Electronic Colloquium on Computational Complexity, Report No. 24 (2010)

On a singular value method in quantum communication complexity

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Abstract. We introduce a new lower bound method for bounded-error quantum communication complexity, the *singular value method (svm)*, based on sums of squared singular values of the communication matrix, and we compare it with existing methods.
The first finding is a constant factor improvement of lower bounds based on the spectral norm. This is exemplified with an $n/2 - \mathcal{O}(1)$ lower bound for the inner product function mod two.
As our main result we exhibit a function based on quasi-random graphs such that svm yields a *linear* lower bound while the spectral norm method only yields a *constant* lower bound. In addition, we discuss the strength of svm and show that the class of languages with a low svm value is as hard as the communication complexity version of the polynomial hierarchy.

Key words: Quantum Communication Complexity, Singular Value Method, Lower Bound Method

1 Introduction

In communication complexity theory [1] communication models are studied where several players want to cooperatively solve a problem. The resource under consideration is communication, i.e. the number of communicated (quantum) bits. In general, the players have to communicate because the input is distributed among them. The arguably simplest communication model is Yao's model [2] where two players Alice and Bob want to compute the value $f(x, y)$ of a function $f\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. Here, Alice has $x \in \mathcal{X}$ and Bob has $y \in \mathcal{Y}$, and they may send each other messages according to a fixed protocol. Several variants of this deterministic model exist: In the public-coin randomized model the players are allowed to use a public source of randomness; in the private-coin model each player has his own source of randomness unknown to the other player. The deterministic model can also be enriched with guess strings. Here, the players want to solve a decision problem. They may guess bits and the acceptance of an input is determined by an acceptance mode. Yao [3] also introduced a quantum model where the players can send quantum bits instead of classical bits. Two variants are distinguished depending on whether the players share entangled states (e.g. Einstein-Podolsky-Rosen pairs) prior to communication or not.

All these models induce communication complexity measures. In particular, $\mathrm{R}_\varepsilon^{\mathrm{priv}}(f)$ is the ε-error private-coin randomized communication complexity of f; $\mathrm{Q}_\varepsilon(f)$ is the ε-error quantum communication complexity of f without prior shared entanglement, and $\mathrm{Q}_\varepsilon^*(f)$ is the one where prior shared entanglement is allowed.

For these complexities many lower bound methods have been developed. Some of them are based on notions of approximate rank, see e.g. Lee and Shraibman [4]. In this paper we use the Frobenius rank, which lower bounds several known approximate ranks. The advantage of this notion of approximate rank is a characterization via an expression in the singular values of the corresponding matrix that can be computed efficiently. We call

ISSN 1433-8092

this expression the *singular value method (svm)*. It has not appeared in the published literature before[1] as a lower bound method for bounded-error randomized or quantum communication complexity.

The purpose of this contribution is to explore the capabilities of svm as a new method in communication complexity, that is, to derive high lower bounds with it, to study the strength of this method, and to compare it with existing other methods. Our main findings are the following:

We obtain constant factor improvements of existing lower bounds for randomized and quantum communication complexity. First of all, we improve the spectral norm method of Krause for bounded-error private-coin randomized communication complexity by a factor of eight. Secondly, several lower bound methods for bounded-error quantum communication complexity are based on spectral norms. We improve such bounds by a factor of two. In particular, we show with ease an $n/2 - \mathcal{O}(1)$ lower bound for the bounded-error quantum communication complexity of the inner product function mod two.

Using singular values to lower bound quantum communication complexity is not a new idea. For example, almost a decade ago Klauck [6] introduced several such methods, one uses the entropy of the squared normalized singular values, another one uses Ky Fan norms. The novelty in our approach is *how* singular values are used in the computation of a lower bound. As our main result we present gaps between svm and previous approaches, i.e. we exhibit an explicit function F such that svm yields a *linear* lower bound on the quantum communication complexity of F while the spectral norm method and Klauck's Ky Fan method only yield *constant* lower bounds. Klauck's entropy method yields a bound of $\Theta(n/\log n)$. This function F is not an exception to the rule, but a representative of a large class of functions based on communication games on quasi-random graph families that lead to similar lower bounds.

We also study the strength of svm with structural means. (For definitions of communication complexity classes see, e.g., Babai et al. [7].) If problems with a low, i.e. polylog, singular value bound are collected in a communication complexity class $\mathbf{Fr}^{\mathrm{cc}}$ – let us call it *Frobenius class* –, then we show that this class is as hard as the the communication complexity version $\mathbf{PH}^{\mathrm{cc}}$ of the polynomial hierarchy. This shows that on the one hand, svm does not characterize randomized or quantum communication complexity. Instead, it gives high lower bounds only for problems far away from communication complexity classes like $\mathbf{BQP}^{\mathrm{cc}}$ or $\mathbf{NP}^{\mathrm{cc}}$. On the other hand, svm might turn out to be a useful tool in separating complexity classes not amenable to current methods, for example separating the polynomial hierarchy $\mathbf{PH}^{\mathrm{cc}}$ from polynomial space $\mathbf{PSPACE}^{\mathrm{cc}}$.

Note that in a few places of this contribution some of the details had to be omitted due to limitations in space. In these cases, appropriate references are given.

2 Preliminaries

We mainly work over the Boolean alphabet $\mathbb{B} := \{0, 1\}$ and the sign alphabet $\mathbb{S} := \{-1, +1\}$. Accordingly, we call functions with range $\mathbb{B}$ Boolean, and functions with range $\mathbb{S}$ sign

[1] We recently noticed that Lokam defined Frobenius rank and rigidity in a survey paper [5] under the names "ℓ_2-rank" and "ℓ_2-rigidity", respectively, and gave a characterization of ℓ_2-rigidity via singular values. Lemma 3.5 of his contribution is a reformulation of the Theorem of Eckart and Young. Lokam compared ℓ_2-rigidity with geometric rigidity, but did not give any applications of ℓ_2-rigidity. In particular, he did not develop from this a lower bound method in communication complexity as is done in the paper at hand.

functions. We adopt the same terminology for matrices. By $\delta_{\alpha,\beta}$ we denote the Kronecker Delta, which is defined as $\delta_{\alpha,\beta} := 1$ if $\alpha = \beta$, and $\delta_{\alpha,\beta} := 0$ otherwise. By $|\cdot|$ we denote the Hamming weight of Boolean vectors. More generally, for arbitrary matrices A we define the Hamming weight $\mathrm{wt}(A)$ as the number of nonzero entries in A. As usual, $[n] := \{1, \ldots, n\}$. Occasionally, in order to avoid ugly case distinctions we use *Iverson's bracket* $[P]$ defined on predicates P, which evaluates to 1, if P is true, and to 0 otherwise.

