $Q_1$. Then, $\mathsf{F}$ retrieves the secret key $sk_\mathsf{G}$ as it computes $f_{t_k}^{-1}(sk_\mathsf{G})$. $\mathsf{F}$ computes pseudo-random strings $(r_{12}, r_{22}, r_{32}, r_{42}, r_{52})$ by executing $\mathsf{P}_{sk_\mathsf{F}}(Q_2)$. Thus, $\mathsf{F}$ outputs $(k, \mathsf{P}_{sk_\mathsf{G}}(r_{32}), r_{32}, r_{42}, r_{52})$.

- Upon the input $(k, \mathsf{P}_{sk_\mathsf{G}}(r_{32}), r_{32}, r_{42}, r_{52})$ from $\mathsf{F}$, function $\mathsf{G}$ obtains the secret key $sk_\mathsf{G}$ by computing $\mathsf{P}_{k_\mathsf{G}}(k)$. Then, it check if the second coordinate of the input is the encryption of the third coordinate with repect to the obtained $sk_\mathsf{G}$. Since this truely holds, $\mathsf{G}$ is now convinced that the input from $\mathsf{F}$ was adaptively generated. Function $\mathsf{G}$ outputs $(f_k(sk_\mathsf{G}), \mathsf{P}_{sk_\mathsf{G}}(s_{32}), s_{32}, s_{42}, \mathsf{P}_{sk_\mathsf{G}}(k_\mathsf{G}^{'}))$ where $(s_{12}, s_{22}, s_{32}, s_{42}, s_{52})$ is computed from the excution of $\mathsf{P}_{k_\mathsf{G}}(k, \mathsf{P}_{sk_\mathsf{G}}(r_{32}), r_{32}, r_{42}, r_{52})$.

All the individual outputs of $\mathsf{F}$ and $\mathsf{G}$ as a part of sequential composition are described as follows:

$$Q_2 \xrightarrow{\mathsf{F}} (k, \mathsf{P}_{sk_\mathsf{G}}(r_{32}), r_{32}, r_{42}, r_{52}) \xrightarrow{\mathsf{G}} (f_k(sk_\mathsf{G}), \mathsf{P}_{sk_\mathsf{G}}(s_{32}), s_{32}, s_{42}, \mathsf{P}_{sk_\mathsf{G}}(k_\mathsf{G}^{'})).$$

Now we define our final adaptive query $Q_3$ to be $(u^*, f_k(sk_\mathsf{G}), \mathsf{P}_{sk_\mathsf{G}}(s_{32}), s_{32}, \mathsf{P}_{sk_\mathsf{G}}(k_\mathsf{G}^{'}))$ which is the mixture of the first adaptive query and the outputs on the previous queries. Functions $\mathsf{F}$ and $\mathsf{G}$ are defined as follows.

- Again, $\mathsf{F}$ obtains the same pair of key and trapdoor $(k, t_k)$ by $\mathsf{Gen}(1^n, \mathsf{P}_{k_\mathsf{F}}(u^*))$. $\mathsf{F}$ retrieves the secret key $sk_\mathsf{G}$ by computing $f_{t_k}^{-1}(sk_\mathsf{G})$. Then, by using the retrieved secret key $sk_\mathsf{G}$, $\mathsf{F}$ verifies that the third coordinate of $Q_3$ is the encryption of the four coordinate. Therefore, function $\mathsf{F}$ is also convinced that $Q_3$ is adaptively contructed and outputs $(k'_\mathsf{G}, k_\mathsf{F}, k'_\mathsf{F}, 0^n, 0^n)$ where $k'_\mathsf{G}$ is obtained from the inversion of the final coordinate using $sk_\mathsf{G}$.
- Upon the input $(k'_\mathsf{G}, k_\mathsf{F}, k'_\mathsf{F}, 0^n, 0^n)$ from $\mathsf{F}$, function $\mathsf{G}$ confirms that the first coordinate of the input equals its one of the secret keys. Hence, function $\mathsf{G}$ outputs $(k_\mathsf{G}, k'_\mathsf{G}, k_\mathsf{F}, k'_\mathsf{F}, 0^n)$ which reveals all the secret keys of $\mathsf{F}$ and $\mathsf{G}$.

Below, the individual outputs of $\mathsf{F}$ and $\mathsf{G}$ as a part of sequential composition are described:

$$Q_3 \xrightarrow{\mathsf{F}} (k'_\mathsf{G}, k_\mathsf{F}, k'_\mathsf{F}, 0^n, 0^n) \xrightarrow{\mathsf{G}} (k_\mathsf{G}, k'_\mathsf{G}, k_\mathsf{F}, k'_\mathsf{F}, 0^n).$$

### 3.2.2 Formal Construction of Non-Adaptively Secure Function F

We first provie the underlying primitives and their standard specifications. In the following formal construction, we have two PRPs: $\pi_k : \{0,1\}^n \to \{0,1\}^n$ and $\tilde{\pi}_k : \{0,1\}^{5n} \to \{0,1\}^{5n}$. Also, we are given a family of DTP $(\mathsf{Gen}(\cdot,\cdot),\ f,\ f^{-1})$ which is indentical to the one given in Section 3.1.2. Function $\mathsf{F}$ is defined over $\{0,1\}^{5n}$ and internally contains two $n$-bit secret seeds $k_\mathsf{F}$ and $k'_\mathsf{F}$. The formal construction of $\mathsf{F}$ is given in Algorithm 3.

<u>**Construction of F**</u>

```
1. For any $I = (u_1, u_2, u_3, u_4, u_5)$, then
     Output $(v_1, v_2, v_3, v_4, v_5)$ where
     $r_\mathsf{F} \leftarrow \pi_{k_\mathsf{F}}(u_1)$
     $(k, t_k) \leftarrow \mathsf{Gen}(1^n, r_\mathsf{F})$
     $sk_\mathsf{F} \leftarrow f_{t_k}^{-1}(u_2)$
     $(r_1, r_2, r_3, r_4, r_5) \leftarrow \tilde{\pi}_{k_F}(u_1, u_2, u_3, u_4, u_5)$
   (a) If $\pi_{sk_\mathsf{F}}^{-1}(u_3) = u_4$, then
         $(v_1, v_2, v_3, v_4, v_5) \leftarrow (\pi_{sk_\mathsf{F}}^{-1}(u_5), k_\mathsf{F}, k'_\mathsf{F}, 0^n, 0^n)$
   (b) else
         $(v_1, v_2, v_3, v_4, v_5) \leftarrow (k, \pi_{sk_\mathsf{F}}(r_3), r_3, r_4, r_5)$
```

**Algorithm 3:** The algorithm of function $\mathsf{F}$

**Claim 3.4.** *The function* $\mathsf{F}$ *is secure against any non-adaptive PPT adversary* $\mathcal{A}(q,t)$*, running in time* $t$ *and making at most* $q$ *non-adaptive queries, where* $t$ *and* $q$ *are any polynomials of security parameter* $n$*.*

*Proof.* We need to prove that

$$\mathbf{Adv}_\mathcal{A}^\mathsf{F}(q,t) \le \mathbf{Adv}_\mathcal{A}^{\tilde{\pi}}(q,t') + \mathbf{Adv}_\mathcal{A}^{\pi}(q,t') + \mathbf{Adv}_\mathcal{A}^{\mathsf{Gen}}(q,t') + \frac{q}{2^n} \tag{7}$$

where $t' = t + poly(n,q)$, which accounts for the extra time costs resulting from our reduction. Let $Q_i \in (\{0,1\}^n)^{2n+3}$ for $i = 1, \ldots, q$ denote the $i$th query that PPT non-adaptive adversary $\mathcal{A}$

generates. Then, towards a contradiction, assume that $\mathcal{A}$ distinguishes the outputs of $\mathsf{F}$ on $Q_i$'s from uniform random.

First we consider the case that a PPT adversary non-adaptively generate a query satisfying condition 1(a) that reveals all the secret keys of $\mathsf{F}$. To satisfy the condition, the adversary must find a pair $(u_3, u_4)$ such that $\pi^{-1}_{sk_\mathsf{F}}(u_3) = u_4$ where $sk_\mathsf{F}$ is a pseudo-random string of $\mathsf{F}$. Finding such pairs is equivalent to quessing $sk_\mathsf{F}$ with fixing $(u_3, u_4)$. Therefore, by using $q$ queries, the probability is $q/2^n$ which constitutes the final term on the right hand side of (7).

Consider the queries that invoke the case 1(b). Unising the identical hybrid argument in the proofs of the previous section, we can prove that any PPT adversary, which distinguishes the output of case 1(b) from uniform random, also distinguishes from unform random the outputs of either $\tilde{\pi}$, $\pi$, or $\mathsf{Gen}$. This constitutes the first three terms on the right hand side of (7), completing the proof. $\square$

### 3.2.3 Formal Construction of Non-Adaptively Secure Function G

Similarly to function $\mathsf{F}$, function $\mathsf{G}$ is also defined over $\{0,1\}^{5n}$ and contains two $n$-bit secret keys $k_\mathsf{G}$ and $k^{'}_\mathsf{G}$. The underlying primitives and standard notations are identical to those used to construct $\mathsf{F}$. The formal definition of function $\mathsf{G}$ is provided in Algorithm 4.

**Construction of G**

```
1. For any $I = (u_1, u_2, u_3, u_4, u_5)$, then
       Output $(v_1, v_2, v_3, v_4, v_5)$ where
       $sk_\mathsf{G} \leftarrow \pi_{k_\mathsf{G}}(u_1)$
       $(s_1, s_2, s_3, s_4, s_5) \leftarrow \tilde{\pi}_{k_G}(u_1, u_2, u_3, u_4, u_5)$
   (a) If $u_1 = k^{'}_\mathsf{G}$, then
           $(v_1, v_2, v_3, v_4, v_5) \leftarrow (k_\mathsf{G}, k^{'}_\mathsf{G}, u_2, u_3, 0^n)$
   (b) Else If $\pi^{-1}_{sk_\mathsf{G}}(u_2) = u_3$, then
           $(v_1, v_2, v_3, v_4, v_5) \leftarrow (f_{u_1}(sk_\mathsf{G}), \pi_{sk_\mathsf{G}}(s_3), s_3, s_4, \pi_{sk_\mathsf{G}}(k^{'}_\mathsf{G}))$
   (c) else
           $(v_1, v_2, v_3, v_4, v_5) \leftarrow (f_{u_1}(sk_\mathsf{G}), s_2, s_3, s_4, s_5)$
```

**Algorithm 4:** The algorithm of function $\mathsf{G}$

**Claim 3.5.** *The function* $\mathsf{G}$ *is secure against any non-adaptive PPT adversary* $\mathcal{A}(q,t)$*, running in time* $t$ *and making at most* $q$ *non-adaptive queries, where* $t$ *and* $q$ *are any polynomials of security parameter* $n$*.*

*Proof.* To prove the claim, we will show the following inequality

$$\mathbf{Adv}^{\mathsf{G}}_{\mathcal{A}}(q,t) \leq \mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t') + \mathbf{Adv}^{\pi}_{\mathcal{A}}(q,t') + \frac{q}{2^{n-1}} \tag{8}$$

where $t' = t + poly(n,q)$, which accounts for the extra time costs resulting from our reduction. Let $Q_i = (u_{i1}, u_{i2}, \cdots, u_{i(2n+3)}) \in (\{0,1\}^n)^{2n+3}$ for $i = 1, 2, \ldots, q$ denote the $i$th query that PPT non-adaptive adversary $\mathcal{A}$ generates. Towards a contradiction, assume that PPT non-adaptive adversary $\mathcal{A}$ distinguishes $\mathsf{G}$ from a uniform random function.

Consider the cases that a PPT adversary makes non-adaptive queries that satisfies conditions at 1(a) or 1(b) of Algorithm 4. Any output from these cases are not random. The outputs of 1(a) reveal all secret keys of function $\mathsf{G}$. Notice that any output of 1(b) is not uniform random. That is, an adversary has an access to the dense trapdoor famaily, so the adversary generates its own pair of a public key and corresponding trapdoor. On any output of 1(b), the adversary can invert the first coordinate of the output by using its trapdoor and retrieve the secret key. It is easy to see that any speical relationships in the outputs will be revealed by using this obtained secret key. Similar to the previous section, the probability that $q$ non-adaptive queries evoke one of these cases is $2(q/2^n)$ which is the final term on the right hand side of inequality (8).

Now, consider the case 1(c) to which any non-adaptive query belongs with overwhelming probability. By the identical hyprid argument in the prior sections, any PPT adversary that distinguishes the output of 1(c) from uniform random can be transformed to a PPT adversary that distinguishes the outputs of $\tilde{\pi}$ or $\pi$, constituting the first two terms on the right hand side of inequality (8). Therefore, the proof is complete. $\square$

### 3.2.4 Adaptive Insecurity of Sequential Composition of F and G

**Claim 3.6.** *The sequential composition $\mathsf{G}(\mathsf{F}(\cdot))$ is breakable by three adaptive queries.*

*Proof.* We will present three adaptive queries which reveals all the secrets of $\mathsf{F}$ and $\mathsf{G}$ (i.e., $k_{\mathsf{F}}$ and $k_{\mathsf{G}}$). In the following proof, let $r_{ij}$ denote the $i$th randomness of $\mathsf{F}$ generated in $j$th query, and $s_{ij}$ denote the $i$th randomness of $\mathsf{G}$ on the $j$th adaptive query. Also, for clarity, we denote $sk^i_{\mathsf{F}}$ (resp. $sk^i_{\mathsf{G}}$) as a shared secret key computed by $\mathsf{F}$ (resp. $\mathsf{G}$) on the $i$-th adaptive query.

We define the first adaptive query $Q_1$ to be $(u^*, 0^n, 0^n, 0^n, 0^n)$ where $u^*$ can be any string of length $n$. Then, on $Q_1$, $\mathsf{F}$ first computes a pseudo-random string $r_{\mathsf{F}}$ by $\pi_{k_{\mathsf{F}}}(u_1)$. Using this pseudo-random string, $\mathsf{F}$ obtains a key and its corresponding trapdoor $(k, t_k)$ as it executes $\mathsf{Gen}(1^n, r_{\mathsf{F}})$. Then, $\mathsf{F}$ retrieves an suppposed secret key $sk^1_{\mathsf{F}}$ by computing the inverse permutation on the third coordinate of $Q_1$ as $f^{-1}_{t_k}(0^n) = sk^1_{\mathsf{F}}$. Since the condition at 1(a) does not hold with overwhelming probability, $\mathsf{F}$ outputs $(k, \pi_{sk^1_{\mathsf{F}}}(r_{31}), r_{31}, r_{41}, r_{51})$.

On $(k, \pi_{sk^1_{\mathsf{F}}}(r_{31}), r_{31}, r_{41}, r_{51})$ from $\mathsf{F}$, function $\mathsf{G}$ computes its shared key $sk^1_{\mathsf{G}}$ as $\pi_{k_{\mathsf{G}}}(k) = sk^1_{\mathsf{G}}$. Then, $\mathsf{G}$ checks if $\pi^{-1}_{sk^1_{\mathsf{G}}}(\pi_{sk^1_{\mathsf{F}}}(r_{31})) = r_{31}$. Since $sk^1_{\mathsf{G}} \neq sk^1_{\mathsf{F}}$ (with overwhelming probability), $\mathsf{G}$ outputs $(f_k(sk^1_{\mathsf{G}}), s_{21}, s_{31}, s_{41}, s_{51})$ where $s_{21}$, $s_{31}$, $s_{41}$ and $s_{51}$ are pseudo-random strings computed by $\mathsf{G}$ on the input.

The second adaptive query $Q_2$ is defined to be $(u^*, f_k(sk^1_{\mathsf{G}}), 0^n, 0^n, 0^n)$. As upon $Q_1$, $\mathsf{F}$ obtains $(k, t_k)$ since $u^*$ is provided in $Q_2$ as well. $\mathsf{F}$ retrieves the shared key $sk^2_{\mathsf{F}}$ by executing $f^{-1}_{t_k}(f_k(sk^1_{\mathsf{G}}))$. That is, $sk^2_{\mathsf{F}} = sk^1_{\mathsf{G}}$. Then, $\mathsf{F}$ checks if $\pi^{-1}_{sk^2_{\mathsf{F}}}(0^n) = 0^n$, which does not hold. Therefore, $\mathsf{F}$ outputs $(k, \pi_{sk^2_{\mathsf{F}}}(r_{32}), r_{32}, r_{42}, r_{52})$.

$\mathsf{G}$ computes the shared key $sk^2_{\mathsf{G}}$ as it computes $\pi_{k_{\mathsf{G}}}(k)$. Note that since $k$ remains same, $sk^2_{\mathsf{G}} = sk^1_{\mathsf{G}}$. In addition, as $sk^2_{\mathsf{F}} = sk^1_{\mathsf{G}}$, $sk^2_{\mathsf{F}} = sk^2_{\mathsf{G}}$. Thus, $\mathsf{G}$ confirms that $\pi^{-1}_{sk^2_{\mathsf{G}}}(\pi_{sk^2_{\mathsf{F}}}(r_{32})) = r_{32}$. Hence, $\mathsf{G}$ outputs $(f_k(sk^2_{\mathsf{G}}), \pi_{sk^2_{\mathsf{G}}}(s_{32}), s_{32}, s_{42}, \pi_{sk^2_{\mathsf{G}}}(k'_{\mathsf{G}}))$.

We define the final adaptive query $Q_3$ to be $(k, f_k(sk^2_{\mathsf{G}}), \pi_{sk^2_{\mathsf{G}}}(s_{32}), s_{32}, \pi_{sk^2_{\mathsf{G}}}(k'_{\mathsf{G}}))$. On $Q_3$, $\mathsf{F}$ computes the shared key $sk_{\mathsf{F}^3}$ which is simply $sk^2_{\mathsf{G}}$ as in the previous inputs. $\mathsf{F}$ verifies that $\pi^{-1}_{sk^3_{\mathsf{F}}}(\pi_{sk^2_{\mathsf{G}}}(s_{32})) = s_{32}$. Therefore, $\mathsf{F}$ outputs $(\pi^{-1}_{sk^3_{\mathsf{F}}}(\pi_{sk^2_{\mathsf{G}}}(k'_{\mathsf{G}})), k_{\mathsf{F}}, k'_{\mathsf{F}}, 0^n, 0^n)$ which is $(k'_{\mathsf{G}}, k_{\mathsf{F}}, k'_{\mathsf{F}}, 0^n, 0^n)$.

On the input from $\mathsf{F}$, function $\mathsf{G}$ finds out that the first coordinate equals one of its secret seeds. Therefore, $\mathsf{G}$ outputs $(k_{\mathsf{G}}, k'_{\mathsf{G}}, k_{\mathsf{F}}, k'_{\mathsf{F}}, 0^n)$ which reveals all of the secret seeds of $\mathsf{F}$ and $\mathsf{G}$. $\square$

Proving Claim 3.4, 3.5, and 3.6 substantiates the following lemma.

