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Verifying Computations with Streaming Interactive Proofs

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Abstract

Applications based on outsourcing computation require guarantees to the data owner that the desired computation has been performed correctly by the service provider. Methods based on proof systems can give the data owner the necessary assurance, but previous work does not give a sufficiently scalable and practical solution, requiring a lot of time, space or computational power for both parties. In this paper, we develop new proof protocols for verifying computations which are streaming in nature: the verifier (data owner) needs only a single pass over the input storing a logarithmic amount of information, and follows a simple protocol with a prover (service provider) that takes a logarithmic number of rounds. A dishonest prover fools the verifier with only polynomially small probability, while an honest prover's answer is always accepted.

We first observe that some existing constructions for interactive proof systems can be modified to work with streaming verifiers. The consequences are powerful: these constructions imply that all problems in the complexity class NP have *computationally sound* streaming protocols requiring a polylogarithmic communication and space, and that all problems in log-space uniform NC have *statistically sound* protocols with the same space and communication requirements.

We then seek to bridge the gap between theory and practice by developing improved and simplified protocols for a variety of problems of central importance in streaming and database processing. All of our protocols achieve statistical soundness and most require only logarithmic communication between prover and verifier. We also experimentally demonstrate their practicality and scalability. All these problems require linear space in the traditional streaming model, showing that adding a prover can exponentially reduce the effort needed by the verifier.

1 Introduction

Efficient proof verification has long played a central role in complexity theory. For example, the class NP can be equivalently defined as the set of languages with proofs of membership that can be verified in polynomial time. The most general proof verification model is the *interactive proof system* where there is a resource-limited *verifier* $\mathcal{V}$ and an all-powerful *prover* $\mathcal{P}$. To solve a problem, the verifier initiates a conversation with the prover, who solves the problem and *proves* the validity of his answer, following an established (randomized) protocol.

This model is directly relevant to the setting of *outsourcing computations* to a (potentially untrusted) service provider. A wide variety of scenarios fit this template: in one extreme, a large business outsources its data to another company to store and process; at the other end of the scale, a hardware co-processor performs some computations within an embedded system. In both these situations, the data owner (the verifier in our

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model) wants to be assured that the computations performed by the service provider (the prover) are correct, without having to take the effort to perform the computation himself. A natural approach is to use a proof protocol to *prove* the correctness of the answer. However, existing protocols in complexity theory are mostly of theoretical interest, requiring a lot of time and space for both parties. Historically, protocols have required that the verifier must retain the full input, whereas in many practical situations the verifier can only process the input in a one-pass streaming fashion.

In this paper we introduce a proof system over data streams. That is, the verifier sees a data stream and tries to solve a (potentially difficult) problem with the help of a more powerful prover who sees the same stream. At the end of the stream, they conduct a conversation following an established protocol, through which an honest prover will always solve the problem and make the verifier accept its results, whereas any dishonest prover will not be able to fool the verifier into accepting a wrong answer with probability more than $\frac{1}{3}$.[1] In the streaming setting, we are most interested in the space complexity (of the verifier). At the same time we want the other costs of the verification process to be bounded. So we define an (s,t)*-protocol* to be one where the space usage of $\mathcal{V}$ is $O(s)$ and the total communication cost of the conversation between $\mathcal{P}$ and $\mathcal{V}$ is $O(t)$. We will measure both s and t in terms of words, where each word can represent quantities polynomial in u, a measure of the size of universe of the computations.

Note that if $t = 0$ the model degenerates to the standard streaming model. We are interested in whether it is possible to increase the computing power by communicating with a third party, and verifiably solve some problems that are known to be hard in the standard streaming model. This paper presents positive answers to a suite of problems, all of which require linear space in the streaming model. We begin by observing that a variation of Probabilistically Checkable Proofs (PCPs) due to Kilian (Universal Arguments) [15] as well as a construction of Goldwasser, Kalai, and Rothblum [12] can be modified to work with streaming verifiers. This implies that all problems in the complexity class NP have *computationally sound* $(\text{poly}\log u, \text{poly}\log u)$ protocols, meaning that a computationally bounded dishonest prover cannot fool the verifier under standard cryptographic assumptions. It also implies that all problems in NC have *statistically sound* $(\text{poly}\log u, \text{poly}\log u)$ protocols. The power and generality of these results can be contrasted with most results in the streaming literature, which normally apply only to one or a few problems at a time. These results demonstrate in principle the power of the streaming interactive proof model, but they may not yield practical verification protocols.

We then improve upon the construction of Goldwasser et al. by providing protocols that are not only asymptotically more efficient in both space and communication, but also easy to implement and highly practical, for the following problems: *self join size*, *inner product*, *frequency moments*, various *sketches*, *range query*, *range-sum query*, *dictionary*, *predecessor*, and *index*. These problems are all of considerable importance and many have been studied extensively in the standard streaming model and shown to require linear space. As a result, approximations have to be allowed if sub-linear space is desired (for the first 3 problems); some of the problems do not have even approximate streaming algorithms (the last 5 problems). On the other hand, we solve them all *exactly* in our model. Formal definitions of these problems are given in Section 1.3.

Although our focus is to minimize the space usage of the verifier and the communication cost of the verification, these new protocols are also very efficient in terms of both parties' running time. In particular, when processing the stream, the verifier spends $O(\log u)$ time per element. During verification the verifier spends $O(\log u)$ time while the (honest) prover runs in near-linear time. So although the model allows a prover with unlimited power, an honest prover can execute our protocols efficiently. This makes our protocols simple enough to be deployed in real computation-outsourcing situations. Meanwhile, the model

[1] In fact, our protocols let this probability be set arbitrarily small.

as defined ensures that even a computationally unbounded dishonest prover will not be able to fool the resource-limited verifier.

1.1 Theoretical significance

Proof verification lies at the heart of complexity theory. Many classical results show that the variations of this verifier-prover setting characterize many complexity classes. In the most general case where there are no restrictions on the interaction, beyond that the verification (hence also communication) is done in polynomial time, the resulting class IP, can be shown to be equivalent to PSPACE [22]. If only a constant number of rounds of interaction between prover and verifier are allowed, the class AM (short for *Arthur-Merlin protocols*) results.[2] It is known that $\mathsf{AM} \subseteq \Pi_2\mathsf{P} \subseteq \mathsf{PSPACE} = \mathsf{IP}$. The (possibly) smaller class $\mathsf{MA} \subseteq \mathsf{AM}$ restricts the communication to be one-way from the prover to the verifier. Finally, if there is no probability of failure on the verifier's side allowed, the class boils down to NP. These classes all generalize the case without communication, captured by RP or P, depending on whether a probability of failure is allowed. The concept of *probabilistically checkable proofs* [3] is also closely related: here, the prover is replaced by a *proof* in redundant form, and the verifier need access only a few (randomly chosen) bits of the proof.

All these classical results are for the non-streaming setting; indeed, most data streaming results are concerned with the complexity of solving or approximating problems known to be in P. The difficulty comes from the additional constraints imposed by the streaming model: sublinear space, ideally logarithmic or polylogarithmic, and a single pass over the data. This severely reduces the computational power, and there are many seemingly easy problems (such as the problems considered in this paper) that are provably "hard" for this model, i.e., they require at least linear space if only one pass is allowed. Although prior work on interactive proofs has studied the verifier's space complexity [10, 12], these constructions did not restrict the verifier to make only one pass over the data. Such extra passes can make a significant difference in terms of the difficulty of the problem. For example, streaming problems considered in this paper, such as *index, dictionary, range query, range-sum query, predecessor*, become trivial if just one extra pass is allowed over the whole input data.

Therefore, our goal is to extend the computational power of the streaming model by considering it in the interactive proof framework, hoping for an improvement as significant as from RP to PSPACE. Our results show that by allowing only a logarithmic amount of interaction between the verifier and the prover, the space cost of many problems can be improved exponentially: from linear to logarithmic for the problems we focus on. We therefore obtain a strong separation result, since the proof that these problems require linear space in the standard streaming model is independent of any complexity-theoretic conjectures. Further characterizing the class of problems that can be solved in the streaming interactive proof model remains a challenging open question.

Our work is motivated by prior work [24, 6] on verification of streaming computations that had stronger constraints. In the first model [24], the prover may send only the answer to the computation, which must be verified by $\mathcal{V}$ using a small sketch computed from the input stream of size n. Protocols were defined to verify identity and near-identity, and so because of the size of the answer, had $s = 1$ and $t = n$. Subsequent work showed that problems of showing a matching and connectedness in a graph could be solved in the same bounds, in a model where the prover's message was restricted to be a *permutation* of the input alone [20].

[6] introduced the notion of a streaming verifier, who must read first the input and then the proof under

[2] Strictly speaking, in AM the verifier is required to reveal his random bits, but it is known that this does not affect the model's power [13, 4].

space constraints. This can be thought of as the MA model in the streaming setting. However, this does not dramatically improve the computational power. In this model, INDEX (see the definition in Section 1.3) can be solved using a $(\sqrt{n},\sqrt{n})$-protocol and there is also a matching lower bound of $st = \Omega(n)$ [6]; note that both $(n,1)$- and $(1,n)$-protocols are trivial, so the contribution of [6] is achieving a tradeoff between s and t. In this paper, we show that allowing more interaction between the prover and the verifier exponentially reduces st for this and other problems that are hard in the standard streaming model.

1.2 Practical implications

Our study is motivated by developing applications in data outsourcing and trustworthy computing in general. In the increasingly popular model of "cloud computing", individuals or small businesses delegate the storage and processing of their data to a powerful but potentially untrusted third party, the "cloud". This results in cost-savings, since the data owner no longer has to maintain expensive data storage and processing infrastructure. However, it is important that the data owner is assured that their data is processed accurately and completely by the cloud. Already, there has been work on "proofs of retrievability" in the cryptography community to verify that the data is stored correctly by the cloud [14]. In this paper, we provide "proofs of queries" which allow the cloud to demonstrate that the results of queries are correct while keeping the data owner's computational effort minimal.

Our "proofs of queries" only need the data owner (taking the role of verifier $\mathcal{V}$) to make a single streaming pass over the original data. This fits the cloud setting well: the pass over the input can take place incrementally as the verifier uploads data to the cloud. So the verifier may never need to hold the entirety of the data, since it can be shipped up to the cloud to store as it is collected. Without these new protocols, the verifier would either need to store the data in full, or retrieve the whole data from the cloud for each query: either way negates the benefits of the cloud model. Instead, our methods require the verifier to track only a logarithmic amount of information and follow a simple protocol with logarithmic communication to verify each query.