2.1 Matrix theory

The aim of this subsection is to clarify our notation concerning matrices and to recall some definitions and results of major importance for our work. For a thorough introduction to matrix theory we refer the reader to [8–11].

For complex matrices $A \in \mathbb{C}^{m\times n}$, A^* denotes the conjugate transpose of A; entrywise complex conjugation is indicated by a bar, as in $\bar{A}$. For n-square matrices A we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A, including repeated ones. In case A is a Hermitian matrix, its eigenvalues are real numbers, and we order them such that $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. By $\mathrm{tr}(A)$ we denote the trace of A.

In particular, A^*A is Hermitian and positive semidefinite for all $A \in \mathbb{C}^{m\times n}$. This justifies the definition of the singular values of an $m \times n$ matrix A as $\sigma_i(A) := \sqrt{\lambda_i(A^*A)}$, $i \in [n]$. Thus, we have $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$ by definition. We denote by $\sigma(A) := (\sigma_1(A), \ldots, \sigma_n(A))$ the row vector containing the singular values of A in decreasing order. Every complex $m \times n$ matrix A has a singular value decomposition $A = U\Sigma V$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, and $\Sigma \in \mathbb{R}^{m\times n}$ is diagonal with entries $\sigma_1(A), \ldots, \sigma_{\min\{m,n\}}(A)$.

Let $A \in \mathbb{C}^{m\times n}$ be a matrix. We denote by $\|A\|_{\ell_p} := \left(\sum_{i,j} |A_{i,j}|^p\right)^{1/p}$, $p \geq 1$, the ℓ_p vector norm of A. In particular, the Frobenius or Hilbert-Schmidt norm is defined as $\|A\|_{\mathrm{F}} := \|A\|_{\ell_2}$. It has the additional property of being a matrix norm, and it can also be regarded as the norm derived from the Frobenius or Hilbert-Schmidt inner product for complex $m \times n$ matrices A and B, which by definition reads $\langle A, B\rangle := \mathrm{tr}(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{A}_{i,j}B_{i,j}$. An important class of matrix norms on $\mathbb{C}^{m\times n}$ are the Schatten p-norms. They are defined as the ℓ_p norms of the singular values: $\|A\|_p := \|\sigma(A)\|_{\ell_p} = \left(\sum_{i=1}^{n} \sigma_i^p(A)\right)^{1/p}$, $p \geq 1$. Two examples of Schatten norms are the trace norm $\|A\|_1 = \sigma_1(A) + \cdots + \sigma_n(A)$ and the spectral norm $\|A\|_\infty = \sigma_1(A)$. It is straightforward to show that $\|A\|_2 = \|A\|_{\mathrm{F}}$ and $\|A\|_\infty = \|A\|$, where $\|A\|$ denotes the operator norm $\|A\| := \sup_{x\in\mathbb{C}^n, \|x\|_{\ell_2}=1} \|Ax\|_{\ell_2}$. We will use the Theorem of Eckart and Young [12] to characterize a variant of rigidity that we define in the next section. According to Horn and Johnson [9], an analogon [13] of the theorem was discovered by E. Schmidt 30 years earlier in the context of integral equations.

Fact 2.1 (Eckart and Young [12]). *For every real $m \times n$ matrix A we have*

$$\min_{\substack{B\in\mathbb{R}^{m\times n} \\ \mathrm{rank}(B)\leq r}} \|A-B\|_{\mathrm{F}}^2 = \sum_{i=r+1}^{n} \sigma_i^2(A)\,.$$

This fact will allow us to express lower bounds on communication complexity in terms of singular values.

2.2 Fourier transform

For an overview of Fourier analysis on $\mathbb{B}^n$ and its applications in computer science we refer the reader to [14]. Here, we give the definition of Fourier coefficients, which will be

sufficient in this contribution. On the 2^n-dimensional vector space of functions $f\colon \mathbb{B}^n \to \mathbb{C}$, an inner product is given by $\langle f, g\rangle := \frac{1}{2^n}\sum_{x\in\mathbb{B}^n} \overline{f(x)}g(x)$. For each $\alpha \in \mathbb{B}^n$, the character $\chi_\alpha\colon \mathbb{B}^n \to \mathbb{S}$ is the function $\chi_\alpha(x) := (-1)^{\alpha_1 x_1 + \ldots + \alpha_n x_n}$. It is straightforward to show that $\langle \chi_\alpha, \chi_\beta\rangle = \delta_{\alpha,\beta}$, that is, the 2^n characters make up an orthonormal basis – the so-called Fourier basis – for the space of all complex-valued functions on $\mathbb{B}^n$. The expansion of f in this basis reads $f(x) = \sum_{\alpha\in\mathbb{B}^n} \langle \chi_\alpha, f\rangle\ \chi_\alpha(x) = \sum_{\alpha\in\mathbb{B}^n} \hat{f}(\alpha)\,\chi_\alpha(x)$, where the coefficients $\hat{f}(\alpha) := \langle \chi_\alpha, f\rangle$ are known as the Fourier coefficients of f.

3 Approximate ranks and rigidities as lower bounds

The concept of (matrix) rigidity was introduced by Valiant [15] as a tool to derive lower bounds in circuit complexity. A matrix has high rigidity, if small perturbations do not lower the rank much. Thereby, "small perturbations" means that only a small number of the matrix entries may be modified. Proving a strong enough lower bound on the rigidity of a matrix implies a non-trivial lower bound, i.e. a superlinear size or a superlogarithmic depth, on the complexity of any linear circuit computing the set of linear forms associated with it. Although it has been shown that most matrices have high rigidity, despite considerable efforts by many researchers no explicit construction of a rigid family of matrices over finite fields is known. For infinite fields Lokam [16] was able to derive quadratic lower bounds for the rigidity of explicit matrix families.

The formal definition of matrix rigidity is given below for the sake of completeness.