**Lemma 2.** *Suppose that a dense trapdoor permutation exists. Then, there exist non-adaptively secure functions* $\mathsf{F}$ *and* $\mathsf{G}$ *whose sequential composition* $\mathsf{G}(\mathsf{F}(\cdot))$ *is breakable by three adaptive queries.*

Therefore, by Lemma 1 and 2, we immediately obtains the following theorem which concludes the impossibility of adaptively secure composition under the existence of DTP.

**Theorem 6.** *If a dense trapdoor permutation exists, then the composition of non-adaptively secure functions does not imply the adaptive security.*

## 4 Composition Insecurity vs. Uniform Transcript Key Agreement

In this section, we prove our main result: the existence of UTKA protocol implies the impossibility of obtaining adaptive security by the general composition of non-adaptively secure functions. Moreover, Pietrzak showed that the insecurity of sequential composition implies the existence of key agreement protocol. In fact, the key agreement protocol satisfies the property of *uniform-transcript* even though Pietrzak did not mention it in [Pie06]. For the whole equality between the impossibility of general adaptively secure composition and UTKA, we prove that the parallel composition insecurity also achieves a UTKA by using the similar technique to that in [Pie06].

### 4.1 Parallel Composition Insecurity vs. Uniform Transcript Key Agreement

#### 4.1.1 Constructing UTKA from the Adaptive Insecurity of $\mathsf{F} \oplus \mathsf{G}$

We present the parallel version of the result by using the technique originally presented by [Pie06]. That is, for $k \geq 2$, if the parallel composition of two $k-1$ adaptively secure functions is not $k$-adaptively secure, then a $(2k-1)$-pass key agreement exists. For clarity, we rather present a special case where $k = 2$. Following the technique of [Pie06], we construct a $(2k-1)$-pass uniform-transcript bit agreement (UTBA) with $\epsilon$-correlation and $\delta$-security where $\epsilon$ is *non-negligible* and $\delta$ is *overwhelming*. It is known that $n$ parallel repetitions of bit agreement with $\epsilon$-correlation and $\delta$-security achieves a $n$-bit key agreement without increasing the round complexity when $\epsilon$ is *noticeable* and $\delta$ is *overwhelming* [Hol05]. With non-negligible $\epsilon$, a bit agreement still realizes a key agreement which achieves correctness for (infinitely many) $n$ such that for any $c$, $\epsilon \geq 1/n^c$.

We present the pictorial description of a $(2k-1)$-pass UTBA from two adaptively pseudo-random functions whose parallel composition is not $k$-adaptively secure when $k = 2$ in Protocol 2. The 3-pass UTBA in Protocol 2 may be easily extended to the $(2k-1)$-pass UTBA for arbitrary $k$ and general adaptive distinguisher $\mathcal{D}$. We describe the above protocol and the extension in detail in the proof of following theorem.

**Theorem 7.** *Let* $\mathsf{F}$ *and* $\mathsf{G}$ *be* $(k-1)$*-adaptively secure pseudo-random functions. If the parallel composition* $\mathsf{F} \oplus \mathsf{G}$ *is NOT* $k$*-adaptively secure, then a* $(2k-1)$*-pass UTKA exists for* $k \geq 2$*.*

*Proof.* We present the special case where $k = 2$. Then, we explain how to generalize the technique for arbitrary $k$. Let $\mathsf{F}$ and $\mathsf{G}$ be non-adaptively (2-adaptively) secure pseudo-random functions from $K \times \{0,1\}^n \to \{0,1\}^n$ where $K$ is the key space of $\mathsf{F}$ and $\mathsf{G}$. Without loss of generality, we let $K$ be $\{0,1\}^n$. Also, let $\mathsf{Gen}_\mathsf{F}$ and $\mathsf{Gen}_\mathsf{G}$ be the key generation algorithms from $\{1\}^l$ to $\{0,1\}^l$ defined as $\mathsf{Gen}_\mathsf{F}(1^l) \to x$ for $x \in \{0,1\}^l$. In this proof, $l = n$ since $K = \{0,1\}^n$.

In Protocol 2, $\mathcal{D}$ is a 2-adaptive distinguisher distinguishing $\mathsf{F} \oplus \mathsf{G}$ from a (uniform) random function $\mathsf{R}^n : \{0,1\}^n \to \{0,1\}^n$. Without loss of generality, $\mathcal{D}$, upon an input $(y_1, y_2)$ for $y_1, y_2 \in \{0,1\}^n$, outputs 1 if and only if $\mathcal{D}$ determines that the input is the output of a uniform random of length $n$. Now we want to show that the protocol Bit-Agreement$(1^n)$ in Protocol 2 has correlation

**Protocol Bit-Agreement**$(1^n)$

| Alice | Transcript | Bob |
|---|---|---|
| $b_{\mathsf{A}} \leftarrow_{\$} \{0,1\}^n$ | | |
| $k_{\mathsf{A}} \leftarrow \mathsf{Gen}_{\mathsf{F}}(1^n)$ | | $k_{\mathsf{B}} \leftarrow \mathsf{Gen}_{\mathsf{G}}(1^n)$ |
| $x_1 \leftarrow \mathcal{D}(1^n)$ | | $x_1 \leftarrow \mathcal{D}(1^n)$ |
| If $b_{\mathsf{A}} = 0$, | | |
| then $z_1 \leftarrow \mathsf{F}_{k_{\mathsf{A}}}(x_1)$ | | |
| else $z_1 \leftarrow_{\$} \{0,1\}^n$ | $\xrightarrow{z_1}$ | |
| | $\xleftarrow{y_1}$ | $y_1 \leftarrow z_1 \oplus \mathsf{G}_{k_{\mathsf{B}}}(x_1)$ |
| $x_2 \leftarrow \mathcal{D}(y_1)$ | | $x_2 \leftarrow \mathcal{D}(y_1)$ |
| If $b_{\mathsf{A}} = 0$, | | |
| then $z_2 \leftarrow \mathsf{F}_{k_{\mathsf{A}}}(x_2)$ | | |
| else $z_2 \leftarrow_{\$} \{0,1\}^n$ | $\xrightarrow{z_2}$ | $y_2 \leftarrow z_2 \oplus \mathsf{G}_{k_{\mathsf{B}}}(x_2)$ |
| | | $b_{\mathsf{B}} \leftarrow \mathcal{D}(y_1, y_2)$ |

**Protocol 2:** 3-pass uniform-transcript bit agreement based on 2-adaptive distinguisher $\mathcal{D}$

$\epsilon(n)$ and is secure with probability $\delta(n)$ where $\epsilon$ is non-negligible and $\delta$ is overwhelming given the security parameter $n$. Furthermore, we want to prove that Bit-Agreement$(1^n)$ satisfies the property of uniform-transcript. Therefore, $n$ parallel repetitions of Bit-Agreement$(1^n)$ will achieve the n-bit

**Claim 4.1** (Non-negligible Correctness)**.** *The protocol Bit-Agreement*$(1^n)$ *has non-negligible* $\epsilon$*.*

*Proof.* Let $b_1, b_2$, and $b_3$ be bits defined as in the following three cases.

| case 1 | case 2 | case 3 |
|---|---|---|
| $b_1 \leftarrow \mathcal{D}(y_1, y_2)$ where, | $b_2 \leftarrow \mathcal{D}(y_1, y_2)$ where, | $b_3 \leftarrow \mathcal{D}(y_1, y_2)$ where, |
| $k_1 \leftarrow \mathsf{Gen}_{\mathsf{F}}(1^n)$ | $x_1 \leftarrow \mathcal{D}(1^n)$ | $y_1 \leftarrow_{\$} \{0,1\}^n$ |
| $k_2 \leftarrow \mathsf{Gen}_{\mathsf{G}}(1^n)$ | $y_1 \leftarrow \mathsf{R}^n(x_1)$ | $y_2 \leftarrow_{\$} \{0,1\}^n$ |
| $x_1 \leftarrow \mathcal{D}(1^n)$ | $x_2 \leftarrow \mathcal{D}(y_1)$ | |
| $y_1 \leftarrow \mathsf{F}_{k_1}(x_1) \oplus \mathsf{G}_{k_2}(x_1)$ | $y_2 \leftarrow \mathsf{R}^n(x_2)$ | |
| $x_2 \leftarrow \mathcal{D}(y_1)$ | | |
| $y_2 \leftarrow \mathsf{F}_{k_1}(x_2) \oplus \mathsf{G}_{k_2}(x_2)$ | | |

$\mathcal{D}$ is a 2-adaptive distinguisher for $\mathsf{F} \oplus \mathsf{G}$. This implies that there exists a non-negligible function $\epsilon(n)$ for $n$ such that

$$|Pr[b_1 = 1] - Pr[b_2 = 1]| \geq \epsilon(n). \tag{9}$$

Since $\mathsf{R}^n$ is a uniformly random function from $\{0,1\}^n$ to $\{0,1\}^n$, for any $x_1$ and $x_2 \in \{0,1\}^n$, the distribution of $y_1 \leftarrow \mathsf{R}^n(x_1)$ and $y_2 \leftarrow \mathsf{R}^n(x_2)$ is equivalent to the uniform random of length $n$. In other words, the distribution of $b_2$ and $b_3$ are equivalent to one another:

$$|Pr[b_2 = 1] - Pr[b_3 = 1]| = 0. \tag{10}$$

Consider one more case described as follows.

$$\begin{aligned}
&b_4 \leftarrow \mathcal{D}(y_1, y_2) \text{ where,}\\
&\quad k_1 \leftarrow \mathsf{Gen}_{\mathsf{G}}(1^n)\\
&\quad x_1 \leftarrow \mathcal{D}(1^n)\\
&\quad z_1 \leftarrow_{\$} \{0,1\}^n\\
&\quad y_1 \leftarrow z_1 \oplus \mathsf{G}_{k_1}(x_1)\\
&\quad x_2 \leftarrow \mathcal{D}(1^n)\\
&\quad z_2 \leftarrow_{\$} \{0,1\}^n\\
&\quad y_2 \leftarrow z_2 \oplus \mathsf{G}_{k_1}(x_2)
\end{aligned}$$

For any $x$ and key $k$ in $\{0,1\}^n$, the distribution of $y$ is uniform if $z \leftarrow_{\$} \{0,1\}^n$ and $y \leftarrow z \oplus \mathsf{G}_k(x)$. Thus, the distribution of $y_1$ and $y_2$ are uniform, which implies that the distribution of $b_4$ and $b_3$ are equal to each other. Hence, by (10),

$$|Pr[b_2 = 1] - Pr[b_3 = 1]| = |Pr[b_2 = 1] - Pr[b_4 = 1]| = 0. \tag{11}$$

Finally, we have

$$\begin{aligned}
Pr[b_{\mathsf{A}} = b_{\mathsf{B}}] &= Pr[b_{\mathsf{A}} = 1] \cdot Pr[b_{\mathsf{B}} = 1 : b_{\mathsf{A}} = 1] + Pr[b_{\mathsf{A}} = 0] \cdot Pr[b_{\mathsf{B}} = 0 : b_{\mathsf{A}} = 0]\\
&= \frac{1}{2} \cdot (1 - Pr[b_1 = 1]) + \frac{1}{2} \cdot Pr[b_4 = 1]\\
&= \frac{1}{2} \cdot (1 + (Pr[b_4 = 1] - Pr[b_1 = 1]))\\
&\le \frac{1}{2} \cdot (1 + \epsilon(n)) \qquad \text{by (9) and (11)}\\
&= \frac{1}{2} + \frac{\epsilon(n)}{2}.
\end{aligned}$$

Therefore, the protocol Bit-Agreement($1^n$) has non-negligible correlation in $\epsilon(n)$. □

**Claim 4.2** (Security with overwhelming probability $\delta$)**.** *The protocol Bit-Agreement*($1^n$) *has overwhelming $\delta$.*

*Proof.* We want to show that there exists an overwhelming function $\delta(n)$ such that for all efficient distinguishers $\mathcal{D} \in$ PPT and $\mathcal{D}$'s own randomness $r$,

$$\Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}} : b_{\mathsf{A}} = b_{\mathsf{B}}] \le 1 - \frac{\delta(n)}{2}.$$

Since

$$\Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}}] = \Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}} : b_{\mathsf{A}} = b_{\mathsf{B}}] + \Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}} : b_{\mathsf{A}} \neq b_{\mathsf{B}}],$$

we have

$$\Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}} : b_{\mathsf{A}} = b_{\mathsf{B}}] \le \Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}}].$$

Hence, it suffices to show that there exists an overwhelming $\delta(n)$ such that

$$\Pr[\mathcal{D}_r(z_1, y_1, z_2) \to b_{\mathsf{A}}] \le 1 - \frac{\delta(n)}{2}.$$

Consider the following five cases, which define the distributions of the transcript triplet $(z_1, y_1, z_2)$.

| case 1 | case 2 | case 3 | case 4 | case 5 |
|---|---|---|---|---|
| $k_1 \leftarrow \mathsf{Gen}_\mathsf{F}(1^n)$ | $k_1 \leftarrow \mathsf{Gen}_\mathsf{F}(1^n)$ | $x_1 \leftarrow \mathcal{D}(1^n)$ | | $k_2 \leftarrow \mathsf{Gen}_\mathsf{G}(1^n)$ |
| $k_2 \leftarrow \mathsf{Gen}_\mathsf{G}(1^n)$ | | $z_1 \leftarrow \mathsf{R}^n(x_1)$ | $x_1 \leftarrow \mathcal{D}(1^n)$ | $x_1 \leftarrow \mathcal{D}(1^n)$ |
| $x_1 \leftarrow \mathcal{D}(1^n)$ | $x_1 \leftarrow \mathcal{D}(1^n)$ | $y_1 \leftarrow z_1 \oplus \mathsf{R}^n(x_1)$ | $z_1 \leftarrow_\$ \{0,1\}^n$ | $z_1 \leftarrow_\$ \{0,1\}^n$ |
| $z_1 \leftarrow \mathsf{F}_{k_1}(x_1)$ | $z_1 \leftarrow \mathsf{F}_{k_1}(x_1)$ | $x_2 \leftarrow \mathcal{D}(y_1)$ | $y_1 \leftarrow z_1 \oplus \mathsf{R}^n(x_1)$ | $y_1 \leftarrow z_1 \oplus \mathsf{G}_{k_2}(x_1)$ |
| $y_1 \leftarrow z_1 \oplus \mathsf{G}_{k_2}(x_1)$ | $y_1 \leftarrow z_1 \oplus \mathsf{R}^n(x_1)$ | $z_2 \leftarrow \mathsf{R}^n(x_2)$ | $x_2 \leftarrow \mathcal{D}(y_1)$ | $x_2 \leftarrow \mathcal{D}(y_1)$ |
| $x_2 \leftarrow \mathcal{D}(y_1)$ | $x_2 \leftarrow \mathcal{D}(y_1)$ | | $z_2 \leftarrow_\$ \{0,1\}^n$ | $z_2 \leftarrow_\$ \{0,1\}^n$ |
| $z_2 \leftarrow \mathsf{F}_{k_1}(x_2)$ | $z_2 \leftarrow \mathsf{F}_{k_1}(x_2)$ | | | |

We define $(z_1, y_1, z_2)_i$ to be the transcript triplet from the $i$th case. Then, $\epsilon_{ij}$ is defined as

$$|\Pr[\mathcal{D}(z_1, y_1, z_2)_i \to 1] - \Pr[\mathcal{D}(z_1, y_1, z_2)_j \to 1]| = \epsilon_{ij}.$$

By the non-adaptive security of $\mathsf{G}$, $\epsilon_{12}$ is negligible. Also, $\epsilon_{23}$ is negligible due to the non-adaptive security of $\mathsf{F}$. As we have seen in the proof of claim 4.1, case 3 is equivalent to case 4. Hence, $\epsilon_{34} = 0$. The non-adaptive security of $\mathsf{G}$ implies that $\epsilon_{45}$ is negligible. Then we have, by triangle inequality,

$$|\Pr[\mathcal{D}(z_1, y_1, z_2)_5 \to 1] - \Pr[\mathcal{D}(z_1, y_1, z_2)_1] \to 1| \leq \sum_{i=1}^{4} \epsilon_{i(i+1)}$$

$$= \epsilon_{12} + \epsilon_{23} + \epsilon_{34} + \epsilon_{45} = \epsilon_{12} + \epsilon_{23} + \epsilon_{45} \overset{def}{=} \epsilon. \qquad (12)$$

Since $\epsilon_{12}, \epsilon_{23}$, and $\epsilon_{45}$ are negligible, $\epsilon$ is negligible in $n$. We define $\delta$ to be $1 - \epsilon$. It is easy to see that $\delta$ is overwhelming in $n$ since $\epsilon$ is negligible. Finally, we complete the proof of claim 4.2 as

$$\begin{aligned}
&\Pr[\mathcal{D}(z_1, y_1, z_2) \to b_\mathsf{A}] \\
&= \Pr[b_\mathsf{A} = 0] \cdot \Pr[\mathcal{D}(z_1, y_1, z_2) \to 0 : b_\mathsf{A} = 0] + \Pr[b_\mathsf{A} = 1] \cdot \Pr[\mathcal{D}(z_1, y_1, z_2) \to 1 : b_\mathsf{A} = 1] \\
&= \frac{1}{2} \cdot (\Pr[\mathcal{D}(z_1, y_1, z_2) \to 0 : b_\mathsf{A} = 0] + \Pr[\mathcal{D}(z_1, y_1, z_2) \to 1 : b_\mathsf{A} = 1]) \\
&= \frac{1}{2} \cdot (1 - \Pr[\mathcal{D}(z_1, y_1, z_2) \to 1 : b_\mathsf{A} = 0] + \Pr[\mathcal{D}(z_1, y_1, z_2) \to 1 : b_\mathsf{A} = 1]) \\
&= \frac{1}{2} \cdot (1 + (\Pr[\mathcal{D}(z_1, y_1, z_2)_5 \to 1] - \Pr[\mathcal{D}(z_1, y_1, z_2)_1 \to 1])) \\
&\leq \frac{1}{2} \cdot (1 + \epsilon) \qquad \text{by (12)} \\
&= \frac{1}{2} \cdot (1 + (1 - \delta)) \\
&= 1 - \frac{\delta(n)}{2}.
\end{aligned}$$

□

**Claim 4.3** (Uniform-transcript)**.** *The protocol Bit-Agreement*$(1^n)$ *is a uniform-transcript bit agreement.*

*Proof.* Notice that the value of $y_1$ is related to the value of $z_1$ in the protocol, and as long as $z_1$ is uniformly random, $y_1$ is also uniformly random. Let $\mathsf{R}_i$ denote a uniformly random vector in $(\{0,1\}^n)^i$. Therefore, we want to show the following inequality for any PPT adversary (distinguisher) $\mathcal{A}$.

$$\left| \Pr_{b_A \leftarrow_\$ \{0,1\}}[\mathcal{A}_r(\mathsf{z}_1, \mathsf{y}_1 \mathsf{z}_2) = 1] - \Pr_{b_A \leftarrow_\$ \{0,1\}}[\mathcal{A}_r(\mathsf{R}_3) = 1 : \mathsf{R}_3 \leftarrow_\$ (\{0,1\}^n)^3] \right| \leq \epsilon(n) \tag{13}$$

Then we define two games in which adversary $\mathcal{A}$ distinguishes a transcript from uniform random. Game $\mathcal{G}_0$ is for an adversary to distinguish $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ while $b_\mathsf{A} = 0$, and game $\mathcal{G}_1$ is for an adversary to distinguish $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ while $b_\mathsf{A} = 1$. We denote the advantage of adversary $\mathcal{A}$ in $\mathcal{G}_0$ and $\mathcal{G}_1$ as $\mathbf{Adv}_{\mathcal{G}_0}$ and $\mathbf{Adv}_{\mathcal{G}_1}$, respectively. Winning $\mathcal{G}_0$ (or $\mathcal{G}_1$) means that one distinguishes $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ with a non-negligible advantage in $\mathcal{G}_0$ (or $\mathcal{G}_1$). Finally, suppose that there exists a PPT adversary $\mathcal{A}$ that wins either of the games.

