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A strong law of computationally weak subsets

Bjørn Kjos-Hanssen
University of Hawai'i at Mānoa *

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Abstract

We show that in the setting of fair-coin measure on the power set of the natural numbers, each sufficiently random set has an infinite subset that computes no random set. That is, there is an almost sure event $\mathcal{A}$ such that if $X \in \mathcal{A}$ then X has an infinite subset Y such that no element of $\mathcal{A}$ is Turing computable from Y.

1 Introduction

We view randomness of an infinite set of positive integers X as a valuable resource. The question arises whether if a set Y is close to X in some sense, then Y retains some of the value of X in that, even if Y is not itself random, one can compute a random sequence from Y.

There are various ways in which Y and X could be considered "close"; a natural one is to assume $Y \subseteq X$ and Y is infinite. In this article we shall prove that under the fair-coin Lebesgue measure there is an almost sure event $\mathcal{A}$ such that if $X \in \mathcal{A}$ then X has an infinite subset Y such that no element of $\mathcal{A}$ is Turing reducible to Y. This confirms the intuition one may have that a subset of a random set should not generally be able to compute a random set. This "strong law of computationally weak subsets" is a probabilistic law in the same sense as the strong law of large numbers; it gives an almost sure property. A key to the proof will be Bienaymé's extinction criterion for Galton-Watson processes, Theorem 4.3.

Our results improve upon an earlier result [7] to the effect that there simply exists a Martin-Löf random set X and an infinite subset Y of X such that no Martin-Löf random set is Turing reducible to Y. A different proof of that result may be deduced from work of Greenberg and Miller [3].

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2 Bushy trees

Our overall plan is to apply Bienaymé's theorem to a *bushy* infinite tree, each path through which obeys a construction like that of Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1].

Definition 2.1 (Kumabe and Lewis [6])**.** A finite set of incomparable strings in $\omega^{<\omega}$ is called a *tree*. (Note that this differs from some common notions of tree.) Given $n \in \omega$, a nonempty tree T is called *n-bushy from $\sigma \in \omega^{<\omega}$* if

(1) every string $\tau \in T$ extends σ, and

(2) for each $\tau \in \omega^{<\omega}$, if there exists $\rho \in T$ with $\sigma \subseteq \tau \subset \rho$, then there are at least n many immediate successors of τ which are substrings of elements of T.

If T is n-bushy from σ and $T \subseteq P \subseteq \omega^{<\omega}$, then T is called *n-bushy from σ for P*.

Definition 2.2. A set $C \subseteq \omega^{<\omega}$ is *n-perfectly bushy* if the empty string is in C and every element of C has at least n many immediate extensions in C.

An n-perfectly bushy set C is in particular a tree in the sense of being closed under substrings, and the set of infinite paths through C is a perfect set $[C] \subset \omega^\omega$.

Lemma 2.3 (an extension of [1, Lemma 2.5])**.** *Let $n \geq 1$. Given a tree T that is $(a+b-1)$-bushy from a string α and given a set $P \subseteq T$, there is a subset S of T which is a-bushy for P or b-bushy for $T - P$.*

Proof. Give the elements of T the label 1 (0) if they are in P (not in P, respectively). Inductively, suppose β extends α and is a proper substring of an element of T. Suppose all the immediate successors of β that are substrings of elements of T have received a label. Give β the label 1 if at least a many of its labelled immediate successors are labelled 1; otherwise, give β the label 0. (In this case, at least $(a+b-1)-(a-1) = b$ many immediate successors are labelled 0.) This process ends after finitely many steps when α is given some label $i \in \{0,1\}$. Let S be the set of i-labelled strings in T. If $i = 1$ then S is contained in P, and if $i = 0$ then S is contained in $T - P$, so it only remains to show that S is $a\mathbf{1}_{\{1\}}(i) + b\mathbf{1}_{\{0\}}(i) = ai + b(1-i)$-bushy.[1]

Let L be the set of all labelled strings. Note that L is the set of strings extending α that are substrings of elements of T. For any $\beta \in L - T$, let k be the number of immediate successors of β that are in L. Since T is $(a+b-1)$-bushy, $k \geq a+b-1$. Let $p \leq k$ be the number of immediate successors of β that have the same label as β. By construction, $p \geq ai + b(1-i)$. It follows that S is $ai + b(1-i)$-bushy. $\square$

[1] Here $\mathbf{1}_A(n) = 1$ if $n \in A$, and $\mathbf{1}_A(n) = 0$ otherwise.