Definition 3.1 (Rigidity). *Let M be a matrix over a field $\mathbb{F}$. The (matrix) rigidity $\mathrm{R}_M^{\mathbb{F}}$ of M is defined as $\mathrm{R}_M^{\mathbb{F}}(r) := \min\left\{\mathrm{wt}(\tilde{M}-M) \;\middle|\; \mathbb{F}\text{-rank}(\tilde{M}) \le r, \tilde{M} \text{ a matrix over } \mathbb{F}\right\}$. In other words, the rigidity of M is the minimum number of entries that must be changed in order to reduce the rank to r.*

The first connection between matrix rigidity and communication complexity was established by Razborov [17]. He showed that high lower bounds for the rigidity over a finite field of an explicit matrix family would yield a language outside the communication complexity theoretic analogon $\mathbf{PH}^{\mathrm{cc}}$ of the polynomial hierarchy. It was shown in [18,19] that this result is a corollary of a slight generalization of Toda's First Theorem in communication complexity, $\mathbf{PH}^{\mathrm{cc}} \subseteq \mathrm{BP}\cdot\mathrm{MOD}_p\cdot\mathbf{P}^{\mathrm{cc}}$, because rigidity over the finite field $\mathbb{F}_p$ is a lower bound for the $\mathrm{BP}\cdot\mathrm{MOD}_p\cdot\mathbf{P}^{\mathrm{cc}}$ communication complexity.

Lokam [20] defined weak variants of matrix rigidity over the field of real numbers $\mathbb{R}$ and was able to complement and strengthen the result of Razborov.

To each notion of rigidity there is an equivalent notion of approximate rank, and vice versa. Several notions of approximate rank have been defined to lower bound communication complexities. The most important ones are

Definition 3.2 (Approximate rank). *Let A be a real $m\times n$ matrix, and let $\alpha \ge 1$. We define*

$$\mathrm{rank}^{\alpha}(A) := \min\{\mathrm{rank}(B) \mid B\in\mathbb{R}^{m\times n}, 1 \le A_{i,j}B_{i,j} \le \alpha\},$$
$$\mathrm{rank}^{\infty}(A) := \min\{\mathrm{rank}(B) \mid B\in\mathbb{R}^{m\times n}, 1 \le A_{i,j}B_{i,j}\}.$$

The former is called α-approximate rank of A, the latter sign rank of A.

Obviously, rank^{α} is a monotonically decreasing function with respect to α. Paturi and Simon [21] gave a characterization of unbounded-error randomized communication complexity $\mathrm{U}(f) := \inf_{\varepsilon<1/2}\mathrm{R}_{\varepsilon}^{\mathrm{priv}}(f)$ via the sign rank of the communication matrix $M^f := (f(x,y))_{x,y}$ of f.

Fact 3.3 (Paturi and Simon [21]).
For every sign function f, $\mathrm{U}(f) = \log_2 \mathrm{rank}^{\infty}(M^f) + \mathcal{O}(1)$.

Let us define the spectral norm method as follows:

Definition 3.4 (Spectral norm method). *We define* $\mathrm{spec}(A) \coloneqq \|A\|_{\mathrm{F}}/\|A\|$ *for every* $m \times n$ *matrix* A. *For functions* f *we introduce the abbreviation* $\mathrm{spec}(f) \coloneqq \mathrm{spec}\,(M^f)$.

Krause [22] defined a slight variant of rank^{α} and proved a lower bound for the bounded-error private-coin randomized communication complexity of a sign function. In particular, he showed that the $(1/2 - \delta)$-error private-coin randomized communication complexity of a sign function f is at least $(1/4)(\log_2 \mathrm{spec}\,(f) - (1/2)\log_2(1/\delta) - 2)$.

An adaptation to the α-approximate rank, $\mathrm{rank}^{\alpha}(M^f)$, yields

Fact 3.5 (Krause [4,22]). *For every sign function f and every $\varepsilon \in [0, 1/2)$,*

$$\mathrm{R}^{\mathrm{priv}}_{\varepsilon}(f) \geq \log_2 \mathrm{rank}^{\alpha_\varepsilon}(M^f), \quad \alpha_\varepsilon \coloneqq 1/(1-2\varepsilon)\,.$$

In a breakthrough work Forster [23] showed that the spectral norm method is even a lower bound for unbounded-error randomized communication complexity.

Fact 3.6 (Forster [23]). *For every $m \times n$ sign matrix A we have*

$$\mathrm{rank}^{\infty}(A) \geq \mathrm{spec}(A) = \frac{\sqrt{mn}}{\|A\|}\,. \tag{1}$$

In particular, by (3.3) *for every sign function f we have* $\mathrm{U}(f) \geq \log_2 \mathrm{spec}\,(f)$.

For $A \in \mathbb{R}^{m\times n}$ and $\varepsilon \geq 0$, Buhrman and de Wolf [24] defined a notion of approximate rank by $\widetilde{\mathrm{rank}}_{\varepsilon}(A) \coloneqq \min\{\,\mathrm{rank}(B) \mid B \in \mathbb{R}^{m\times n},\ \|A - B\|_{\ell_\infty} \leq \varepsilon\,\}$. They showed that for every Boolean function f, their approximate rank is a lower bound for the bounded-error quantum communication complexity, $\mathrm{Q}_{\varepsilon}(f) \geq \frac{1}{2}\log_2 \widetilde{\mathrm{rank}}_{\varepsilon}(M^f)$. An adaptation to the α-approximate rank, $\mathrm{rank}^{\alpha}(M^f)$, of sign functions f yields

Fact 3.7 (Buhrman and de Wolf [4,24]). *For every sign function f and $\varepsilon \in [0, 1/2)$,*

$$\mathrm{Q}_{\varepsilon}(f) \geq \frac{1}{2}\log_2 \mathrm{rank}^{\alpha_\varepsilon}(M^f), \quad \alpha_\varepsilon \coloneqq 1/(1-2\varepsilon)\,.$$

4 The singular value method

In this section, we introduce our *singular value method*. For sign matrices this method lower bounds bounded-error quantum communication complexity without prior entanglement.