In game $\mathcal{G}_1$, since both $z_1$ and $z_2$ are randomly chosen from $\{0,1\}^n$, distinguishing $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ is equivalent to distinguishing $(r_1, G(r_2), r_3)$ from $\mathsf{R}_3$. Hence, if $\mathcal{A}$ wins $\mathcal{G}_1$, it also distinguishes $\mathsf{G}$ from uniform random functions with non-negligible advantage. This is clearly impossible by the non-adaptive security of $\mathsf{G}$.

Assume that $\mathcal{A}$ wins game $\mathcal{G}_0$ by distinguishing, with non-negligible probability,

$$Dist(0) : (z_1, y_1, z_2) = (\mathsf{F}(r), \mathsf{F}(r) \oplus \mathsf{G}(r), \mathsf{F}(\mathcal{D}(\mathsf{F}(r) \oplus \mathsf{G}(r))) \oplus \mathsf{D}(\mathsf{G}(\mathsf{F}(r) \oplus \mathsf{G}(r))))$$

from a uniform random triplet $\mathsf{R}_3$ for $r \leftarrow_\$ \{0,1\}^n$. Consider the following distributions.

$$Dist(1) : (\mathsf{F}(r_1), r_2, \mathsf{F}(\mathcal{D}(\mathsf{F}(r_1))) \oplus \mathsf{G}(\mathcal{D}(\mathsf{F}(r_1))))$$

for $r_1$ and $r_2 \leftarrow_\$ \{0,1\}^n$.

$$Dist(2) : (r_1, r_2, \mathsf{F}(\mathcal{D}(r_1 \oplus r_2)) \oplus \mathsf{G}(\mathcal{D}(r_1 \oplus r_2)))$$

for $r_1$ and $r_2 \leftarrow_\$ \{0,1\}^n$.

$$Dist(3) : (r_1, r_2, r_3 \oplus r_4) = (r_1, r_2, r_5) = \mathsf{R}_3$$

where $r_5 = r_3 \oplus r_4$ for $r_1$, $r_2$, $r_3$ and $r_4 \leftarrow_\$ \{0,1\}^n$.

Since $\mathcal{A}$ distinguishes Dist(0) from Dist(3) with non-negligible advantage, $\mathcal{A}$ distinguishes Dist($i$) from Dist($j$) for $i \neq j$ with non-negligible probability. This is clearly a contradiction since the above distributions only negligibly deviate from each other by the non-adaptive security of both $\mathsf{F}$ and $\mathsf{G}$.

Therefore, the advantage of $\mathcal{A}$ to distinguish the transcript from $\mathsf{R}_3$ is $\mathbf{Adv}_{\mathcal{G}_0}/2 + \mathbf{Adv}_{\mathcal{G}_1}/2$ which is negligible since both $\mathbf{Adv}_{\mathcal{G}_0}$ and $\mathbf{Adv}_{\mathcal{G}_1}$ are negligible. This directly validates the inequality (13). □

With Claims 4.1 and 4.2, we showed that the protocol Bit-Agreement($1^n$) has non-negligible correlation $\epsilon$ and is secure with overwhelming probability $\delta$. Also, we show the indistinguishability of messages from randoms with Claim 4.3. Thus, this implies the existence of a uniform-transcript key agreement. We eventually show that the $(k-1)$-adaptively secure parallel composition implies the existence of the $(2k-1)$-pass uniform-transcript key agreement for the case $k = 2$ with respect to a 2-adaptive distinguisher. We will generalize the same technique to the arbitrary $k$.

We just showed that the $(k-1)$-adaptively secure parallel composition implies the $(2k-1)$-pass uniform-transcript bit agreement for the case $k = 2$. However, notice that the 2-adaptive distinguisher $\mathcal{D}$, which we use to build Bit-Agreement($1^n$), is not a general 2-adaptive distinguisher since $\mathcal{D}$ makes two adaptive queries. However, it is easy to see that we can construct the same 3-pass bit agreement protocol based on a general 2-adaptive distinguisher denoted by $\mathcal{D}_q$, where $q$ is the any polynomial size of blocks queried by the distinguisher. Then, we build the same 3-pass uniform-transcript bit-agreement by replacing $x_1, x_2, y_1, y_2, z_1, z_2$ with $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in the

```
                    Protocol Bit-Agreement$(1^n)$
          Alice                          Transcript     Bob
$b_A \leftarrow_\$ \{0,1\}^n$
$k_A \leftarrow Gen_F(1^n)$                             $k_B \leftarrow Gen_G(1^n)$
FOR $i = 1$ to $k-1$ DO
    $X_i \leftarrow \mathcal{D}_q^k(Y_1, Y_2, \cdots, Y_{i-1})$
    If $b_A = 0$,
        then $Z_i \leftarrow F_{k_A}(X_i)$                   $X_i \leftarrow \mathcal{D}_q^k(Y_1, Y_2, \cdots, Y_{i-1})$
        else $Z_i \leftarrow_\$ (\{0,1\}^n)^q$   $\xrightarrow{Z_i}$
                                         $\xleftarrow{Y_i}$     $Y_i \leftarrow Z_i \oplus G_{k_B}(X_i)$
ENDFOR
    $X_k \leftarrow \mathcal{D}_q^k(Y_1, Y_2, \cdots, Y_{k-1})$
    If $b_A = 0$,
        then $Z_k \leftarrow F_{k_A}(X_k)$                   $X_k \leftarrow \mathcal{D}_q^k(Y_1, Y_2, \cdots, Y_{k-1})$
        else $Z_k \leftarrow_\$ (\{0,1\}^n)^q$   $\xrightarrow{Z_k}$     $Y_k \leftarrow Z_k \oplus G_{k_B}(X_k)$
                                                        $b_B \leftarrow \mathcal{D}_q^k(Y_1, Y_2, \cdots, Y_k)$
```

**Protocol 3:** $(2k-1)$-pass uniform-transcript bit agreement based on a general $k$-adaptive distinguisher

Bit-Agreement$(1^n)$, where $X_i, Y_i, Z_i$ are $q$ tuples. That is, $x_i = (x_{1i}, x_{2i}, \cdots, x_{qi})$ for $i$. Now we are ready to generalize the construction of Bit-Agreement$(1^n)$ to the arbitrary $k \geq 2$. Denote $\mathcal{D}_q^k$ as a $k$-adaptive distinguisher with the query block of size $q$. Then, $X_i, Y_i, Z_i$ are defined by $q$-tuples for all $i$ as before. We provide the construction in Protocol 3. Obviously, we can extend the arguments of Claim 4.1, 4.2 and 4.3 to the $(2k-1)$-pass uniform transcript bit-agreement. To prove that the above bit agreement is secure with overwhelming probability, only the number of intermediate cases between the distributions of transcripts is increased according to the increased number of rounds. This completes the proof of Theorem 7. □

### 4.1.2 Intuitions of Adaptively Insecure Parallel Composition of F and G under UTKA

A $\gamma$-round uniform-transcript key agreement protocol ($\gamma$-UTKA), denoted by $\Phi_u^\gamma = (\mathsf{A}, \mathsf{B})$, is a uniform-transcript key agreement protocol consisting of two sub-protocols A and B, in which Alice (using A) and Bob (using B) exchange $2\gamma$ messages to each other ($\gamma$ messages from each party) in order to share a secret key $sk$.

We first provide the intuitive description of the parallel version of $\gamma$-UTKA in Protocol 4, which we will use to construct counter-example functions. Notice that Alice and Bob are *symmetric* to each other in Protocol 4. In particular, Bob's first message is completely independent of Alice's first message and is only dependent on his own private randomness.[2]

Now, we provide a high-level overview of our pseudo-random functions F and G from $\gamma$-UTKA

[2]The main reason for using this parallel version of $\gamma$-UTKA is that it is easier to emulate the key agreement protocol in the context of parallel composition of our proposed counter-example pseudo-random functions F and G. Also, it provides us with a tighter bound on the number of adaptive queries required to break the adaptive security of the parallel composition. It is possible to construct the counter-example functions to show the same composition insecurity result by using $\gamma - UTKA$ in which Bob's first message is *dependent* on Alice's first message. However, it requires more adaptive queries to break the parallel composition of such functions.

| Alice | Transcript | Bob |
|---|---|---|
| $r_{\mathsf{A}} \leftarrow_{\$} \{0,1\}^n$ | | $r_{\mathsf{B}} \leftarrow_{\$} \{0,1\}^n$ |
| $\alpha_1 \leftarrow \mathsf{A}_1(r_{\mathsf{A}})$ | | $\beta_1 \leftarrow \mathsf{B}_1(r_{\mathsf{B}})$ |
| | $\xrightarrow{\alpha_1}$ | |
| | $\xleftarrow{\beta_1}$ | |
| $\alpha_2 \leftarrow \mathsf{A}_2(r_{\mathsf{A}}, \beta_1)$ | | $\beta_2 \leftarrow \mathsf{B}_2(r_{\mathsf{B}}, \alpha_1)$ |
| | $\xrightarrow{\alpha_2}$ | |
| | $\xleftarrow{\beta_2}$ | |
| | $\vdots$ | |
| $\alpha_\gamma \leftarrow \mathsf{A}_\gamma(r_{\mathsf{A}}, \beta_1, \cdots, \beta_{\gamma-1})$ | | $\beta_\gamma \leftarrow \mathsf{B}_\gamma(r_{\mathsf{B}}, \alpha_1, \cdots, \alpha_{\gamma-1})$ |
| | $\xrightarrow{\alpha_\gamma}$ | |
| | $\xleftarrow{\beta_\gamma}$ | |
| secret key $sk \leftarrow \mathsf{A}_{\gamma+1}(r_{\mathsf{A}}, \beta_1, \cdots, \beta_\gamma)$ | | secret key $sk \leftarrow \mathsf{B}_{\gamma+1}(r_{\mathsf{B}}, \alpha_1, \cdots, \alpha_\gamma)$ |

**Protocol 4:** Parallel version of $\gamma$-UTKA

and describe how to break the adaptive security of their parallel composition. For underlying primitives, we have a black-box access to $\Phi_u = (\mathsf{A}, \mathsf{B})$, $\gamma$-UTKA described in Protocol 2. $\alpha_i$ and $\beta_i$ denote the $i$th message computed by $\mathsf{A}$ and $\mathsf{B}$ respectively. We are give a pseudo-random private-key encryption scheme $(\mathsf{Enc}, \mathsf{Dec})$ such that $\mathsf{Dec}_k(\mathsf{Enc}_k(x)) = x$. Finally, let $\mathsf{P}$ be any given adaptively secure PRP.

Intuitively, $\mathsf{F}$ utilizes $\mathsf{A}$ as its subroutine as well as $\mathsf{G}$ utilizes $\mathsf{B}$ as its subroutine in order for them to share a secret key via input and outputs. Then, $\mathsf{F}$ and $\mathsf{G}$ create a specially related pseudo-random strings with respect to the shared secret key. As we input the specially related pseudo-random strings to the parallel composition, the functions retrieve the shared key, verify the special relation hidden in the input query, and reveal their secret keys in their outputs. $\mathsf{F}$ and $\mathsf{G}$ internally contain secret keys $k_{\mathsf{F}}$ and $k_{\mathsf{G}}$. $\mathsf{F}$ and $\mathsf{G}$ are defined over $(\{0,1\}^n)^{\gamma+2}$.

First, we define $\mathsf{F}$ and $\mathsf{G}$ upon the first adaptive (fixed) query $Q_1 = (0^n, 0^n, \cdots, 0^n)$ as:

- $\mathsf{F}$ generates $\gamma + 2$ pseudo-random strings $r_{\mathsf{F}}$, $r_{21}$, $\cdots$ , $r_{(\gamma+2)1}$ by $\mathsf{P}_{k_{\mathsf{F}}}(Q_1)$. $\mathsf{F}$ creates Alice's first message $\alpha_1$ by $\mathsf{A}_1(r_{\mathsf{F}})$ and then outputs $(\alpha_1, r_{21}, \cdots, r_{(\gamma+2)1})$.
- $\mathsf{G}$ does the same as it generates $s_{\mathsf{G}}$, $s_{21}$, $\cdots$ , $s_{(\gamma+2)1}$ by $\mathsf{P}_{k_{\mathsf{F}}}(Q_1)$, and then computes Bob's first message $\beta_1$ by $\mathsf{B}_1(s_{\mathsf{G}})$, and outputs $(\beta_1, s_{21}, \cdots, s_{(\gamma+2)1})$.

Below we depict the individual outputs of $\mathsf{F}$ and $\mathsf{G}$ on $Q_1$ and their parallel composition:

$$Q_1 \rightarrow \begin{bmatrix} \mathsf{F} \rightarrow (\alpha_1, r_{21}, \cdots, r_{(\gamma+2)1}) \\ \mathsf{G} \rightarrow (\beta_1, s_{21}, \cdots, s_{(\gamma+2)1}) \end{bmatrix} \rightarrow (\alpha_1 \oplus \beta_1, r_{21} \oplus s_{21}, \cdots, r_{(\gamma+2)1} \oplus s_{(\gamma+2)1})$$

Inductively, for $2 \leq i \leq \gamma$, we define $\mathsf{F}$ and $\mathsf{G}$ to process the $i$-th adaptive query $Q_i = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_{i-1} \oplus \beta_{i-1}, 0^n, \cdots, 0^n)$ as follows.

- $\mathsf{F}$ first regenerates $r_{\mathsf{F}}$ and $\alpha_1$ by simulating the first-round computation. That is, $\mathsf{F}$ first computes $\mathsf{P}_{k_{\mathsf{F}}}(Q_1)$ to obtain $r_{\mathsf{F}}$ and then executes $\mathsf{A}(r_{\mathsf{F}})$. Then, $\mathsf{F}$ processes the following *chain of computations* in the direction of left-to-right and top-to-bottom with $r_{\mathsf{F}}$, $\alpha_1$ and $Q_i$,

$$\begin{array}{ll} \beta_1 \leftarrow (\alpha_1 \oplus u_1) & \alpha_2 \leftarrow \mathsf{A}_2(r_{\mathsf{F}}, \beta_1) \\ \beta_2 \leftarrow (\alpha_2 \oplus u_2) & \alpha_3 \leftarrow \mathsf{A}_3(r_{\mathsf{F}}, \beta_1, \beta_2) \\ \vdots & \vdots \\ \beta_{i-1} \leftarrow (\alpha_{i-1} \oplus u_{i-1}) & \alpha_i \leftarrow \mathsf{A}_i(r_{\mathsf{F}}, \beta_1, \beta_2, \ldots, \beta_{i-1}) \end{array}$$

Finally, $\mathsf{F}$ outputs ($\alpha_i$, $r_{2i}$, $\cdots$ , $r_{(\gamma+2)i}$) where $r_{2i}$, $\cdots$ , $r_{(\gamma+2)i}$ are fresh pseudo-random strings generated by $\mathsf{P}_{k_{\mathsf{F}}}(Q_i)$.

- $\mathsf{G}$ is symmetrically defined. That is, we have the description $\mathsf{G}$ on $Q_i$ by replacing all of $\mathsf{F}$, $r$, $\mathsf{A}$, and $\alpha$ in the above description with $\mathsf{G}$, $s$, $\mathsf{B}$, and $\beta$. Hence, $\mathsf{G}$ outputs ($\beta_i$, $s_{2i}$, $\cdots$ , $s_{(\gamma+2)i}$) where $s_{2i}$, $\cdots$ , $s_{(\gamma+2)i}$ are generated by $\mathsf{P}_{k_{\mathsf{G}}}(Q_i)$.

On $Q_i$ for $2 \le i \le \gamma$, we demonstrate the individual outputs of $\mathsf{F}$ and $\mathsf{G}$ and the output of their parallel composition below.

$$Q_i \to \begin{bmatrix} \mathsf{F} \to (\alpha_i, r_{2i}, \cdots, r_{(\gamma+2)i}) \\ \mathsf{G} \to (\beta_i, s_{2i}, \cdots, s_{(\gamma+2)i}) \end{bmatrix} \to (\alpha_i \oplus \beta_i, r_{2i} \oplus s_{2i}, \cdots, r_{(\gamma+2)i} \oplus s_{(\gamma+2)i})$$

Hence, we obtain $\alpha_\gamma \oplus \beta_\gamma$ by feeding the parallel composition of $\mathsf{F}$ and $\mathsf{G}$ with $Q_\gamma$ to be $(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_{\gamma-1} \oplus \beta_{\gamma-1}, 0^n, 0^n)$.

The $(\gamma+1)$th adaptive query is defined to be $Q_{\gamma+1} = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, 0^n, 0^n)$. Then, we define our functions $\mathsf{F}$ and $\mathsf{G}$ on $Q_{\gamma+1}$ as follows.

- $\mathsf{F}$ first regenerates $r_{\mathsf{F}}$ and $\alpha_1$ by simulating the first-round computation as before. Then, $\mathsf{F}$ performs the chain of computations described above, and so obtains $\beta_1, \beta_2, \cdots, \beta_\gamma$. Hence, $\mathsf{F}$ can generate a shared key $sk$ by $\mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, \beta_1, \beta_2, \ldots, \beta_\gamma)$. $\mathsf{F}$ generates pseudo-random strings $r_{1(\gamma+1)}$, $r_{2(\gamma+1)}$, $\cdots$, $r_{(\gamma+2)(\gamma+1)}$ by $\mathsf{P}_{k_{\mathsf{F}}}(Q_{\gamma+1})$. $\mathsf{F}$ creates an (pseudo-random) encryption $\mathsf{Enc}_{sk}(r_{1(\gamma+1)})$. Finally, $\mathsf{F}$ outputs ($\mathsf{Enc}_{sk}(r_{1(\gamma+1)})$, $r_{1(\gamma+1)}$, $r_{3(\gamma+1)}$, $\cdots$, $r_{(\gamma+2)(\gamma+1)}$).
- $\mathsf{G}$ is symmetrically defined. So, $\mathsf{G}$ outputs ($\mathsf{Enc}_{sk}(s_{1(\gamma+1)})$, $s_{1(\gamma+1)}$, $s_{3(\gamma+1)}$, $\cdots$, $s_{(\gamma+2)(\gamma+1)}$).