Query verification/authentication for data outsourcing has been a popular topic recently in the database community. The majority of the work is in the non-streaming setting; see [23] and the references therein. More recently, there have been a few works which adopt a streaming-like model for the verifier, although they still require linear memory resources. For example, maintaining a Merkle tree [18] (a binary tree where each internal node is a cryptographic hash of its children) takes space linear in the size of the tree. Li et al. [17] considered verifying queries on a data stream with sliding windows via Merkle trees, hence the verifier's space is proportional to the window size. The protocol of Papadopoulos et al. [19] verifies a *continuous* query over streaming data, again requiring linear space on the verifier's side in the worst case. Lastly, we distinguish the two types of security guarantees provided: guarantees based on cryptographic assumptions (termed *computational soundness*), and probabilistic guarantees (termed *statistical soundness*). Our main results guarantee statistical soundness: even if a (dishonest) prover has unlimited computing power, it is not possible to fool the verifier with high probability because one cannot ensure collisions for *randomly* chosen hash functions known only to the verifier [24, 6]. In contrast, the cryptographic setting assumes the prover is unable to break certain cryptosystems (typically, is unable to find collisions under cryptographic hash functions such as SHA, as in [17, 19]). This is weaker than the statistical case: a determined adversary could eventually find hash collisions and fool the verifier.

Although interactive proof systems and other notions of proof verification have been extensively studied, they are primarily used to establish complexity results and hardness of approximation. Because they are usually concerned with answering "hard" problems, the (honest) prover's time cost is usually super-polynomial. As such, these schemes have had very little practical impact thus far. More recently, there has been work aimed at reducing the cost of the prover to polynomial [12]. As mentioned above, the protocols designed there allow the verifier to make multiple passes over the data, but it turns out that this construction also works with a streaming (one pass) verifier. Although of striking generality, the protocols that result are still complex, and require (polylogarithmically) many words of space and rounds of interaction between prover and verifier. In contrast, our protocols for the problems defined in Section 1.3 require only logarithmic space and communication (and nearly linear running time for both prover and verifier). Thus we claim that they are highly practical for use in verifying outsourced query processing.

1.3 Problems

We now define some canonical problems, chosen to demonstrate the power of the interactive proof setting, and are suitably abstract to capture a wide variety of possible applications. Let $[u] = \{0, \ldots, u-1\}$ be the universe from which all data elements are drawn.

INDEX: Given a stream of u bits $b_1, \ldots, b_u$, followed by an index q, the answer is b_q.

PREDECESSOR: Given a stream of n elements in $[u]$, followed by a query $q \in [u]$, the answer is the largest p in the stream such that $p \leq q$. We assume that 0 always appears in the stream.

DICTIONARY: The input is a stream of $n \leq u$ (key, value) pairs, where both the key and the value are drawn from the universe $[u]$, and all keys are distinct. The stream is followed by a query $q \in [u]$. If q is one of the keys, then the answer is the corresponding value; otherwise the answer is "not found".

RANGE QUERY: Given a stream of n elements in $[u]$, followed by a range query $[q_L, q_R]$, the answer is the set of all elements in the stream between q_L and q_R inclusive.

RANGE-SUM: The input is a stream of n (key, value) pairs, where both the key and the value are drawn from the universe $[u]$, and all keys are distinct. The stream is followed by a range query $[q_L, q_R]$. The answer is the sum of all the values with keys between q_L and q_R inclusive.

SELF-JOIN SIZE: Given a stream of n elements from $[u]$, compute $\sum_{i \in [u]} a_i^2$ where a_i is the number of occurrences of i in the stream. This is also known as the *second frequency moment*.

FREQUENCY MOMENTS: In general, for any integer $k \geq 1$, $\sum_{i \in [u]} a_i^k$ is called the *k-th frequency moment* of the vector $\mathbf{a}$, written $F_k(\mathbf{a})$.

INNER PRODUCT (or JOIN SIZE): Given two streams A and B with frequency vectors $(a_1, \ldots, a_u)$ and $(b_1, \ldots, b_u)$, respectively, compute $\sum_{i \in [u]} a_i b_i$.

These queries are broken into two groups. The first four are *reporting* queries, which ask for elements from the input to be returned. INDEX is a classical problem that in the streaming model requires $\Omega(u)$ space [16]. It is clear that PREDECESSOR, DICTIONARY, RANGE QUERY, RANGE-SUM are all more general than INDEX and hence, also require linear space. These problems would be trivial if the query were fixed before the data is seen. But in most applications, the user (the verifier) forms queries in response to other information that is only known after the data has arrived.

The remaining queries are *aggregation* queries, computations that combine multiple elements from the input. SELF-JOIN SIZE requires linear space in the streaming model [1] to solve exactly (although there are space-efficient approximation algorithms). Since FREQUENCY MOMENTS and INNER PRODUCT are more general than SELF-JOIN SIZE, they also require linear space.

Outline. In Section 2, we describe how the Universal Arguments of Kilian as well as the construction of Goldwasser, Kalai, and Rothblum can be modified to work with streaming verifiers, thereby providing space- and communication-efficient streaming protocols for all of NP and NC respectively. Subsequently, we improve upon these protocols for many problems of central importance in streaming and database processing. In Section 3 we give more efficient protocols to solve the aggregation queries (exactly), and in Section 4 we provide protocols for the reporting queries. In both cases, our protocols require only $O(\log u)$ space for the verifier $\mathcal{V}$, and the total size of the interaction between the two parties is $O(\log u)$ over $\log u$ rounds. In Section 5 we extend this approach to a class of *frequency-based functions*; this results in protocols requiring $O(\log u)$ space and $\log u$ rounds, at the cost of more communication. An experimental study in Section 6 shows that these protocols are practical. Concluding remarks are made in Section 7.

2 Proofs and Streams

Arithmetization and Low-Degree Extensions. Given a function f', arithmetization involves extending the domain of f' to a field and replacing f' with its *low-degree extension* (LDE) f as a polynomial over the field. The low-degree extension f can be interpreted as a high-distance encoding of f', and the error-detecting properties of this code typically give the verifier considerable power over the prover.

More precisely, let integer ℓ be a parameter, and assume for presentation purposes that $u = \ell^d$ is a power of ℓ. Let $\mathbf{a} = (a_1, \ldots, a_u)$ be a vector in $[u]^u$. Later, we will think of $\mathbf{a}$ as the frequency-vector of a stream, or sometimes as the (unaggregated) stream itself. We may interpret $\mathbf{a}$ as a function $f' : [\ell]^d \to [u]$ as follows: letting $(i)^\ell_k$ denote the k-th least significant digit of i in base-ℓ representation, we associate each $i \in [u]$ with a vector $((i)^\ell_1, (i)^\ell_2, \ldots, (i)^\ell_d) \in [\ell]^d$, and define $f'(i) = a_i$.

Pick a prime p such that $u \leq p$. The *low-degree extension* (LDE) of f' is a d-variate polynomial f over the field $\mathbb{Z}_p$ so that $f(\mathbf{x}) = f'(\mathbf{x})$ for all $\mathbf{x} \in [\ell]^d$; we alternatively refer to f as the LDE of $\mathbf{a}$. Notice since f is a polynomial over the field $\mathbb{Z}_p$, $f(\mathbf{x})$ is defined for all $\mathbf{x} \in [p]^d$; f essentially extends the domain of f' from $[\ell]^d$ to $[p]^d$. We can define the polynomial $f : [p]^d \to \mathbb{Z}_p$ in a constructive manner as follows. Let $\mathbf{x} = (x_1, \ldots, x_d) \in [p]^d$. First, note that the polynomial corresponding to a function which is 1 at location $\mathbf{v} = (v_1, \ldots, v_d) \in [\ell]^d$ and zero elsewhere in $[\ell]^d$ is

$$\chi_{\mathbf{v}}(\mathbf{x}) = \prod_{j=1}^{d} \chi_{v_j}(x_j)$$

where $\chi_k(x_j)$ is the Lagrange basis polynomial given by

$$\frac{(x_j - 0)\cdots(x_j - (k-1))(x_j - (k+1))\cdots(x_j - (\ell-1))}{(k-0)\cdots(k-(k-1))(k-(k+1))\cdots(k-(\ell-1))},$$

which has the property that $\chi_k(x_j) = 1$ if $x_j = k$ and 0 for all $x_j \neq k, x_j \in [\ell]$. We then define

$$f(\mathbf{x}) = \sum_{\mathbf{v} \in [\ell]^d} a_{\mathbf{v}} \chi_{\mathbf{v}}(\mathbf{x}).$$

One can easily verify that such an f satisfies the requirement above.

Streaming Computation. Let $\mathbf{a} \in [u]^u$ be an input; in the context of this paper, $\mathbf{a}$ will be a data stream, and we seek to compute a query or statistical aggregate of $\mathbf{a}$. Our first observation is that, in a number of important constructions in the theory of proof systems, the verifier only needs access to $\mathbf{a}$ in order to evaluate one or several randomly chosen locations in the LDE f of $\mathbf{a}$ (we explain in more detail why this is true later). That is, the only information $\mathcal{V}$ need extract from the input is $f(\mathbf{r})$ for a small number of randomly chosen locations $\mathbf{r} \in [p]^d$. In particular, this is the case for a suitable instantiation of Kilian's Universal Arguments [15] and the "Interactive Proofs for Muggles" construction of Goldwasser et al. [12]. Moreover, in both cases, the location(s) $\mathbf{r}$ only depend on the random coin tosses of $\mathcal{V}$, and do *not* depend on the interaction with $\mathcal{P}$. Therefore, to determine the value(s) $\mathbf{r}$, $\mathcal{V}$ may toss all coins before observing the input $\mathbf{a}$ ($\mathcal{V}$ remembers the coin tosses and keeps them private from $\mathcal{P}$). $\mathcal{V}$ may then compute the values $f(\mathbf{r})$ while observing the input, and only afterward does $\mathcal{V}$ need to communicate with $\mathcal{P}$.

We now make a second observation: given one-way access to $\mathbf{a}$, a constant space verifier can compute $f(\mathbf{r})$ in a streaming fashion. Indeed, we may write $f(\mathbf{r}) = \sum_{\mathbf{v} \in [\ell]^d} a_{\mathbf{v}} \chi_{\mathbf{v}}(\mathbf{r})$.

$\mathcal{V}$ can therefore make one pass over the vector $\mathbf{a}$, and each time $\mathcal{V}$ observes a new entry $a_{\mathbf{v}}$ of the input, $\mathcal{V}$ may update

$$f(\mathbf{r}) \leftarrow f(\mathbf{r}) + a_{\mathbf{v}} \cdot \chi_{\mathbf{v}}(\mathbf{r}).$$

Note that in order to maintain $f(\mathbf{r})$, $\mathcal{V}$ only needs to keep $f(\mathbf{r})$ and $\mathbf{r}$, which takes $d+1$ words in $[p]$.