Definition 4.1 (Frobenius rigidity and rank). *Let A be a real $m\times n$ matrix. We define the Frobenius rigidity of A by*

$$\mathrm{rigidity}^{\mathrm{F}}_A(r) \coloneqq \min_{\substack{B\in\mathbb{R}^{m\times n}\\ \mathrm{rank}(B)\leq r}} \frac{\|A-B\|^2_{\mathrm{F}}}{\|A\|^2_{\mathrm{F}}}\,.$$

The Frobenius rank is analogously defined by

$$\mathrm{rank}^{\mathrm{F}}_{\varepsilon}(A) \coloneqq \min\{\,\mathrm{rank}(B) \mid B \in \mathbb{R}^{m\times n}, \|A - B\|^2_{\mathrm{F}} \leq \varepsilon\|A\|^2_{\mathrm{F}}\,\}\ .$$

Thus, unlike the rigidity $\mathrm{R}_M^{\mathbb{F}}$ and the α-approximate and sign ranks, the Frobenius rigidity and rank employ $\|A–B\|_{\mathrm{F}}$ to define a notion of closeness of the matrices A and B. From the definition of approximate rank and rigidity it follows that the two concepts are essentially equivalent:

$$\operatorname{rank}_\varepsilon^{\mathrm{F}}(A) \leq r \iff \operatorname{rigidity}_A^{\mathrm{F}}(r) \leq \varepsilon\,. \tag{2}$$

The Theorem of Eckart and Young immediately yields the following characterization of Frobenius rigidity:

$$\operatorname{rigidity}_A^{\mathrm{F}}(r) = \frac{1}{\|A\|_{\mathrm{F}}^2} \sum_{i=r+1}^{n} \sigma_i^2(A)\,. \tag{3}$$

Fortunately, there are simple connections between Frobenius rank and different notions of approximate ranks.

Proposition 4.2. *For every* sign *matrix A and every $\varepsilon \geq 0$ we have*

$$\widetilde{\operatorname{rank}}_\varepsilon(A) \geq \operatorname{rank}_{\varepsilon^2}^{\mathrm{F}}(A)\,. \tag{4}$$

Proof. Let A be an $m \times n$ sign matrix, and let B be an $m \times n$ real matrix such that $\widetilde{\operatorname{rank}}_\varepsilon(A) = \operatorname{rank}(B)$ and $\|A-B\|_{\ell_\infty} \leq \varepsilon$. Note that $\|A-B\|_{\mathrm{F}}^2 \leq \varepsilon^2 \cdot mn = \varepsilon^2 \cdot \|A\|_{\mathrm{F}}^2$. This shows $\operatorname{rank}_{\varepsilon^2}^{\mathrm{F}}(A) \leq \operatorname{rank}(B)$. □

Proposition 4.3. *For every* sign *matrix A and every $\alpha \geq 1$ we have*

$$\operatorname{rank}^{\alpha}(A) \geq \operatorname{rank}_{\alpha^2-1}^{\mathrm{F}}(A)\,. \tag{5}$$

Proof. Let A be an $m \times n$ sign matrix, and let B be an $m \times n$ real matrix such that $\operatorname{rank}^{\alpha}(A) = \operatorname{rank}(B)$ and $1 \leq A_{i,j}B_{i,j} \leq \alpha$. We have $\|A-B\|_{\mathrm{F}}^2 = \sum_{i,j}(A_{i,j} - B_{i,j})^2 = \sum_{i,j} A_{i,j}^2 + \sum_{i,j} B_{i,j}^2 - 2\sum_{i,j} A_{i,j}B_{i,j} \leq mn + \alpha^2 mn - 2mn = (\alpha^2 - 1)\,\|A\|_{\mathrm{F}}^2$. This shows $\operatorname{rank}_{\alpha^2-1}^{\mathrm{F}}(A) \leq \operatorname{rank}(B)$. □

Definition 4.4 (Singular value method). *Let A be a real $m \times n$ matrix, and let $\varepsilon \in [0,1]$. We define the singular value method (svm) by*

$$\begin{aligned}
\operatorname{svm}_\varepsilon(A) &:= \max\left\{ r \geq 1 \;\middle|\; \operatorname{rigidity}_A^{\mathrm{F}}(r-1) > \varepsilon \right\} \\
&= \min\left\{ r \geq 1 \;\middle|\; \operatorname{rigidity}_A^{\mathrm{F}}(r) \leq \varepsilon \right\} \\
&= \min\left\{ r \geq 1 \;\middle|\; \sum_{i=1}^{r} \sigma_i^2(A) \geq (1-\varepsilon) \cdot \|A\|_{\mathrm{F}}^2 \right\}.
\end{aligned}$$

For a function f we define $\operatorname{svm}_\varepsilon(f) := \operatorname{svm}_\varepsilon(M^f)$.

Note that the maximum in the definition of the singular value method always exists, since $\sigma_1^2(A) + \cdots + \sigma_n^2(A) = \|A\|_{\mathrm{F}}^2$. If A is a sign matrix, we have $\|A\|_{\mathrm{F}}^2 = mn$, whereas $\|A\|_{\mathrm{F}}^2$ is the number of ones in A if A is Boolean.

Observation 4.5. *Due to the connection* (2) *between Frobenius rigidity and Frobenius rank, the singular value method coincides with Frobenius rank, i.e. we have* $\operatorname{svm}_\varepsilon(A) = \operatorname{rank}_\varepsilon^{\mathrm{F}}(A)$ *for all real matrices A.*

The singular value method is in some respect similar to a lower bound of Lokam [20, Lemma 3.4] for the minimum size of a depth d linear circuit computing a linear transformation. While many methods for randomized or quantum communication complexity have been developed that involve singular values, to the authors' knowledge our singular value method has not appeared in the published literature before.

