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Green's Theorem and Isolation in Planar Graphs

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Abstract

We show a simple application of Green's theorem from multivariable calculus to the isolation problem in planar graphs. In particular, we construct a skew-symmetric, polynomially bounded, edge weight function for a directed planar graph in logspace such that the weight of any simple cycle in the graph is non-zero with respect to this weight function. As a direct consequence of the above weight function, we are able to isolate a directed path between two fixed vertices, in a directed planar graph. We also show that given a bipartite planar graph, we can obtain an edge weight function (using the above function) in logspace, which isolates a perfect matching in the given graph. Earlier this was known to be true only for grid graphs - which is a proper subclass of planar graphs.

We also look at the problem of obtaining a straight line embedding of a planar graph in logspace. Although we do not quite achieve this goal, we give a piecewise straight line embedding of the given planar graph in logspace.

1 Introduction

We show a simple application of a celebrated theorem due to 19th century British mathematician George Green to the isolation problems in planar graphs. Green's theorem, stated below, relates certain line integral over a closed curve on the plane to a related double integral over the region enclosed by this curve.

Theorem 1 (Green's Theorem)**.** *Let C be a closed, piecewise smooth, simple curve on the plane which is oriented counterclockwise. Let R_C be the region bounded by C. Let P and Q be functions of (x, y) defined on a region containing R_C and having continuous partial derivatives in the region. Then*

$$\oint_C (P\,dx + Q\,dy) = \iint_{R_C} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

This fundamental theorem and its generalizations (such as Stokes' theorem) have deeply influenced the development of several areas of physics and mathematics. Strikingly, Green's Theorem also has a very immediate and elegant practical application in calculating the area of an arbitrary two-dimensional shape. The device known as planimeter, used to calculate the area of an arbitrary shape (such as a region in a map) is based on the following instantiation of Green's theorem, which we also use in this paper. If we substitute $Q(x, y) = x$ and $P(x, y) = 0$ in Green's theorem we get the following theorem.

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Theorem 2 (Area by line integrals). *Let C be a closed, piecewise smooth, simple curve on the plane which is oriented counterclockwise. Let R_C be the region bounded by C. Then,*

$$\text{Area}(R_C) = \oint_C x\, dy$$

Refer to any standard text books on calculus (such as [13]) to know more about Green's and other related theorems.

Distinguishing a solution out of a set of solutions is a basic algorithmic problem with many applications. The well-known *isolating lemma* due to Mulmuley, Vazirani, and Vazirani provides a general randomized solution to this problem. Let $\mathcal{F}$ be a non-empty set system on a universe $U = \{1, \ldots, n\}$. Then *isolating lemma* says, for a random weight function on U (bounded by $n^{O(1)}$), with high probability there is a *unique* set in $\mathcal{F}$ of minimum weight [9]. This lemma, originally used to give an elegant RNC algorithm for constructing a maximum matching (by isolating a minimum weight perfect matching) in general graphs, has found many applications, mostly in discovering new randomized and non-uniform upper bounds, via isolating minimum weight solutions [9, 11, 8, 2]. Clearly, derandomizing the isolating lemma in sufficient generality will improve these upper bounds to their deterministic counterparts and hence will be a major result. Unfortunately, recently it was shown that such a derandomization will imply certain circuit lower bounds and hence is a difficult task [3]. However, such negative results do not rule out the possibility of bypassing the isolating lemma altogether by directly prescribing efficient deterministic weight functions for specific situations so that the minimum weight solutions become unique. In fact recently simple logspace computable weight functions were prescribed for directed reachability and bipartite perfect matching problems over *grid graphs* to yield new deterministic upper bounds [4, 5]. Grid graphs are a restricted class of planar graphs where the graph completely lies on the two dimensional grid. It was not clear how to extend these weight functions to planar graphs. In this paper we settle this question.