It is already known that the construction of [12] (respectively, Universal Arguments) yield small-space *non*-streaming verifiers and polylogarithmic communication for all problems in log-space uniform NC (respectively, NP), and achieve statistical (respectively, computational) soundness. Below, we describe the insights necessary to show that these results also hold for a streaming verifier.[3]

Streaming Universal Arguments. Recall that a probabilistically checkable proof (PCP) is a proof in redundant form, such that the verifier need access only a few (randomly chosen) bits of the proof before deciding whether to accept or reject. A Universal Argument effectively simulates a PCP while ensuring $\mathcal{P}$ need not send the entire proof to $\mathcal{V}$. We first describe this simulation, before describing a particular PCP system that, when simulated by a Universal Argument, can be executed by a streaming verifier.

For a language L on input $\mathbf{a}$, a Universal Argument consists of four messages: First, $\mathcal{V}$ sends $\mathcal{P}$ a collision-resistant hash function h. Next, an honest $\mathcal{P}$ constructs a PCP π for $\mathbf{a}$, and then constructs a Merkle tree of π using h (the leaves of the tree are the bits of π) [18]. $\mathcal{P}$ then sends the value of the root of the tree to $\mathcal{V}$. This effectively "commits" $\mathcal{P}$ to the proof π; $\mathcal{P}$ cannot subsequently alter it without finding collisions for h. Third, $\mathcal{V}$ sends $\mathcal{P}$ a list of the locations of π he needs to query. Finally, for each bit b_i that is queried, $\mathcal{P}$ responds with the value of all nodes on b_i's authentication path in the Merkle tree (note this path has only logarithmic length). $\mathcal{V}$ checks, for each bit b_i that the authentication path is correct relative to the value of the root; if so, $\mathcal{V}$ is convinced $\mathcal{P}$ returned the correct value for b_i as long as $\mathcal{P}$ cannot find a collision for h. The theorem follows by combining this construction with the fact that there exist PCP systems in which $\mathcal{V}$ only needs access to $\mathbf{a}$ in order to evaluate $O(1)$ locations in the LDE f of $\mathbf{a}$. We now justify this last claim by describing such a PCP system.

In [5], Ben-Sasson et al. describe for any language in NP a PCP system in which $\mathcal{V}$ is not given explicit access to the input; instead, $\mathcal{V}$ has oracle access to an encoding of the input $\mathbf{a}$ under an *arbitrary* error-correcting code (to simplify a little). In their PCP system, $\mathcal{V}$ runs in polylogarithmic time and queries only $O(1)$ bits of the encoded input, and $O(1)$ bits of the proof π. Moreover, these bits are determined non-adaptively (specifically, they do not depend on $\mathbf{a}$). We show this implies a PCP system that satisfies the claim for any $L \in \mathsf{NP}$. Indeed, let $LDE(\mathbf{a})$ denote the truth-table of f; i.e. $LDE(\mathbf{a})$ is a list of elements in the field $\mathbb{Z}_p$, one for each $\mathbf{r} \in \mathbb{Z}_p^d$. There are (two-stage) *concatenated codes* whose first stage applies the *LDE* operation to the input $\mathbf{a}$ (and whose second stage applies a code to turn the field elements in $LDE(\mathbf{a})$ into bits) that suffice as encodings of $\mathbf{a}$ [2]. Therefore, a streaming verifier with explicit access to the input $\mathbf{a}$ may simulate the verifier $\mathcal{V}$ in the PCP system of Ben-Sasson et al: each time $\mathcal{V}$ queries a bit b_i of the

[3]This fact was observed by Guy Rothblum; here, we present the details of the construction for completeness.

encoded input, there is a location $\mathbf{r}$ such that b_i can be extracted from $f(\mathbf{r})$.

A Universal Argument based on the PCP of the previous paragraph has two additional properties worth mentioning. First, since $\mathcal{V}$ need only query $O(1)$ bits of $LDE(\mathbf{a})$ and otherwise runs in poly log time, we obtain a streaming verifier that runs in *near-linear* time. Second, since $\mathcal{V}$ need only query $O(1)$ bits of the proof, and the authentication path of each bit in the Merkle tree is of length $O(\log u)$, it follows that the communication complexity of the Universal Argument is $O(\log u)$ words. Putting all these pieces together yields the following theorem:

Theorem 1. *There are computationally sound* $(\text{poly}\log u, \log u)$ *protocols for any problem in* NP.

Theorem 1 implies that Universal Arguments can be implemented with a streaming verifier, but we are not suggesting that this actually yields a practical proof system, even if we are satisfied with security guarantees based on cryptographic assumptions. Indeed, even ignoring the complexity of constructing a PCP, the prover in a Universal Argument may need to solve an NP-hard problem just to determine the correct answer! However, Theorem 1 does demonstrate that in principle it is possible to have extremely efficient verification systems with streaming verifiers even for problems that are computationally difficult in a *non*-streaming setting.

Streaming "Interactive Proofs for Muggles". In [12], $\mathcal{V}$ and $\mathcal{P}$ first agree on a circuit C of fan-in 2 that computes the function of interest; C is assumed to be in layered form. $\mathcal{P}$ begins by claiming a value for the output gate of the circuit. The protocol then proceeds iteratively from the output layer of C to the input layer, with one iteration for each layer. Let $\mathbf{v}^{(i)}$ be the vector of values that the gates in i-th layer of C take on input x, with layer 1 corresponding to the output layer, and let $f^{(i)}$ be the LDE of $\mathbf{v}^{(i)}$.

At a high level, in iteration 1, $\mathcal{V}$ reduces verifying the claimed value of the output gate to verifying $f^{(2)}(\mathbf{r})$ for a random location $\mathbf{r}$. Likewise, in iteration i, $\mathcal{V}$ reduces verifying $f^{(i)}(\mathbf{r})$ to verifying $f^{(i+1)}(\mathbf{r}')$ for a random $\mathbf{r}'$. Critically, the verifier's final test requires only $f^{(d)}(\mathbf{r})$, the low-degree extension of the input at the random location $\mathbf{r}$, which can be chosen at random independent of the data or the circuit, and hence computed by a streaming verifier. Note that each iteration takes logarithmically many rounds, with a constant number of words of communication in each round. Therefore the protocol requires $O(d \log u)$ communication in total. In particular, all problems that can be solved in log-space by non-streaming algorithms (i.e. algorithms that can make multiple passes over the input) possess circuits of depth $\log^2 u$, and hence there are $(\log^3 u, \log^3 u)$ protocols for these problems.

Theorem 2 (Extending Theorem 3 in [12])**.** *There are statistically sound (*$\text{poly}\log u$, $\text{poly}\log u$*) protocols for any problem in log-space uniform* NC.

Here, NC is the class of all problems decidable by circuits of polynomial size and polylogarithmic depth; equivalently, the class of problems decidable in polylogarithmic time on a parallel computer with a polynomial number of processors. This class includes, for example, many fundamental matrix problems (e.g. determinant, product, inverse), and graph problems (e.g. minimum spanning tree, shortest paths) (see [2, Chapter 6]).

Despite its powerful generality, the protocol implied by Theorem 2 is not optimal for many of the low-complexity functions of most importance in streaming and database applications. The remainder of this paper obtains improved, practical protocols for the fundamental problems listed in Section 1.3.

3 Interactive Proofs for Aggregation Queries

We describe a protocol for the aggregation queries with a quadratic improvement over that obtained from Theorem 2. We describe our protocol over a more general stream where each element in the stream is an (i,δ) pair. Initialize the vector $\mathbf{a} = (a_0, \dots, a_{u-1})$ to be $\mathbf{0}$. A pair (i,δ) in the stream updates $a_i \leftarrow a_i + \delta$.

3.1 SELF-JOIN SIZE

We first explain the case of SELF-JOIN SIZE, which is $F_2 = \sum_{i\in[u]} a_i^2$. In the SELF-JOIN SIZE problem we have $\delta = 1$ for all updates, but our protocol allows any δ, positive or negative. This generality is useful for other queries considered later.

As in Section 2, let integer $\ell \geq 2$ be a parameter to be determined. We assume that u is a power of ℓ for ease of presentation. Pick a prime p such that $u \leq p \leq 2u$ (by Bertrand's Postulate, such a p always exists). We also assume that p is chosen so that $F_2 = O(p)$, to keep the analysis simple. The protocol we propose is similar to sum-check protocols in interactive proofs (see [2, Chapter 8]); given any d-variate polynomial g over $\mathbb{Z}_p$, a sum-check protocol allows a polynomial-time verifier $\mathcal{V}$ to compute $\sum_{\mathbf{z}\in H^d} g(\mathbf{z})$ for any $H \subseteq \mathbb{Z}_p$, as long as $\mathcal{V}$ can evaluate g at a randomly-chosen location in polynomial time. A sum-check protocol requires d rounds of interaction, and the length of the i'th message from $\mathcal{P}$ to $\mathcal{V}$ is equal to $\deg_i(g)$, the degree of g in the i'th variable.

Let $\mathbf{a}^2$ denote the entry-wise square of $\mathbf{a}$. A natural first attempt at a protocol for F_2 is to apply a sum-check protocol to the LDE g of $\mathbf{a}^2$ i.e. $g = \sum_{\mathbf{v}\in[l]^d} a_{\mathbf{v}}^2 \chi_{\mathbf{v}}$. However, a streaming verifier cannot evaluate g at a random location because $\mathbf{a}^2$ is not a linear transform of the input. The key observation for our protocol is that a streaming verifier *can* work with a different polynomial of slightly higher degree that also agrees with $\mathbf{a}^2$ on $[\ell]^d$. Specifically, this polynomial is $f^2 = \left(\sum_{\mathbf{v}\in[l]^d} a_{\mathbf{v}}\chi_{\mathbf{v}}\right)^2$, where as usual f is the LDE of $\mathbf{a}$. That is, $\mathcal{V}$ can evaluate the polynomial f^2 at a random location $\mathbf{r}$: $\mathcal{V}$ computes $f(\mathbf{r})$ as in Section 2, and uses the identity $f^2(\mathbf{r}) = f(\mathbf{r})^2$. We are therefore able to apply a sum-check protocol to f^2 in our model; details follow.

The protocol. Before observing the stream, the verifier picks a random location $\mathbf{r} = (r_1, \dots, r_d) \in [p]^d$. Both the prover and the verifier observe the stream which defines $\mathbf{a}$. The verifier $\mathcal{V}$ evaluates $f(\mathbf{r})$ in incremental fashion, as described in Section 2.

After observing the stream, the verification protocol proceeds in d rounds as follows. In the first round, the prover sends a polynomial $g_1(x_1)$, and claims that

$$g_1(x_1) = \sum_{x_2,\dots,x_d\in[\ell]^{d-1}} f^2(x_1, x_2, \dots, x_d). \tag{1}$$

Observe that if g_1 is as claimed, then

$$F_2(\mathbf{a}) = \sum_{x_1\in[\ell]} g_1(x_1).$$

Since the polynomial $g_1(x_1)$ has degree $2(\ell-1)$, it can be described in $2(\ell-1)+1$ words.