We now need a simple but crucial strengthening of [1, Lemma 2.10]; the difference is that nonemptiness is replaced by bushiness. From now on fix the integer $\Delta = 3$.

Lemma 2.4. *Suppose we are given α and n and a set $P \subseteq \omega^{<\omega}$ such that there is no n-bushy tree from α for P. If V is an $n + \Delta - 1$-bushy tree from α then there exists a Δ-bushy set of strings T such that for each $\beta \in T$,*

1. *β extends an element of V, and*
2. *there is no n-bushy tree from β for P.*

Proof. We claim in fact that there exists a Δ-bushy set of strings T such that for each $\beta \in T$,

1. $\beta \in V$, and
2. there is no n-bushy tree from β for P.

Suppose otherwise. By Lemma 2.3 there is an n-bushy set $B \subseteq V$ such that for all $\beta \in B$, there *is* an n-bushy tree V_β from β for P; then

$$V^* = \bigcup_{\beta \supseteq \alpha,\, \beta \in B} V_\beta$$

would be n-bushy from α for P. □

3 Diagonalization

Diagonally non-recursive functions will be our bridge between randomness and bushy trees. To a certain extent this section follows Ambos-Spies et al. [1].

Definition 3.1. If $\sigma, \tau \in \omega^{<\omega}$ then σ is called a *substring* of τ, $\sigma \subseteq \tau$, if for all x in the domain of σ, $\sigma(x) = \tau(x)$. The length of a string σ is denoted by $|\sigma|$. A string $\langle a_1, \ldots, a_n \rangle \in \omega^n$ is denoted $(a_1, \ldots, a_n)$ when we find this more natural. The *concatenation* of $\langle a_1, \ldots, a_n \rangle$ by $\langle a_n \rangle$ on the right is denoted $\langle a_1, \ldots, a_n \rangle * \langle a_{n+1} \rangle = \langle a_1, \ldots, a_n \rangle * a_{n+1}$. If $G \in \omega^\omega$ then σ is a substring of G if for all x in the domain of σ, $\sigma(x) = G(x)$.

Let Φ_n, $n \in \omega$, be a standard list of the Turing functionals. So if A is recursive in B then for some n, $A = \Phi_n^B$. For convenience, if Φ is a Turing functional and for all B and x, the computation of $\Phi^B(x)$ is independent of x, we sometimes write Φ^B instead of $\Phi^B(x)$. Let $\Phi_{n,t}$ be the modification of Φ_n which goes into an infinite loop after t computation steps if the computation has not ended after t steps. We abbreviate $\Phi_n^\emptyset$ by Φ_n. If the computation $\Phi_e(x)$ terminates we write $\Phi_e(x) \downarrow$, otherwise $\Phi_e(x) \uparrow$.

Definition 3.2 (Fix)**.** Let $F = \mathrm{Fix}$ be a computable function such that for all $a \in \omega$, $\mathrm{Fix}(a)$ is the fixed-point of Φ_a produced by the Recursion Theorem; thus, if $e = \mathrm{Fix}(a)$ then

$$\Phi_e(x) = \Phi_{\Phi_a(e)}(x)$$

for all $x \in \omega$.

Definition 3.3 (Search)**.** Given a string $\alpha \in \omega^{<\omega}$, $c \in \omega$, and $n \in \omega$, let $f = f_{\alpha,c,n} = \Phi_{\mathrm{Search}(\alpha,c,n)}$ be defined by the condition:

> $\Phi_{\Phi_{\mathrm{Search}(\alpha,c,n)}(e)}(x) = i$ if
> there is a finite tree T and a number $i < h(e)$ such that T is n-bushy from α for $\{\beta : \Phi_c^\beta(e) = i\}$ (and i is the i occurring for the first such tree found). If such T and i do not exist then $\Phi_{f(e)}(x)\uparrow$.

If we let $e = \mathrm{Fix}(\mathrm{Search}(\alpha, c, n))$ then consequently

> $\Phi_e(x) = i$ if
> there is a finite tree T and a number $i < h(e)$ are found such that T is n-bushy from α for $\{\beta : \Phi_c^\beta(e) = i\}$ (and i is the i occurring for the first such tree found).

