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Improved bounds for the randomized decision tree complexity of recursive majority*

Frédéric Magniez[1], Ashwin Nayak[†2], Miklos Santha[‡1,3],
Jonah Sherman[4], Gábor Tardos[§5], and David Xiao[1]

[1]CNRS, LIAFA, Univ Paris Diderot, Sorbonne Paris-Cité, 75205 Paris, France
[2]C&O and IQC, U. Waterloo; and Perimeter Institute, Waterloo, ON, Canada
[3]Centre for Quantum Technologies, National U. of Singapore, Singapore 117543
[4]University of California, Berkeley, USA
[5]Rényi Institute, Budapest, Hungary

Abstract

We consider the randomized decision tree complexity of the recursive 3-majority function. For evaluating height h formulae, we prove a lower bound for the δ-two-sided-error randomized decision tree complexity of $(1/2-\delta)\cdot 2.57143^h$, improving the lower bound of $(1-2\delta)(7/3)^h$ given by Jayram, Kumar, and Sivakumar (STOC'03), and the one of $(1-2\delta)\cdot 2.55^h$ given by Leonardos (ICALP'13). Second, we improve the upper bound by giving a new zero-error randomized decision tree algorithm that has complexity at most $(1.007)\cdot 2.64944^h$. The previous best known algorithm achieved complexity $(1.004)\cdot 2.65622^h$. The new lower bound follows from a better analysis of the base case of the recursion of Jayram *et al.* The new algorithm uses a novel "interleaving" of two recursive algorithms.

1 Introduction

Decision trees form a simple model for computing boolean functions by successively reading the input bits until the value of the function can be determined. In this model, the only cost is the number of input bits queried. Formally, a *deterministic decision tree algorithm* A on n variables is a binary tree in which each internal node is labeled with an input variable x_i, and the leaves of the tree are labeled by either 0 or 1. Each internal node has two outgoing edges, one labelled with 0, the other with 1. Every input $x = x_1 \dots x_n$ determines a unique path in the tree leading from

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the root to a leaf: if an internal node is labeled by x_i, we follow either the 0 or the 1 outgoing edge according to the value of x_i. The value of the algorithm A on input x, denoted by $A(x)$, is the label of the leaf on this unique path. Thus, the algorithm A *computes* a boolean function $A : \{0,1\}^n \to \{0,1\}$.

We define the *cost* $\mathrm{C}(A,x)$ of a deterministic decision tree algorithm A on input x as the number of input bits queried by A on x. Let $\mathcal{P}_f$ be the set of all deterministic decision tree algorithms which compute f. The *deterministic complexity* of f is $\mathrm{D}(f) = \min_{A\in\mathcal{P}_f} \max_{x\in\{0,1\}^n} \mathrm{C}(A,x)$. Since every function can be evaluated after reading all the input variables, $\mathrm{D}(f) \le n$.

In an extension of the deterministic model, we can also permit randomization in the computation. A *randomized decision tree algorithm* A on n variables is a distribution over all deterministic decision tree algorithms on n variables. Given an input x, the algorithm first samples a deterministic tree $B \in_{\mathrm{R}} A$, then evaluates $B(x)$. The error probability of A in computing f is given by $\max_{x\in\{0,1\}^n} \Pr_{B\in_{\mathrm{R}}A}[B(x) \neq f(x)]$. The *cost* of a randomized algorithm A on input x, denoted also by $\mathrm{C}(A,x)$, is the expected number of input bits queried by A on x. Let $\mathcal{P}_f^\delta$ be the set of randomized decision tree algorithms computing f with error at most δ. The two-sided bounded error *randomized complexity* of f with error $\delta \in [0,1/2)$ is $\mathrm{R}_\delta(f) = \min_{A\in\mathcal{P}_f^\delta} \max_{x\in\{0,1\}^n} \mathrm{C}(A,x)$.

We write $\mathrm{R}(f)$ for $\mathrm{R}_0(f)$. By definition, for all $0 \le \delta < 1/2$, it holds that $\mathrm{R}_\delta(f) \le \mathrm{R}(f) \le \mathrm{D}(f)$, and it is also known [BI87, HH87, Tar90] that $\mathrm{D}(f) \le \mathrm{R}(f)^2$, and that for all constant $\delta \in (0,1/2)$, $\mathrm{D}(f) \in \mathrm{O}(\mathrm{R}_\delta(f)^3)$ [Nis89].