Then, in round $j > 1$, the verifier sends r_{j-1} to the prover. In return, the prover sends a polynomial $g_j(x_j)$, and claims that

$$g_j(x_j) = \sum_{x_{j+1},\dots,x_d\in[\ell]^{d-j}} f^2(r_1, \dots, r_{j-1}, x_j, x_{j+1}, \dots, x_d). \tag{2}$$

The verifier compares the two most recent polynomials by checking

$$g_{j-1}(r_{j-1}) = \sum_{x_j \in [\ell]} g_j(x_j)$$

and rejecting otherwise. The verifier also rejects if the degree of g is too high: each g should have degree $2(\ell-1)$.
In the final round, the prover has sent g_d which is claimed to be

$$g_d(x_d) = f^2(r_1,\ldots,r_{d-1},x_d)$$

The verifier can now check that $g_d(r_d) = f^2(\mathbf{r})$ (recall that the verifier tracked $f(\mathbf{r})$ incrementally in the stream). If this test succeeds (and so do all previous tests), then the verifier accepts, and is convinced that $F_2(\mathbf{a}) = \sum_{x_1 \in [\ell]} g_1(x_1)$.

Proof of correctness. We now argue in detail that the verifier is unlikely to be fooled by a dishonest prover.

Lemma 1. *If the prover follows the above protocol then the verifier will accept with certainty. However, if the prover sends any polynomial which does not meet the required property, then the verifier will accept with probability at most $2dl/p$, where this probability is over the random coin tosses of $\mathcal{V}$.*

Proof. The first part is immediate from the following discussion: if each g_j is as claimed, then the verifier can easily ensure that each g_j is consistent with g_{j-1}.

For the second part, the proof proceeds from the dth round back to the first round. In the final round, the prover has sent g_d, of degree $2\ell-2$, and the verifier checks that it agrees with a precomputed value at $x_d = r_d$. This is an instance of the Schwartz-Zippel polynomial equality testing procedure [21]. If g_d is indeed as claimed, then the test will always be passed, no matter what the value of r_d. But if g_d does not satisfy the equality, then $\Pr[g_d(r_d) = f^2(\mathbf{r})] \le \frac{2\ell-2}{p}$. Therefore, if p was chosen so that $p \gg \ell$, then the verifier is unlikely to be fooled.

The argument now proceeds inductively. Suppose that the verifier is convinced (with some small probability of error) that $g_{j+1}(x_{j+1})$ is indeed as claimed, and wants to be sure that $g_j(x_j)$ is also as claimed. The prover has claimed that

$$g_j(x_j) = \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} f^2(r_1,\ldots,r_{j-1},x_j,x_{j+1},\ldots,x_d).$$

We again verify this by a Schwartz-Zippel polynomial test: we evaluate $g_j(x_j)$ at a randomly chosen point r_j, and ensure that the result is correct. Observe that

$$\begin{aligned} g_j(r_j) &= \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} f^2(r_1,\ldots,r_j,x_{j+1},\ldots x_d) \\ &= \sum_{x_{j+1}\in[\ell]} \sum_{x_{j+2},\ldots,x_d \in [\ell]^{d-j-1}} f^2(r_1,\ldots,r_j,x_{j+1},x_{j+2},\ldots,x_d) \\ &= \sum_{x_{j+1}\in[\ell]} g_{j+1}(x_{j+1}). \end{aligned}$$

Therefore, if the verifier $\mathcal{V}$ believes that g_{j+1} is as claimed, then (provided the test passes) $\mathcal{V}$ has enough confidence to believe that g_j is also as claimed. More formally,

$$\Pr\left[g_j \not\equiv \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} f^2(r_1,\ldots r_{j-1},x_j,\ldots,x_d) \quad \middle|\, g_{j+1} \equiv \sum_{x_{j+2},\ldots,x_d \in [\ell]^{d-j+1}} f^2(r_1,\ldots r_j,x_{j+1},\ldots,x_d)\right] < \frac{2\ell}{p}.$$

In the final step, the verifier is satisfied that g_1 is consistent with g_2, and so g_1 is as claimed. The probability that g_1 is not as claimed can be bounded as the probability that the verifier was fooled in any intervening step. This is at most $2d\ell/p$, by a union bound.

Intuitively, the key reason for the prover's inability to fool the verifier is that the prover must *commit* to a particular g_j before r_j is revealed to him. So while the prover could then choose a g_{j+1} which causes the test on that pair to pass, g_{j+1} is also "dishonest". But ultimately, the prover must provide g_d, which $\mathcal{V}$ can check based on information that is known to $\mathcal{V}$ alone. The prover is very unlikely to have included a dishonest g_j along the way and passed all the subsequent tests to generate a g_d which is consistent with the final test using r_d (which remains unknown to $\mathcal{P}$). $\square$

Analysis of space and communication. The communication cost of the protocol is dominated by the polynomials being sent by the prover. Each polynomial can be sent in $O(\ell)$ words, so over the d rounds, the total cost is $O(d\ell)$ communication. The space required by the verifier is bounded by having to remember $\mathbf{r}$, $f(\mathbf{r})$ and a constant number of polynomials (the verifier can "forget" intermediate polynomials once they have been checked). The total cost of this is $O(d+\ell)$ words. Probably the most economical tradeoff is reached by picking $\ell = 2$ and $d = \log u$, yielding both communication and space cost for $\mathcal{V}$ of $O(\log u)$ words. [4] Combining these settings with Lemma 1, we have:

Theorem 3. *There is a* $(\log u, \log u)$*-protocol for* SELF-JOIN SIZE *with probability of failure* $O(\frac{\log u}{u})$.

Remarks. The failure probability can be set as low as $O(\frac{\log u}{u^c})$ for any constant c, by choosing p larger than u^c, which does not affect the space and communication by more than a constant factor.

Notice that the smallest-depth circuit computing F_2 has depth $\Theta(\log u)$, as any function that depends on all bits of the input requires at least logarithmic depth. Therefore, Theorem 2 yields a $(\log^2 u, \log^2 u)$-protocol for F_2, and our protocol represents a quadratic improvement in both parameters.

Analysis of other costs. Besides the primary concerns of the verifier's space and communication, this protocol is also quite efficient in terms of the other costs. Let us fix $\ell = 2$. As the stream is being processed the verifier has to update $f(\mathbf{r})$. The updates are very simple, since $\chi_0(x) = 1 - x$ and $\chi_1(x) = x$, so

$$\chi_{\mathbf{v}}(\mathbf{r}) = \prod_{j=1}^{d}((1-v_j)(1-r_j) + v_j r_j).$$

Thus processing each update in the stream $O(d) = O(\log u)$ time.

The prover has to retain the input vector $\mathbf{a}$, which can be done efficiently in space $O(\min(u,n))$. In the verification process it is clear that the verifier spends $O(1)$ time per round evaluating a degree-2 polynomial, so the total time is $O(\log u)$. On the prover side, it might appear costly to compute each $g_j(x_j)$ naively following the definition. But observe that $g_j(x_j)$ is a polynomial of degree 2, so it is sufficient to evaluate $g_j(x_j)$ at three locations, say at $x_j = 0, 1, 2$, to determine $g_j(x_j)$. For a location $x_j = c$, we rewrite

$$\begin{aligned} g_j(c) &= \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} f^2(r_1,\ldots,r_{j-1},c,x_{j+1},\ldots x_d) = \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} \left(\sum_{\mathbf{v} \in [\ell]^d} a_{\mathbf{v}} \chi_{\mathbf{v}}(r_1,\ldots,r_{j-1},c,x_{j+1},\ldots x_d) \right)^2 \\ &= \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} \sum_{\mathbf{v}_1,\mathbf{v}_2 \in [\ell]^d} a_{\mathbf{v}_1} a_{\mathbf{v}_2} \chi_{\mathbf{v}_1}(r_1,\ldots,r_{j-1},c,x_{j+1},\ldots x_d) \cdot \chi_{\mathbf{v}_2}(r_1,\ldots,r_{j-1},c,x_{j+1},\ldots x_d) \end{aligned}$$

[4] It is possible to tradeoff smaller space for more communication by, say, setting $\ell = \log^{\varepsilon} u$ and $d = \frac{\log u}{\varepsilon \log\log u}$ for any small constant $\varepsilon > 0$, which yields a protocol with $O(\frac{\log u}{\log\log u})$ space and $O(\log^{1+\varepsilon} u)$ communication.

$$= \sum_{\mathbf{v}_1,\mathbf{v}_2\in[\ell]^d} \left(a_{\mathbf{v}_1} a_{\mathbf{v}_2} \prod_{k=1}^{j-1} \chi_{v_{1,k}}(r_k)\cdot\chi_{v_{1,j}}(c)\cdot\prod_{k=1}^{j-1}\chi_{v_{2,k}}(r_k)\cdot\chi_{v_{2,j}}(c)\cdot \sum_{x_{j+1}\dots x_d\in[\ell]^{d-j}} \Big(\prod_{k=j+1}^{d}\chi_{v_{1,k}}(x_k)\chi_{v_{2,k}}(x_k)\Big)\right).$$

Note that $\chi_{v_k}(x_k) = 1$ iff $x_k = v_k$ and 0 for any other value in $[\ell]$, for any pair of $\mathbf{v}_1, \mathbf{v}_2$, we have

$$\sum_{x_{j+1},\dots,x_d\in[\ell]^{d-j}} \left(\prod_{k=j+1}^{d}\chi_{v_{1,k}}(x_k)\chi_{v_{2,k}}(x_k)\right) = 1$$

if and only if $\forall j+1 \le k \le d : v_{1,k} = v_{2,k}$, and 0 otherwise. Thus,

$$g_j(c) = \sum_{\mathbf{v}_1,\mathbf{v}_2\in[\ell]^d, \forall j+1\le k\le d: v_{1,k}=v_{2,k}} \left(a_{\mathbf{v}_1} a_{\mathbf{v}_2} \prod_{k=1}^{j-1}\chi_{v_{1,k}}(r_k)\cdot\chi_{v_{1,j}}(c)\prod_{k=1}^{j-1}\chi_{v_{2,k}}(r_k)\chi_{v_{2,j}}(c)\right)$$

$$= \sum_{v_{j+1},\dots,v_d\in[\ell]^{d-j}} \left(\sum_{v_1,\dots,v_j\in[\ell]^j} \Big(a_{\mathbf{v}}\chi_{v_j}(c)\prod_{k=1}^{j-1}\chi_{v_k}(r_k)\Big)\right)^2.$$

$\mathcal{P}$ maintains $a_{\mathbf{v}}\prod_{k=1}^{j-1}\chi_{v_k}(r_k)$ for each nonzero $a_{\mathbf{v}}$, updating with the new r_k in each round as it is revealed in constant time. Thus the total time spent by the prover for the verification process can be bounded via $O(n\log u)$, where n is the number of nonzero $a_{\mathbf{v}}$'s.