Theorem 4.6 (Quantum lower bound). *For every sign function f and every $\varepsilon \in [0, 1/2 - 1/\sqrt{8}]$ we have*

$$\mathrm{Q}_\varepsilon(f) \geq \frac{1}{2} \log_2 \mathrm{svm}_{\alpha_\varepsilon^2 - 1}(f)\,, \quad \alpha_\varepsilon := \frac{1}{1-2\varepsilon}\,. \tag{6}$$

Proof. Combining Fact 3.7, Proposition 4.3 and Observation 4.5 we conclude

$$\mathrm{Q}_\varepsilon(f) \geq \tfrac{1}{2} \log_2 \mathrm{rank}^{\alpha_\varepsilon}\big(M^f\big) \geq \tfrac{1}{2} \log_2 \mathrm{rank}^{\mathrm{F}}_{\alpha_\varepsilon^2 - 1}\big(M^f\big) = \tfrac{1}{2} \log_2 \mathrm{svm}_{\alpha_\varepsilon^2 - 1}(f)\,.$$

□

Theorem 4.7. *For every $m \times n$ sign matrix A and all $\varepsilon \in [0,1]$, svm is lower bounded by*

$$\mathrm{svm}_\varepsilon(A) \geq (1-\varepsilon) \cdot \big(\mathrm{spec}(A)\big)^2\,. \tag{7}$$

Furthermore, the approximate rank satisfies the inequality

$$\mathrm{rank}^\alpha(A) \geq (2 - \alpha^2) \cdot \big(\mathrm{spec}(A)\big)^2 \tag{8}$$

for all $\alpha \in [1, \sqrt{2}]$.

Proof. By the definition of the singular value method and the spectral method, in conjunction with the monotonicity of the singular values, we obtain (7):

$$\frac{\mathrm{svm}_\varepsilon(A)}{(\mathrm{spec}(A))^2} = \mathrm{svm}_\varepsilon(A)\, \frac{\sigma_1^2(A)}{\|A\|_{\mathrm{F}}^2} \geq \sum_{i=1}^{\mathrm{svm}_\varepsilon(A)} \frac{\sigma_i^2(A)}{\|A\|_{\mathrm{F}}^2} \geq 1 - \varepsilon\,.$$

We have seen in the proof of Theorem 4.6 that $\mathrm{svm}_{\alpha^2-1}$ is a lower bound for rank^α. Thus, (8) is a consequence of (7). □

We note that (8) in conjunction with Fact 3.5 implies $\mathrm{R}_\varepsilon^{\mathrm{priv}}(f) \geq 2\log_2 \mathrm{spec}(f) + \log_2(2 - \alpha_\varepsilon^2)$, improving the original bound of Krause by a factor of 8. Furthermore, inequality (8) provides an additional exponent of two, compared to the statement $\mathrm{rank}^\alpha(A) \geq \mathrm{spec}(A)$ that can be obtained from lower bounding rank^α by rank^∞ and then applying (1).

5 Lower bound for the inner product function mod two

The inner product function mod two by definition reads $\mathrm{IP}_n \colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$, $\mathrm{IP}_n(x,y) := 1 - 2\,[\sum_{i=1}^n x_i y_i \bmod 2 = 1]$. We will recall the lower bounds on $\mathrm{Q}_\varepsilon(\mathrm{IP}_n)$ obtainable by previously known methods, and subsequently show that the singular value method also leads to the best of these bounds.

The lower bound by Buhrman and de Wolf (Fact 3.7), in conjunction with the monotonicity of rank^α and the result of Forster (Fact 3.6), leads to

$$\mathrm{Q}_\varepsilon(\mathrm{IP}_n) \geq \frac{1}{2}\, \log_2 \mathrm{rank}^\infty \left(M^{\mathrm{IP}_n}\right) \geq \frac{1}{2}\, \log_2 \mathrm{spec}\left(\mathrm{IP}_n\right)\,.$$

It is easily verified that all singular values of M^{IP_n} are equal. Thus, $\mathrm{spec}\left(\mathrm{IP}_n\right) = 2^{n/2}$, so that the old results only lead to a bound of $\mathrm{Q}_\varepsilon(\mathrm{IP}_n) \geq n/4$.

In contrast, using svm, we obtain the following

Theorem 5.1. *For arbitrary* $\varepsilon \in \left[0, 1/2 - 1/\sqrt{8}\right[$*, the quantum communication complexity of* IP_n *without prior entanglement is lower bounded by* $\mathrm{Q}_\varepsilon(\mathrm{IP}_n) \geq \frac{n}{2} + \frac{1}{2}\log_2(2-\alpha_\varepsilon^2)$.

Proof. The proof is based on Theorems 4.6 and 4.7:

$$\begin{aligned}\mathrm{Q}_\varepsilon(\mathrm{IP}_n) &\geq \frac{1}{2}\log_2 \mathrm{svm}_{\alpha_\varepsilon^2-1}(\mathrm{IP}_n) \geq \frac{1}{2}\log_2\left(\left(2-\alpha_\varepsilon^2\right)\left(\mathrm{spec}(\mathrm{IP}_n)\right)^2\right)\\ &= \log_2 \mathrm{spec}(\mathrm{IP}_n) + \frac{1}{2}\log_2\left(2-\alpha_\varepsilon^2\right).\end{aligned}$$

As stated above, the spectral norm method for IP_n is $\mathrm{spec}(\mathrm{IP}_n) = 2^{n/2}$. □

Now consider the factorization norm lower bound of Linial and Shraibman [25]. This method is in fact a lower bound on Q^*_ε. Since $\mathrm{Q}^*_\varepsilon(f) \leq \lceil n/2\rceil + 1$ for all functions $f\colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$ by superdense coding, the factorization norm method cannot lead to a lower bound greater than $\lceil n/2\rceil + 1$. The same holds for the discrepancy lower bound [26], since it is subsumed by the factorization norm method [25].

6 Comparison with other singular value methods

In order to prove large gaps between svm and other methods based on singular values, we consider the following communication game on a graph $G \coloneqq (V, E)$. Alice has $x \in V$, Bob has $y \in V$ and they want to know if $\{x, y\}$ is an edge in G. Clearly, if $\mathrm{EDGE}_G(x, y) \coloneqq 1 - 2\,[\{x, y\} \in E]$ denotes the sign version of this game, then $M^{\mathrm{EDGE}_G} = J - 2A^G$, where A^G is the adjacency matrix of G and J is the all-one matrix.

We get large gaps for graphs with a single large eigenvalue and and a high spectral gap. For this we consider quasi-random graphs [27] and choose as a representative the following one: For a prime power q and a natural number $k \leq q$, the *Delsarte-Goethals-Turyn graph*, $G_{q,k}$, as defined in [27, p. 23–24, 5.], is a regular graph of degree $D(G_{q,k}) = k(q-1)$ on $|V(G_{q,k})| = q^2$ nodes, and the eigenvalues of its adjacency matrix are $\lambda_1\big(A^{G_{q,k}}\big) = D\left(G_{q,k}\right)$ and $\lambda_i\big(A^{G_{q,k}}\big) \in \{-k, q-k\}$ for $i \geq 2$.