Definition 3.4. Let $\epsilon : \omega \to \omega$ be a finite partial function and write $e_t = \epsilon(t)$ for each t in the domain of ϵ.

Let Φ be any Turing functional such that for all $G : \omega \to \omega$,

$$\Phi^G(\epsilon)\downarrow \leftrightarrow \exists t \in \mathrm{dom}(\epsilon)\,[\Phi_t^G(e_t)\downarrow < h(e_t)].$$

Given $n \in \omega$ and ϵ, let $g(n, \epsilon) = 2^a n$ where

$$a = \sum_{t\in\mathrm{dom}(\epsilon)} h(e_t).$$

Lemma 3.5 ([1, [Lemma 2.8])**.** *Let $n \geq 1$, let ϵ be a finite partial function from ω to ω, and let g be the function defined in Definition 3.4. For each pair (t, i) satisfying $i < h(e_t)$ (where h is as discussed above) and $t \in \mathrm{dom}(\epsilon)$, let $Q_{(t,i)} = \{\beta : \Phi_t^\beta(e_t) = i\}$. Let $Q = \{\beta : \Phi^\beta(\epsilon)\downarrow\}$. If there is a $g(n,\epsilon)$-bushy tree for Q from some string α, then for some (t, i), there is an n-bushy tree from α for $Q_{(t,i)}$.*

Definition 3.6. Given functions $H, G : \omega \to \omega$, we say H is DNR (*diagonally nonrecursive*) if for all $x \in \omega$, $H(x) \neq \Phi_x^G(x)$ or $\Phi_x^G(x)\uparrow$. Given $h : \omega \to \omega$, we say H is h-DNR if in addition for all n, $H(n) < h(n)$. (This necessitates that $h(n) > 0$ for all n.) If H is DNR and σ is a substring of H then σ is called a DNR *string*.

Throughout the rest of this article, fix a recursive function $h : \omega \to \omega$ satisfying Theorem 4.1; for example, $h(n) = n^2$ works. If $C \subseteq \omega^{<\omega}$ and $G \in \omega^\omega$ then we say $G \in [C]$ if for all n, $G \restriction n \in C$.

Definition 3.7. *The Construction.*

At any stage $s + 1$, the finite set D_{s+1} will consist of indices $t \leq s$ for computations Φ_t^G that we want to ensure to be divergent. The set A_{s+1} will consist of what we think of as acceptable strings. The numbers $n[s]$ and $n[s + \frac{1}{2}]$ will measure the amount of bushiness required.
Stage 0.

Let $G[0] = \emptyset$, the empty string, and $\epsilon[0] = \emptyset$. Let $n[0] = 2$. Let $D_0 = \emptyset$ and $A_0 = \omega^{<\omega}$.
Stage $s+1$, $s \geq 0$.

Let $n[s+\frac{1}{2}] = g(n[s], \epsilon[s])$, with g as in Definition 3.4. Let $n[s+1] = n[s+\frac{1}{2}] + \Delta - 1$. Below we will define D_{s+1}. Given D_{s+1}, A_{s+1} will be

$$A_{s+1} = \{\tau \supset G[s] \mid \neg(\exists t \in D_{s+1})(\exists i < h(e_t)(\exists T)$$

$$(T \text{ is a finite } n[s+1]\text{-bushy tree from } \tau \text{ for } Q_{(t,i)})\}$$

Let e be the fixed point of $f = f_{G[s],s,n[s+1]}$ (as defined in Definition 3.3) produced by the Recursion Theorem, i. e., $\Phi_e = \Phi_{f(e)}$.
Case 1. $\Phi_e(e)\downarrow$.

Fix T as in Definition 3.3. Let $D_{s+1} = D_s$.

$$\text{Let } G[s+1] \text{ be an extension of } G[s] \text{ with } G[s+1] \in T \cap A_{s+1}. \qquad (*)$$

Case 2. $\Phi_e(e)\uparrow$. Let $D_{s+1} = D_s \cup \{s\}$. Let $\epsilon[s+1] = \epsilon[s] \cup \{(s,e)\}$. In other words, $e_s = \epsilon(s)$ exists and equals e.

$$\text{Let } G[s+1] \text{ be any element of } A_{s+1}. \qquad (+)$$

Let $G = \bigcup_{s\in\omega} G[s]$.
End of Construction.