Our results

Given a directed graph G with a planar embedding, we prescribe a skew-symmetric, logspace computable, polynomially bounded weight function w with the property that, with respect to w, the weight of any simple cycle in G is non-zero. We then use arguments identical to that in [4] to show that such weight functions isolate directed paths - that is, with respect to such weight functions, between any pair of nodes if there is a path, then there is a unique minimum weight path. We also show that such weight functions isolate a matching in bipartite graphs (by appropriately directing edges). Our weight function is based on the line integral on the right hand side of Theorem 2.

The weighting scheme that we prescribe works for any "nice" embedding of the graph on the plane. For ease of presentation we will assume that the graph is presented as a straight line embedding, which means each vertex v is given as a point, (x_v, y_v) on the coordinate axes, and an edge (u, v) is a line between points (x_u, y_u) and (x_v, y_v) so that no such lines intersect other than at the vertices. Moreover, we will assume that the coordinates are integer points with values bounded by $\text{poly}(n)$. Existence of such embeddings were known earlier [7, 6, 12].

Typically for algorithmic purposes planar graphs are presented in terms of a *combinatorial embedding*. Time efficient algorithms are known that can compute a straight line embedding of a planar graph [6, 12] from a combinatorial embedding. Unfortunately, these algorithms require linear space and at present we do not know how to get a space efficient implementation of them. In Section 4 we give a logspace algorithm that gives a *piecewise straight line embedding* of the given planar graph from a combinatorial embedding. This is the first logspace construction known to us, of a piecewise straight line embedding of a given planar graph and might be of independent interest. It will be very clear how the weight function for a straight line embedding can be extended to a piecewise straight line embedding also.

We do not get any new upper bounds for directed planar reachability or planar bipartite matching problems (other than simplified proofs of existing results) since it is known that these problems over planar graphs reduce to their counter parts in grid graphs [1, 5] and hence the weight functions known for grid graphs suffice to derive upper bounds for planar versions of these problems. However, we feel that the application of Green's Theorem to the isolation problem gives it a new dimension and might yield potential strategies to solve the more general cases.

2 The weight function

Let $G = (V, E)$ be a graph with a straight line embedding. Let $e = (u, v)$ be a directed edge directed from u to v where u is identified with the point (x_u, y_u) and v is identified with (x_v, y_v). For such a directed edge, define a weight function w as follows (if e is piecewise straight, we calculate the integral over each piece and sum them up):

$$w(e) = 2 \times \oint_e x\,dy = (y_v - y_u)(x_v + x_u)$$

In order to calculate the second equality, we can use the parametric equation of the line segment which is given by $x(t) = (x_v - x_u)t + x_u$ and $y(t) = (y_v - y_u)t + y_u$ where $t \in [0, 1]$. Notice that if the coordinates of the vertices are polynomially bounded, this weight function is also polynomially bounded. For any cycle C in G, weight of C, $w(C)$ is defined as the sum of the weights of the edges in C.

An important property of this weight function is that it is *skew-symmetric*, that is, $w(u, v) = -w(v, u)$. We use this skew-symmetry property in our proofs. We will first show the following lemma which is crucial in proving that this weight function has the required isolation property.

Lemma 3. *Let G be a directed planar graph and let C be any directed simple cycle in G. Let R_C be the region enclosed by C. Then $|w(C)| = 2 \times \mathrm{Area(R_c)}$. In particular, $w(C)$ is non-zero.*

Proof. Let $C = (e_1, e_2, \ldots, e_l)$ be a directed cycle oriented counterclockwise. Then we have

$$\begin{aligned} w(C) &= \sum_i w(e_i) \\ &= 2 \times \sum_i \oint_{e_i} x\,dy \\ &= 2 \times \oint_C x\,dy \\ &= 2 \times \mathrm{Area}(R_C) \end{aligned}$$

The last equality follows from Theorem 2. If C is oriented clockwise, we get that $w(C) = -2 \times \text{Area}(R_C)$. Hence the lemma.