For $n \geq 2$ we define $q = q(n) \coloneqq 2^n$, $k = k(n) := q/4$, and finally $F_{2n} \coloneqq \mathrm{EDGE}_{G_{q(n),k(n)}}$. Then $\sigma_1\big(M^{F_{2n}}\big) = \lambda_1\big(M^{F_{2n}}\big) = |V(G_{q,k})| - 2\lambda_1\big(A^{G_{q,k}}\big) = (q^2+q)/2$ and $\lambda_i\big(M^{F_{2n}}\big) \in \{q/2, -3q/2\}$, $i \geq 2$. Thus, $q/2 \leq \sigma_i\big(M^{F_{2n}}\big) \leq 3q/2$ for $i \geq 2$. In addition, $\|M^{F_{2n}}\|_F^2 = |V(G_{q,k})|^2 = q^4$.

First of all, we calculate the spectral norm method for F_{2n}: We have $\mathrm{spec}\left(F_{2n}\right) = 2q^2/(q^2+q) < 2$.

In contrast, the singular value method yields a linear lower bound: Let $r \geq 1$ be minimal such that $\sum_{i=1}^r \sigma_i^2\big(M^{F_{2n}}\big) \geq (1-\varepsilon)\cdot\|M^{F_{2n}}\|_F^2$. Then $\frac{1}{4}(q^2+q)^2 + (r-1)\frac{9}{4}q^2 \geq (1-\varepsilon)q^4$, and thus $r \geq \frac{4}{9}\left(\frac{1}{2}-\varepsilon\right)q^2$. For e.g. $\varepsilon = 1/3$ we obtain $\log_2 \mathrm{svm}_{1/3}\left(F_{2n}\right) = 2n - \mathcal{O}(1)$.

Theorem 6.1 (Arbitrary gap). *There exists an explicit function* $F_{2n}\colon \mathbb{B}^{2n} \times \mathbb{B}^{2n} \to \mathbb{S}$*,* $n \geq 2$*, such that the singular value method yields* $\log_2 \mathrm{svm}_{1/3}\left(F_{2n}\right) = 2n - \mathcal{O}(1)$ *while the spectral norm method only yields* $\log_2 \mathrm{spec}\left(F_{2n}\right) = \mathcal{O}(1)$.

Note that in order to obtain this statement, we could have chosen any D-regular dense quasi-random graph on N nodes with high spectral gap such that D is bounded away from $N/2$.

Klauck [6, Thm. 6.10] defined two lower bound methods for bounded-error quantum communication complexity via singular values. For a function $f\colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$, he defined

normalized singular values $\tilde{\sigma}_i(f) \coloneqq \sigma_i(M^f)/2^n, i \in [2^n]$. Then $\tilde{\sigma}^2(f) \coloneqq \big(\tilde{\sigma}_1^2(f), \ldots, \tilde{\sigma}_{2^n}^2(f)\big)$ is a probability distribution. Let $H(p_1, \ldots, p_m)$ denote the Shannon entropy of a probability distribution $p_1, \ldots, p_m$, and let $\kappa_l(f) \coloneqq \sum_{i=1}^{l} \tilde{\sigma}_i(f)$ denote the l-th Ky Fan norm. Klauck proved the following result:

Theorem 6.2 (Klauck [6]). *For a sign function $f\colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$ we have*

$$\mathrm{Q}_\varepsilon(f) = \Omega\big(H(\tilde{\sigma}^2(f))/\log n\big) \ . \tag{9}$$

Let $\kappa_l \coloneqq \kappa_l(f)$. If $\kappa_l \geq \Omega\big(\sqrt{l}\big)$, then $\mathrm{Q}_\varepsilon(f) \geq \Omega\big(\log(\kappa_l)\big)$.
If $\kappa_l \leq \mathcal{O}\big(\sqrt{l}\big)$, then $\mathrm{Q}_\varepsilon(f) \geq \Omega\big(\log(\kappa_l)/(\log(\sqrt{l}) - \log(\kappa_l) + 1)\big)$.

We note that Klauck's entropy bound achieves $H(\tilde{\sigma}^2(F_{2n})) = \Theta(n/\log n)$, because of

$$\begin{aligned} H\left(\tilde{\sigma}^2\left(F_{2n}\right)\right) &\leq -\frac{1}{4}\left(1+\frac{1}{q}\right)^2 \log_2\left(\frac{1}{4}\left(1+\frac{1}{q}\right)^2\right) + \left(q^2-1\right)\cdot\left(\frac{9}{4q^2}\right)\log_2\left(4q^2\right) \\ &= \mathcal{O}\left(\log q\right) \ = \ \mathcal{O}\left(n\right) \ , \end{aligned}$$

while his Ky Fan bound $\kappa_l\left(F_{2n}\right) \leq \frac{1}{2} + \frac{3}{2}\frac{l}{q}$ only yields a constant lower bound.

7 On the strength of svm

In this section, we show for which functions one can expect the singular value method to provide high lower bounds. In particular, we show that all problems in $\mathbf{PH}^{\mathrm{cc}}$ can be solved by protocols in $\mathbf{P}^{\mathrm{cc}}$ if we admit oracle queries to a function with low svm value. To keep notation concise, we introduce the communication complexity class of languages that exhibit an svm value at most polylogarithmic in n. Here, a language $L \coloneqq (L_n)_{n\in\mathbb{N}}$ is defined as a family of functions $L_n\colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$. We define the Frobenius class $\mathbf{Fr}^{\mathrm{cc}}$ as

$$\mathbf{Fr}^{\mathrm{cc}} \coloneqq \left\{L \;\middle|\; L = (L_n)_{n\in\mathbb{N}},\ \log_2 \mathrm{svm}_{\frac{1}{2}-\Omega(1)}(L_n) = \mathrm{polylog}(n)\right\} \ .$$

Theorem 7.1 (Strength of svm). *The polynomial hierarchy is contained in the Turing closure of the Frobenius class, $\mathbf{PH}^{\mathrm{cc}} \subseteq \mathbf{P}^{\mathrm{cc}}(\mathbf{Fr}^{\mathrm{cc}})$, i.e. the Frobenius class is as hard as the polynomial hierarchy.*

This result shows that on the one hand Frobenius rank does not characterize bounded-error randomized or quantum communication complexity. It only gives high lower bounds for problems far away from communication complexity classes like $\mathbf{BQP}^{\mathrm{cc}}$ or $\mathbf{NP}^{\mathrm{cc}}$. On the other hand, the singular value method might turn out to be a useful tool in separating complexity classes not amenable to current methods. Recall that Lokam's results [20], in particular high lower bounds for weak notions of rigidity, were derived using singular values. Thus, there might be a connection between svm and these rigidity variants.