We now prove that the Construction satisfies Theorem 3.13 in a sequence of lemmas.

Lemma 3.8. *For each $s, t \in \omega$ with $t \leq s$, $n_t[s] \geq 2$.*

Proof. For $s = 0$, we have $n[0] = 2$. For $s+1$, we have $n[s+1] = g(n[s], \epsilon[s]) = 2^a n[s]$ for a certain $a \geq 0$, by Definition 3.5, hence the lemma follows. □

Lemma 3.9. *For each $s \geq 0$ the following holds.*

(1) The Construction at stage s is well-defined and $G[s] \in A_s$. In particular, if $s > 0$ then in Case 2, A_s is nonempty, and in Case 1, A_s contains at least one element of T.

(2) There is no $n[s+\frac{1}{2}]$-bushy tree for $Q = \{\beta : \Phi^\beta(\epsilon[s])\downarrow\}$ from $G[s]$.

(3) Every tree V which is $n[s+1]$-bushy from $G[s]$, and is not just the singleton of $G[s]$, contains a Δ-bushy set of elements of A_{s+1}.

Proof. It suffices to show that (1) holds for $s = 0$, and that for each $s \geq 0$, (1) implies (2) which implies (3), and moreover that (3) for s implies (1) for $s+1$. (1) holds for $s = 0$ because $G[0] = \emptyset \in \omega^{<\omega} = A_0$.
(1) implies (2):

By definition of A_s and the fact that $G[s] \in A_s$ by (1) for s, we have that for each $t \in D_s$, and each $i < h(e_t)$, there is no $n[s]$-bushy tree from $G[s]$ for

$Q_{(t,i)} = \{\beta : \Phi_t^\beta(e_t) \downarrow = i\}$. Hence by Lemma 3.5, there is no $n[s+\frac{1}{2}]$-bushy tree for $Q = \{\beta : \Phi^\beta(\epsilon[s]) \downarrow\}$ from $G[s]$.

(2) implies (3):

Since V is $n[s+1]$-bushy, by Lemma 2.4 there is a Δ-bushy set of elements β of V from which there is no $n[s+\frac{1}{2}]$-bushy tree for Q, and hence no $n[s+1]$-bushy tree for $Q_{(t,i)}$ either, since $n[s+\frac{1}{2}] \leq n[s+1]$ and $Q_{(t,i)} \subseteq Q$. Moreover, each such β properly extends $G[s]$, since V is an antichain and is not the singleton of $G[s]$. Hence by definition of A_{s+1}, each such element β belongs to A_{s+1}.

(3) for s implies (1) for $s+1$:

If Case 1 holds, let T be the tree found by Φ_e, i. e., T is $n[s+1]$-bushy from $G[s]$ (for $Q_{(s,i)}$ for some i). If T is not just the singleton of $G[s]$, and Case 1 holds, then apply (3) for s to T.

If T is just the singleton of $G[s]$ or if Case 2 holds, then apply (3) for s to any $n[s+1]$-bushy non-singleton tree from $G[s]$. For example, this could be the set of immediate extensions $G[s] * k$, $k < n[s+1]$. □

Lemma 3.10. *For any $s \geq 0$, if $s \in D_{s+1}$ then $\Phi_s^G(e_s) \uparrow$ or $\Phi_s^G(e_s) \geq h(e_s)$.*

Proof. Otherwise for some $t \in \omega$, $\Phi_s^{G[t]}(e_s) \downarrow < h(e_s)$. Since the singleton tree $T = \{G[t]\}$ is n-bushy from $G[t]$ for all n, hence in particular $n[t]$-bushy, this contradicts the fact that by Lemma 3.9(1), $G[t] \in A_t$. □

Lemma 3.11. *G is a total function, i.e., $G \in \omega^\omega$.*

Proof. By Lemma 3.9(3), $G[s+1] \in A_{s+1}$ for each $s \geq 0$, and hence by definition of A_{s+1}, $G[s+1]$ is a proper extension of $G[s]$. From this the lemma immediately follows. □

Lemma 3.12. *G computes no h-DNR function.*

Proof. If Case 1 of the construction is followed then $\Phi_s^G(e) = \Phi^{G[s+1]}(e) = \Phi_e(e)$ because $G[s+1] \in T$. So Φ_s^G is not h-DNR. If Case 2 of the construction is followed then $s \in D_{s+1}$ and so $\Phi_s^G(e) \uparrow$ or $\Phi_s^G(e) \geq h(e)$ by Lemma 3.10. Thus again Φ_s^G is not h-DNR. □

Let $0'$ denote the halting problem for Turing machines.