$\square$

3 Isolating paths and matchings in planar graphs

Theorem 4. *Let G be a planar directed graph with a straight line embedding. Then with respect to the weight function w, for every pair of nodes u and v, if there is a directed path from u to v, then there is a* unique *path from u to v of minimum weight.*

Proof. Suppose there are u, v so that there are two u to v paths P_1 and P_2 of minimum weight. We will assume that the paths do not intersect on vertices other than the end points (otherwise we can find two vertices u' and v' along these paths that satisfies this property using a standard cut-and-paste argument and use these vertices instead). We have $w(P_1) = w(P_2)$. Now consider the graph G' which is same as G except that the path P_2 is reversed so that the set of edges (P_1, P_2^r) becomes a simple cycle in G' (P_2^r denotes the reversed path). Let C denote this cycle. Then $w(C) = w(P_1) + w(P_2^r) = w(P_1) - w(P_2) = 0$. The second equality holds because of the skew-symmetry of the weight function. This contradicts Lemma 3. $\square$

Now we will consider isolation of matchings in bipartite planar graphs. Since for matching we have undirected graphs, we need to give directions to the edges in order to assign weights. Let G be a bipartite graph. First we compute the bipartition. This can be achieved in logspace by Reingold's reachability algorithm (say using a universal exploration sequence) for undirected graphs [10]. Thus given a vertex u, we can decide in logspace whether $u \in L$ or $u \in R$, where (L, R) is a bipartition of G. For any undirected edge $\{u, v\}$ so that $u \in L$ and $v \in R$, we first assign direction from u to v. Thus in the corresponding directed graph, denoted by $\vec{G}$, all the edges go from L to R. Then weight of an undirected edge $e = w(e) = (y_v - y_u)(x_v + x_u)$ with respect to the above-mentioned direction.

Theorem 5. *Let G be a planar undirected bipartite graph. Then with respect to the weight function w, if there is a perfect matching in G, the minimum weight perfect matching in G is unique.*

Proof. Suppose the theorem is not true and let M_1 and M_2 be two matchings so that $w(M_1) = w(M_2)$. Consider $M_1 \oplus M_2$, the symmetric difference of M_1 and M_1. This is nonempty and is a collection of simple alternating (between M_1 and M_2) cycles. Let C be one of the cycles. Let $C_1 = C \cap M_1$ and $C_2 = C \cap M_2$. Then we claim that $w(C_1) = w(C_2)$. Suppose $w(C_1) < w(C_2)$ then $(M_2 \setminus C_2) \cup C_1$ will be a matching of weight smaller than that of M_2. Let $\vec{C_1}$ and $\vec{C_2}$ be the corresponding set of directed edges. Now consider a directed planar graph $\vec{G}'$ which is same as $\vec{G}$ except that the directions of all the edges in C_2 is reversed. Thus edges of $\vec{C_1}, \vec{C_2}^r$ form a directed cycle $\vec{C}$ in $\vec{G}'$. But $w(\vec{C}) = w(\vec{C_1}) + w(\vec{C_2}^r) = w(\vec{C_1}) - w(\vec{C_2}) = 0$. This contradicts Lemma 3. $\square$

3.1 A sufficient condition for isolating bipartite matching

Note that the above isolation theorems follow, using simple arguments, from a weight function w for directed graphs with the property that weight of any directed cycle is non-zero with respect to w. We state a general result that captures the essentials of the above argument for bipartite matching. A similar theorem holds for isolating directed paths also.

Definition 1. For a class of directed graphs $\mathcal{G}$, we call a subset $\mathcal{G}'$ of $\mathcal{G}$ *semi-isomorphic* if the underlying graph of every graph in $\mathcal{G}'$ is isomorphic.

Consider an arbitrary but fixed labeling of the vertices of the underlying graph of a semi-isomorphic set $\mathcal{G}'$. This induces a labeling on the vertices of every graph in $\mathcal{G}'$. For a graph $G \in \mathcal{G}'$, we denote the ith vertex in G as v_i^G. Note that if G_1 and G_2 are two graphs in $\mathcal{G}'$, then $(v_i^{G_1}, v_j^{G_1})$ is an edge in G_1 iff either $(v_i^{G_2}, v_j^{G_2})$ or $(v_j^{G_2}, v_i^{G_2})$ or both are edges in G_2.