Considerable attention in the literature has been given to the randomized complexity of functions computable by read-once formulae, which are boolean formulae in which every input variable appears only once. For a large class of well balanced formulae with NAND gates the exact randomized complexity is known. In particular, let NAND_h denote the *complete* binary tree of height h with NAND gates, where the inputs are at the $n = 2^h$ leaves. Snir [Sni95] has shown that $\mathrm{R}(\mathsf{NAND}_h) \in \mathrm{O}(n^c)$ where $c = \log_2\left(\frac{1+\sqrt{33}}{4}\right) \approx 0.753$. A matching $\Omega(n^c)$ lower bound was obtained by Saks and Wigderson [SW86], and extended to Monte Carlo algorithms (i.e., with constant error $\delta < 1/2$) by Santha [San95]. Since $\mathrm{D}(\mathsf{NAND}_h) = 2^h = n$ this implies that $\mathrm{R}(\mathsf{NAND}_h) \in \Theta(\mathrm{D}(\mathsf{NAND}_h)^c)$. Saks and Wigderson conjectured that *for every* boolean function f and constant $\delta \in [0,1/2)$, $\mathrm{R}_\delta(f) \in \Omega(\mathrm{D}(f)^c)$.

After further progress due to Heiman, Newman, and Wigderson [HNW90] and Heiman and Wigderson [HW91], one would have hoped that the simple model of decision tree algorithms might shed more light on the power of randomness. But surprisingly, we know the exact randomized complexity of very few boolean functions. In particular, the randomized complexity of the recursive 3-majority function ($\mathsf{3\text{-}MAJ}_h$) is still open. This function, proposed by Boppana, was one of the earliest examples where randomized algorithms were found to be more powerful than deterministic decision trees [SW86]. It is a read-once formula on 3^h variables given by the complete ternary tree of height h whose internal vertices are majority gates. It is easy to check that $\mathrm{D}(\mathsf{3\text{-}MAJ}_h) = 3^h$, but there is a naive randomized recursive algorithm for $\mathsf{3\text{-}MAJ}_h$ that performs better: pick two random children of the root and recursively evaluate them, then evaluate the third child if the value is not yet determined. This has zero-error randomized complexity $(8/3)^h$. However, it was already observed by Saks and Wigderson [SW86] that one can do even better than this naive algorithm. As for lower bounds, that reading 2^h variables is necessary for zero-error algorithms is easy to show. In spite of some similarities with the NAND_h function, no progress was reported on the randomized complexity of $\mathsf{3\text{-}MAJ}_h$ for 17 years. In 2003, Jayram, Kumar, and Sivakumar [JKS03] proposed an explicit randomized algorithm that achieves complexity $(1.004) \cdot 2.65622^h$, and beats the naive

recursion. (Note, however, that the recurrence they derive in [JKS03, Appendix B] is incorrect.) They also prove a $(1-2\delta)(7/3)^h$ lower bound for the δ-error randomized decision tree complexity of $\mathsf{3\text{-}MAJ}_h$. In doing so, they introduce a powerful combinatorial technique for proving decision tree lower bounds.

In this paper, we considerably improve the lower bound obtained in [JKS03], first by proving that $\mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h) \geq (1-2\delta)(5/2)^h$, then further improving the base $5/2$. We also improve the upper bound by giving a new zero-error randomized decision tree algorithm.

Theorem 1.1. *For all $\delta \in [0,1/2]$, we have $(1/2-\delta)\cdot 2.57143^h \leq \mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h) \leq (1.007)\cdot 2.64944^h$.*

In contrast to the randomized case, the bounded-error *quantum* query complexity of $\mathsf{3\text{-}MAJ}_h$ is known more precisely; it is in $\Theta(2^h)$ [RS08].