Theorem 3.13. *There is a Δ-perfectly bushy set $C \subseteq \omega^{<\omega}$, $C \leq_T 0'$, such that for each $G \in [C]$ and all Turing functionals Φ, Φ^G is not h-DNR.*

Proof. We showed how to construct a single $G \in \omega^\omega$, but since by Lemma 3.9(3) the choice of $G[s+1]$ can be made in a Δ-bushy set of ways, the set C of all functions G obeying $(*)$ and $(+)$ in the construction 3.7 is Δ-perfectly bushy. Routine inspection show that the construction and hence the set C are recursive in $0'$. □

4 A law of weak subsets

A sequence $X \in 2^\omega$ is also considered to be a set $X \subseteq \omega$. For the notions of Martin-Löf random and Schnorr random sets X relative to an oracle A we refer the reader to Nies' book [12]. For $n \in \omega$, a set X is $(n+1)$-random if it is Martin-Löf random relative to the n^{th} iteration of the Turing jump, $0^{(n)}$, and Schnorr $(n+1)$-random if it is Schnorr random relative to $0^{(n)}$.

Theorem 4.1 (Kučera [8] and Kurtz (see Jockusch [5, Proposition 3])**.** *There is a recursive function h such that for each 1-random real R, there is an h-*DNR *function f recursive in R.*

Applying Theorem 3.13, we have

Theorem 4.2. *There is a Δ-perfectly bushy set $C \subseteq \omega^{<\omega}$, $C \leq_T 0'$, such that for each $G \in [C]$ and each 1-random set X, $X \not\leq_T G$.*

The key idea is now to consider the intersection of C with a random set $X \subseteq \omega^{<\omega}$ as a Galton-Watson process. Theorem 4.3 can be considered the fundamental result in the theory of such processes. It was first stated by Bienaymé in 1845; see Heyde and Seneta [4, pp. 116–120] and Lyons and Peres [11, Proposition 5.4]. The first published proof of Bienaymé's theorem appears in Cournot [10, pp. 83–86]. As usual, $\mathbb{P}$ denotes probability.

Theorem 4.3 (Extinction Criterion)**.** *Given numbers $p_k \in [0,1]$ with $p_1 \neq 1$ and $\sum_{k\geq 0} p_k = 1$, let $Z_0 = 1$, let L be a random variable with $\mathbb{P}(L = k) = p_k$, let $\{L_i^{(n)}\}_{n,i\geq 1}$ be independent copies of L, and let*

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_i^{(n+1)}.$$

Let $q = \mathbb{P}((\exists n)\, Z_n = 0)$. Then $q = 1$ iff $\mathbb{E}(L) = \sum_{k\geq 0} kp_k \leq 1$. Moreover, q is the smallest fixed point of $f(s) = \sum_{k\geq 0} p_k s^k$.

We are interested in the case where each person has n children, each with probability p of surviving; then the probability p_k of k children surviving satisfies

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}$$

and $\mathbb{E}(L) = np$. In particular, if $n = \Delta = 3$ and $p = 1/2$ then $q < 1$, i.e., there is a positive probability of non-extinction of the family.

Theorem 4.4 (Law of Computationally Weak Subsets)**.** *For each Schnorr 2-random set R there is an infinite set $S \subseteq R$ such that for all $Z \leq_T S$, Z is not 1-random.*

Proof. Let $R \subseteq \omega$ be Schnorr 2-random, and let $X \subseteq \omega^{<\omega}$ be the image of R under an effective bijection $h : \omega \to \omega^{<\omega}$. In this situation we say that X is a

Schnorr 2-random subset of $\omega^{<\omega}$. Since h induces a map $\hat{h} : \{R : R \subseteq \omega\} \to \{X : X \subseteq \omega^{<\omega}\}$ given by $\hat{h}(R) = \{h(n) : n \in R\}$, that preserves subsets and infinitude, it suffices to show that there is an infinite set $Y \subseteq X$ such that for all $Z \leq_T S$, Z is not 1-random.