Proof. We prove Theorem 7.1 using three Claims. First of all, we give an example of a function in $\mathbf{Fr}^{\mathrm{cc}}$ – the Hamming function HD defined below. Afterwards, it will suffice to show $\mathbf{PH}^{\mathrm{cc}} \subseteq \mathbf{P}^{\mathrm{cc}}(\mathrm{HD})$ in order to prove the theorem.

The majority function $\mathrm{maj}_n\colon \mathbb{B}^n \to \mathbb{S}$ is defined by $\mathrm{maj}_n(x) \coloneqq 1 - 2\left[\sum_{i=1}^{n} x_i \geq n/2\right]$, that is, $\mathrm{maj}_n(x) = -1$ if and only if at least half of the x_i are 1. The Fourier coefficients $a_\alpha \coloneqq \langle \chi_\alpha, \mathrm{maj}_n\rangle$ of the majority function for odd n are

$$a_\alpha = \frac{(-1)^{(|\alpha|-1)/2}}{2^{n-1}} \frac{(|\alpha|-1)!}{\left(\frac{|\alpha|-1}{2}\right)!} \frac{(n-|\alpha|)!}{\left(\frac{n-|\alpha|}{2}\right)!\left(\frac{n-1}{2}\right)!} \tag{10}$$

if $|\alpha|$ is odd, and $a_\alpha = 0$ if $|\alpha|$ is even [28]. We define the Hamming function by means of the majority function, $\mathrm{HD}_n \colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}$, $\mathrm{HD}_n(x,y) \coloneqq \mathrm{maj}_n(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$. Thus, the Hamming function $\mathrm{HD} \coloneqq (\mathrm{HD}_n)_{n\in\mathbb{N}}$ is the function that Alice and Bob have to compute if they are to decide whether their inputs differ in more than half of the entries.

Claim 1: $\mathrm{HD} \in \mathbf{Fr}^{\mathrm{cc}}$. The singular values of the corresponding communication matrix M^{HD_n} are the absolute values of the Fourier coefficients a_α times 2^n [25, Section 6.2]. Therefore by (10) we obtain for odd n the singular values

$$2\frac{(k-1)!}{\left(\frac{k-1}{2}\right)!}\frac{(n-k)!}{\left(\frac{n-k}{2}\right)!\left(\frac{n-1}{2}\right)!} \quad \text{with multiplicity } \binom{n}{k} \quad \text{for } 1 \le k \le n,\, k \text{ odd},$$

and the singular value 0 with multiplicity 2^{n-1}. We will show that this leads to a logarithmic lower bound for $\log_2 \mathrm{svm}_\varepsilon(\mathrm{HD}_n)$. To this end, denote the singular values of M^{HD_n} by $\sigma_\alpha \coloneqq 2^n|a_\alpha|$, and consider the sum of the squared singular values corresponding to $k = 1$,

$$\sum_{|\alpha|=1} \sigma_\alpha^2 = n\left[2\binom{n-1}{\frac{n-1}{2}}\right]^2 \sim 4n\frac{4^{n-1}}{\pi\frac{n-1}{2}} \sim \frac{2}{\pi}\,4^n\,,$$

where we have used the asymptotic behavior of the central binomial coefficient. Since $(2/\pi)\,4^n > (1-\varepsilon)\,\|M^{\mathrm{HD}_n}\|_{\mathrm{F}}^2$ for $\varepsilon > 1-(2/\pi)$, we find that for all $\varepsilon > 1-(2/\pi)$ and sufficiently large values of n, $\mathrm{svm}_\varepsilon(\mathrm{HD}_n) \le n$. Consequently, $\mathrm{HD} \in \mathbf{Fr}^{\mathrm{cc}}$.

The communication version of the majority function $\mathrm{MAJ} \coloneqq (\mathrm{MAJ}_n)_{n\in\mathbb{N}}$ is defined as

$$\mathrm{MAJ}_n \colon \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{S}\,, \quad \mathrm{MAJ}_n(x,y) \coloneqq \mathrm{maj}(x_1y_1, \ldots, x_ny_n) = 1 - 2\left[\textstyle\sum_i x_iy_i \ge n/2\right].$$

Claim 2: $\mathrm{MAJ} \in \mathbf{P}^{\mathrm{cc}}(\mathrm{HD})$. We specify a deterministic protocol for $\mathrm{MAJ}_n(x,y)$ that uses the oracle HD. First, Alice sends $|x|$, then Bob sends $|y|$, which will require at most $2\lceil\log_2 n\rceil$ bits of communication. If they observe that $|x| < n/2$ or $|y| < n/2$, the output of the protocol is $+1$. Otherwise they both compute $t \coloneqq |x| + |y| - n + 1$. Now note that $\sum_{i=1}^n x_iy_i - \frac{n}{2} = \frac{1}{2}\big(|x| + |y| - n - |x \oplus y|\big)$. This implies the equivalence $\mathrm{MAJ}_n(x,y) = -1 \iff |x \oplus y| < t$. Therefore, Alice and Bob can complete the computation by executing the oracle query $\mathrm{HD}_{2(n-t)}(x0^{n-2t}, y1^{n-2t})$ if $t \le n/2$ and $\mathrm{HD}_{2t}(x0^{2t-n}, y0^{2t-n})$ if $t > n/2$, respectively, upon which they output -1 if and only if the query result was $+1$.

Claim 3: MAJ *is* $\mathbf{PP}^{\mathrm{cc}}$*-complete.* See [1, Example 4.45] for a proof that can easily be converted into a proof of the lemma. Thereby note that the problems in $\mathbf{PP}^{\mathrm{cc}}$ can be solved by efficient guess protocols that accept an input if and only if the majority of the guess strings lead to acceptance [7].