The following describes the each output of $\mathsf{F}$ and $\mathsf{G}$, and that of parallel composition on $Q_{\gamma+1}$.

$$Q_{\gamma+1} \to \begin{bmatrix} \mathsf{F} \to (\mathsf{Enc}_{sk}(r_{1(\gamma+1)}), r_{1(\gamma+1)}, r_{3(\gamma+1)}, \cdots, r_{(\gamma+2)(\gamma+1)}) \\ \mathsf{G} \to (\mathsf{Enc}_{sk}(s_{1(\gamma+1)}), s_{1(\gamma+1)}, s_{3(\gamma+1)}, \cdots, s_{(\gamma+2)(\gamma+1)}) \end{bmatrix}$$

$$\to (\mathsf{Enc}_{sk}(r_{1(\gamma+1)}) \oplus \mathsf{Enc}_{sk}(s_{1(\gamma+1)}), r_{1(\gamma+1)} \oplus s_{1(\gamma+1)}, r_{3(\gamma+1)} \oplus s_{3(\gamma+1)}, \cdots, r_{(\gamma+2)(\gamma+1)} \oplus s_{(\gamma+2)(\gamma+1)})$$

The final $(\gamma+2)$th adaptive query is defined to be $Q_{\gamma+2} = (\alpha_1 \oplus \beta_1, \cdots, \alpha_\gamma \oplus \beta_\gamma, \mathsf{Enc}_{sk}(r_{1(\gamma+1)}) \oplus \mathsf{Enc}_{sk}(s_{1(\gamma+1)}), r_{1(\gamma+1)} \oplus s_{1(\gamma+1)})$ which is the combination of all the outputs of the parallel composition on the previous adaptive queries. Then, $\mathsf{F}$ and $\mathsf{G}$ are defined on $Q_{\gamma+2}$ as follows.

- $\mathsf{F}$ executes the chain of computations to retrieve $\beta_1, \beta_2, \cdots, \beta_\gamma$, then computes a shared key $sk$ by $\mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, \beta_1, \beta_2, \ldots, \beta_\gamma)$. Since $Q_{\gamma+1} = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, 0^n, 0^n)$, $\mathsf{F}$ can obtain $\mathsf{Enc}_{sk}(r_{1(\gamma+1)})$ and $r_{1(\gamma+1)}$ generated by the internal *simulation* of $\mathsf{F}(Q_{\gamma+1})$. $\mathsf{F}$ checks to see if equality $\mathsf{Dec}_{sk}(\mathsf{Enc}_{sk}(r_{1(\gamma+1)}) \oplus (\mathsf{Enc}_{sk}(r_{1(\gamma+1)}) \oplus \mathsf{Enc}_{sk}(s_{1(\gamma+1)}))) = r_{1(\gamma+1)} \oplus (r_{1(\gamma+1)} \oplus s_{1(\gamma+1)})$ holds where $(\mathsf{Enc}_{sk}(r_{1(\gamma+1)}) \oplus \mathsf{Enc}_{sk}(s_{1(\gamma+1)}))$ and $(r_{1(\gamma+1)} \oplus s_{1(\gamma+1)})$ are obtained from $Q_{\gamma+2}$. As the equality holds, $\mathsf{F}$ is convinced that $Q_{\gamma+2}$ is indeed an adaptively generated query. Hence, $\mathsf{F}$ outputs ($k_{\mathsf{F}}$, $0^n$, $0^n$, $\cdots$, $0^n$).

- $\mathsf{G}$ is symmetrically defined. Hence, $\mathsf{G}$ similarly outputs $(0^n, k_{\mathsf{G}}, 0^n, \cdots, 0^n)$.

Below we provide the overall picture of the individual computations of $\mathsf{F}$ and $\mathsf{G}$ and the output of their parallel composition.

$$Q_{\gamma+2} \rightarrow \begin{bmatrix} \mathsf{F} \rightarrow (k_{\mathsf{F}}, 0^n, 0^n, \cdots, 0^n) \\ \mathsf{G} \rightarrow (0^n, k_{\mathsf{G}}, 0^n, \cdots, 0^n) \end{bmatrix} \rightarrow (k_{\mathsf{F}}, k_{\mathsf{G}}, 0^n, \cdots, 0^n)$$

### 4.1.3 Formal Construction of Non-Adaptively Secure Functions F and G

The underlying primitives used for the construction of $\mathsf{F}$ and $\mathsf{G}$ are two PRPs, $\tilde{\pi} : K \times (\{0,1\}^n)^{\gamma+2} \rightarrow (\{0,1\}^n)^{\gamma+2}$ and $\pi : K \times \{0,1\}^n \rightarrow \{0,1\}^n$, where $K$ is key space $\{0,1\}^n$. Also, we are given a black-box access to the parallel version of $\gamma$-UTKA denoted by $\Phi_u = (\mathsf{A}, \mathsf{B})$ described in Section 4.1.2.

We formally define our non-adaptively secure pseudo-random function $\mathsf{F}$ to map from $(\{0,1\}^n)^{\gamma+2}$ to $(\{0,1\}^n)^{\gamma+2}$. Furthermore, $\mathsf{F}$ internally possesses a secret key $k_{\mathsf{F}} \in K$. The formal definition of function $\mathsf{F}$ is provided in Algorithm 5.

The construction of $\mathsf{G}$ is symmetric to $\mathsf{F}$. That is, replacing $\mathsf{F}$, $\mathsf{A}$, $r$, and $(k_{\mathsf{F}}, 0^n, \cdots, 0^n)$ in Algorithm 5 with $\mathsf{G}$, $\mathsf{B}$, $s$, and $(0^n, k_{\mathsf{G}}, 0^n, \cdots, 0^n)$ respectively will provide us with the formal construction of $\mathsf{G}$.

**Claim 4.4.** *The functions $\mathsf{F}$ and $\mathsf{G}$ are secure against any non-adaptive PPT adversary $\mathcal{A}(q,t)$, running in time $t$ and making at most $q$ non-adaptive queries, where $t$ and $q$ are any polynomials of security parameter $n$.*

*Proof.* By hybrid argument, we reduce the security of function $\mathsf{F}$ to the indistinguishability of the underlying PRPs and the $\gamma$-UTKA. Let $\mathcal{A}$ be a PPT adversary making $q$ queries and running for time $t$. Notice that $\mathsf{F}$ and $\mathsf{G}$ are structurally identical to one another. Hence, it suffices to show that $\mathsf{F}$ is non-adaptively secure. To prove the claim, we will show the following inequality.

$$\mathbf{Adv}^{\mathsf{F}}_{\mathcal{A}}(q,t) \leq \mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t') + \mathbf{Adv}^{\pi}_{\mathcal{A}}(q,t') + \mathbf{Adv}^{\mathsf{A}}_{\mathcal{A}} + \frac{q}{2^n}.$$

First, assume that to obtain the secret key $k_{\mathsf{F}}$, $\mathcal{A}$ makes $q$ queries of the form in $(u_1 \neq 0^n, \cdots, u_{\gamma+2} \neq 0^n)$ evoking case 2 of Algorithm 5. Then, let $\mathcal{A}$ fix the first $\gamma$ queries to be the same through the $q$ queries. This implies that $sk_{\mathsf{F}}, b_1$, and $b_2$ are fixed in the condition $\pi^{-1}_{sk_{\mathsf{F}}}(b_1 \oplus u_{\gamma+1}) = b_2 \oplus u_{\gamma+2}$ in the case 2(c) of Algorithm 5. This only helps the adversary $\mathcal{A}$ by allowing it to choose $u_{\gamma+1}$ and $u_{\gamma+2}$ to satisfy the condition. Since $\pi$ is a permutation, there uniquely exists $u_{\gamma+1}$ for each $u_{\gamma+2}$, which satisfies the condition. Hence,

$$\Pr[\pi^{-1}_{k'}(\alpha \oplus u_{\gamma+1}) = r_1 \oplus u_{\gamma+2} : u_{\gamma+1}, u_{\gamma+2} \leftarrow_{\$} \{0,1\}^n] \leq \frac{1}{2^n}. \tag{14}$$

Since $\mathcal{A}$ makes $q$ queries, the probability of successfully finding a pair of such $u_{\gamma+1}$ and $u_{\gamma+2}$ is $q/2^n$, which constitutes the final term of the inequality (14).

Consider that $\mathcal{A}$ makes queries that fall into case 2(c), 2(d), and 3 of Algorithm 5. As we showed above, $q$ non-adaptive queries can satisfy the condition in 2(c) with only negligible probability. Thus, assume that all the queries made here do not satisfy the condition. Hence, $\mathsf{F}$ outputs, upon $q$ queries,

$$(r_{11}, r_{21}, \cdots, r_{(\gamma+2)1}), (r_{12}, r_{22}, \cdots, r_{(\gamma+2)2}), \cdots, (r_{1q}, r_{2q}, \cdots, r_{(\gamma+2)q}), \tag{15}$$

where $r_{ij}$ is the $i$th coordinate of the $j$th query for $1 \leq i \leq \gamma+2$ and $1 \leq j \leq q$.

**Construction of F**

```
1. If $I = (u_1 = 0^n, u_2 = 0^n, \cdots, u_{\gamma+2} = 0^n)$, then
       Output $(v_1, v_2, \cdots, v_{\gamma+2})$ where
       $(r_1, r_2, \cdots, r_{\gamma+2}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(0^n, 0^n, \cdots, 0^n)$
       $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (\mathsf{A}_1(r_1), r_2, \cdots, r_{\gamma+2})$

2. If $I = (u_1 \neq 0^n, \cdots, u_i \neq 0^n, u_{i+1} = 0^n, \cdots, u_{\gamma+2} = 0^n)$, then
       Output $(v_1, v_2, \cdots, v_{\gamma+2})$ where
       $(a_1, a_2, \cdots, a_{\gamma+2}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(0^n, 0^n, \cdots, 0^n)$
       $(r_1, r_2, \cdots, r_{\gamma+2}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(u_1, u_2, \cdots, u_{\gamma+2})$
       $\alpha_1 \leftarrow \mathsf{A}_1(a_1)$
       For $j = 1$ to $\gamma$ Do
           $\beta_j \leftarrow \alpha_j \oplus u_j$
           $\alpha_{j+1} \leftarrow \mathsf{A}_{j+1}(a_1, \beta_1, \cdots, \beta_j)$
       END For
       $sk_\mathsf{F} \leftarrow \alpha_{j+1}$

   (a) If $i < \gamma$, then
           $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (\alpha_{i+1}, r_2, \cdots, r_{\gamma+2})$

   (b) Else If $i = \gamma$, then
           $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (\pi_{sk_\mathsf{F}}(r_1), r_1, r_2, \cdots, r_{\gamma+1})$

   (c) Else If $i = \gamma + 2$, then
           $(b_1, b_2, \cdots, b_{\gamma+2}) \leftarrow \mathsf{F}(u_1, u_2, \cdots, u_\gamma, 0^n, 0^n)$
       i. If $\pi^{-1}_{sk_\mathsf{F}}(b_1 \oplus u_{\gamma+1}) = b_2 \oplus u_{\gamma+2}$, then
               $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (k_\mathsf{F}, 0^n, \cdots, 0^n)$
       ii. else $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (r_1, r_2, \cdots, r_{\gamma+2})$

   (d) Else $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow (r_1, r_2, \cdots, r_{\gamma+2})$

3. If the input is not any of above cases, then
       $(v_1, v_2, \cdots, v_{\gamma+2}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(u_1, u_2, \cdots, u_{\gamma+2})$
```

**Algorithm 5:** The algorithm of function $\mathsf{F}$

Proceeding by hybrid argument, assume that $\mathcal{A}$ distinguishes (15) from uniform distributions with a non-negligible probability $\xi$,

$$(r^*_{11}, r^*_{21}, \cdots, r^*_{(\gamma+2)1}), (r^*_{12}, r^*_{22}, \cdots, r^*_{(\gamma+2)2}), \cdots, (r^*_{1q}, r^*_{2q}, \cdots, r^*_{(\gamma+2)q}), \tag{16}$$

where $(r^*_{1i}, r^*_{2i}, \cdots, r^*_{(\gamma+2)i}) \leftarrow_\$ (\{0,1\}^n)^{\gamma+2}$ for $1 \le i \le q$.

Let $H_i$ be the hybrid distribution as

$$H_i = h_1, \cdots, h_i, h^*_{i+1}, \cdots, h^*_q$$

for $h_i = (r_{1i}, \cdots, r_{(\gamma+2)i})$ from (15) and $h^*_j = (r^*_{1j}, \cdots, r^*_{(\gamma+2)j})$ from (16).

Since $H_q - H_0 \ge \xi$, there exists $i$ such that $|H_j - H_{j-1}| \ge \frac{\xi}{q}$. Consider the $i$th hybrid,

$$H_i(x) = h_1, h_2, \cdots, h_{i-1}, x, h^*_{i+1}, \cdots, h^*_q.$$

Then we can build a PPT distinguisher $\mathcal{D}$ using $\mathcal{A}$ as a sub-routine. With an unknown distribution $u$ from $(\{0,1\})^{\gamma+2}$, $\mathcal{D}$ constructs and queries $H_i(u)$ to $\mathcal{A}$. Since $\mathcal{A}$ distinguishes $\mathsf{F}$ from a uniform random function with the probability $\xi$, $\mathcal{D}$ will distinguish $\tilde{\pi}$ from a uniform random function with the probability $\frac{\xi}{q}$, which is non-negligible.

Consider that $\mathcal{A}$ makes $q$ queries in the form of $Q_i = (u_{1i} \neq 0^n, u_{2i} \neq 0^n, \cdots, u_{\gamma i} \neq 0^n, 0^n, 0^n)$, that provoke case 2(b) of $\mathsf{F}$ in Algorithm 5. Then, the corresponding outputs can be written as,

$$(\pi_{k_1}(r_{11}), r_{11}, r_{31}, \cdots, r_{(\gamma+2)1}), (\pi_{k_2}(r_{12}), r_{12}, r_{32}, \cdots, r_{(\gamma+2)2}), \cdots, (\pi_{k_q}(r_{1q}), r_{1q}, r_{3q}, \cdots, r_{(\gamma+2)q}),$$

where $(r_{1i}, r_{2i}, \cdots, r_{(\gamma+2)i}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(Q_i)$ for $1 \le i \le q$ and $k_i$ is computed by $\mathsf{A}_{\gamma+1}(r, \mathsf{Trans}^\mathsf{A})$ for unknown $r$. distinguishing the above distribution from uniform random implies that one of the following two cases must be true. First, the adversary $\mathcal{A}$ distinguishes the last $\gamma+1$ coordinates (all coordinates except for the first coordinate) of the outputs from uniform random. Second, the first two coordinates from uniform random. Assume that the first case is true. Then, $\mathcal{A}$ can distinguish all the output of $\mathsf{F}$ upon queries of case 2(c)ii, 2(d) or 3 by simply ignoring the second coordinate of the outputs. This is clearly a contradiction to the non-adaptive security of $\mathsf{F}$ in those cases proven above. Assume that the second case holds. Distinguishing the first two coordinates of the outputs from uniform random is equivalent to distinguishing,

$$(\pi(r_1), r_1), (\pi(r_2), r_2), \cdots, (\pi(r_q), r_q), \tag{17}$$

where $r_i \leftarrow_\$ \{0,1\}^n$ for $1 \le i \le q$, from the uniform distribution,

$$(a_1, b_1), (a_2, b_2), \cdots, (a_q, b_q), \tag{18}$$

where $a_i$ and $b_i \leftarrow_\$ \{0,1\}^n$ for $1 \le i \le q$.

Suppose that $\mathcal{A}$ distinguishes (17) from (18) with a non-negligible probability $\xi$. Define a hybrid distribution $H_i$ as

$$H_i = (\pi(r_1), r_1), \cdots, (\pi(r_i), r_i), (\pi(r_{i+1}), b_{i+1}), \cdots, (\pi(r_q), b_q), \tag{19}$$

where $r_i$ and $b_i \leftarrow_\$ \{0,1\}^n$ for $1 \le i \le q$.

Since $\mathcal{A}$ distinguishes (17) from (18) with probability $\xi$, $|H_0 - H_q| \ge \xi$, there exists $i$ s.t. $|H_i - H_{i+1}| \ge \frac{\xi}{q}$. Hence, let $H_i(x)$ for $x \in (\{0,1\}^n)^2$ be

$$H_i(x) = (\pi(r_1), r_1), \cdots, (\pi(r_{i-1}), r_{i-1}), x, (\pi(r_{i+1}), b_{i+1}), \cdots, (\pi(r_q), b_q).$$

Upon an unknown distribution $(\alpha, \beta) \in (\{0,1\}^n)^2$, we can build a distinguisher $\mathcal{D}$, which determines whether $(\alpha, \beta)$ comes from (17) or from (18) with a non-negligible probability $\frac{\xi}{q}$ as $\mathcal{D}$ queries $H_i((\alpha, \beta))$ to $\mathcal{A}$. Therefore, $\mathcal{D}$ distinguishes $\pi$ from a uniform random function with a non-negligible probability.

Consider that $\mathcal{A}$ makes $q$ queries in the form of $(u_1 \neq 0^n, \cdots, u_i \neq 0^n, u_{i+1} = 0^n, \cdots, u_{\gamma+2} = 0^n)$ for $i < \gamma$. That is, all the $q$ queries fall into case 1 or 2(a) of Algorithm 5. By replacing $\pi$ in the above hybrid argument with $\mathsf{A}$, we can also show that if $\mathcal{A}$ breaks the indistinguishability of $\mathsf{F}$, then we can construct a PPT distinguisher $\mathcal{D}$, which breaks the indistinguishability of messages. $\square$

### 4.1.4 Adaptive Insecurity of Parallel Composition of F and G

**Claim 4.5.** *The parallel composition $\mathsf{F} \oplus \mathsf{G}$ is breakable by $\gamma+2$ adaptive queries.*

*Proof.* Let $r_{ij}$ denote the $i$th randomness upon the $j$th input. To initiate the key agreement, our first special input is $(0^n, 0^n, \cdots, 0^n)$. Then, $\mathsf{F}$ gets into case 1 of Algorithm 5 and computes its first message $\alpha_1$ and outputs $(\alpha_1, r_{21}, \cdots, r_{(\gamma+2)1})$. So does $\mathsf{G}$. Hence, the output of the parallel composition on the first input query is

$$(\alpha_1 \oplus \beta_1, r_{21} \oplus s_{21}, \cdots, r_{(\gamma+2)1} \oplus s_{(\gamma+2)1}).$$

Our second adaptive query is $Q_2 = (\alpha_1 \oplus \beta_1, 0^n, \cdots, 0^n)$, where $\alpha_1 \oplus \beta_1$ comes from the output of the parallel composition on $Q_1$. $Q_2$ evokes the case 2 of Algorithm 5 unless $\alpha_1 \oplus \beta_1 = 0^n$. The only case that $\alpha_1 \oplus \beta_1 = 0^n$ is $\alpha_1 = \beta_1$ which occurs with negligible probability. By the computations of "For" loop of case 2, $\mathsf{F}$ obtains $\alpha_1$, $\alpha_2$, $\cdots$, $\alpha_{\gamma+1}$ as follows.

$$\begin{array}{lll}
\alpha_1 \leftarrow \mathsf{A}_1(a_1) & \text{Before entering For loop} & \\
\beta_1 \leftarrow \alpha_1 \oplus u_1 = \alpha_1 \oplus (\alpha_1 \oplus \beta_1) & \alpha_2 \leftarrow \mathsf{A}_2(a_1, \beta_1) & j = 1 \\
\beta_2 \leftarrow \alpha_2 \oplus u_2 = \alpha_2 \oplus 0^n & \alpha_3 \leftarrow \mathsf{A}_3(a_1, \beta_1, \alpha_2) & j = 2 \\
\beta_3 \leftarrow \alpha_3 \oplus u_3 = \alpha_3 \oplus 0^n & \alpha_4 \leftarrow \mathsf{A}_4(a_1, \beta_1, \alpha_2, \alpha_3) & j = 3 \\
\vdots & \vdots & \vdots \\
\beta_\gamma \leftarrow \alpha_\gamma \oplus u_\gamma = \alpha_\gamma \oplus 0^n & \alpha_{\gamma+1} \leftarrow \mathsf{A}_{\gamma+1}(a_1, \beta_1, \alpha_2, \cdots, \alpha_\gamma) & j = \gamma
\end{array}$$

Since the first coordinate is the only non-zero coordinate, $Q_2$ evokes case (a) of case 2 after the For loop. Hence, function $\mathsf{F}$ outputs $(\alpha_1, r_{22}, r_{32}, \cdots, r_{(\gamma+2)2})$ where $(r_{12}, r_{22}, \cdots, r_{(\gamma+2)2})$ is generated by $\tilde{\pi}_{k_\mathsf{F}}(Q_2)$. $\mathsf{G}$ undertakes the identical course of computation, so it outputs $(\beta_2, , s_{22}, \cdots, s_{(\gamma+2)2})$.