We make one further simplification. At the heart of the computation is a summation over $[\ell]^j$ for each $v_{j+1},\dots,v_d \in [\ell]^{d-j}$. As we set $\ell = 2$,

$$\sum_{v_1,\dots,v_j\in[\ell]^j}\left(a_{\mathbf{v}}\chi_{v_j}(c)\prod_{k=1}^{j-1}\chi_{v_k}(r_k)\right) = \sum_{v_j=0}^{1}\left(\chi_{v_j}(c)\cdot\sum_{v_1,\dots,v_{j-1}\in[\ell]^{j-1}}\Big(a_{\mathbf{v}}\prod_{k=1}^{j-1}\chi_{v_k}(r_k)\Big)\right)$$

And for each $v_j,\dots,v_d \in [\ell]^{d-j+1}$, we can decompose

$$\sum_{v_1,\dots,v_{j-1}\in[\ell]^{j-1}}\left(a_{\mathbf{v}}\prod_{k=1}^{j-1}\chi_{v_k}(r_k)\right) = \sum_{v_{j-1}=0}^{1}\left(\chi_{v_{j-1}}(r_{j-1})\sum_{v_1,\dots,v_{j-2}\in[\ell]^{j-2}}\Big(a_{\mathbf{v}}\prod_{k=1}^{j-2}\chi_{v_k}(r_k)\Big)\right).$$

By storing $A_j[v_j\dots v_d] = \sum_{v_1\dots v_{j-1}\in[\ell]^{j-1}}\big(\mathbf{a}_{\mathbf{v}}\prod_{k=1}^{j-1}\chi_{v_k}(r_k)\big)$, $\mathcal{P}$ computes

$$A_{j+1}[v_{j+1}\dots v_d] = \chi_0(r_j)A_j[0,v_{j+1}\dots v_d] + \chi_1(r_j)A_j[1,v_{j+1},\dots v_d]$$

in time $O(u/2^j)$. The total time is now linear: $O(\min(n\,\log(u/n),\,u))$. Note that computing the F_2 already takes $\Theta(\min(n,u))$ time, so there is at most a logarithmic factor more work than simply providing the answer.

3.2 Other problems

Our protocol for F_2 can be easily modified to support the other aggregation queries listed in Section 1.3.

Higher frequency moments. The protocol outlined above naturally extends to higher frequency moments, or the sum of any polynomial function of a_i. For example, we can simply replace f^2 with f^k in (1) and (2) to compute the k-th frequency moment F_k (again, assuming u is chosen large enough so $F_k < u$). The communication cost increases to $O(k \log u)$, since each g_j now has degree $O(k)$ and so requires correspondingly more words to describe. However, the verifier's space bound remains at $O(\log u)$ words.

Inner product. Given two streams defining two vectors $\mathbf{a}$ and $\mathbf{b}$, their inner product is defined by $\mathbf{a} \cdot \mathbf{b} = \sum_{i \in [u]} a_i b_i$. Observe that $F_2(\mathbf{a}+\mathbf{b}) = F_2(\mathbf{a}) + F_2(\mathbf{b}) + 2\mathbf{a}\cdot\mathbf{b}$. Hence, the inner product can be verified by verifying three F_2 computations.

More directly, the above protocol for F_2 can be adapted to verify the inner product: instead of providing polynomials which are claimed to be sums of f^2, now define polynomials $f_{\mathbf{a}}$ and $f_{\mathbf{b}}$ which encode $\mathbf{a}$ and $\mathbf{b}$ respectively. The verifier again picks a random $\mathbf{r}$, and evaluates $f_{\mathbf{a}}(\mathbf{r})$ and $f_{\mathbf{b}}(\mathbf{r})$ over the stream. The prover now provides polynomials that are claimed to be sums of $f_{\mathbf{a}} f_{\mathbf{b}}$. This observation is useful for the RANGE-SUM problem and sketches.

Range-sum. It is easy to see that RANGE-SUM is a special case of INNER PRODUCT. Here, every (key, value) pair in the input stream can considered as updating $i =$key with $\delta =$value to generate $\mathbf{a}$. When the query $[q_L, q_R]$ is given, the verifier defines $b_{q_L} = \cdots = b_{q_R} = 1$ and $b_i = 0$ for all other i. One technical issue is that computing $f_{\mathbf{b}}(\mathbf{r})$ directly from the definition requires $O(u \log u)$ time. However, the verifier can compute it much faster. Again fix $\ell = 2$. Decompose the range $[q_L, q_R]$ into $O(\log u)$ canonical intervals where each interval consists of all locations $\mathbf{v}$ where $v_{j+1}, \ldots, v_d$ are fixed while all possible $(v_1, \ldots, v_j) \in [2]^j$ for some j occur. The value of $f_{\mathbf{b}}(\mathbf{r})$ in each such interval is

$$
\begin{aligned}
f_{\mathbf{b}}(\mathbf{r}) &= \sum_{(v_1,\ldots,v_j)\in[2]^j} \chi_{(v_1,\ldots,v_d)}(\mathbf{r}) = \sum_{(v_1,\ldots,v_j)\in[2]^j} \prod_{k=1}^{j} \chi_{v_k}(r_k) \cdot \prod_{k=j+1}^{d} \chi_{v_k}(r_k) = \prod_{k=j+1}^{d} \chi_{v_k}(r_k) \cdot \left(\sum_{(v_1,\ldots,v_j)\in[2]^j} \prod_{k=1}^{j} \chi_{v_j}(r_j) \right) \\
&= \prod_{k=j+1}^{d} \chi_{v_k}(r_k) \cdot \left(\prod_{k=1}^{j} \left(\chi_0(r_j) + \chi_1(r_j) \right) \right) = \prod_{k=j+1}^{d} \chi_{v_k}(r_k),
\end{aligned}
$$

which can be computed in $O(\log u)$ time. The final evaluation is found by summing over the $O(\log u)$ canonical intervals, so the time to compute $f_{\mathbf{b}}(\mathbf{r})$ is $O(\log^2 u)$. This is used to determine whether $g_d(r_d) = f_{\mathbf{a}}(\mathbf{r}) f_{\mathbf{b}}(\mathbf{r})$. Hence, the verifier can continue the rest of the verification process in $O(\log u)$ rounds as before.

4 Interactive Proofs for Reporting Queries

We first present an interactive proof protocol for a class of SUB-VECTOR queries, which is powerful enough to incorporate INDEX, DICTIONARY, PREDECESSOR, and RANGE QUERY as special cases.

4.1 SUB-VECTOR queries

Let $\mathbf{a} = (a_1, \ldots, a_u)$ be a vector in $[u]^u$, initialized to $\mathbf{0}$. We are first given a stream of length n consisting of (i, δ) pairs, which sets $a_i \leftarrow a_i + \delta$. In the end, a SUB-VECTOR query is specified by a range $[q_L, q_R]$, and the required answer is the nonzero entries in the sub-vector $(a_{q_L}, \ldots, a_{q_R})$. Let the number of such nonzero entries be k.

The protocol. Let p be a prime such that $u < p \leq 2u$. The verifier $\mathcal{V}$ *conceptually* builds a tree $\mathcal{T}$ of constant degree ℓ on the vector $\mathbf{a}$. $\mathcal{V}$ first generates $\log u$ independent random numbers $r_1, \ldots, r_{\log_\ell u}$ uniformly from $[p]$. For simplicity, we describe the case for $\ell = 2$. Each node v of the tree is a number defined as follows. For the i-th leaf v, set $v = a_i$. For an internal node v at level j (the leaves are at level 0), define

$$v = v_L + v_R r_j, \tag{3}$$

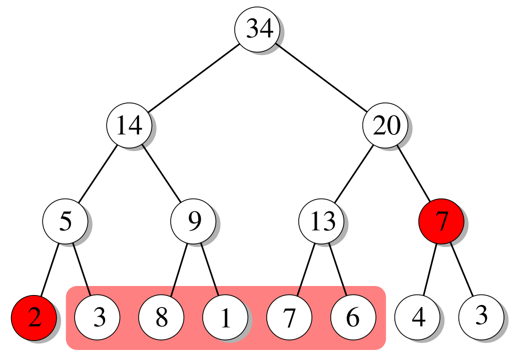


Figure 1: Example tree $\mathcal{T}$ over input vector $[2,3,8,1,7,6,4,3]$ and sub-vector query $(1,5)$.

where v_L and v_R are the left and right child of v, respectively. Additions and multiplications are done over the field $\mathbb{Z}_p$ as in Section 3. Denote the root of the tree by t. The verifier is only required to keep $r_1,\dots,r_{\log u}$ and t. Later we show that $\mathcal{V}$ can compute t without materializing the binary tree $\mathcal{T}$.

We first present the interactive verification protocol between $\mathcal{P}$ and $\mathcal{V}$ after the input has been observed by both. The verifier only needs $r_1,\dots,r_{\log u}$, t, and the query range $[q_L,q_R]$ to carry out the protocol. First $\mathcal{V}$ sends q_L and q_R to $\mathcal{P}$, and $\mathcal{P}$ returns the claimed sub-vector, say, $a'_{q_L},\dots,a'_{q_R}$ ($\mathcal{P}$ actually only needs to return the nonzero entries). In addition, if q_L is even, $\mathcal{P}$ also returns a'_{q_L-1}; if q_R is odd, $\mathcal{P}$ also returns a'_{q_R+1}. Then $\mathcal{V}$ tries to verify whether $a_i = a'_i$ for all $q_L \le i \le q_R$ using the following protocol. The general idea is to reconstruct $\mathcal{T}$ using information provided by $\mathcal{P}$. If $\mathcal{P}$ is honest, the reconstructed root, say t', should be the same as t; otherwise with high probability $t' \neq t$ and $\mathcal{V}$ will reject. Define $\gamma^{(j)}(i)$ to be the ancestor of the i-th leaf of $\mathcal{T}$ on level j. The protocol proceeds in $\log u - 1$ rounds, and maintains the invariant that after the j-th round, $\mathcal{V}$ has reconstructed $\gamma^{(j+1)}(i)$ for all $q_L \le i \le q_R$. The invariant is easily established initially ($j=0$) since $\mathcal{P}$ provides $a'_{q_L},\dots,a'_{q_R}$ and the siblings of a'_{q_L} and a'_{q_R} if needed. In the j-th round, $\mathcal{V}$ sends r_j to $\mathcal{P}$. Having $r_1,\dots,r_j$ to hand, $\mathcal{P}$ can construct the j-th level of $\mathcal{T}$. $\mathcal{P}$ then returns to $\mathcal{V}$ the siblings of $\gamma^{(j)}(q_L)$ and $\gamma^{(j)}(q_R)$ if they are needed by $\mathcal{V}$. Then $\mathcal{V}$ reconstructs $\gamma^{(j+1)}(i)$ for all $q_L \le i \le q_R$. At the end of the $(\log u - 1)$-th round, $\mathcal{V}$ has reconstructed $\gamma^{(\log u)}(i) = t'$, and checks that $t = t'$. If so, then the initial answer provided by $\mathcal{P}$ is accepted, otherwise it is rejected.

