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Parameterized Complexity of First-Order Logic

Correction

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1 Retraction of Main Result

In the previous version of this technical report we claimed the following result.

Theorem. (Corollary 5.7) Let $\mathcal{C}$ be a class of graphs. If $\mathcal{C}$ is nowhere dense, then $\text{MC}(\text{FO}, \mathcal{C})$ is fixed-parameter tractable. For every $\varepsilon > 0$, the running time of the algorithm for deciding whether a formula φ is true in a graph $G \in \mathcal{C}$ can be bounded by $f(|\varphi|) \cdot n^{1+\varepsilon}$, where $f : \mathbb{N} \to \mathbb{N}$ is a computable function.

When preparing a conference submission of the paper we discovered a flaw in the argument, which we have not been able to fix. We therefore retract the claim of the theorem.

Tractability of first-order model checking on nowhere-dense classes was also claimed by Dvořák and Král in [3, Theorem 10] citing an unpublished manuscript by the authors and R. Thomas. The final version of that paper [4] contains only the weaker result for classes of locally bounded expansion. To the best of our knowledge, the question for nowhere-dense classes remains open.

The argument outlined in the technical report does go through for classes of graphs of bounded expansion, which is a weaker statement, so that the following result holds true.

Theorem. First-order model-checking is fixed-parameter tractable by linear time parameterized algorithms on any class of graphs of bounded expansion (and hence on classes which exclude a fixed minor).

A detailed proof of this theorem can be found in [4] and also in [8].

In the previous version of this technical report we also claimed a lower bound, i.e. an intractability result for first-order model-checking on classes of graphs which are not nowhere dense. This result still holds true and we repeat the proof here.

2 Preliminaries

Our graph theoretical notation follows [1]. In particular, if G is a graph we refer to its set of vertices by $V(G)$ and to its set of edges by $E(G)$. All graphs in this paper are undirected and simple, i.e. without self-loops. A *colouring* of a graph G is an assignment of colours to the vertices of G. A colouring is *proper* if whenever $\{u, v\} \in E(G)$, then u and v are assigned different colours.

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We refer to [6, 5] for background on logic. The complexity theoretical framework we use in this paper is *parameterized complexity*. See [2, 7] for details. Let $\mathcal{C}$ be a class of coloured graphs. The *parameterized model-checking problem* $\text{MC}(\text{FO}, \mathcal{C})$ for first-order logic (FO) on $\mathcal{C}$ is defined as the problem to decide, given $G \in \mathcal{C}$ and $\varphi \in \text{FO}$, if $G \models \varphi$. The *parameter* is $|\varphi|$. $\text{MC}(\text{FO}, \mathcal{C})$ is *fixed-parameter tractable* (fpt), if for all $G \in \mathcal{C}$ and $\varphi \in \text{FO}$, $G \models \varphi$ can be decided in time $f(|\varphi|) \cdot |G|^c$, for some computable function $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{N}$. The class FPT is the class of all problems which are fixed-parameter tractable. In parameterized complexity theory it plays a similar role to polynomial time in classical complexity theory. The role of NP as a witness for intractability is played by a class called W$[1]$ and it is a standard assumption in parameterized complexity theory that FPT $\neq$ W$[1]$, similar to P $\neq$ NP in classical complexity. It has been shown that $\text{MC}(\text{FO}, \mathcal{G})$, where $\mathcal{G}$ is the class of all finite graphs, is complete for a parameterized complexity class called AW$[*]$ which is much larger than W$[1]$. Hence, unless FPT $=$ AW$[*]$, an assumtion widely disbelieved in the community, first-order model-checking is not fixed-parameter tractable on the class of all graphs.

Let G be a structure and $v_1, \dots, v_k$ be elements in $V(G)$. For $q \geq 0$, the *first-order q-type* $\text{tp}_q^G(\overline{v})$ of $\overline{v}$ is the class of all FO-formulas $\varphi(\overline{x})$ of quantifier-rank $\leq q$ such that $G \models \varphi(\overline{v})$. The first-order 0-type is referred to as the *atomic type* of $\overline{v}$ and denoted by $\text{atp}^G(\overline{v})$. We will usually omit the superscript G if its is clear from the context. A *first-order q-type* $\tau(\overline{x})$ is a maximally consistent class of formulas $\varphi(\overline{x})$.

By definition, types are infinite. However, it is well known that there are only finitely many FO-formulas of quantifier rank $\leq q$ which are pairwise not equivalent. Furthermore, we can effectively *normalise* formulas in such a way that equivalent formulas are normalised syntactically to the same formula. Hence, we can represent types by their finite set of normalised formulas and we can also check whether a formula belongs to a type. Note, though, that it is undecidable whether a set of formulas is a type as by definition, types are satisfiable.

We refer to [5] for a definition of Ehrenfeucht-Fraïssé games.

Let $\mathcal{C}$ be a class of graphs. The *first-order theory* $\text{Th}_{\text{FO}}(\mathcal{C})$ is defined as the class of first-order formulas true in all graphs $G \in \mathcal{C}$.