Inductively, for $2 \leq i \leq \gamma$, our $i$th adaptive query to the parallel composition is defined as

$$Q_i = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_{i-1} \oplus \beta_{i-1}, 0^n, \cdots, 0^n) \tag{20}$$

where $\alpha_l \oplus \beta_l$ is obtained from the output of $\mathsf{F}$ on $Q_l$. Then, the output of the parallel composition on $Q_i$ is

$$(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_i \oplus \beta_i, r_{(i+1)i} \oplus s_{(i+1)i}, \cdots, r_{(\gamma+2)i} \oplus s_{(\gamma+2)i}) \tag{21}$$

where $(r_{1i}, \cdots, r_{(\gamma+2)i})$ is generated by $\tilde{\pi}_{k_\mathsf{F}}(Q_i)$ and $(s_{1i}, \ldots, s_{(\gamma+2)i})$ is generated by $\tilde{\pi}_{k_\mathsf{G}}(Q_i)$.

Hence, the $\gamma$th adaptive query (the final case of the above inductive cases) is $Q_\gamma = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_{\gamma-1} \oplus \beta_{\gamma-1}, 0^n, 0^n, 0^n)$ by (20), and the corresponding output is $(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, r_{(\gamma+1)i} \oplus s_{(\gamma+1)i}, r_{(\gamma+2)i} \oplus s_{(\gamma+2)i})$ by (21).

Now, we define our $(\gamma+1)$th adaptive query as $Q_{\gamma+1} = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, 0^n, 0^n)$ which evokes case 2. Again, $\mathsf{F}$ retrives $a_1$ by computing $\pi_{k_\mathsf{F}}(Q_1)$. $\mathsf{F}$ performs the computations of For loop as follows.

$$\begin{array}{lll}
\alpha_1 \leftarrow \mathsf{A}_1(a_1) & \text{Before entering For loop} & \\
\beta_1 \leftarrow \alpha_1 \oplus u_1 = \alpha_1 \oplus (\alpha_1 \oplus \beta_1) & \alpha_2 \leftarrow \mathsf{A}_2(a_1, \beta_1) & j=1 \\
\beta_2 \leftarrow \alpha_2 \oplus u_2 = \alpha_2 \oplus (\alpha_2 \oplus \beta_2) & \alpha_3 \leftarrow \mathsf{A}_3(a_1, \beta_1, \beta_2) & j=2 \\
\beta_3 \leftarrow \alpha_3 \oplus u_3 = \alpha_3 \oplus (\alpha_3 \oplus \beta_3) & \alpha_4 \leftarrow \mathsf{A}_4(a_1, \beta_1, \beta_2, \beta_3) & j=3 \\
\vdots & \vdots & \vdots \\
\beta_\gamma \leftarrow \alpha_\gamma \oplus u_\gamma = \alpha_\gamma \oplus (\alpha_\gamma \oplus \beta_\gamma) & \alpha_{\gamma+1} \leftarrow \mathsf{A}_{\gamma+1}(a_1, \beta_1, \beta_2, \cdots, \beta_\gamma) & j=\gamma
\end{array}$$

Since $Q_{\gamma+1}$'s first $\gamma$ coordinates are adaptively generated from the previous adaptive queries and non-zero with overwhelming probability, $\alpha_{\gamma+1}$ is a properly computed shared key, $sk_\mathsf{F}$ and $Q_{\gamma+1}$ evokes case 2(b). Thus, $\mathsf{F}$ outputs $(\pi_{sk_\mathsf{F}}(r_{1(\gamma+1)}), r_{1(\gamma+1)}, r_{2(\gamma+1)}, \cdots, r_{(\gamma+1)(\gamma+1)})$ where $(r_{1(\gamma+1)}, \cdots, r_{(\gamma+2)(\gamma+1)})$ is generated from $\tilde{\pi}_{k_\mathsf{F}}(Q_{\gamma+1})$. On $Q_{\gamma+1}$, $\mathsf{G}$ also performs the identical course of computations, so it outputs $(\pi_{sk_\mathsf{G}}(s_{1(\gamma+1)}), s_{1(\gamma+1)}, s_{2(\gamma+1)}, \cdots, s_{(\gamma+1)(\gamma+1)})$. Hence, the output of the parallel composition on $Q_{\gamma+1}$ is

$$(\pi_{sk_\mathsf{F}}(r_{1(\gamma+1)}) \oplus \pi_{sk_\mathsf{G}}(s_{1(\gamma+1)}), r_{1(\gamma+1)} \oplus s_{1(\gamma+1)}, r_{2(\gamma+1)} \oplus s_{2(\gamma+1)}, \cdots, r_{(\gamma+1)(\gamma+1)} \oplus s_{(\gamma+1)(\gamma+1)})$$

The final $(\gamma+2)$th adaptive query is defined as

$$Q_{\gamma+2} = (\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, \underbrace{\pi_{sk_\mathsf{F}}(r_{1(\gamma+1)}) \oplus \pi_{sk_\mathsf{G}}(s_{1(\gamma+1)})}_{u_{\gamma+1}}, \underbrace{r_{1(\gamma+1)} \oplus s_{1(\gamma+1)}}_{u_{\gamma+2}}). \tag{22}$$

On $Q_{\gamma+2}$, $\mathsf{F}$ obtains the shared key $sk_\mathsf{F}$ by simulating the previous round computation with the first $\gamma$ coordinates of $Q_{\gamma+2}$. $Q_{\gamma+2}$ evokes case 2(c) of Algorithm 5 as the final two coordinates of $Q_{\gamma+2}$ are non-zero with overwhelming probability. Now $\mathsf{F}$ computes $b_1$ and $b_2$ by simulating the output of the previous round of itself such that $(b_1, b_2, \cdots, b_2)$ is the output from $\mathsf{F}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \cdots, \alpha_\gamma \oplus \beta_\gamma, 0^n, 0^n) = \mathsf{F}(Q_{\gamma+1})$. Therefore,

$$b_1 = \pi_{k_{sk_\mathsf{F}}}(r_{1(\gamma+1)}) \tag{23}$$

$$b_2 = r_{1(\gamma+1)}. \tag{24}$$

Then, $\mathsf{F}$ verifies that equality $\pi^{-1}_{sk_\mathsf{F}}(b_1 \oplus u_{\gamma+1}) = b_2 \oplus u_{\gamma+2}$ holds as

$$\begin{aligned}
\pi^{-1}_{sk_\mathsf{F}}(b_1 \oplus u_{\gamma+1}) &= \pi^{-1}_{sk_\mathsf{F}}(\pi_{k_{sk_\mathsf{F}}}(r_{1(\gamma+1)}) \oplus \pi_{sk_\mathsf{F}}(r_{1(\gamma+1)}) \oplus \pi_{sk_\mathsf{G}}(s_{1(\gamma+1)})) && \text{by (23)and (22)} \\
&= \pi^{-1}_{sk_\mathsf{F}}(\pi_{sk_\mathsf{G}}(s_{1(\gamma+1)})) && \\
&= s_{1(\gamma+1)} && \text{since } sk_\mathsf{F} = sk_\mathsf{G} \\
&= (r_{1(\gamma+1)} \oplus r_{1(\gamma+1)}) \oplus s_{1(\gamma+1)} && \\
&= r_{1(\gamma+1)} \oplus (r_{1(\gamma+1)} \oplus s_{1(\gamma+1)}) && \\
&= b_2 \oplus u_{\gamma+2} && \text{by (24)and (22)}
\end{aligned}$$

Therefore, $\mathsf{F}$ outputs $(k_\mathsf{F}, 0^n, \cdots, 0^n)$. Similarly, $\mathsf{G}$ performs the same course of computation as does $\mathsf{F}$, so it outputs $(0^n, k_\mathsf{G}, 0^n, \cdots, 0^n)$. Therefore, the final output of the parallel composition is $(k_\mathsf{F}, k_\mathsf{G}, 0^n, \cdots, 0^n)$ which reveals all the secret keys of $\mathsf{F}$ and $\mathsf{G}$. □

Now, by having proved Claim 4.4 and 4.5, we obtain the following theorem of the impossibility of adaptively secure parallel composition under the existence of $\gamma$-UTKA.

**Theorem 8.** *If $\gamma$-UTKA $\Phi_u = (\mathsf{A}, \mathsf{B})$ exists, then there exist non-adaptively secure pseudo-random functions $\mathsf{F}$ and $\mathsf{G}$ such that their parallel composition over XOR is ($\gamma$+2)-adaptive query breakable.*

Theorem 7 and 8 immediately substantiate the equivalence between the existence of UTKA and the above impossibility result as formally stated below.

**Theorem 9.** *The parallel composition of two pseudo-random functions does not imply adaptive security if and only if the uniform-transcript key agreement exists.*

## 4.2 Sequential Composition Insecurity vs. Uniform Transcript Key Agreement

### 4.2.1 Constructing UTKA from the Adaptive Insecurity of $\mathsf{G}(\mathsf{F}(\cdot))$

In this section, we briefly review the main idea behind Pietrzak's construction of key agreement, which is achieved from two pseudo-random functions whose sequential composition is not adaptively secure. Let us have two secure functions $\mathsf{F}$ and $\mathsf{G}$ such that their sequential composition $\mathsf{G}(\mathsf{F}(\cdot))$ is broken by a distinguisher $\mathcal{D}$ with $k$-adaptive queries. Alice chooses her secret bit $b$ in which Bob will eventually know what bit $b$ is. If $b = 1$, Alice obtains the first (starting) adaptive input from $\mathcal{D}$, and sends Bob the cypher-text $c$ of the first input, encrypted by $\mathsf{F}$. Bob simply computes $\mathsf{G}(c)$ and sends it back to Alice. Then, Alice again uses $\mathcal{D}$ to get an (the second) adaptive input based on $\mathsf{G}(c)$ and repeats the same process. If $b = 0$, then Alice simply chooses and sends a uniformly random string to Bob (every time) and Bob follows the same procedure as described above. As this process is performed repeatedly $k$ times, at the end Bob obtains $k$ adaptive outputs on which $\mathcal{D}$ outputs a bit consistent with Alice's secret bit with non-negligible probability. Hence, Bob obtains $b$ with non-negligible probability. The pictorial description of the protocol is provided in Protocol 5. In fact, his bit agreement protocol satisfies the property of uniform-transcript. We prove this as

**Pietrzak's Protocol Bit-Agreement**$(1^n)$

| Alice | Transcript | Bob |
|---|---|---|
| $b_{\mathsf{A}} \leftarrow_\$ \{0,1\}^n$ | | |
| $k_{\mathsf{A}} \leftarrow \mathsf{Gen}_{\mathsf{F}}(1^n)$ | | $k_{\mathsf{B}} \leftarrow \mathsf{Gen}_{\mathsf{G}}(1^n)$ |
| $x_1 \leftarrow \mathcal{D}(1^n)$ | | |
| If $b_{\mathsf{A}} = 0$, | | |
| then $z_1 \leftarrow \mathsf{F}_{k_{\mathsf{A}}}(x_1)$ | | |
| else $z_1 \leftarrow_\$ \{0,1\}^n$ | $\xrightarrow{z_1}$ | |
| | $\xleftarrow{y_1}$ | $y_1 \leftarrow \mathsf{G}_{k_{\mathsf{B}}}(z_1)$ |
| $x_2 \leftarrow \mathcal{D}(y_1)$ | | |
| If $b_{\mathsf{A}} = 0$, | | |
| then $z_2 \leftarrow \mathsf{F}_{k_{\mathsf{A}}}(x_2)$ | | |
| else $z_2 \leftarrow_\$ \{0,1\}^n$ | $\xrightarrow{z_2}$ | $y_2 \leftarrow \mathsf{G}_{k_{\mathsf{B}}}(z_2)$ |
| | | $b_{\mathsf{B}} \leftarrow \mathcal{D}(y_1, y_2)$ |

**Protocol 5:** [Pie06] 3-pass bit agreement based on 2-adaptive distinguisher $\mathcal{D}$

a separate claim and formally restate Pietrzak's theorem below.

**Claim 4.6.** *The protocol Bit-Agreement($1^n$) in [Pie06] is a uniform transcript bit agreement.*

*Proof.* To show that the protocol Bit-Agreement($1^n$) has uniform transcript, we prove that for any PPT adversary $\mathcal{A}$,

$$\left| \Pr_{b_A \leftarrow_\$ \{0,1\}}[\mathcal{A}_r(\mathsf{z}_1, \mathsf{y}_1\mathsf{z}_2) = 1] - \Pr_{b_A \leftarrow_\$ \{0,1\}}[\mathcal{A}_r(\mathsf{R}_3) = 1 : \mathsf{R}_3 \leftarrow_\$ (\{0,1\}^n)^3] \right| \leq \epsilon(n), \qquad (25)$$

where $\epsilon(n)$ is a negligible function in $n$. First, we define two games for adversary $\mathcal{A}$ to distinguish transcript from random string. Game $\mathcal{G}_0$ is for the adversary to distinguish $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ when $b_\mathsf{A} = 0$, and game $\mathcal{G}_1$ is for the adversary to distinguish $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ when $b_\mathsf{A} = 1$. Then we denote the advantage of adversary $\mathcal{A}$ in $\mathcal{G}_0$ and $\mathcal{G}_1$ as $\mathbf{Adv}_{\mathcal{G}_0}$ and $\mathbf{Adv}_{\mathcal{G}_1}$, respectively. Finally, suppose that there exists a PPT adversary $\mathcal{A}$ winning either of the games with non-negligible probability.

In game $\mathcal{G}_1$, since both $z_1$ and $z_2$ are randomly chosen from $\{0,1\}^n$, the advantage for the adversary to distinguish $(z_1, y_1, z_2)$ from $\mathsf{R}_3$ is equivalent to the advantage of distinguishing $(r^*, \mathsf{G}(r^*))$ from $(r_1, r_2)$ where $r^*, r_1, r_2$ are all uniformly random. Hence, by winning game $\mathcal{G}_1$, $\mathcal{A}$ distinguishes $(r^*, \mathsf{G}(r^*))$ from $(r_1, r_2)$ with non-negligible probability. This is an obvious contraction to the non-adaptive security of $\mathsf{G}$.

Now consider the game $\mathcal{G}_0$. Assume that a PPT adversary $\mathcal{A}$ wins the game by distinguishing, with non-negligible probability,

$$Dist(0) : (z_1, y_1, z_2) = (\mathsf{F}(r), \mathsf{G}(\mathsf{F}(r)), \mathsf{F}(\mathcal{D}(\mathsf{G}(\mathsf{F}(r)))))$$

from a random triplet $\mathsf{R}_3$ for $r \leftarrow_\$ \{0,1\}^n$. Then, consider the following distributions.

$$Dist(1) : (r_1, \mathsf{G}(r_1), \mathsf{F}(\mathcal{D}(\mathsf{G}(r_1))))$$

for $r_1 \leftarrow_\$ \{0,1\}^n$.

$$Dist(2) : (r_1, r_2, \mathsf{F}(\mathcal{D}(r_2)))$$

for $r_1$ and $r_2 \leftarrow_\$ \{0,1\}^n$.

$$Dist(3) : (r_1, r_2, r_3) = \mathsf{R}_3$$

for $r_1$, $r_2$ and $r_3 \leftarrow_\$ \{0,1\}^n$.

Since $\mathcal{A}$ distinguishes Dist(0) from Dist(3) with non-negligible probability, $\mathcal{A}$ distinguishes two of the intermediate distributions, Dist($i$) and Dist($j$), for $i \neq j$, with non-negligible probability. This is clearly a contradiction since the above distributions only negligibly deviate from each other by the the non-adaptive security of $\mathsf{F}$ and $\mathsf{G}$.