Example. Figure 1 shows a small example on the vector $\mathbf{a} = [2,3,8,1,\ 7,6,4,3]$. We fix the hash function parameters $r=[1,1,1]$ to keep the example simple (in practice these parameters are chosen randomly), and show the hash value inside each node. For the range $(2,6)$, in the first round the prover reports the sub-vector $[3,8,1,7,6]$ (shown highlighted). Since the left end of this range is even, $\mathcal{P}$ also reports $a_1 = 2$. From this, $\mathcal{V}$ is able to compute some hashes at the next level: 5, 9 and 13. After sending r_1 to $\mathcal{P}$, $\mathcal{V}$ received the fact that the hash of the range $(7,8)$ is 7. From this, $\mathcal{V}$ can compute the final hash values and check that they agree with the precomputed hash value of t, 34.

Theorem 4. *There is an interactive* $(\log u, \log u + k)$*-protocol for* SUB-VECTOR*, with failure probability* $O(\frac{\log u}{u})$.

Proof. **Correctness.** It is clear that with an honest $\mathcal{P}$, $\mathcal{V}$ always accepts. Next, we argue that if $\mathcal{P}$ returns a wrong value in any round, then $t' \neq t$ with high probability. $\mathcal{P}$ first sends back a'_i for all $q_L \le i \le q_R$ and their siblings (if they are outside of the range). Consider any pair of siblings, say a'_i and a'_{i+1}. Consider the functions $f(x) = a_i + a_{i+1}x$ and $f'(x) = a'_i + a'_{i+1}x$ in the field $\mathbb{Z}_p$. If $a_i \neq a'_i$ or $a_{i+1} \neq a'_{i+1}$, the two linear functions will not be identical, and they will intersect at no more than one point in $[p]$. Since we choose r_1

uniformly randomly from $[p]$, the probability that $f(r_1) = f'(r_1)$ is at most $1/p$. Thus, if $\mathcal{P}$'s first message is not correct, with probability at least $1-1/p$, there will be at least one error in the computed $\gamma^{(1)}(i)$, $q_L \le i \le q_R$. The same argument applies to each of the following $(\log u - 1)$ rounds: if either of the siblings of $\gamma^{(j)}(q_L)$ and $\gamma^{(j)}(q_R)$ returned by $\mathcal{V}$ is wrong or some $\gamma^{(j)}(i), q_L \le i \le q_R$ is already wrong previously, then with probability at most $1/p$, the reconstructed $\gamma^{(j)}(i)$ will be all correct. By the union bound, the probability that an incorrect response from $\mathcal{V}$ will lead to a correct t' is at most $\frac{\log u}{p}$.

Analysis of costs. We first argue that $\mathcal{V}$ can compute t in small space. Expanding t, we have

$$t = \sum_i \big(a_i \prod_{j=1}^{\log u} r_j^{(i-1)_j}\big), \tag{4}$$

where $(i-1)_j$ denotes the j-th least significant bit of the binary representation of $i-1$. Initially when $\mathbf{a} = \mathbf{0}$, we have $t = 0$; when we have $a_i \leftarrow a_i + \delta$, t is incremented (modulo p) by $\Delta t = \delta \cdot \prod_{j=1}^{\log u} r_j^{(i-1)_j}$, which is easily computed in $O(\log u)$ time. Thus $\mathcal{V}$ can maintain t by just keeping $t, r_1, \ldots, r_{\log u}$.

The verifier's space requirement during the protocol is also bounded by $O(\log u)$ words. Given the query range, as the sub-vector result arrives at $\mathcal{V}$, the verifier can keep track of only $O(\log u)$ hash values of internal nodes, corresponding to at most one child of $\gamma^j(q_L)$ and $\gamma^j(q_R)$ for each j. Combining these with the hash values provided by $\mathcal{P}$ will be sufficient to run the checking protocol. Each of these can be maintained in small space in the same manner as the root t via (4) above. Thus the space to carry out the protocol is $O(\log u)$.

The communication cost consists of the initial query result of size k sent by the prover, plus $O(1)$ nodes per level of the binary tree $\mathcal{T}$. So the total communication cost is $O(\log u + k)$.

Now we analyze the prover's cost. As the stream is received the prover clearly needs linear space and $O(1)$ time per element to construct the vector $\mathbf{a}$. At verification time the prover essentially reconstructs the binary tree $\mathcal{T}$. Note that $\mathcal{T}$ has at most n nonzero leaves, so it has size $O(\min(u, n\log(u/n)))$. Computing this tree in a bottom-up fashion costs $O(1)$ time per node, hence $O(\min(u, n\log(u/n)))$ time in total. □

Remarks. As in Section 3 the failure probability can be driven down to $O(\frac{\log u}{u^c})$ for any constant c by picking p greater than u^c, without changing the asymptotic bounds. From the description above a dishonest prover may cause excessive communication by sending more than k nonzero entries in the initial answer. To guarantee the $O(\log u + k)$ bound with any $\mathcal{P}$, we could first verify the value of k, i.e., a RANGE-COUNT query, with $O(\log u)$ communication using the protocol in Section 3. Then if $\mathcal{P}$ sends more than k nonzero entries $\mathcal{V}$ will reject immediately.

We note that by modifying the hash function to $(1-r_j)v_L + r_j v_R$, it is possible to show that t is equivalent to $f(\mathbf{r})$, while the same analysis holds. This provides a connection between the two approaches, although the proofs are quite different in nature.

4.2 Answering reporting queries

We now show how to answer the reporting queries using the solution to SUB-VECTOR.

- It is straightforward to solve RANGE QUERY using SUB-VECTOR: each element i in the stream is interpreted as a vector update with $\delta = 1$, and vector entries with non-zero counts intersecting the range give the required answer.
- INDEX can be interpreted as a special case of RANGE QUERY with $q_L = q_R = q$.

- For DICTIONARY, we need to distinguish between "not found" and a value of 0. This can be done by using a universe size of $[u+1]$ for the values: each value is incremented on insertion. At query time, if the retrieved value is 0, the result is "not found"; otherwise the value is decremented by 1 and returned.
- For PREDECESSOR, we interpret each key in the stream as an update with $\delta = 1$. In the protocol $\mathcal{V}$ first asks for the index of the predecessor of q, say q', and then verifies that the sub-vector $(a_{q'}, \ldots, a_q) = (1, 0, \ldots, 0)$, with communication cost $O(\log u)$ (since $k = 0$).

Corollary 1. *There is a* $(\log u, \log u)$*-protocol for* DICTIONARY*,* INDEX *and* PREDECESSOR*, and a* $(\log u, \log(u) + k)$*-protocol for* RANGE QUERY.

5 Extensions

We next consider how to treat other functions in the streaming interactive proof setting. We first consider some functions which are of interest in streaming, such as heavy hitters, k-largest, and sketch computations. We then extend the framework to handle a more general class of "frequency-based functions".

5.1 Other Specific Functions

Heavy Hitters. The heavy hitters (HHs) are those items whose frequencies exceed a fraction ϕ of the total stream length n. In verifying the claimed set of HHs, $\mathcal{V}$ must ensure that all claimed HHs indeed have high enough frequency, and moreover no HHs are *omitted*. To convince $\mathcal{V}$ of this, $\mathcal{P}$ will combine a succinct witness set with a generalization of the SUB-VECTOR protocol to give a $(1/\phi \log u,\ 1/\phi \log u)$ protocol for verifying the heavy hitters and their frequencies. As in our SUB-VECTOR protocol, $\mathcal{V}$ conceptually builds a binary tree $\mathcal{T}$ with leaves corresponding to entries of $\mathbf{a}$, and a random hash function associated with each level of $\mathcal{T}$. We augment each internal node v with a third child c_v. c_v is a leaf whose value is the sum of the frequencies of all *descendents* of v, the *subtree count* of v. The hash function now takes *three* arguments as input. It is evident that $\mathcal{V}$ can still compute the hash t of the root of this tree in logarithmic space.

In the lth round, the prover lists all leaves at level l whose sub-tree count is at least ϕn, their siblings, as well as their hash value and their subtree counts (so the hash of their parent can be computed). In addition, $\mathcal{P}$ provides all leaves whose subtree count is less than ϕn but whose parent has subtree count at least ϕn; these nodes serve as *witnesses* to the fact that none of their descendants are heavy hitters, enabling $\mathcal{V}$ to ensure that no heavy hitters are omitted. This procedure is repeated for each level of $\mathcal{T}$; note that for each node v whose value $\mathcal{P}$ provides, all ancestors of v and their siblings (i.e. all nodes on v's "authentication path") are also provided, because the subtree count of any ancestor is at least as high as the subtree count of v. Therefore, $\mathcal{V}$ can compare the hash of the root (calculated while observing the stream) to the value provided by $\mathcal{P}$, and the proof of soundness is analogous to that for the SUB-VECTOR protocol.

In total, there are at most $O(1/\phi \log u)$ nodes provided by $\mathcal{P}$: for each level l, the sum of the sub-tree counts of nodes at level l is n, and therefore there are $O(1/\phi)$ nodes at each level which have sub-tree count exceeding ϕn or whose parent has subtree count exceeding ϕn. Hence, the size of the proof is at most $O(1/\phi \log u)$.

With a little effort, the protocol cost is improved to $(\log u,\ 1/\phi \log u)$, i.e. we do not require $\mathcal{V}$ to store the heavy hitter nodes. This is accomplished by having the prover, at each level of $\mathcal{T}$, "replay" the hash values of all nodes listed in the previous round. $\mathcal{V}$ can keep a simple fingerprint of the identities and hash values of all nodes listed in each round (computing their hash values internally), and compare this to a fingerprint of the hash values and identities listed by $\mathcal{P}$. If these fingerprints match for each level, $\mathcal{V}$ is assured that the correct information was presented. Note each node is repeated just once, so this only doubles the communication cost. This reduced cost protocol is used in Section 5.2.

k-largest. Given the same set up as the PREDECESSOR query, the k-th largest problem is to find the largest p in the stream such that there are at least $k-1$ larger values p' also present in the stream. This can be solved by the prover claiming that the kth largest item occurs at location j, and performing the range query protocol with the range (j,u), allowing $\mathcal{V}$ to check that there are exactly k distinct items present in the range. This has a cost of $(\log u, k+\log u)$. For large values of k, alternative approaches via range sum (assuming all keys are distinct) can reduce the cost to $(\log u, \log u)$.

Sketch computations. In situations where the verifier only requires an approximate answer to the query, the prover can use a *sketch* algorithm which produces a data structure that is more compact than storing the entire stream. For a given approximation error ε, these sketches typically have size $O(\varepsilon^{-c} \cdot \text{poly}\log(u))$ for some constant $c>0$. Many of the sketches defined in the streaming literature map every element in $[u]$ to a smaller space $[v]$ by some compact hash function h, and compute the frequency vector for the reduced universe $[v]$, i.e., $a'_j = \sum_{h(i)=j} a_i$, for all $j \in [v]$. Then the desired quantity can be derived from $\mathbf{a}' = (a'_0, \ldots, a'_{v-1})$. Examples include the (fast) *AMS sketch* for F_2 [1], the *FM sketch* for the number of distinct items [9], the *Count-Min sketch* [8], etc. A technical issue is that some of these sketches run the basic scheme multiple times with different h's. This can be treated as a single h that maps each $i \in [u]$ to multiple values in a (larger) $[v]$.