Definition 2. Let $\mathcal{G}$ be a class of directed graphs. An edge weight function, w, is said to be *consistent* for $\mathcal{G}$, if for every semi-isomorphic subset $\mathcal{G}'$ of $\mathcal{G}$ and any two graphs G_1 and G_2 in $\mathcal{G}'$, if $(v_i^{G_1}, v_j^{G_1})$ is an edge in G_1, then $w(v_i^{G_1}, v_j^{G_1}) = -w(v_j^{G_1}, v_i^{G_1}) = w(v_i^{G_2}, v_j^{G_2}) = -w(v_j^{G_2}, v_i^{G_2})$, if the respective edges are present.

Definition 3. Given a undirected graph H, let $\mathcal{O}_H$ be the set of all directed graphs H', such that the underlying graph of H' is H. For a class of undirected graphs $\mathcal{H}$, define $\mathcal{O}_{\mathcal{H}} = \cup_{H \in \mathcal{H}} \mathcal{O}_H$.

Theorem 6. *Let $\mathcal{G}$ be a class of directed graphs and let w be a consistent, edge weight function defined for every $G \in \mathcal{G}$ such that for any cycle C in G, $w(C) \neq 0$. Let $\mathcal{H}$ be a class of undirected bipartite graphs, such that $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{G}$. Then we can construct a weight function w' in logspace, such that for every graph $H \in \mathcal{H}$, the minimum weight perfect matching in H with respect to w' is unique.*

Proof. Given H, use Reingold's undirected reachability algorithm [10], to construct a bipartition of H, say L and R. Now orient the edges of H as follows to get the graph H': for every edge $e = \{u, v\}$ in H, where $u \in L$ and $v \in R$, replace e with the directed edge $e' = (u, v)$. By definition $H' \in \mathcal{G}$ and thus $w(H')$ is well defined. We now use w to define a weight on H. For every edge $e \in H$, let $w'(e) = w(e')$.

Now suppose H has two distinct minimum weight perfect matchings, M_1 and M_2, with respect to w'. Then the symmetric difference of M_1 and M_2 is a collection of disjoint, even length, simple cycles, where the edges of the cycle alternate between the matchings M_1 and M_2. Since M_1 and M_2 are distinct, there is at least one cycle. Let $C = (v_1, v_2, \ldots v_{2k}, v_1)$ be such one such cycle. Let $e_i = (v_i, v_{(i+1) \bmod k})$ for $i \in [k]$. Without loss of generality assume, $v_1 \in L$ and the edge e_1 is in M_1. Therefore if i is odd (resp. even), then $e_i \in M_1$ (resp $e_i \in M_2$) and e'_i is directed from L to R (resp from R to L). Thus $w'(e_{2i-1}) = w(e'_{2i-1})$ and $w'(e_{2i}) = -w(e'_{2i})$ for $i \in [k]$, due to skew symmetry of w.

The weight of the restriction of M_1 to C, $w'(M_1 \cap C) = \sum_{i=1}^{k} w'(e_{2i-1})$. Similarly $w'(M_2 \cap C) = \sum_{i=1}^{k} w'(e_{2i})$. Now,

$$
\begin{aligned}
w'(M_1 \cap C) - w'(M_2 \cap C) &= \sum_{i=1}^{k} w'(e_{2i-1}) - \sum_{i=1}^{k} w'(e_{2i}) \\
&= \sum_{i=1}^{k} w(e'_{2i-1}) + \sum_{i=1}^{k} w(e'_{2i}) = \sum_{i=1}^{2k} w(e'_i) \\
&\neq 0.
\end{aligned}
$$

Therefore either $M_1 \cap C$ or $M_2 \cap C$ has higher weight with respect to w'. Without loss of generality assume its M_2. Thus we get a perfect matching $M' = M_2 \setminus (M_2 \cap C) \cup (M_1 \cap C)$ in H of lesser weight, which is a contradiction. $\square$

As an application, we get a simpler proof of the following result from Datta, Kulkarni and Roy [5].

Corollary 7 ([5])**.** *There is a logspace computable weight function, with respect to which there is a unique minimum weight perfect matching in every bipartite, planar, undirected graph.*

Lemma 3 gives a weight function for the class of directed planar graphs that is consistent. Then we apply Theorem 6 to get the above Corollary.