The advantage of $\mathcal{A}$ to distinguish the transcript from $\mathsf{R}_3$ is $\mathbf{Adv}_{\mathcal{G}_0}/2 + \mathbf{Adv}_{\mathcal{G}_1}/2$ which is negligible since both $\mathbf{Adv}_{\mathcal{G}_0}$ and $\mathbf{Adv}_{\mathcal{G}_1}$ are negligible. Thus, inequality (25) holds. ☐

**Theorem 10** ([Pie06])**.** *Let $\mathsf{F}$ and $\mathsf{G}$ be $(k-1)$-adaptively secure pseudo-random functions. If the sequential composition $\mathsf{G}(\mathsf{F}(\cdot))$ is NOT k-adaptively secure, then a (2k-1)-pass UTKA exists for $k \geq 2$.*

### 4.2.2 Intuitions of Adaptively Insecure Sequential Composition of F and G under UTKA

In the following description of building our counter-example functions, we use the sequential version of $\gamma$-UTKA in which Bob's first message is *dependent* on Alice's first message. That is, Bob must wait for the first message $\alpha_1$ from Alice in order to compute his first message $\beta_1$. See Protocol 6 for the overview of sequetial $\gamma$-UTKA.

| Alice | Transcript | Bob |
|---|---|---|
| $r_{\mathsf{A}} \leftarrow_{\$} \{0,1\}^n$ | | $r_{\mathsf{B}} \leftarrow_{\$} \{0,1\}^n$ |
| $\alpha_1 \leftarrow \mathsf{A}_1(r_{\mathsf{A}})$ | $\xrightarrow{\alpha_1}$ | |
| | $\xleftarrow{\beta_1}$ | $\beta_1 \leftarrow \mathsf{B}_1(r_{\mathsf{B}}, \alpha_1)$ |
| $\alpha_2 \leftarrow \mathsf{A}_2(r_{\mathsf{A}}, \beta_1)$ | $\xrightarrow{\alpha_2}$ | |
| | $\xleftarrow{\beta_2}$ | $\beta_2 \leftarrow \mathsf{B}_2(r_{\mathsf{B}}, \alpha_1, \alpha_2)$ |
| | $\vdots$ | |
| $\alpha_\gamma \leftarrow \mathsf{A}_\gamma(r_{\mathsf{A}}, \beta_1, \cdots, \beta_{\gamma-1})$ | $\xrightarrow{\alpha_\gamma}$ | |
| | $\xleftarrow{\beta_\gamma}$ | $\beta_\gamma \leftarrow \mathsf{B}_\gamma(r_{\mathsf{B}}, \alpha_1, \cdots, \alpha_\gamma)$ |
| secret key $sk \leftarrow \mathsf{A}_{\gamma+1}(r_{\mathsf{A}}, \beta_1, \cdots, \beta_\gamma)$ | | secret key $sk \leftarrow \mathsf{B}_{\gamma+1}(r_{\mathsf{B}}, \alpha_1, \cdots, \alpha_\gamma)$ |

**Protocol 6:** Sequential version of $\gamma$-UTKA

Now, we present the high-level overview on our constructions of counter-example functions $\mathsf{F}$ and $\mathsf{G}$ based on $\gamma$-UTKA described above. For the building blocks, we are given a sequential version of $\gamma$-UTKA, $\Phi_u = (\mathsf{A}, \mathsf{B})$ and all the other primitives remain identical to the ones in Section 4.1. $\mathsf{F}$ (resp. $\mathsf{G}$) is defined over $(\{0,1\}^n)^{\gamma+3}$ and internally possesses two secret keys $k_{\mathsf{F}}$ and $k_{\mathsf{F}'}$ (resp. $k_{\mathsf{G}}$ and $k_{\mathsf{G}'}$).

Our first adaptive query is an arbitrary vector in $(\{0,1\}^n)^{\gamma+3}$ as $Q_1 = (u_1, u_2, \cdots, u_{\gamma+2}, u^*)$ for $u_1, u_2, \cdots, u_{\gamma+2}, u^* \leftarrow_{\$} \{0,1\}^n$. On $Q_1$, we define $\mathsf{F}$ and $\mathsf{G}$ as follows.

- $\mathsf{F}$ computes a pseudo-random string $r_{\mathsf{F}}$ by $\mathsf{P}_{k_{\mathsf{F}}}(u^*)$. Then, $\mathsf{F}$ generates the first message $\alpha_1$ by executing $\mathsf{A}_1(r_{\mathsf{F}})$. $\mathsf{F}$ continues to compute $r_{21}, \cdots, r_{\gamma 1}$ by executing $\mathsf{A}_2(r_{\mathsf{F}}, u_1)$, $\cdots$, $\mathsf{A}_\gamma(r_{\mathsf{F}}, u_1, \cdots, u_{\gamma-1})$. Notice that $Q_1$ is an arbitrarily chosen input so that running $\mathsf{A}$ (Alice) on $Q_1$ produces only pseudo-random strings except for the first message $\alpha_1$. $\mathsf{F}$ computes its first $n$-bit shared key $sk_{\mathsf{F}}^1$ from $\mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, u_1, \cdots, u_\gamma)$. $\mathsf{F}$ tests if $\mathsf{Dec}_{sk_{\mathsf{F}}^1}(u_{\gamma+1}) = u_{\gamma+2}$. The equality is satisfied only negligible probability since $u_{\gamma+1}$ and $u_{\gamma+2}$ are arbitrary chosen. Hence, with overwhelming probability, $\mathsf{F}$ concludes its computation by outputting $(\alpha_1, r_{21}, r_{31}, \cdots, \mathsf{Enc}_{sk_{\mathsf{F}}^1}(r_{(\gamma+1)1}), r_{(\gamma+1)1}, r_{(\gamma+3)1})$ where $r_{(\gamma+1)1}$, $r_{(\gamma+2)1}$ and $r_{(\gamma+3)1}$ are generated from $\mathsf{P}_{k_{\mathsf{F}}}(u_{\gamma+1}, u_{\gamma+2}, u_{\gamma+3})$.
- On $\mathsf{F}(Q_1)$, $\mathsf{G}$ is defined to compute $\beta_1$ by $\mathsf{B}_1(s_{\mathsf{G}}, \alpha_1)$ where $s_{\mathsf{G}}$ is generated by $\mathsf{P}_{k_{\mathsf{G}}}(u_1)$ and $\alpha_1$ is the first message validly generated by $\mathsf{F}$. $\mathsf{G}$ continues to compute $s_{21}, \cdots, s_{\gamma 1}$ by executing $\mathsf{B}_2(s_{\mathsf{G}}, \alpha_1, r_{21})$, $\cdots$, $\mathsf{B}_\gamma(s_{\mathsf{G}}, \alpha_1, r_{21}, \cdots, r_{\gamma 1})$. Since $r_{21}, \cdots, r_{\gamma 1}$ are pseudo-random strings computed by $\mathsf{F}$ upon non-adaptive query $Q_1$, $s_{21}, \cdots, s_{\gamma 1}$ are pseudo-random strings. $\mathsf{G}$ computes $sk_{\mathsf{G}}^1$ from $\mathsf{B}_{\gamma+1}(r_{\mathsf{G}}, u_1, \cdots, u_\gamma)$ and then tests if $\mathsf{Dec}_{sk_{\mathsf{G}}^1}(u_{\gamma+1}) = u_{\gamma+2}$ holds. This equality holds with only negligible probability. $\mathsf{G}$ computes pseudo-random strings $s_{(\gamma+1)1}$, $s_{(\gamma+2)1}$ and $s_{(\gamma+3)1}$ from $\mathsf{P}_{k_{\mathsf{G}}}(\pi_{sk_{\mathsf{F}}^1}(r_{(\gamma+1)1}), r_{(\gamma+1)1}, r_{(\gamma+3)1})$. $\mathsf{G}$ outputs $(\beta_1, s_{21}, s_{31}, \cdots, s_{(\gamma+1)1}, s_{(\gamma+2)1}, s_{(\gamma+3)1})$.

We describe the outputs of $\mathsf{F}$ and $\mathsf{G}$ in the computation of their sequential composition on $Q_1$:

$$Q_1 \xrightarrow{\mathsf{F}} (\alpha_1, r_{21}, r_{31}, \cdots, \mathsf{Enc}_{sk^1_{\mathsf{F}}}(r_{(\gamma+1)1}), r_{(\gamma+1)1}, r_{(\gamma+3)1})$$
$$\xrightarrow{\mathsf{G}} (\beta_1, s_{21}, s_{31}, \cdots, s_{(\gamma+1)1}, s_{(\gamma+2)1}, s_{(\gamma+3)1}).$$

Inductively, for $2 \le i \le \gamma - 1$, the $i$th adaptive query $Q_i$ is in the form of $(\beta_1, \cdots, \beta_{i-1}, s_{i(i-1)}, \cdots, s_{\gamma(i-1)}, s_{(\gamma+1)(i-1)}, s_{(\gamma+2)(i-1)}, u^*)$ where $u^*$ is the final coordinate of $Q_1$ and the rest of coordinates are the first $2\gamma+2$ coordinates in the output of $\mathsf{G}(\mathsf{F}(Q_{i-1}))$. Then, $\mathsf{F}$ computes all the messages $\alpha_1$ to $\alpha_\gamma$ and shared key $sk^i_{\mathsf{F}}$ based on $Q_i$ as described above. $\mathsf{F}$ tests if $\mathsf{Dec}_{sk^i_{\mathsf{F}}}(\mathsf{Enc}_{sk^{i-1}_{\mathsf{G}}}(s_{(\gamma+1)(i-1)})) = s_{(\gamma+1)(i-1)}$. Obviously, $sk^i_{\mathsf{F}} \neq sk^{i-1}_{\mathsf{G}}$ with overwhelming probability since the keys are computed based on insufficient number of valid messages. Hence, $\mathsf{F}$ outputs $(\alpha_1, \cdots, \alpha_i, r_{(i+1)i}, \cdots, r_{(\gamma)i}, \mathsf{Enc}_{sk^i_{\mathsf{F}}}(r_{(\gamma+1)i}), r_{(\gamma+1)i}, r_{(\gamma+3)i})$. Similarly, $\mathsf{G}$ undertakes the same course of computations: $\mathsf{G}$ computes messages and shared key, tests the equality and finally outputs $(\beta_1, \cdots, \beta_i, s_{(i+1)i}, \cdots, s_{(\gamma)i}, s_{(\gamma+1)i}, s_{(\gamma+2)i}, s_{(\gamma+3)i})$. The individual output of $\mathsf{F}$ and the output of $\mathsf{G}$ in their sequential composition on $Q_i$ are described as follows:

$$Q_i \xrightarrow{\mathsf{F}} (\alpha_1, \cdots, \alpha_i, r_{(i+1)i}, \cdots, r_{(\gamma)i}, \mathsf{Enc}_{sk^i_{\mathsf{F}}}(r_{(\gamma+1)i}), r_{(\gamma+1)i}, r_{(\gamma+3)i})$$
$$\xrightarrow{\mathsf{G}} (\beta_1, \cdots, \beta_i, s_{(i+1)i}, \cdots, s_{(\gamma)i}, s_{(\gamma+1)i}, s_{(\gamma+2)i}, s_{(\gamma+3)i}).$$

Hence, after the $(\gamma-1)$th adaptive query, our $\gamma$th adaptive query $Q_\gamma$ is $(\beta_1, \beta_2, \cdots, \beta_{\gamma-1}, s_{\gamma(\gamma-1)}, s_{(\gamma+1)(\gamma-1)}, s_{(\gamma+2)(\gamma-1)}, u^*)$. On $Q_\gamma$, we define $\mathsf{F}$ and $\mathsf{G}$ as follows.

- $\mathsf{F}$ computes $r_{\mathsf{F}}$ from $\mathsf{P}_{k_{\mathsf{F}}}(u^*)$. Then, $\mathsf{F}$ internally regenerates all $\alpha_i$ by $\mathsf{A}_i(r_{\mathsf{F}}, \beta_1, \cdots, \beta_{i-1})$ for $1 \le i \le \gamma$ and shared key $sk^\gamma_{\mathsf{F}}$ by $\mathsf{A}_i(r_{\mathsf{F}}, \beta_1, \cdots, \beta_{i-1}, s_{\gamma(\gamma-1)})$. $sk^\gamma_{\mathsf{F}}$ is still a merely pseudo-random string since $s_{\gamma(\gamma-1)}$ is not a proper message. $\mathsf{F}$ performs the equality test $\mathsf{Dec}_{sk^\gamma_{\mathsf{F}}}(\mathsf{Enc}_{sk^{\gamma-1}_{\mathsf{G}}}(s_{(\gamma+1)(\gamma-1)})) = s_{(\gamma+1)(\gamma-1)}$ which fails with overwhelming probability. Hence, $\mathsf{F}$ outputs $(\alpha_1, \cdots, \alpha_\gamma, \mathsf{Enc}_{sk^\gamma_{\mathsf{F}}}(r_{(\gamma+1)\gamma}), r_{(\gamma+1)\gamma}, r_{(\gamma+3)\gamma})$ as $(r_{(\gamma+1)\gamma}, r_{(\gamma+2)\gamma}, r_{(\gamma+3)\gamma})$ is generated by $\mathsf{P}_{k_{\mathsf{F}}}(\mathsf{Enc}_{sk^{\gamma-1}_{\mathsf{G}}}(s_{(\gamma+1)(\gamma-1)}), s_{(\gamma+1)(\gamma-1)}, u^*)$.
- $\mathsf{G}$ obtains $r_{\mathsf{G}}$ by $\mathsf{P}_{k_{\mathsf{G}}}(\alpha_1)$. Then, since $\mathsf{G}$ obtains its complete set of $\gamma$ messages $\alpha_i$'s from $\mathsf{F}$, function $\mathsf{G}$ correctly generates all the messages $\beta_i$'s by executing $\mathsf{B}_i(r_{\mathsf{G}}, \alpha_1, \cdots, \alpha_i)$ for all $1 \le i \le \gamma$. In addition, $\mathsf{G}$ computes the shared key $sk^\gamma_{\mathsf{G}}$ from executing $\mathsf{B}_{\gamma+1}(r_{\mathsf{G}}, \alpha_1, \cdots, \alpha_\gamma)$. Notice that the obtained shared key $sk^\gamma_{\mathsf{G}}$ is a correct key which is generated upon $\gamma$ valid messages. However, when $\mathsf{G}$ tests if $\mathsf{Dec}_{sk^\gamma_{\mathsf{G}}}(\mathsf{Enc}_{sk^\gamma_{\mathsf{F}}}(r_{(\gamma+1)(\gamma)})) = r_{(\gamma+1)(\gamma)}$, the test fails with overwhelming probability since $sk^\gamma_{\mathsf{F}}$ is not valid. Finally, $\mathsf{G}$ outputs $(\beta_1, \cdots, \beta_\gamma, s_{(\gamma+1)\gamma}, s_{(\gamma+2)\gamma}, s_{(\gamma+3)\gamma})$ where $(s_{(\gamma+1)\gamma}, s_{(\gamma+2)\gamma}, s_{(\gamma+3)\gamma})$ is generated by $\mathsf{P}_{k_{\mathsf{G}}}(\mathsf{Enc}_{sk^\gamma_{\mathsf{F}}}(r_{(\gamma+1)\gamma}), r_{(\gamma+1)\gamma}, r_{(\gamma+3)\gamma})$.

We describe the overall picture of $\mathsf{F}$ and $\mathsf{G}$ in their sequential composition on input $Q_\gamma$ below:

$$Q_\gamma \xrightarrow{\mathsf{F}} (\alpha_1, \cdots, \alpha_\gamma, \mathsf{Enc}_{sk^\gamma_{\mathsf{F}}}(r_{(\gamma+1)\gamma}), r_{(\gamma+1)\gamma}, r_{(\gamma+3)\gamma}) \xrightarrow{\mathsf{G}} (\beta_1, \cdots, \beta_\gamma, s_{(\gamma+1)\gamma}, s_{(\gamma+2)\gamma}, s_{(\gamma+3)\gamma}).$$

The $(\gamma+1)$th adaptive query $Q_{\gamma+1}$ is defined to be $(\beta_1, \cdots, \beta_\gamma, s_{(\gamma+1)\gamma}, s_{(\gamma+2)\gamma}, u^*)$. On $Q_{\gamma+1}$, we define functions $\mathsf{F}$ and $\mathsf{G}$ on $Q_{\gamma+1}$ as:

- $\mathsf{F}$ now obtains all the messages $\beta_i$'s from $Q_{\gamma+1}$ so that it can compute all the messages $\alpha_1, \cdots, \alpha_\gamma$ and the shared key $sk^{\gamma+1}_{\mathsf{F}}$ by executing $\mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, \beta_1, \cdots, \beta_\gamma)$. $\mathsf{F}$ tests if the following equality is satisfied: $\mathsf{Dec}_{sk^{\gamma+1}_{\mathsf{F}}}(\mathsf{Enc}_{sk^\gamma_{\mathsf{G}}}(s_{(\gamma+1)(\gamma)})) = s_{(\gamma+2)(\gamma)}$. Notice that $sk^{\gamma+1}_{\mathsf{F}} = sk^\gamma_{\mathsf{G}}$ since both keys are computed on each complete set of messages. However, since $s_{(\gamma+1)(\gamma)}$ and $s_{(\gamma+2)(\gamma)}$ are both mere pseudo-random strings, the test fails. Finally, $\mathsf{F}$ outputs $(\alpha_1, \cdots, \alpha_\gamma, \pi_{sk_{\mathsf{F}}^{\gamma+1}}(r_{(\gamma+1)(\gamma+1)}), r_{(\gamma+1)(\gamma+1)}, k_{\mathsf{F}})$ where $r_{(\gamma+1)(\gamma+1)}, r_{(\gamma+2)(\gamma+1)}$ and $r_{(\gamma+3)(\gamma+1)}$ are generated from $\mathsf{P}_{k_{\mathsf{F}}}(\mathsf{Enc}_{sk_{\mathsf{G}}^{\gamma}}(s_{(\gamma+1)\gamma}), s_{(\gamma+1)\gamma}, u^*)$.

- On $(\alpha_1, \cdots, \alpha_\gamma, \pi_{sk_{\mathsf{F}}^{\gamma+1}}(r_{(\gamma+1)(\gamma+1)}), r_{(\gamma+1)(\gamma+1)}, k_{\mathsf{F}})$, $\mathsf{G}$ also computes all of the messages and shared key $sk_{\mathsf{G}}^{\gamma+1}$. Clearly, $sk_{\mathsf{F}}^{\gamma+1} = sk_{\mathsf{G}}^{\gamma+1}$ since both keys are computed based on the complete set of interatively generated messages $\alpha_1 \cdots \alpha_\gamma$. Then $\mathsf{G}$ tests if $\mathsf{Dec}_{sk_{\mathsf{G}}^{\gamma+1}}(\mathsf{Enc}_{sk_{\mathsf{F}}^{\gamma+1}}(r_{(\gamma+1)(\gamma+1)}))$ $= r_{(\gamma+1)(\gamma+1)}$. Since both $sk_{\mathsf{F}}^{\gamma+1}$ and $sk_{\mathsf{G}}^{\gamma+1}$ are computed from the complete sets of messages, they must be equal. $\mathsf{G}$ is convinced that the query from $\mathsf{F}$ is adaptively generated. Therefore, $\mathsf{G}$ outputs $(k'_{\mathsf{G}}, \mathsf{Enc}_{sk_{\mathsf{G}}^{\gamma+1}}(k'_{\mathsf{G}}), 0^n, \cdots, 0^n)$.