For a pre-agreed ε (hence v) and h, both the prover and verifier can do exactly the same as before, except that now the calculations are over the transformed stream whose elements are the $h(i)$'s. The verifier still only needs space $O(\log u)$. At the time of verification the exact function to verify depends on the sketch. We very briefly outline some of the variations, and postpone further discussion.

- If the desired quantity is a sum of a polynomial function of the a'_j's, such as the AMS sketch, then the verification process follows the outline of Section 3, and the communication cost is $O(\log v)$.
- For the Count-Min sketch, the query probes a small number of entries in the sketch and finds the minimum: these can be recovered efficiently using the protocols described in Section 4 in $O(\log v)$ communication.
- For the FM sketch, a large fraction of the vector $\mathbf{a}'$ may need to be recovered and checked by the verifier. In this case the communication cost is $O(v)$.

5.2 Frequency-based functions

Given the approach described in Section 3, it is natural to ask what other functions can be computed via sum-check protocols applied to carefully chosen polynomials. By extending the ideas from the protocol of Section 3, we get protocols for any statistic F of the form $F(\mathbf{a}) = \sum_{i\in[u]} h(\mathbf{a}_i)$. Here, $h : \mathbb{N}_0 \to \mathbb{N}_0$ is a function over frequencies. Any statistic F of this form is called a *frequency-based function*. Such functions occupy an important place in the streaming world. For example, setting $h(x) = x^2$ gives the self-join size. We will subsequently show that using functions of this form we can obtain non-trivial protocols for problems including:

- F_0, the number of distinct items in stream A.
- F_{max}, the frequency of the most-frequent item in A.

- Point queries on the inverse distribution of A. That is, for any i, we will obtain protocols for determining the number of tokens with frequency exactly i.

The Protocol. A natural first attempt to extend the protocols of Section 3 to this more general case is to have $\mathcal{V}$ compute $f(\mathbf{r})$ as in Section 3, then have $\mathcal{P}$ send polynomials which are claimed to match sums over $h(f(\mathbf{x}))$. In principal, this approach will work: for the F_2 protocol, this is essentially the outline with $h(x) = x^2$. However, recall that when this technique was generalized to F_k for larger values of k, the cost increased with k. This is because the degree of the polynomial h increased. In general, this approach yields a solution with cost $\deg(h)\log n$. This does not yet yield interesting results, since in general, the degree of h can grow arbitrarily high, and the resulting protocol is worse than the trivial protocol which simply sends the entire vector $\mathbf{a}$ at a cost of $O(\min(n,u))$.

To overcome this obstacle, we modify this approach to use a polynomial function $\tilde{h}$ with bounded degree that is sublinear in n and u. At a high level, we "remove" any very heavy elements from the stream A before running the protocol of Section 3.1, with f^2 replaced by $\tilde{h} \circ f$ for a suitably chosen polynomial $\tilde{h}$. By removing all heavy elements from the stream, we keep the degree of $\tilde{h}$ (relatively) low, thereby controlling the communication cost. We now make this intuition precise.

Assume $n = O(u)$ and let $\phi = u^{-1/2}$. The first step is to identify the set H of ϕ-heavy hitters (i.e. the set of elements with frequency at least $u^{1/2}$) and their frequencies. We accomplish this via the $(\log u, 1/\phi \log u)$ protocol described in Section 5.1. $\mathcal{V}$ runs this protocol and, as the heavy hitters are reported, $\mathcal{V}$ incrementally computes $F' := \sum_{i \in H} h(\mathbf{a}_i)$, which can be understood as the contribution of the heaviest elements to F, the statistic of interest.

In parallel with the heavy hitters protocol, $\mathcal{V}$ also runs the first part of the protocol of Section 3.1 with $d = \log u$. That is, $\mathcal{V}$ chooses a random location $\mathbf{r} = (r_1, \ldots, r_d) \in [p]^d$ (where p is a prime chosen larger then the maximum possible value of F), and while observing the stream $\mathcal{V}$ incrementally evaluates $f(\mathbf{r})$. As in Sections 2 and 3.1, this requires only $O(d)$ additional words of memory.

As the heavy hitters are reported, $\mathcal{V}$ "removes" their contribution to f by subtracting $\mathbf{a}_{\mathbf{v}}\chi_{\mathbf{v}}(\mathbf{r})$ from $f(\mathbf{r})$ for each $\mathbf{v} \in H$. That is, let $\widetilde{f}$ denote the polynomial implied by the derived stream obtained by removing all occurrences of all ϕ-heavy hitters from A. Then $\mathcal{V}$ may compute $\widetilde{f}(\mathbf{r})$ via the identity $\widetilde{f}(\mathbf{r}) = f(\mathbf{r}) - \sum_{\mathbf{v} \in H} \chi_{\mathbf{v}}(\mathbf{r})$. Crucially, $\mathcal{V}$ need not store the items in H to compute this value; instead, $\mathcal{V}$ subtracts $\chi_{\mathbf{v}}(\mathbf{r})$ each time a heavy hitter $\mathbf{v}$ is reported, and then immediately "forgets" the identity of $\mathbf{v}$.

Now let $\tilde{h}$ be the unique polynomial of degree at most $u^{1/2}$ such that $\tilde{h}(i) = h(i)$ for $i = 0, \ldots, u^{1/2}$; $\mathcal{V}$ next computes $\tilde{h}(\widetilde{f}(\mathbf{r}))$ in small space. Note that this computation can be performed without explicitly storing $\tilde{h}$, since we can compute

$$\tilde{h}(x) = \sum_{i=0,\ldots u^{1/2}} h(i)\chi_i(x)$$

(assuming $h()$ has a compact description as in the examples below).

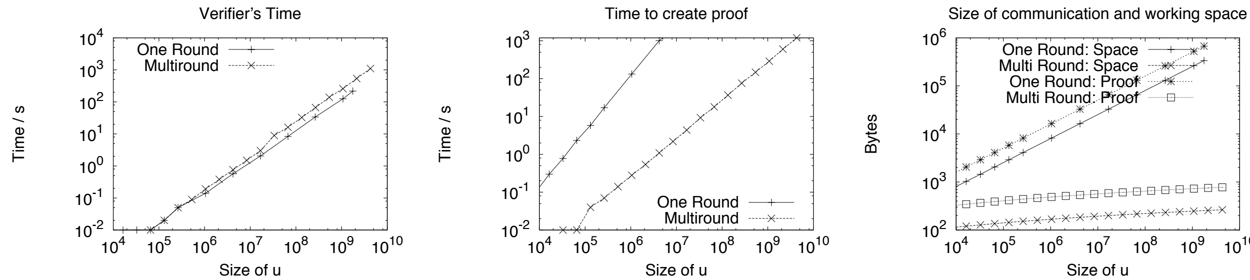
The second part of the verification protocol can proceed in parallel with the first part. In the first round, the prover sends a polynomial $g_1(x_1)$ claimed to be

$$g_1(x_1) = \sum_{x_2,\ldots,x_d \in [\ell]^{d-1}} \tilde{h} \circ \widetilde{f}(x_1, x_2, \ldots, x_d).$$

Observe that if g_1 is as claimed, and we assume (without loss of generality) that $h(0) = 0$, then

$$F(\mathbf{a}) = \sum_{x_1 \in [\ell]} g_1(x_1) + F'.$$

Since the polynomial $g_1(x_1)$ has degree at most $u^{1/2}$, it can be described in $u^{1/2}$ words.



(a) Verifier's time (b) Prover's time (c) Space complexity

Figure 2: Experimental results

Then, as in Section 3.1, $\mathcal{V}$ sends r_{j-1} to $\mathcal{P}$ in round $j > 1$. In return, the prover sends a polynomial $g_j(x_j)$, and claims

$$g_j(x_j) = \sum_{x_{j+1},\ldots,x_d \in [\ell]^{d-j}} \tilde{h} \circ \widetilde{f}(r_1,\ldots,r_{j-1},x_j,x_{j+1},\ldots,x_d).$$

The verifier conducts tests for correctness that are completely analogous to those in Section 3.1, which completes the description of the protocol. The proof of completeness and soundness of this protocol is analogous to those in Section 3.1 as well.

Analysis of space and communication. $\mathcal{V}$ requires $\log u$ words to run the heavy hitters protocols, and $O(d) = O(\log u)$ space to store $r_1,\ldots,r_d$, $f(\mathbf{r})$, $\widetilde{f}(\mathbf{r})$, and to compute and store $\tilde{h}(\widetilde{f}(\mathbf{r}))$. The communication cost of the heavy hitters protocol is $u^{1/2}\log u$, while the communication cost of the rest of the protocol is bounded by the $du^{1/2} = u^{1/2}\log u$ words used by $\mathcal{P}$ to send a polynomial of degree at most $u^{1/2}$ in each round. Thus, we have the following theorem:

Theorem 5. *There is a* $(\log u, u^{1/2}\log u)$*-protocol for any statistic* F *of the form* $F(\mathbf{a}) = \sum_{i\in[u]} h(\mathbf{a}_i)$*, with probability of failure* $O(\frac{\log u}{u})$*. The protocol requires* $\log u$ *rounds of interaction.*

Using this approach yields protocols for the following problems:

- F_0, the number of items with non-zero count. This follows by observing that F_0 is equivalent to computing $\sum_{i\in[u]} h(\mathbf{a}_i)$ for $h(0) = 0$ and $h(i) = 1$ for $i > 0$.
- More generally, we can compute functions on the inverse distribution, i.e. queries of the form "how many items occur exactly k times in the stream" by setting, for any fixed k, $h(k) = 1$ and $h(i) = 0$ for $i \neq j$. One can build on this to compute, e.g. the number of items which occurred between k and k' times, the median of this distribution, etc.
- We obtain a protocol for $F_{max} = \max_i \mathbf{a}_i$, with a little more work. $\mathcal{P}$ first claims a lower bound lb on F_{max} by providing the index of an item with frequency F_{max}, which $\mathcal{V}$ verifies by running the INDEX protocol from Section 4. Then $\mathcal{V}$ runs the above protocol with $h(i) = 0$ for $i \leq lb$ and $h(i) = 1$ for $i > lb$; if $\sum_{i\in[u]} h(\mathbf{a}_i) = 0$, then $\mathcal{V}$ is convinced no item has frequency higher than lb, and concludes that $F_{max} = lb$.