3 Nowhere Dense Classes of Graphs

In this section we present the concept of *nowhere dense* classes of graphs introduced in [10, 11].

A graph H is a *minor* of G (written $H \preccurlyeq G$) if H can be obtained from a sub-graph of G by contracting edges. An equivalent characterisation (see [1]) states that H is a minor of G if there is a map that associates to each vertex v of H a non-empty tree $G_v \subseteq G$ such that G_u and G_v are disjoint for $u \neq v$ and whenever there is an edge between u and v in H there is an edge in G between some node in G_u and some node in G_v. The sub-graphs G_v are called *branch sets*.

We say that H is a *minor at depth r* of G (and write $H \preccurlyeq_r G$) if H is a minor of G and this is witnessed by a collection of branch sets $\{G_v \mid v \in V(H)\}$, each of which induces a graph G_v of radius at most r. That is, for each $v \in V(H)$, there is a $w \in V(G)$ such that $G_v \subseteq N_r^{G_v}(w)$.

The following definition is due to Nešetřil and Ossona de Mendez [11].

3.1 Definition (nowhere dense classes). *A class of graphs $\mathcal{C}$ is said to be* nowhere dense *if for every r there is a graph H such that $H \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$.*

$\mathcal{C}$ is called somewhere dense *if it is not nowhere dense.*

It follows immediately from the definitions that if a class $\mathcal{C}$ of graphs which is not nowhere dense then there is a radius r such that every graph H is a depth r minor of some graph $G_H \in \mathcal{C}$. If, furthermore, $\mathcal{C}$ is closed under taking sub-graphs, then the depth-d image I_H of H in G_H is itself a graph in $\mathcal{C}$. Note that the size of I_H is polynomially bounded in H (for fixed r). Classes which are not nowhere dense are called *somewhere dense* in [11]. Let us call a class *effectively somewhere dense* if, given a graph H, a depth-d image $I_H \in \mathcal{C}$ of H in a graph $G_H \in \mathcal{C}$ can be computed in polynomial time.

4 Graph Classes which are Somewhere Dense

In this section we will show that essentially first-order model-checking is not fixed-parameter tractable on classes of graphs closed under sub-graphs which are somewhere dense..

Recall the definition of effectively somewhere dense classes of graphs in Section 2. If a class $\mathcal{C}$ of graphs is not nowhere dense then there is a radius r such that every graph H is a depth r minor of some graph $G_H \in \mathcal{C}$. If, furthermore, $\mathcal{C}$ is closed under taking sub-graphs, then the depth-d image I_H of H in G_H is itself a graph in $\mathcal{C}$. Note that the size of I_H is polynomially bounded in H (for fixed r). Classes which are not nowhere dense are called *somewhere dense* in [11]. Let us call a class *effectively somewhere dense* if, given a graph H, a depth-d image $I_H \in \mathcal{C}$ of H in a graph $G_H \in \mathcal{C}$ can be computed in polynomial time.

4.1 Theorem. *If $\mathcal{C}$ is closed under sub-graphs and effectively somewhere dense then* $\text{MC}(\text{FO}, \mathcal{C}) \notin \text{FPT}$ *unless* FPT= $\text{AW}[*]$.

To prove the theorem we will show that first-order model-checking on the class of all graphs, which is $\text{AW}[*]$ complete, is parameterized reducible to first-order model-checking on any effectively somewhere dense class closed under sub-graphs. We find it convenient to state this in terms of a first-order interpretations. See e.g. [9].

4.2 Definition. *Let $\sigma := \{E\}$ be the signature of graphs, where E is a binary relation symbol. A (one-dimensional)* interpretation *from σ-structures to σ-structures is a triple $\Gamma := (\varphi_{univ}(x), \varphi_{valid}, \varphi_E(x, y))$ of* $\text{FO}[\sigma]$*-formulas.*

For every σ-structure T with $T \models \varphi_{valid}$ we define a graph $G := \Gamma(T)$ as the graph with vertex set $V(G) := \{u \in V(T) : T \models \varphi_{univ}(v)\}$ and edge set $E(G) := \{\{u, v\} \in V(G) : T \models \varphi_E(u, v)\}$.

If $\mathcal{C}$ is a class of σ-structures we define $\Gamma(\mathcal{C}) := \{\Gamma(T) : T \in \mathcal{C}, T \models \varphi_{valid}\}$.

Every interpretation naturally defines a mapping from $\text{FO}[\sigma]$-formulas φ to $\text{FO}[\sigma]$-formulas $\varphi^* := \Gamma(\varphi)$. Here, φ^* is obtained from φ by recursively replacing

- first-order quantifiers $\exists x\varphi$ and $\forall x\varphi$ by $\exists x(\varphi_{univ}(x) \wedge \varphi^*)$ and $\forall x(\varphi_{univ}(x) \rightarrow \varphi^*)$ respectively, and
- atoms $E(x, y)$ by $\varphi_E(x, y)$.

The following lemma is easily proved (see [9]).