4 Piecewise straight line embedding of a planar graph

In this section we give a logspace algorithm to compute a piecewise straight line embedding of a planar graph. All graphs considered in this section are undirected, unless otherwise specified.

Definition 4. For $p_i \in \mathbb{R}^2$, $(p_1, \ldots, p_{k+1})$ is said to be a piecewise straight line segment, if there is a straight line segment connecting p_i with p_{i+1} for every $i \in [k]$.

Definition 5. For $k \geq 1$, a k-piecewise straight line embedding of a graph $G = (V, E)$ is a function $f : V \to \mathbb{R}^2$ and a collection of $(k-1)$ functions $g_i : E \to \mathbb{R}^2$ for $i \in [k-1]$, such that every edge $e = (u, v) \in E$ is a piecewise straight line segment, $(f(u), g_1(e), \ldots, g_{l_e-1}(e), f(v))$ for some $l_e \leq k$ and no two embedded edges intersect except possibly at the end points.

Theorem 8. *Given a combinatorial embedding of a planar graph G, there is a logspace algorithm that computes a 4-piecewise straight line embedding of G.*

We will give an embedding of G in the first quadrant of the coordinate plane. We first use Reingold's undirected reachability algorithm [10] to compute a spanning tree T of G rooted at r. Now from G we create a new graph G_T by "cutting" every non-tree edge into two edges. Thus G' would be a tree. We then give a straight line embedding of G' in the first quadrant of the two dimensional Cartesian coordinate system (we shall just refer to it as the coordinate system from now on), such that the leaf end of every "split edge" lies on a circle centered at the origin and containing G'. Next we reconnect the split edges appropriately to avoid intersections. Below we give a more formal description of the algorithm.

We create $G_T = (V_T, E_T)$ from G as follows. For each edge $e = (u, v)$ in $E \setminus T$, we introduce two new vertices w_e^u and w_e^v. Now replace e with the edges (u, w_e^u) and (v, w_e^v). Denote the newly introduced set of vertices and edges as V_T' and E_T'. Thus $V_T = V \cup V_T'$ and $E_T = T \cup E_T'$. Note that G_T is a tree and every vertex in V_T' is a leaf. We shall think of G_T as a tree rooted at r as well. Next we define the height function, h for every vertex in G_T. For the root node $h(r) = 0$. For every vertex $v \neq r$ in V, $h(v) = h(p) + 1$, where p is the parent node of v in G_T and for every vertex v in V_T', $h(v) = \max\{h(v) : v \in V\}$. Define $h(G_T) = \max\{h(v) : v \in V_T\}$. For a vertex v, let $A(v)$ be the set of leaves u in G_T, such that u is not present in the subtree rooted at v and the path from u to r lies to the left of the path from v to r. Let L be the set of leaves in G_T. Then $\theta(v) = \frac{|A(v)|}{|L|}\frac{\pi}{2}$.

The coordinates of a vertex v, in our embedding would be $F(v) = (h(v)\cos(\theta(v)), h(v)\sin(\theta(v)))$. For every edge $e = (u, v) \in E_T$ draw a straight line segment between $F(u)$ and $F(v)$ to represent the edge. We shall denote the embedding of this edge (line segment) by $F(e)$. Note that, since the function h is defined to be equal to the maximum over all values of h, for vertices in the set V_T', therefore the vertices in V_T' lie on the concentric circle of radius $h(G_T)$, which(the circle) by definition contains the entire embedded graph G_T.

Next we compare the sets $A(u)$ and $A(v)$ for two vertices u and v.

Lemma 9. *Let u and v be two distinct vertices in G. (a) If u is an ancestor of v then $A(u) \subseteq A(v)$. (b) If u lies to the left of v, then $A(u) \subsetneq A(v)$. (c) For any descendent w of u, $A(w) \subsetneq A(v)$.*

Proof. (a) follows from the observation that any vertex to the left of a node also lies to the left of any of its descendent. Similarly, if u lies to the left of v, then any node to the left of u also lies to the left of v. This proves (b). (c) follows since any descendent of u lies to the left of v. □

In Lemma 10 we show that distinct vertices get mapped to distinct coordinates by F. In Lemma 11 we prove that no two edges of G_T intersect at an intermediate point.