Finally, recall Toda's Theorem, $\mathbf{PH}^{\mathrm{cc}} \subseteq \mathbf{P}^{\mathrm{cc}}(\mathbf{PP}^{\mathrm{cc}})$, in the communication complexity setting [29]. This completes our proof of Theorem 7.1:

$$\mathbf{PH}^{\mathrm{cc}} \subseteq \mathbf{P}^{\mathrm{cc}}(\mathbf{PP}^{\mathrm{cc}}) = \mathbf{P}^{\mathrm{cc}}(\mathrm{MAJ}) \subseteq \mathbf{P}^{\mathrm{cc}}(\mathrm{HD}) \subseteq \mathbf{P}^{\mathrm{cc}}(\mathbf{Fr}^{\mathrm{cc}})\,.$$

□

Acknowledgement

We would like to sincerely thank Jacobo Torán for many fruitful discussions.

References

1. Kushilevitz, E., Nisan, N.: Communication Complexity. Cambridge University Press (1997)
2. Yao, A.C.C.: Some Complexity Questions Related to Distributive Computing (Preliminary Report). In: Conference Record of the Eleventh Annual ACM Symposium on Theory of Computing, 30 April-2 May, 1979, Atlanta, Georgia, USA, ACM (1979) 209–213
3. Yao, A.C.C.: Quantum Circuit Complexity. In: FOCS, IEEE (1993) 352–361
4. Lee, T., Shraibman, A.: Lower Bounds in Communication Complexity. Foundations and Trends in Theoretical Computer Science **3**(4) (2009) 263–398
5. Lokam, S.V.: Complexity lower bounds using linear algebra. Foundations and Trends in Theoretical Computer Science **4**(1-2) (2009) 1–155
6. Klauck, H.: Lower bounds for quantum communication complexity. In: 42nd Annual Symposium on Foundations of Computer Science, October 14–17, Las Vegas Nevada, USA, IEEE (2001) 288–297
7. Babai, L., Frankl, P., Simon, J.: Complexity classes in communication complexity theory (preliminary version). In: 27th Annual Symposium on Foundations of Computer Science, FOCS 1986, 27–29 October 1986, Toronto, Ontario, Canada, IEEE Computer Society (1986) 337–347
8. Horn, R.A., Johnson, C.R.: Matrix analysis. Cambridge University Press (1985)
9. Horn, R.A., Johnson, C.R.: Topics in matrix analysis. Cambridge University Press (1991)
10. Zhang, F.: Matrix theory: basic results and techniques. Universitext. Springer-Verlag New York Inc. (1999)
11. Zhan, X.: Matrix inequalities. Volume 1790 of Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg (2002)
12. Eckart, C., Young, G.: The approximation of one matrix by another of lower rank. Psychometrika **1** (1936) 211–218
13. Schmidt, E.: Zur Theorie der linearen und nichtlinearen Integralgleichungen. Mathematische Annalen **63**(4) (1907) 433–476
14. de Wolf, R.: A Brief Introduction to Fourier Analysis on the Boolean Cube. Theory of Computing (1) (2008) 1–20
15. Valiant, L.G.: Graph-Theoretic Arguments in Low-Level Complexity. In Gruska, J., ed.: Mathematical Foundations of Computer Science 1977, 6th Symposium, Tatranska Lomnica, Czechoslovakia, September 5–9, 1977, Proceedings. Volume 53 of Lecture Notes in Computer Science., Springer-Verlag (1977) 162–176
16. Lokam, S.V.: Quadratic Lower Bounds on Matrix Rigidity. In Cai, J.Y., Cooper, S.B., Li, A., eds.: Theory and Applications of Models of Computation, Third International Conference, TAMC 2006, Beijing, China, May 15–20, 2006, Proceedings. Volume 3959 of Lecture Notes in Computer Science., Springer-Verlag (2006) 295–307
17. Razborov, A.: On Rigid Matrices (in Russian). Technical report, Steklov Mathematical Institute (1989)
18. Wunderlich, H.: Contributions to Structural Communication Complexity. PhD thesis, Technische Universität Ilmenau (2009)
19. Wunderlich, H.: On a Theorem of Razborov. submitted (2009)
20. Lokam, S.V.: Spectral Methods for Matrix Rigidity with Applications to Size-Depth Trade-offs and Communication Complexity. J. Comput. Syst. Sci. **63**(3) (2001) 449–473
21. Paturi, R., Simon, J.: Probabilistic Communication Complexity. J. Comput. Syst. Sci. **33**(1) (1986) 106–123
22. Krause, M.: Geometric Arguments Yield Better Bounds for Threshold Circuits and Distributed Computing. Theor. Comput. Sci. **156**(1&2) (1996) 99–117
23. Forster, J.: A linear lower bound on the unbounded error probabilistic communication complexity. J. Comput. Syst. Sci. **65**(4) (2002) 612–625
24. Buhrman, H., de Wolf, R.: Communication Complexity Lower Bounds by Polynomials. In: IEEE Conference on Computational Complexity. (2001) 120–130
25. Linial, N., Shraibman, A.: Lower bounds in communication complexity based on factorization norms. Random Structures and Algorithms **34**(3) (2009) 368–394
26. Kremer, I.: Quantum Communication. Master's thesis, Computer Science Department, The Hebrew University of Jerusalem (1995)
27. Krivelevich, M., Sudakov, B.: Pseudo-random graphs. In Györi, E., Katona, G.O.H., Lovász, L., eds.: More sets, graphs and numbers. Volume 15 of Bolyai Soc. Math. Studies., Springer-Verlag (2006) 199–262
28. Brandman, Y., Orlitsky, A., Hennessy, J.L.: A Spectral Lower Bound Technique for the Size of Decision Trees and Two Level AND/OR Circuits. IEEE Trans. Computers **39**(2) (1990) 282–287

29. Wunderlich, H.: On Toda's Theorem in Structural Communication Complexity. In Nielsen, M., Kucera, A., Miltersen, P.B., Palamidessi, C., Tuma, P., Valencia, F.D., eds.: SOFSEM. Volume 5404 of Lecture Notes in Computer Science., Springer (2009) 609–620

ECCC ISSN 1433-8092

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