4.3 Lemma (interpretation lemma). *Let Γ be an* FO*-interpretation from σ-structures to σ-structures. Then for all* FO*-formulas and all σ-structures $G \models \varphi_{valid}$*

$$G \models \Gamma(\varphi) \quad \iff \quad \Gamma(G) \models \varphi.$$

4.4 Definition. *Let $\mathcal{C}, \mathcal{D}$ be classes of σ-structures. A* first-order reduction *(Γ, f) from $\mathcal{C}$ to $\mathcal{D}$ consists of a first-order interpretation Γ of $\mathcal{C}$ in $\mathcal{D}$ together with a polynomial-time computable function $f : \mathcal{C} \rightarrow \mathcal{D}$ such that for all $G \in \mathcal{C}$ and all $\varphi \in$* $\text{FO}[\sigma]$*,*

$$G \models \varphi \text{ if, and only if, } f(G) \models \Gamma(\varphi).$$

The following lemma follows immediately from the definitions.

4.5 Lemma. *Let $\mathcal{C}, \mathcal{D}$ be two classes of graphs and let (Γ, f) be a first-order reduction from $\mathcal{C}$ to $\mathcal{D}$. Then (Γ, f) is a parameterized reduction from* $\text{MC}(\text{FO}, \mathcal{C})$ *to* $\text{MC}(\text{FO}, \mathcal{D})$. *In particular, if* $\text{MC}(\text{FO}, \mathcal{D}) \in \text{FPT}$ *then* $\text{MC}(\text{FO}, \mathcal{C}) \in \text{FPT}$.

Let $\mathcal{G}$ be the class of all graphs and let $\mathcal{C}$ be an effectively somewhere dense class of graphs closed under sub-graphs. Let r be the radius as above such that every graph occurs as a depth r minor of some graph in $\mathcal{G}$. We first define the function $f : \mathcal{G} \to \mathcal{C}$.

Let $H \in \mathcal{G}$ be a graph. We construct a graph H' as follows. Let $I \subseteq V(H)$ be the set of isolated vertices in H and let $V := V(H) \setminus I$.

For every vertex $v \in V$ we add the following gadget $\rho(v) := (V_v, E_v)$ to H': $V_v := \{v, v_1, v_2\}$ and $E_v := \big\{\{v, v_1\}, \{v, v_2\}\big\}$. Hence, essentially, we take v and add two new neighbours of degree 1. For every edge $\{u, v\} \in E(H)$ we add a path of length $2r$ linking v and u in H'. Formally, we fix an ordering $\leq_H$ on $V(H)$ and let

$$\begin{aligned} V(H') := \quad & V(H) \dot\cup \{v_1, v_2 : v \in V(H) \setminus I\} \;\dot\cup \\ & \{e^i_{(v,w)} : 1 \leq i \leq 2r, \{u, v\} \in E(H), u \leq_H v\} \end{aligned}$$

and

$$\begin{aligned} E(H') := \quad & \Big\{\{v, e^1_{(v,w)}\}, \{w, e^{2r}_{(v,w)}\}, \{e^i_{(v,w)}, e^{i+1}_{(v,w)}\} : \begin{array}{l} 1 \leq i < 2r, v \leq_H w, \\ \{v, w\} \in E(H) \end{array} \Big\} \cup \\ & \Big\{\{v, v1\}, \{v, v_2\} : v \in V(H) \setminus I\Big\} \end{aligned}$$

Now, let $G_{H'}$ be a depth d image of H' in a graph $G \in \mathcal{C}$. As $\mathcal{C}$ is closed under sub-graphs, $G_{H'} \in \mathcal{C}$ and, as $\mathcal{C}$ is effectively somewhere dense, given H, we can compute $G_{H'}$ in polynomial time. We define $f(H) := G_{H'}$.

To complete the reduction we define a first-order interpretation of $\mathcal{G}$ in $\mathcal{C}$. For this, we let $\varphi_{\text{univ}}(x)$ be the formula that says x *is an isolated vertex* or x *has degree at least* 3 and *there are two disjoint paths of length at most r from x to vertices of degree* 1. Now let H be a graph and let $G := f(H)$ be the image of H' in $\mathcal{C}$. Then $\varphi_{\text{univ}}(x)$ will be true at all vertices in G which are copies of vertices $v \in V(H)$. Now to define the edges we take the formula $\varphi_E(x, y)$ which says that x, y satisfy φ_{univ} and there is a path between x and y of length at most $2r^2$. Finally, we let φ_{valid} be the formula that says *every vertex either satisfies φ_{univ} or lies on a path of length at most $4r^2$ between two vertices satisfying φ_{univ} and has degree* 2.

Now clearly, for all graphs $G \in \mathcal{G}$, $\Gamma(f(G)) \cong G$ and hence, by the interpretation lemma, $G \models \varphi$ if, and only if, $f(G) \models \Gamma(\varphi)$.

Theorem 4.1 now follows immediately from the fact that $\text{MC}(\text{FO}, \mathcal{G})$ is AW[$*$]-complete (see e.g.[7]).

A further consequence of this construction is the following

4.6 Corollary. *If $\mathcal{C}$ is a somewhere dense class of graphs closed under sub-graphs then* $\text{Th}_{\text{FO}}(\mathcal{C})$ *is undecidable.*

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