The overall description of outputs of $\mathsf{F}$ and $\mathsf{G}$ on the final adaptive query is provided below:

$$Q_{\gamma+1} \xrightarrow{\mathsf{F}} (\alpha_1, \cdots, \alpha_\gamma, \mathsf{Enc}_{sk_{\mathsf{F}}^{\gamma+1}}(r_{(\gamma+1)(\gamma+1)}), r_{(\gamma+1)(\gamma+1)}, k_{\mathsf{F}}) \xrightarrow{\mathsf{G}} (k'_{\mathsf{G}}, \mathsf{Enc}_{sk_{\mathsf{G}}^{\gamma+1}}(k'_{\mathsf{G}}), 0^n, \cdots, 0^n).$$

We define the final $(\gamma+2)$th adaptive query $Q_{\gamma+2}$ to be defined by $(\beta_1, \cdots, \beta_\gamma, \mathsf{Enc}_{sk_{\mathsf{G}}^{\gamma+1}}(k'_{\mathsf{G}}), k'_{\mathsf{G}}, u^*)$. On $Q_{\gamma+2}$, Functions $\mathsf{F}$ and $\mathsf{G}$ on $Q_{\gamma+1}$ are defined as follows:

- $\mathsf{F}$ can compute $\alpha_1, \cdots, \alpha_\gamma$ as it did upon $Q_{\gamma+1}$ and the shared key $sk_{\mathsf{F}}^{\gamma+1}$ by executing $\mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, \beta_1, \cdots, \beta_\gamma)$. $\mathsf{F}$ checks if the following equality is satisfied: $\mathsf{Dec}_{sk_{\mathsf{F}}^{\gamma+2}}(\mathsf{Enc}_{sk_{\mathsf{G}}^{\gamma+1}}(k'_{\mathsf{G}})) = k'_{\mathsf{G}})$. Since $sk_{\mathsf{F}}^{\gamma+1} = sk_{\mathsf{G}}^{\gamma}$ in which both keys are computed on each complete set of messages, so the equality holds. $\mathsf{F}$ is now convinced that $Q_2$ is an adaptive query. Therefore, $\mathsf{F}$ outputs $(k'_{\mathsf{G}}, k_{\mathsf{G}}, k'_{\mathsf{F}}, 0^n, \ldots, 0^n)$.
- On $(k'_{\mathsf{G}}, k_{\mathsf{G}}, k'_{\mathsf{F}}, 0^n, \ldots, 0^n)$, $\mathsf{G}$ discovers that the first coordinate of the input equals $k'_{\mathsf{G}}$, which is one of its own secret keys. Therefore, $\mathsf{G}$ outputs $(k_{\mathsf{G}}, k'_{\mathsf{G}}, k_{\mathsf{G}}, k'_{\mathsf{F}}, 0^n, \ldots, 0^n)$ which reveals all of the secret keys of $\mathsf{F}$ and $\mathsf{G}$.

The overall description of outputs of $\mathsf{F}$ and $\mathsf{G}$ on the final adaptive query is provided below:

$$Q_{\gamma+1} \xrightarrow{\mathsf{F}} (k'_{\mathsf{G}}, k_{\mathsf{G}}, k'_{\mathsf{F}}, 0^n, \ldots, 0^n) \xrightarrow{\mathsf{G}} (k_{\mathsf{G}}, k'_{\mathsf{G}}, k_{\mathsf{G}}, k'_{\mathsf{F}}, 0^n, \ldots, 0^n).$$

### 4.2.3 Formal Construction of Non-Adaptively Secure Function F

We use three underlying primitives for the construction of $\mathsf{F}$: PRP $\pi : K \times \{0,1\}^n \to \{0,1\}^n$, PRP $\tilde{\pi} : K \times (\{0,1\}^n)^3 \to (\{0,1\}^n)^3$ where key space $K$ is $\{0,1\}^n$ and $\gamma$-UTKA $\Phi_u = (\mathsf{A}, \mathsf{B})$ described in Section 4.2.2. We define function $\mathsf{F} : (\{0,1\}^n)^{\gamma+3} \to (\{0,1\}^n)^{\gamma+3}$ to emulate the $\mathsf{A}$ of $\Phi_u$ via a black-box access to $\mathsf{A}$. In addition, we define $\mathsf{F}$ to internally retain two private keys $k_{\mathsf{F}}$ and $k'_{\mathsf{F}}$of length $n$. The formal construction is presented in Algorithm 6.

**Claim 4.7.** *The function $\mathsf{F}$ is secure against any non-adaptive PPT adversary $\mathcal{A}(q,t)$, running in time $t$ and making at most $q$ non-adaptive queries, where $t$ and $q$ are any polynomials of security parameter $n$.*

*Proof.* We prove this claim by showing that

$$\mathbf{Adv}_{\mathcal{A}}^{\mathsf{F}}(q,t) \leq \mathbf{Adv}_{\mathcal{A}}^{\tilde{\pi}}(q,t') + \mathbf{Adv}_{\mathcal{A}}^{\pi}(q,t') + \mathbf{Adv}_{\mathcal{A}}^{\mathsf{A}}(q,t') + \frac{q}{2^n} \tag{26}$$

**Construction of F**

```
1. For any $I = (u_1, u_2, \ldots, u_{\gamma+3})$, then
        Output $(v_1, v_2, \ldots, v_{\gamma+3})$ where
        $r_{\mathsf{F}} \leftarrow \pi_{k_{\mathsf{F}}}(u_{\gamma+3})$
        $\alpha_1 \leftarrow \mathsf{A}_1(r_{\mathsf{F}})$
        FOR $i$ from 2 to $\gamma$ DO
            $\alpha_i \leftarrow \mathsf{A}_i(r_{\mathsf{F}}, u_1, u_2, \cdots, u_{i-1})$
        ENDFOR
        $sk \leftarrow \mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, u_1, u_2, \cdots, u_\gamma)$
        $(a_1, a_2, a_3) \leftarrow \tilde{\pi}_{k_{\mathsf{F}}}(u_{\gamma+1}, u_{\gamma+2}, u_{\gamma+3})$
    (a) If $u_{\gamma+2} = \pi_{sk}^{-1}(u_{\gamma+1})$
        i. then $(v_1, v_2, \ldots, v_{\gamma+3}) \leftarrow (\alpha_1, \alpha_2, \cdots, \alpha_\gamma, u_{\gamma+2}, k_{\mathsf{F}}, k_{\mathsf{F}}^{'})$
        ii. else $(v_1, v_2, \ldots, v_{\gamma+3}) \leftarrow (\alpha_1, \alpha_2, \cdots, \alpha_\gamma, \pi_{sk}(a_1), a_1, a_3)$
```

**Algorithm 6:** The algorithm of function $\mathsf{F}$

where $t' = t + poly(q)$, which accounts for the extra time costs resulting from our reduction.

Let $\mathcal{A}$ be a PPT adversary making non-adaptive $q$ queries to function $\mathsf{F}$. Since $\mathsf{F}$ on the same input query outputs the same output vector, assume that $\mathcal{A}$ does not input the same query twice. Let $Q_1$, $Q_2$, $\cdots$, $Q_q$ denote the $q$ non-adaptive queries that $\mathcal{A}$ generates.

Consider the event that a non-adaptive query evokes case 1(a)i of Algorithm 6. That is, for some $i \in [1, q]$, $Q_i = (u_1, u_2, \cdots, u_{\gamma+3})$ satisfies the condition of 'if statement' at the beginning of 1(a): $u_{\gamma+2} = \pi_{sk}^{-1}(u_{\gamma+1})$. $sk$ is computed by $\mathsf{A}$ on $u_1$, $u_2$, $\cdots$, $u_\alpha$ and an initial random string denoted by $r_{\mathsf{F}}$ in Algorithm 6. Since $\mathsf{F}$ picks a pseudo-random string generated from $\pi_{k_{\mathsf{F}}}(u_{\gamma+3})$ for the initial random string, $\mathcal{A}$ does not know what $r_{\mathsf{F}}$ is as well as what the shared key $sk$ is. Therefore, in order to evoke case 1(a)i (so the corresponding output reveals the secret key of $\mathsf{F}$), $\mathcal{A}$ merely guesses $u_{\gamma+1}$ and $u_{\gamma+2}$ such that $u_{\gamma+2} = \pi_{sk}^{-1}(u_{\gamma+1})$. Since $\pi$ is a permutation, there uniquely exists $u_{\gamma+1}$ for given $u_{\gamma+2}$ satisfying the condition. Therefore, the probability of each $Q_i$ succeeding to evoke case 1(a)i is $1/2^n$. Consequently, the success probability would be $q/2^n$ with asking $q$ non-adaptive queries, which is negligible in security parameter $n$. This case analysis accounts for the final term of inequality (26).

Consider the case that a non-adaptive query evokes case 1(a)i of Algorithm 6, which happens with overwhelming probability. We point out that the distribution of the outputs of case 1(a)i of Algorithm 6 is identical to a mixture of the distribution of the outputs of 2(a) and 2(b) of Algorithm 5. By the identical hybrid argument, it is easy to see that any PPT adversary, distinguishing outputs of case 1(a)i of Algorithm 6 from uniform random, can also distinguish $\tilde{\pi}$, $\pi$ or $\mathsf{A}$ from uniform random functions. This constitutes the first three terms on the right hand side of inequality (26), which completes the proof. □

### 4.2.4 Formal Construction of Non-Adaptively Secure Function G

The function $\mathsf{G}$ defined over $(\{0,1\}^n)^{\gamma+3}$ contains two secret keys $k_{\mathsf{G}}$ and $k_{\mathsf{G}}^{'}$and runs $\mathsf{B}$ of UTKA $\Phi_u$ described in Section 4.2.2 as a subroutine to emulate interactive communication (as Bob) with $\mathsf{F}$ (as Alice) via its inputs and outputs. The underlying primitives for the construction of function $\mathsf{G}$ are identical to those for the construction of function $\mathsf{F}$. The formal construction of function $\mathsf{G}$

is given in Algorithm 7 below.

```
                              Construction of G

1. For any $I = (u_1, u_2, \ldots, u_{\gamma+3})$, then
       Output $(v_1, v_2, \ldots, v_{\gamma+3})$ where
       $r_{\mathsf{G}} \leftarrow \pi_{k_{\mathsf{G}}}(u_1)$
       FOR $i$ from 1 to $\gamma$ DO
           $\beta_i \leftarrow \mathsf{B}_i(r_{\mathsf{G}}, u_1, u_2, \cdots, u_i)$
       ENDFOR
       $sk \leftarrow \mathsf{B}_{\gamma+1}(r_{\mathsf{G}}, u_1, u_2, \cdots, u_\gamma)$
       $(b_1, b_2, b_3) \leftarrow \tilde{\pi}_{k_{\mathsf{G}}}(u_{\gamma+1}, u_{\gamma+2}, u_{\gamma+3})$
   (a) If $k'_{\mathsf{G}} = u_{\gamma+1}$,
           i. then $(v_1, v_2, \ldots, v_{\gamma+3}) \leftarrow (k_{\mathsf{G}}, k'_{\mathsf{G}}, u_{\gamma+2}, u_{\gamma+3}, 0^n, \ldots, 0^n)$
   (b) Else If $u_{\gamma+2} = \pi^{-1}_{sk}(u_{\gamma+1})$,
           i. then $(v_1, v_2, \ldots, v_{\gamma+3}) \leftarrow (k'_{\mathsf{G}}, \pi_{sk}(k'_{\mathsf{G}}), 0^n, \cdots, 0^n)$
          ii. else  $(v_1, v_2, \ldots, v_{\gamma+3}) \leftarrow (\beta_1, \beta_2, \cdots, \beta_\gamma, b_1, b_2, b_3)$
```

**Algorithm 7:** The algorithm of function $\mathsf{G}$

**Claim 4.8.** *The function* $\mathsf{G}$ *is secure against any non-adaptive PPT adversary* $\mathcal{A}(q,t)$*, running in time* $t$ *and making at most* $q$ *non-adaptive queries, where* $t$ *and* $q$ *are any polynomials of security parameter* $n$*.*

*Proof.* We reduce the security of function $\mathsf{G}$ to the indistinguishability of $\pi$, $\tilde{\pi}$, the security of $\gamma$-UTKA $\Phi_u$, and the probability of guessing secret key $k_G$ by showing the following inequality:

$$\mathbf{Adv}^{\mathsf{G}}_{\mathcal{A}}(q,t) \leq \mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t') + \mathbf{Adv}^{\mathsf{B}}_{\mathcal{A}}(q,t') + \frac{q}{2^{n-1}} \tag{27}$$

where $t' = t + poly(q)$, which accounts for the extra time costs resulting from our reduction.

Suppose that a PPT adversary $\mathcal{A}$ generates non-adaptive $q$ queries over $(\{0,1\}^n)^{\gamma+3}$. Then, consider the case that a non-adaptive query $Q_i = (u_1, u_2, \cdots, u_{\gamma+3})$ evokes case 1(a)i or 1(b)i of Algorithm 7, which reveals the secret keys of $\mathsf{G}$. It is easy to see that $\mathsf{A}$ can make only a mere guess on $u_{\gamma+1}$ and $u_{\gamma+2}$ such that $u_{\gamma+2} = \pi^{-1}_{sk}(u_{\gamma+1})$, which succeeds with probability $1/2^n$. Also, the probability that the first coordinate of a non-adaptive query equals $k_{\mathsf{G}'}$ is $1/2^n$. Hence, the total probability that $q$ non-adaptive query evoke case 1(a)i or 1(b)i is $q(1/2^n + 1/2^n)$, which is the final term on the right hand side of inequality (27).

Hence, any non-adaptive query evokes 1(b)ii of Algorithm 7 with overwhelming probability. Notice that the distribution of the outputs in this case is generated only by either $\mathsf{B}$ or $\tilde{\pi}$. As the hybrid argument is used similarly to the previous section, we can easily prove that distinguishing the outputs of case 1(b)ii of Algorithm 7 from uniform random implies distinguishing $\mathsf{B}$ and $\tilde{\pi}$ from uniform random functions, which account for $\mathbf{Adv}^{\tilde{\pi}}_{\mathcal{A}}(q,t')$ and $\mathbf{Adv}^{\pi}_{\mathcal{A}}(q,t')$ of inequality (27). ☐

### 4.2.5 Adaptive Insecurity of Sequential Composition of F and G

**Claim 4.9.** *The sequential composition* $\mathsf{G}(\mathsf{F}(\cdot))$ *is breakable by* $\gamma+2$ *adaptive queries.*

*Proof.* To prove the claim, we will present a particular sequence of $\gamma + 1$ adaptive queries and describe how to adaptively build them. For the standard notation, let $Q_i = (u_{1i}, u_{2i}, \cdots, r_{(\gamma+2)i})$ be the $i$th adaptive input query. For the clarity of proof, we define 'valid' and 'invalid' messages and shared keys as follows. We define the the first 'valid' message generated from function $\mathsf{F}$ (as Alice by using $\mathsf{A}$), denoted by $\alpha_1^{val}$ to be the output of $\mathsf{A}_1(r_\mathsf{F})$ where $r_\mathsf{F}$ is the initial random string picked by $\mathsf{F}$ as $\mathsf{F}$ executes PRP $\pi_{k_\mathsf{F}}$ on a coordinate of the input query. Similarly, we define the first 'valid' message generated from function $\mathsf{G}$ (as Bob by using $\mathsf{B}$) denoted by $\beta_i^{val}$ to be $\mathsf{B}_1(r_\mathsf{G}, \alpha_1^{val})$. Inductively for $2 \leq i \leq \gamma$, we define the $i$th valid message message of $\mathsf{F}$ (resp. $\mathsf{G}$) to be the output of $\mathsf{A}_i(r_\mathsf{F}, \beta_1^{val}, \cdots, \beta_{i-1}^{val})$ (resp. $\mathsf{B}_i(r_\mathsf{G}, \alpha_1^{val}, \cdots, \alpha_i^{val})$). Finally, we define a valid shared key of $\mathsf{F}$ (resp. $\mathsf{G}$) denoted by $sk_\mathsf{F}^{val}$ (resp. $sk_\mathsf{G}^{val}$) to be $\mathsf{A}_{\gamma+1}(r_\mathsf{F}, \beta_1^{val}, \cdots, \beta_\gamma^{val})$ (resp. $\mathsf{B}_{\gamma+1}(r_\mathsf{G}, \beta_1^{val}, \cdots, \beta_\gamma^{val})$). We define the $i$th 'invalid' messages and 'invalid' shared key of $\mathsf{F}$ to be the output of $\mathsf{A}_i(r_\mathsf{F}, u_1, \cdots, u_{i-1})$ for $2 \leq i \leq \gamma + 1$ where there exists $i$ such that $u_i \neq \beta_i$. We define the $i$th 'invalid' messages and 'invalid' shared key of $\mathsf{F}$, respectively denoted by $\alpha_i^{inval}$ and $sk_\mathsf{F}^{inval}$, to be the output of $\mathsf{A}_i(r_\mathsf{F}, u_1, \cdots, u_{i-1})$ for $2 \leq i \leq \gamma + 1$ where there exists $i$ such that $u_i \neq \beta_i$. Similarly, we define the $i$th 'invalid' messages of $\mathsf{G}$, denoted by $\beta_i^{inval}$, to be the output of $\mathsf{B}_i(r_\mathsf{G}, u_1, \cdots, u_i)$ for $1 \leq i \leq \gamma$ where there exists $i$ such that $u_i \neq \alpha_i$. Finally, we define the 'invalid' shared key of $\mathsf{G}$, denoted by $sk_\mathsf{G}^{inval}$, to be the output of $\mathsf{B}_{\gamma+1}(r_\mathsf{G}, u_1, \cdots, u_\gamma)$ where there exists $1 \leq i \leq \gamma$ such that $u_i \neq \alpha_i$. Recall that all the messages generated by $\mathsf{A}$ and $\mathsf{B}$ are indistinguishable from uniform random over $\{0,1\}^n$ due to the uniform transcript property of UTKA $\Phi_u = (\mathsf{A}, \mathsf{B})$ regardless of the validity of the messages.