Lemma 10. *Let u, v be two vertices in G_T. Then $u = v$ if and only if $F(u) = F(v)$.*

Proof. Let u and v be two distinct vertices. If $h(u) \neq h(v)$ then $F(u) \neq F(v)$ since they lie in different concentric cycles around the origin by definition of F. If $h(u) = h(v)$, then it follows from Lemma 9. □

Lemma 11. *Let e_1 and e_2 be two edges in G_T. Then $F(e_1)$ and $F(e_2)$ do not intersect except possibly at end points.*

Proof. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ such that u_i is the parent of v_i. If $u_1 = u_2$ then since $v_1 \neq v_2$, e_1 and e_2 do not intersect non-trivially. Also if v_1 is an ancestor of u_2 then v_1 is an ancestor of v_2 as well and therefore they cannot intersect since they lie in concentric circles of different length around the origin.

We now consider the case when u_1 is not an ancestor or descendent of u_2. Without loss of generality assume u_1 is to the left of u_2, which implies that $A(u_1) \subsetneq A(u_2)$. From Lemma 9 we get $\theta(u_1) \leq \theta(v_1) < \theta(u_2) \leq \theta(v_2)$. Therefore the line segments $(F(u_1), F(v_1))$ and $(F(u_2), F(v_2))$ do not intersect. □

Next we rejoin the split edges to get back the original graph. After joining, a split edge would be embedded as a piecewise straight line as we describe below. Recall that precisely the non-tree edges in G are the edges that were split.

Suppose $e = (u, v)$ was a non-tree edge in G. Then e was replaced by the edges (u, w_e^u) and (v, w_e^v) by the introduction of two new vertices w_e^u and w_e^v. We remove the vertices w_e^u and w_e^v and the edges (u, w_e^u) and (v, w_e^v) and draw the piecewise straight line segment $(F(u), F(w_e^u), \max\{F(w_e^u), F(w_e^v)\}, F(w_e^v), F(v))$ to represent edge e. (where the max function is defined as $\max\{(a_1, b_1), (a_2, b_2)\} \triangleq (\max\{a_1, a_2\}, \max\{b_1, b_2\})$) We shall denote the embedding of this edge (piecewise line segment) by $F(e)$. In Lemma 12 we show that non-tree edges do not intersect non-trivially, to complete the proof of Theorem 8.

Lemma 12. *Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two non-tree edges in G. Then the edges $F(e_1)$ and $F(e_2)$ do not intersect non-trivially.*

Proof. We only need to show that the piecewise line segments $(F(w^{u_1}_{e_1}), \max\{F(w^{u_1}_{e_1}), F(w^{v_1}_{e_1})\}, F(w^{v_1}_{e_1}))$ and $(F(w^{u_2}_{e_2}), \max\{F(w^{u_2}_{e_2}), F(w^{v_2}_{e_2})\}, F(w^{v_2}_{e_2}))$ do not intersect.
Case 1 (One end point of e_1 *and* e_2 *is common)*: Without loss of generality assume $u_1 = u_2 = u(say)$ and $\theta(v_1) \leq \theta(v_2)$. Thus in G_T, $w^u_{e_2}$ lies to the left of $w^u_{e_1}$ which implies $\theta(w^u_{e_2}) < \theta(w^u_{e_1})$ by Lemma 9. Also $\theta(w^{v_1}_{e_1}) < \theta(w^{v_2}_{e_2})$ since $w^{v_1}_{e_1}$ and $w^{v_2}_{e_2}$ are children of v_1 and v_2 respectively. This shows the Lemma for Case 1.
Case 1 (All end points of e_1 *and* e_2 *are distinct)*: Without loss of generality assume $\theta(u_1) \leq \theta(u_2)$ and $\theta(u_i) \leq \theta(v_i)$ for $i \in \{1, 2\}$. Since e_1 and e_2 cannot intersect, therefore if $\theta(v_1) \geq \theta(u_2)$, then $\theta(u_1) \leq \theta(v_2) \leq \theta(v_1)$, and if $\theta(v_1) < \theta(u_2)$, then either $\theta(v_2) \geq \theta(v_1)$. This implies that either $\theta(w^{u_1}_{e_1}) < \theta(w^{u_2}_{e_2}) < \theta(w^{v_2}_{e_2}) < \theta(w^{v_1}_{e_1})$ or $\theta(w^{u_1}_{e_1}) < \theta(w^{v_1}_{e_1}) < \theta(w^{u_2}_{e_2}) < \theta(w^{v_2}_{e_2})$. Hence the Lemma holds for this case too. □