By Theorem 4.2, let C be a 3-perfectly bushy subset of $\omega^{<\omega}$, $C \leq_T 0'$, such that for each $Y \in [C]$ and each 1-random set Z, $Z \not\leq_T Y$.

We claim that there is a $Y \in [C]$ such that $Y \subseteq^* X$, i.e., $X \setminus Y$ is finite. This will suffice for the theorem, because then some finite modification of Y is our desired set.

We first argue that for almost all X, there is a $Y \in [C]$ such that $Y \subseteq^* X$. For this it suffices to show that for positive-measure many X, there is a $Y \in [C]$ such that $Y \subseteq X$.

Recall that $\sigma \subseteq \tau$ means that σ is a substring of τ. Let[2]

$$G_X = \{\sigma \in \omega^{<\omega} : (\forall \tau \subseteq \sigma) \tau \in C \cap X\}.$$

To connect with the Extinction Criterion 4.3, first write $\{\sigma \in G_X : |\sigma| = n\} = \{\sigma_0^{(n)}, \dots, \sigma_{Z_n}^{(n)}\}$, where for each $0 \leq t < Z_n$, $\sigma_t^{(n)}$ precedes $\sigma_{t+1}^{(n)}$ in some fixed computable linear order (say, the lexicographical order). Then for $i \leq Z_n$, let $L_i^{(n)}$ be the cardinality of $\{k : (\sigma_i^{(n)}) * k \in C \cap X\}$.

Note that if we consider X as the value of a fair-coin random variable on the power set of $\omega^{<\omega}$, then $L_i^{(n)}$ is a binomial random variable with parameters $p = 1/2$ and $n = 3$. That is, we have a birth-death process where everyone has 3 children, each with a 50% chance of surviving and themselves having 3 children.

Since the branching rate of C is exactly 3, we have a kind of C-effective compactness making the event of extinction

$$\{X : [G_X] = \varnothing\} = \{X : (\exists n)(\forall \sigma \in \omega^n)(\sigma \notin G_X)\}$$

into a $\Sigma_1^0(C)$ class. We produce independent copies of this class by letting $X^n = \{\sigma : 0^n \sigma \in X\}$ and

$$\mathcal{E}_n = \{X : [G_{X^n}] = \varnothing\}.$$

Then $\mathcal{E}_n$ is $\Sigma_1^0(C)$, the events $\mathcal{E}_n$, $n \in \omega$, are mutually independent, and $\mathbb{P}(\mathcal{E}_n)$ does not depend on n. By Theorem 4.3, $q := \mathbb{P}((\exists n) Z_n = 0) = \mathbb{P}(\mathcal{E}_0)$ is the smallest positive fixed point of $f(s) = \sum_{k \geq 0} p_k s^k = \frac{1}{8} + \frac{3}{8}s + \frac{3}{8}s^2 + \frac{1}{8}s^3$. Solving a quadratic show that the equation

$$\frac{1}{8} + \frac{3}{8}s + \frac{3}{8}s^2 + \frac{1}{8}s^3 = s$$

has its smallest positive solution at $s = \sqrt{5} - 2$. So

$$\mathbb{P}(\cap_{k<n} \mathcal{E}_k) = (\mathcal{P}(\mathcal{E}_0))^n = (\sqrt{5} - 2)^n$$

which is computable and converges to 0 effectively. Thus Schnorr randomness relative to C is enough to guarantee $X \notin \cap_n \mathcal{E}_n$. $\square$

[2] G_X can be thought of as a Galton-Watson family tree.

Corollary 4.5. There is an almost sure event $\mathcal{A}$ such that if $X \in \mathcal{A}$ then X has an infinite subset Y such that no element of $\mathcal{A}$ is Turing reducible to Y.

Proof. Let
$$\mathcal{A} = \{X \mid X \text{ is Schnorr 2-random}\}$$
and apply Theorem 4.4. □

It is of interest for the study of Ramsey's theorem in Reverse Mathematics to know how far the Law of Weak Subsets can be effectivized. This subject is discussed in an earlier paper [7] and studied in detail by Dzhafarov [2].

Question 4.6. Does Corollary 4.5 hold with $\mathcal{A} = \{X \mid X \text{ is 1-random}\}$? That is, does every 1-random set X have an infinite subset $Y \subseteq X$ such that Y does not compute any 1-random set?

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