Our first adaptive query to $\mathsf{G}(\mathsf{F}(\cdot))$ can be any random vector in $(\{0,1\}^n)^{\gamma+3}$. Let us define the first query to be

$$Q_1 = (u_1, u_2, u_3, \cdots, u_{\gamma+2}, u^*). \tag{28}$$

On $Q_1$, $\mathsf{F}$ computes a random seed $r_\mathsf{F} \leftarrow \pi_{k_\mathsf{F}}(u^*)$ for the initiation of key agreement. Then, $\mathsf{F}$ first computes $\alpha_1^{val} \leftarrow \mathsf{A}_1(r_\mathsf{F})$ and then computes the following.

$$\begin{aligned}
\alpha_2^{inval} &\leftarrow \mathsf{A}_2(r_\mathsf{F}, u_1)\\
\alpha_3^{inval} &\leftarrow \mathsf{A}_3(r_\mathsf{F}, u_1, u_2)\\
&\vdots\\
\alpha_\gamma^{inval} &\leftarrow \mathsf{A}_\gamma(r_\mathsf{F}, u_1, u_2, \cdots, u_{\gamma-1})\\
sk_\mathsf{F}^{inval} &\leftarrow \mathsf{A}_{\gamma+1}(r_\mathsf{F}, u_1, u_2, \cdots, u_\gamma)
\end{aligned}$$

The above computations produce only invalid messages and shared key since $u_1, u_2, \cdots, u_\gamma$ are randomly chosen. That is, $u_i = \beta_i^{inval}$ for all $i$ with overwhelming probability. $\mathsf{F}$ checks if $\pi^{-1}_{sk_\mathsf{F}^{inval}}(u_{\gamma+1}) = u_{\gamma+2}$. The condition happens to be satisfied with only negligible probability. Hence, $\mathsf{F}$ computes $(r_{11}, r_{21}, r_{31}) \leftarrow \tilde{\pi}_{k_\mathsf{F}}(u_{\gamma+1}, u_{\gamma+2}, u^*)$ outputs

$$(\alpha_1^{val}, \alpha_2^{inval}, \cdots, \alpha_\gamma^{inval}, \pi_{sk_\mathsf{F}^{inval}}(r_{11}), r_{11}, r_{21}). \tag{29}$$

Taking (29) as an input query, $\mathsf{G}$ computes $r_\mathsf{G} \leftarrow \pi_{k_\mathsf{G}}(\alpha_1^{val})$. Then $\mathsf{G}$ computes the following.

$$\begin{aligned}
\beta_1^{val} &\leftarrow \mathsf{B}_1(r_\mathsf{G}, \alpha_1^{val})\\
\beta_2^{inval} &\leftarrow \mathsf{B}_2(r_\mathsf{G}, \alpha_1^{val}, \alpha_2^{inval})\\
&\vdots\\
\beta_\gamma^{inval} &\leftarrow \mathsf{B}_\gamma(r_\mathsf{G}, \alpha_1^{val}, \alpha_2^{inval}, \cdots, \alpha_\gamma^{inval})\\
sk_\mathsf{G}^{inval} &\leftarrow \mathsf{B}_{\gamma+1}(r_\mathsf{G}, \alpha_1^{val}, \alpha_2^{inval}, \cdots, \alpha_\gamma^{inval})
\end{aligned}$$

Then $\mathsf{G}$ checks to see if $\pi^{-1}_{sk_{\mathsf{F}}^{inval}}(\pi_{sk_{\mathsf{F}}^{inval}}(r_{11})) = r_{11}$ where $\pi_{sk_{\mathsf{F}}^{inval}}(r_{11})$ and $r_{11}$ from (29). It is easy to see that the equality holds with only negligible probability since $sk_{\mathsf{F}}^{inval} = sk_{\mathsf{G}}^{inval}$ with only negligible probability. $\mathsf{G}$ computes $(s_{11}, s_{21}, s_{31}) \leftarrow \tilde{\pi}_{k_{\mathsf{G}}}(\pi_{sk_{\mathsf{F}}^{inval}}(r_{11}), r_{11}, r_{21})$. Then, $\mathsf{G}$ finalizes the first round computation by outputting

$$(\beta_1^{val}, \beta_2^{inval}, \cdots, \beta_\gamma^{inval}, s_{11}, s_{21}, s_{31}). \tag{30}$$

Inductively, for all $2 \le i \le \gamma$, the $i$th adaptive query is of the following form:

$$Q_i = \underbrace{(\beta_1^{val}, \cdots, \beta_{i-1}^{val}, \beta_i^{inval}, \cdots, \beta_\gamma^{inval}, s_{1(i-1)}, s_{2(i-1)}}_{from\ the\ output\ of\ the\ (i-1)\ round} \underbrace{u_*}_{from\ (28)}). \tag{31}$$

On $Q_i$, $\mathsf{F}$ undertakes the course of computations identical to those on $Q_1$. In the same way, $\mathsf{G}$ performs the same computations as on $\mathsf{F}(Q_1)$ described above. Hence, the output of the sequential composition on $Q_i$ is

$$(\beta_1^{val}, \cdots, \beta_i^{val}, \beta_{i+1}^{inval}, \cdots, \beta_\gamma^{inval}, \pi_{sk_{\mathsf{G}}^{inval}}(s_{1i}), s_{1i}, s_{2i}). \tag{32}$$

By (31), the $\gamma$th adaptive query $Q_\gamma$ is

$$Q_\gamma = \underbrace{(\beta_1^{val}, \cdots, \beta_{\gamma-1}^{val}, \beta_\gamma^{inval}, s_{1(\gamma-1)}, s_{2(\gamma-1)},}_{from\ the\ output\ of\ the\ (\gamma-1)\ round} \underbrace{u^*}_{from\ (28)}). \tag{33}$$

On $Q_\gamma$, $\mathsf{F}$ obtains the initial randomness $r_{\mathsf{F}} \leftarrow \pi_{k_{\mathsf{F}}}(u^*)$. Then, $\mathsf{F}$ computes all the messages and shared key based on $Q_\gamma$ as follows:

$$\begin{aligned}
\alpha_1^{val} &\leftarrow \mathsf{A}_1(r_{\mathsf{F}}) \\
\alpha_2^{val} &\leftarrow \mathsf{A}_2(r_{\mathsf{F}}, \beta_1^{val}) \\
&\vdots \\
\alpha_\gamma^{val} &\leftarrow \mathsf{A}_\gamma(r_{\mathsf{F}}, \beta_1^{val}, \beta_2^{val}, \cdots, \beta_{\gamma-1}^{val}) \\
sk_{\mathsf{F}}^{inval} &\leftarrow \mathsf{A}_{\gamma+1}(r_{\mathsf{F}}, \beta_1^{val}, \beta_2^{val}, \cdots, \beta_{\gamma-1}^{val}, \beta_\gamma^{inval}).
\end{aligned}$$

Again, $\mathsf{F}$ checks if $\pi^{-1}_{sk_{\mathsf{F}}^{inval}}(\pi_{sk_{\mathsf{G}}^{val}}(s_{1(\gamma-1)})) = s_{1(\gamma-1)}$. Since the equality is not true with overwhelming probability, $\mathsf{F}$ computes $(r_{1\gamma}, r_{2\gamma}, r_{3\gamma}) \leftarrow \tilde{\pi}_{k_{\mathsf{F}}}(\pi_{sk_{\mathsf{G}}^{inval}}(s_{1(\gamma-1)}), s_{1(\gamma-1)}, s_{2(\gamma-1)})$. Finally, $\mathsf{F}$ outputs the following:

$$(\alpha_1^{val}, \alpha_2^{val}, \cdots, \alpha_\gamma^{val}, \pi_{sk_{\mathsf{F}}^{inval}}(r_{1\gamma}), r_{1\gamma}, r_{2\gamma}). \tag{34}$$

On (34), $\mathsf{G}$ also obtains its initial randomness $r_{\mathsf{G}} \leftarrow \pi_{k_{\mathsf{G}}}(\alpha_1)$ for key agreement. Notice that (34) contains all the valid messages $\alpha_1^{val}, \alpha_2^{val}, \cdots, \alpha_\gamma^{val}$, $\mathsf{G}$ can compute all the valid messages and shared key as follows:

$$\begin{aligned}
\beta_1^{val} &\leftarrow \mathsf{B}_1(r_{\mathsf{G}}, \alpha_1^{val}) \\
\beta_2^{val} &\leftarrow \mathsf{B}_2(r_{\mathsf{G}}, \alpha_1^{val}, \alpha_2^{val}) \\
&\vdots \\
\beta_\gamma^{val} &\leftarrow \mathsf{B}_\gamma(r_{\mathsf{G}}, \alpha_1^{val}, \alpha_2^{val}, \cdots, \alpha_\gamma^{val}) \\
sk_{\mathsf{G}}^{val} &\leftarrow \mathsf{B}_{\gamma+1}(r_{\mathsf{G}}, \alpha_1^{val}, \alpha_2^{val}, \cdots, \alpha_\gamma^{val})
\end{aligned}$$

Again, $\mathsf{G}$ checks to see if $\pi^{-1}_{sk^{val}_{\mathsf{G}}}(\pi_{sk^{inval}_{\mathsf{F}}}(r_{1\gamma})) = r_{1\gamma}$ where $\pi_{sk^{inval}_{\mathsf{F}}}(r_{1\gamma})$ and $r_{1\gamma}$ from (34). The equality is satisfied with only non-negligible probability due to the invalidity of $sk^{inval}_{\mathsf{F}}$ even if $sk^{val}_{\mathsf{G}}$ is the correctly shared key of $\mathsf{G}$. Therefore, $\mathsf{G}$ computes $(s_{1\gamma}, s_{2\gamma}, s_{3\gamma}) \leftarrow \tilde{\pi}_{k_{\mathsf{G}}}(\pi_{sk^{inval}_{\mathsf{F}}}(r_{1\gamma}), r_{1\gamma}, r_{2\gamma})$. Then, $\mathsf{G}$ outputs:

$$(\beta^{val}_1, \beta^{val}_2, \cdots, \beta^{val}_{\gamma}, \pi_{sk^{val}_{\mathsf{G}}}(s_{1\gamma}), s_{1\gamma}, s_{2\gamma}). \tag{35}$$

Given (35), we define the final $(\gamma + 1)$th adaptive input query $Q_{\gamma+1}$ as

$$Q_{\gamma+1} = (\underbrace{\beta^{val}_1, \beta^{val}_2, \cdots, \beta^{val}_{\gamma}, s_{1\gamma}, s_{2\gamma},}_{from(35)} \underbrace{u_*}_{from(28)}).$$

On $Q_{\gamma+1}$, $\mathsf{F}$ retrieves $r_{\mathsf{F}}$ from $\pi_{k_{\mathsf{F}}}(u^*)$. Now, $\mathsf{F}$ obtains all the valid messages $\beta^{val}_i$ for all $1 \leq i \leq \gamma$ from $Q_{\gamma+1}$ so that $\mathsf{G}$ can compute all the valid messages $\alpha^{val}_i$ for all $1 \leq i \leq \gamma$ and shared key $sk^{val}_{\mathsf{F}}$ by the computations described above. $\mathsf{F}$ checks to see if $\pi^{-1}_{sk^{val}_{\mathsf{F}}}(s_{1(\gamma-1)}) = s_{2(\gamma-1)}$ which does not hold. Computing $(r_{1(\gamma+1)}, r_{2(\gamma+1)}, r_{3(\gamma+1)}) \leftarrow \tilde{\pi}_{k_{\mathsf{F}}}(\pi_{sk^{val}_{\mathsf{G}}}(s_{1\gamma}), s_{1\gamma}, s_{2\gamma})$, $\mathsf{F}$ outputs a vector as:

$$(\alpha^{val}_1, \alpha^{val}_2, \cdots, \alpha^{val}_{\gamma}, \pi_{sk^{val}_{\mathsf{F}}}(r_{1(\gamma+1)}), r_{1(\gamma+1)}, r_{2(\gamma+1)}). \tag{36}$$

On (36), $\mathsf{G}$ can compute $sk^{val}_{\mathsf{G}}$ as in the previous rounds. $\mathsf{G}$ checks to see if $\pi^{-1}_{sk^{val}_{\mathsf{G}}}(\pi_{sk^{val}_{\mathsf{F}}}(r_{1(\gamma+1)}))$ $= r_{1(\gamma+1)}$. As the equality obviously holds, $\mathsf{G}$ is now convinced that (36) is adaptively generated. Therefore, $\mathsf{G}$ outputs the following vector:

$$(k^{'}_{\mathsf{G}}, \pi_{sk^{val}_{\mathsf{G}}}(k^{'}_{\mathsf{G}}), 0^n, \cdots, 0^n). \tag{37}$$

Now, we define the final adaptive query to be:

$$Q_{\gamma+1} = (\underbrace{\beta^{val}_1, \beta^{val}_2, \cdots, \beta^{val}_{\gamma}, \pi_{sk^{val}_{\mathsf{G}}}(k^{'}_{\mathsf{G}}), k^{'}_{\mathsf{G}},}_{from(37)} \underbrace{u_*}_{from(28)}).$$

On $Q_{\gamma+2}$, $\mathsf{F}$ retrieves shared key $sk^{val}_{\mathsf{F}}$ by the computations as upon $Q_{\gamma+1}$. $\mathsf{F}$ checks to see if $\pi^{-1}_{sk^{val}_{\mathsf{F}}}(\pi_{sk^{val}_{\mathsf{G}}}(k^{'}_{\mathsf{G}})) = k^{'}_{\mathsf{G}}$ which clearly holds. Hence, $\mathsf{F}$ outputs

$$(k^{'}_{\mathsf{G}}, k_{\mathsf{F}}, k^{'}_{\mathsf{F}}, 0^n, \ldots, 0^n). \tag{38}$$

$\mathsf{G}$ now discovers that the first coordinate of (38) equals $k^{'}_{\mathsf{G}}$. $\mathsf{G}$ reveals both secret keys of $\mathsf{G}$ and $\mathsf{F}$ by outputting a vector:

$$(k_{\mathsf{G}}, k^{'}_{\mathsf{G}}, k_{\mathsf{F}}, k^{'}_{\mathsf{F}}, 0^n, \ldots, 0^n).$$

□

By Claim 4.7, 4.8, and 4.9, we obtain the following impossibility of adaptively secure sequential composition under the existence of UTKA.

**Theorem 11.** *If $\gamma$-UTKA $\Phi_u = (\mathsf{A}, \mathsf{B})$ exists, then there exist non-adaptively secure functions $\mathsf{F}$ and $\mathsf{G}$ such that the sequential composition $\mathsf{G}(\mathsf{F}(\cdot))$ is $(\gamma+2)$-adaptive query breakable.*

Thus, putting the above theorem together with Pietrzak's result [Pie06], we have the equivalence of the above impossibility and the existence of UTKA as formally state below.

**Theorem 12.** *The sequential composition of two pseudo-random functions does not imply adaptive security if and only if the uniform-transcript key agreement exists.*

As we have proved the equivalence in the contexts of both parallel and sequential compositions by Theorem 9 and Theorem 12 respectively, we immediately obtain the following theorem which is our main theorem.

**Theorem 13.** *The composition of two pseudo-random functions does not imply adaptive security if and only if the uniform-transcript key agreement exists.*

## 5 Impossibility of Adaptively Secure Self-Composition

Self-composition is a composition of two or more copies of a single function. For instance, we call $\mathsf{F}(\mathsf{F}(\cdot))$ the sequential self-composition of function $\mathsf{F}$, and $\mathsf{F} \oplus \mathsf{F}$ the parallel self-composition of function $\mathsf{F}$. Note that several copies of identical $\mathsf{F}$'s must contain independent secret seeds. That is, each copy of $\mathsf{F}$'s must be allowed to be independently drawn from its function family.

So far, we proved the equivalence relation between the insecurity of composition and UTKA protocols. In fact, when we mention the insecurity of composition in previous sections, the main argument is rather that, given a non-adaptively secure function, there might be another non-adaptively secure function such that their composition is adaptively insecure. We call this type of composition *general-composition.* Hence, we still have a lingering unanswered question of whether the self-composition of a non-adaptively secure function implies the unconditional adaptive security. We answered the question negatively as follows.

Suppose that we are given non-adaptively secure pseudo-random functions $\mathsf{F}_k$ and $\mathsf{G}_{k'}$, without loss of generality, both defined over $\{0,1\}^n$ such that their parallel (general-)composition $(\mathsf{F} \oplus \mathsf{G})(\cdot)$ is adaptively insecure. Note that $k$ and $k'$ are independently chosen secret seeds for pseudo-random functions. That is, there exists a PPT adversary $\mathcal{A}$ with an adaptive adversarial strategy which succeeds in breaking the security of $(\mathsf{F} \oplus \mathsf{G})(\cdot)$ with non-negligible probability $\delta$. Now, we define a function family $\mathcal{F}_{(b,s)} : \{0,1\}^n \to \{0,1\}^n$ on input string $u$ by

$$\mathcal{F}_{(b,s)}(u) = \begin{cases} \mathsf{F}_s(u) & \text{if b} = 0 \\ \mathsf{G}_s(u) & \text{if b} = 1 \end{cases} \tag{$*$}$$

where $b$ and $s$ are private seeds.

It is easy to see that function $\mathcal{F}(\cdot)$ is also non-adaptively secure due to the non-adaptive security of functions $\mathsf{F}$ and $\mathsf{G}$. This trivially leads to

$$\mathbf{Adv}_{\mathcal{A}}^{\mathcal{F}} \leq \mathbf{Adv}_{\mathcal{A}}^{\mathsf{F}} + \mathbf{Adv}_{\mathcal{A}}^{\mathsf{G}}.$$

To break the adaptive security of $(\mathcal{F} \oplus \mathcal{F})(\cdot)$, it suffices to draw two copies of functions from the family at random and then use the same adaptively adversarial strategy of $\mathcal{A}$ as follows: the first bit of seeds of $\mathsf{F}$ and $\mathsf{G}$ differ in their first bit with probability 1/2. Therefore, if we draw two independent $\mathcal{F}$'s, then $\mathcal{F} \oplus \mathcal{F}$ is equivalent to $\mathsf{F} \oplus \mathsf{G}$ with probability 1/4 which is adaptively insecure.

Informally, by the above construction of $\mathcal{F}$ from any two non-adaptively secure functions $\mathsf{F}$ and $\mathsf{G}$ such that their parallel composition is not adaptively secure, we actually show that the adaptive insecurity of the parallel general-composition implies the adaptive insecurity of the parallel self-composition. We formally state this as follows.

**Theorem 14.** *Suppose there are two non-adaptively secure functions* $\mathsf{F}$ *and* $\mathsf{G}$ *such that the parallel composition* $(\mathsf{F} \oplus \mathsf{G})(\cdot)$ *is adaptively insecure. Then, there exists a non-adaptively secure function* $\mathcal{F}$ *such that the parallel self-composition is adaptively insecure.*

Combining the above theorem with the previous results of this paper in Sections 3.1 and 4.1 related to parallel composition insecurity from DTP and $\gamma$-UTKA, we obtain the following corollaries.

**Corollary 15.** *If a family of dense trapdoor permutations exists, then the parallel self-composition of a non-adaptively secure function does not imply adaptive security.*

**Corollary 16.** *If a UTKA exists, then the parallel self-composition of a non-adaptively secure function does not imply adaptive security.*

In addition, the above construction of $\mathcal{F}$ defined in $(*)$ can be applied to non-adaptively secure pseudo-random functions $\mathsf{F}$ and $\mathsf{G}$ such that their sequential general-composition is adaptively insecure. In particular, $\mathcal{F}$ is also non-adaptively secure while $\mathcal{F}(\mathcal{F}(\cdot))$ is equal to $\mathsf{G}(\mathsf{F}(\cdot))$ with probability $1/4$ when we draw two independent $\mathcal{F}$'s from its function family. That is, $\mathcal{F}$ is the same as $(*)$ and we measure the probability to be equally $1/2$ that the outer function $\mathcal{F}$ has the first bit of seed equal to 0 and the first bit of seed equal to 1. Thus, $\mathcal{F}(\mathcal{F}(\cdot))$ is also adaptively insecure. Consequently, we easily obtain the following theorem.

**Theorem 17.** *Suppose there are two non-adaptively secure functions* $\mathsf{F}$ *and* $\mathsf{G}$ *such that the sequential composition* $\mathsf{G}(\mathsf{F}(\cdot))$ *is adaptively insecure. Then, there exists a non-adaptively secure function* $\mathcal{F}$ *such that the self-composition is adaptively insecure.*

Again, combining the above theorem with the previous results of this paper in Sections 3 and 4 relevant to sequential composition insecurity from DTP and $\gamma$-UTKA, we derive the following corollaries.

**Corollary 18.** *If a family of dense trapdoor permutations exists, then the sequential self-composition of a non-adaptively secure function does not imply adaptive security.*

**Corollary 19.** *If a UTKA exists, then the sequential self-composition of a non-adaptively secure function does not imply adaptive security.*

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