Note that the coordinates that we assign are real numbers and need not be computable in logspace. To take of this problem we can "inflate" the entire mapping by multiplying each coordinate with a suitable large number (say $|V|^5$) and then taking the floor of each point to get an integral embedding.

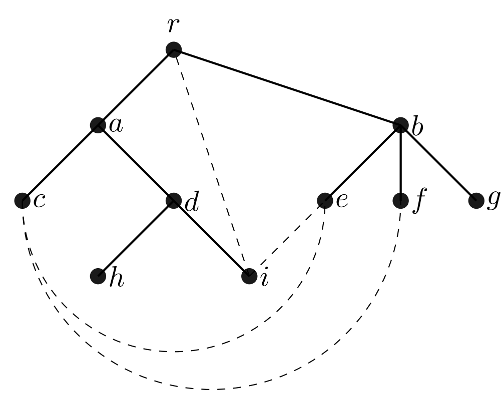

Figure 1: Example of a graph G containing a spanning tree T(solid edges) rooted at r. The dashed edges are the non-tree edges.

5 Final remarks

It is clear that there are many other weight functions that will work. In fact any "nice" solution to the differential equation $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$ will yield isolating weight functions. In particular, setting $P(x, y) = \frac{-y}{2}$ and $Q(x, y) = \frac{x}{2}$ to the left hand side of Green's theorem yields the weight function $w(e) = (x_u y_v - x_v y_u)$ which is isolating.

One can easily verify that the weight function we give here is a true extension of the following weight function prescribed in [4] for isolating paths in grid graphs: east and west edges are given 0 weight, a north edge at $((i, j), (i, j + 1))$ is given a weight i, and a south edge at $((i, j), (i, j - 1))$ is given a weight $-i$. However, if we apply our theorem for the case of isolating matching in grid graphs, we get a different (slightly simpler) weight function than the one prescribed in [5]. We believe that this paper better explains the reason behind why these weight functions work.

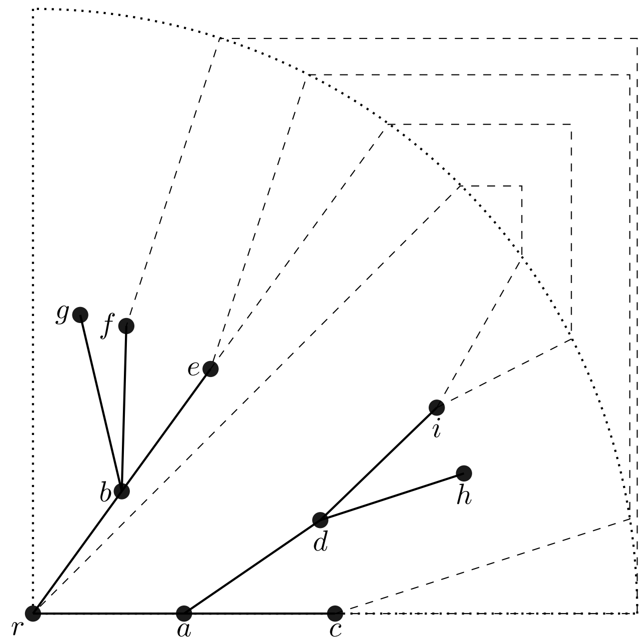

Figure 2: Piecewise straight line embedding of G.

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