Corollary 2. *There is a* $(\log u, u^{1/2}\log u)$*-protocol that requires just* $\log u$ *rounds of interaction for* F_0*,* F_{max}*, and queries on the inverse distribution.*

Comparison. Compared to the previous protocols, the methods above increase the amount of communication between the two parties by a $u^{\frac{1}{2}}$ factor. The number of rounds of interaction remains $\log u$, equivalent to $\mathcal{V}$'s space requirement. So arguably these bounds are still good from the verifier's perspective. In contrast, the construction of [12] requires $\Omega(\log^2 u)$ rounds of interaction and communication, which may be large enough to be offputting. To make this concrete, for a terabyte-size input, $\log u$ rounds is of the order of 40, while $\log^2 u$ is of the order of thousands. Meanwhile, the $u^{\frac{1}{2}}$ communication is of the order of a megabyte. So although the total communication cost is higher, one can easily imagine scenarios where the latency of network communications make it more desirable to have fewer rounds with more communication in each.

6 Experimental Study

We performed a brief experimental study to evaluate the effectiveness of the protocols we have described in practice. We compared the multi-round protocols for F_2 we described in Section 3 to the single round protocol given in [6], which can be seen as a protocol in our setting with $d = 2$ and $\ell = \sqrt{u}$. A prototype implementation was made in C++: it simulated the computations of both parties, and measured the time and resources consumed by the protocols. For the data, we generated synthetic streams where the number of occurrences of each item was picked uniformly in the range $[0, 1000]$. Note that the choice of data does not affect the behavior of the protocols: their guarantees do not depend on the data, but rather on the random choices of the verifier. The computations were made over the field of size $p = 2^{61} - 1$, implying a very low probability of the verifier being fooled by a dishonest prover.

We evaluated the protocols on a single core of a multi-core machine with 64-bit AMD Opteron processors and 32 GB of memory available. The large amount of memory allowed us to experiment with universes of size several billion, with the prover able to store the entire frequency vector in memory. We measured the time for $\mathcal{V}$ to compute the check information from the stream, for $\mathcal{P}$ to generate the proof, and for $\mathcal{V}$ to verify this proof. We also measured the space required by $\mathcal{V}$, and the size of the proof provided by $\mathcal{P}$.

6.1 Experimental Results

When the prover was honest, both protocols always accepted the proof. We also tried modifying the prover's messages, by changing some pieces of the proof, or computing the proof for a slightly modified stream. In all cases, the protocols caught the error, and rejected the proof. We conclude that the protocols work as analyzed, and the focus of our experimental study is to understand how they scale to potentially large volumes of data.

Figure 2 shows the behavior of the protocols as the size of the domain u varies. First, Figure 2(a) shows the time for the verifier to process the stream as the domain size increases. Both show a linear trend (here, plotted on log scale). Moreover, both take roughly the same time, with the multi-round verifier processing 4-5 million updates per second, and the single round verifier processing 7.5-8.5 million. The similarity is not surprising: both methods are taking each element of the stream and computing the product of the frequency with a function of the element's index i and the random parameter r. The effort in computing this function is roughly similar in both cases. The single round verifier has a slight advantage, since it can compute and use lookup tables within the $O(\sqrt{u})$ space bound, while the multi-round verifier limited to logarithmic space must recompute some values multiple times. The time to check the proof is essentially negligible: less than a millisecond across all data sizes. Hence, we do not consider this to be a significant cost.

Figure 2(b) shows a clear separation between the two methods in $\mathcal{P}$'s effort in generating the proof. Here, we measure the total time across all rounds in the multi-round case, against the effort to generate the single round proof. The cost in the multiround case is dramatically lower than the single round case: it takes minutes to process input with $u = 2^{20}$ in the single round case, whereas the same data requires less than a third of a second when using the multi-round approach. Worse, this cost grows with $u^{3/2}$, as seen with the steeper line: doubling the input size increases the cost by a factor of 2.8. In contrast, the multiround cost grew only linearly with u. Across all values of u, the multiround prover processed 3-4 million updates per second. Meanwhile, at $u = 2^{16}$, the single-round prover processed roughly $30,000$ updates per second, while with $u = 2^{20}$, $\mathcal{P}$ processed only $8,000$. Thus the chief bottleneck of these protocols seems to be $\mathcal{P}$'s time cost to make the proof.

The trend is similar for the space resources required to execute the protocol. In the single round case, both the verifier's space and size of the proof grow proportional to $\sqrt{u}$. This is not impossibly large: Figure 2(c) shows that for u of the order of 1 billion, both these quantities are comfortably under a megabyte. Nevertheless, it is still orders of magnitude larger than the sizes seen in the multiround protocol: there, the space required and proof size are never more than 1KB even when handling gigabytes of data.

In summary, we observe that the methods we have developed are applicable to genuinely large data sets, defined over a domain of size hundreds of millions to billions. Our implementation is capable of processing such datasets within a matter of minutes.

7 Concluding Remarks

We have presented interactive proof protocols for various problems that are known to be hard in the streaming model. By delegating the hard computation task to a possibly dishonest prover, the verifier's space complexity is reduced to $O(\log u)$. We now outline directions for future study.

Multiple Queries. Many of the problems considered are parameterized by values that are only specified at query time. The results of these queries could cause the verifier to ask new queries with different parameters. However, re-running the protocols for a new query with the same choices of random numbers does not provide the same security guarantees. The guarantees rely on $\mathcal{P}$ not knowing these values; with this knowledge a dishonest prover could potentially find collisions under the polynomials, and fool the verifier.

Two simple solutions partially remedy this issue: firstly, it is safe to run multiple queries *in parallel* round-by-round using the same randomly chosen values, and obtain the same guarantees for each query. This can be thought of as a 'direct sum' result, and holds also for the Goldwasser *et al.* construction [12]. Secondly, $\mathcal{V}$ can just carry out multiple independent copies of the protocol. Since each copy requires only $O(\log u)$ space (more precisely $\log u + 1$ integers), the cost per query is low. Nevertheless, it remains of some practical interest to find protocols which can be used repeatedly to support an larger number of queries. Related work based on strong cryptographic assumptions has recently appeared [7, 11] but is currently impractical.

Distributed Computation. A motivation for studying this model arises from the case of cloud computation, which outsources comptutation to the more powerful "cloud". In practice, the cloud may in fact be a distributed cluster of machines, implementing a model such as Map-Reduce. We have so far assumed that the prover operates a traditional centralized computational entity. The next step is to study how to create proofs over large data in the distributed model. A first observation is that the proof protocols we give here naturally lend themselves to this setting: observe that the prover's message in each round can be written as the inner product of the input data with a function defined by the values of r_j revealed so far. Thus, we claim that these protocols fit into Map-Reduce settings very naturally; it remains to demonstrate this empirically, and to establish similar results for other protocols.

Complexity Questions. From a complexity perspective, the main open problem, as indicated in Section 1.1, is to more precisely characterize the class of problems that are solvable in this interactive proof model. We have shown how to modify the construction of [12] to obtain $(\text{poly}\log u, \text{poly}\log u)$ streaming protocols for all of NC, and we showed that a wide class of reporting and aggregation queries possess $(\log u, \log u)$ protocols. It is of interest to establish what other natural queries possess $(\log u, \log u)$ protocols: F_0 and $F_{\max}$ are the prime candidates to resolve. Determining whether problems outside NC possess interactive proofs (streaming or otherwise) with $\text{poly}\log u$ communication and a verifier that runs in nearly linear time is a more challenging problem of considerable interest. This question asks, in essence, whether *parallelizable* computation is more easily verified than sequential computation.

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References

[1] N. Alon, Y. Matias, and M. Szegedy. The space complexity of approximating the frequency moments. *J. Comp Sys Sci*, 58:137–147, 1999.

[2] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

[3] S. Arora and S. Safra. Probabilistic checking of proofs: A new characterization of NP. *J. ACM*, 45(1):70–122, 1998.

[4] L. Babai and S. Moran. Arthur-merlin games: a randomized proof system, and a hierarchy of complexity class. *J. Comput. Syst. Sci.*, 36(2):254–276, 1988.

[5] E. Ben-Sasson, O. Goldreich, P. Harsha, M. Sudan, and S. Vadhan. Short PCPs verifiable in polylogarithmic time. In *CCC* 2005.

[6] A. Chakrabarti, G. Cormode, and A. McGregor. Annotations in data streams. In *ICALP*, 2009.

[7] K.-M. Chung, Y. Kalai, and S. Vadhan. Improved delegation of computation using fully homomorphic encryption. In *CRYPTO*, 2010.

[8] G. Cormode and S. Muthukrishnan. An improved data stream summary: The count-min sketch and its applications. *J. Algorithms*, 55(1):58–75, 2005.

[9] P. Flajolet and G. N. Martin. Probabilistic counting algorithms for data base applications. *J. of Comp Sys Sci*, 31(2):182–209, 1985.

[10] L. Fortnow and C. Lund. Interactive proof systems and alternating time-space complexity. *Theoretical Computer Science*, 113(1):55–73, 1993.

[11] R. Gennaro, C. Gentry, and B. Parno. Non-interactive verifiable computing: Outsourcing computation to untrusted workers. In *CRYPTO*, 2010.

[12] S. Goldwasser, Y. T. Kalai, and G. N. Rothblum. Delegating computation: Interactive proofs for Muggles. In *STOC*, 2008.

[13] S. Goldwasser and M. Sipser. Private coins versus public coins in interactive proof systems. In *STOC*, 1986.

[14] A. Juels and B. Kaliski. PORs: Proofs of retrievability for large files. In *Computer and Communications Security*, 2007.

[15] J. Kilian. A note on efficient zero-knowledge proofs and arguments (extended abstract). In *STOC*, 1992.

[16] E. Kushilevitz and N. Nisan. *Communication Complexity*. Cambridge University Press, 1997.

[17] F. Li, K. Yi, M. Hadjieleftheriou, and G. Kollios. Proof-infused streams: Enabling authentication of sliding window queries on streams. In *VLDB*, 2007.

[18] R. Merkle. *Secrecy, authentication, and public key systems*. PhD thesis, Electrical Engineering, Stanford, 1979.

[19] S. Papadopoulos, Y. Yang, and D. Papadias. Continuous authentication on relational streams. *VLDB J.*, 19(2):161–180, 2010

[20] A. D. Sarma, R. J. Lipton, and D. Nanongkai. Best-order streaming model. In *TAMC*, 2009.

[21] J. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *J. ACM*, 27(4):701–717, 1980.

[22] A. Shamir. $\mathsf{IP} = \mathsf{PSPACE}$. *J. ACM*, 39(4):869–877, 1992.

[23] Y. Yang, S. Papadopoulos, D. Papadias, and G. Kollios. Authenticated indexing for outsourced spatial databases. *VLDB J.*, 18(3):631–648, 2009.

[24] K. Yi, F. Li, G. Cormode, M. Hadjieleftheriou, G. Kollios, and D. Srivastava. Small synopses for group-by query verification on outsourced data streams. *ACM Transactions on Database Systems*, 34(3), article 15, 2009.

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