New lower bound. For the lower bound Jayram *et al.* consider a complexity measure related to the distributional complexity of $\mathsf{3\text{-}MAJ}_h$ with respect to a specific hard distribution (cf. Section 2.3). The focus of the proof is a relationship between the complexity of evaluating formulae of height h to that of evaluating formulae of height $h-1$. They derive a sophisticated recurrence relation between these two quantities, that finally implies that $\mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h) \geq \alpha(2+q)^h$, where αq^h is a lower bound on the probability p_h^δ that a randomized algorithm with error at most δ queries a special variable, called the "absolute minority", on inputs drawn from the hard distribution. They observe that any randomized decision tree with error at most δ must query at least one variable with probability $1-2\delta$. This variable has probability 3^{-h} of being the absolute minority, so $q = 1/3$ and $\alpha = 1-2\delta$ satisfies the conditions and their lower bound follows.

We obtain new lower bounds by improving the lower bound p_h^δ. We start by proving that $p_h^\delta \geq (1-2\delta)2^{-h}$, i.e., increasing q to $1/2$, which immediately implies an improved lower bound for $\mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h)$. We examine the relationship between p_h^δ and p_{h-1}^δ, by encoding a height $h-1$ instance into a height h instance, and using an algorithm for the latter. Analyzing our encoding requires understanding the behavior of all decision trees on 3 variables, and this can be done by exhaustively considering all such trees.

We then improve this lower bound by encoding height $h-2$ instances into height h instances, and prove $p_h^\delta \geq \alpha q^h$ for $q = \sqrt{7/24} > 0.54006$. For technical reasons we set $\alpha = 1/2-\delta$ (half the value considered by Jayram *et al.*) in their bound). For encoding respectively heights $h-3$ and $h-4$ instances into height h instances, we use a computer to get the same estimate with $q = (2203/12231)^{1/3} > 0.56474$ and $q = (216164/2027349)^{1/4} > 0.57143$.

Independently Leonardos [Leo13] improved the $(1-2\delta)(5/2)^h$ lower bound we got in the preliminary version of this work [MNSX11] by giving a lower bound of $\mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h) \geq (1-2\delta)\cdot 2.55^h$. His approach is different than ours and it is based on the method of generalized costs of Saks and Wigderson [SW86]. Nonetheless, the last lower bound we obtain improves the bound of Leonardos.

New algorithm. The naive algorithm and the algorithm of Jayram *et al.* are examples of *depth-k* recursive algorithms for $\mathsf{3\text{-}MAJ}_h$, for $k = 1, 2$, respectively. A depth-k recursive algorithm is a collection of subroutines, where each subroutine evaluates a node (possibly using information about other previously evaluated nodes), satisfying the following constraint: when a subroutine evaluates a node v, it is only allowed to call other subroutines to evaluate children of v at depth at most k, but is not allowed to call subroutines or otherwise evaluate children that are deeper than k. (Our notion of depth-one is identical to the terminology "directional" that appears in the literature. In particular, the naive recursive algorithm is a directional algorithm.)

We present an improved depth-two recursive algorithm. To evaluate the root of the majority formula, we recursively evaluate one grandchild from each of two distinct children of the root.

The grandchildren "give an opinion" about the values of their parents. The opinion guides the remaining computation in a natural manner: if the opinion indicates that the children are likely to agree, we evaluate the two children in sequence to confirm the opinion, otherwise we evaluate the third child. If at any point the opinion of the nodes evaluated so far changes, we modify our future computations accordingly. A key innovation is the use of an algorithm optimized to compute the value of a *partially evaluated* formula. In our analysis, we recognize when incorrect opinions are formed, and take advantage of the fact that this happens with smaller probability.

We do not believe that the algorithm we present here is optimal. Indeed, we conjecture that even better algorithms exist that follow the same high level intuition applied for depth-k recursion for $k > 2$. However, it seems new insights are required to analyze the performance of deeper recursions, as the formulas describing their complexity become unmanageable for $k > 2$.

Organization. We prepare the background for our main results Section 2. In Section 3.1 we prove our new lower bounds for $\mathsf{3\text{-}MAJ}_h$. The new algorithm for the problem is described and analyzed in Section 4.

2 Preliminaries

We write $u \in_{\mathrm{R}} D$ to state that u is sampled from the distribution D. If X is a finite set, we identify X with the uniform distribution over X, and so, for instance, $u \in_{\mathrm{R}} X$ denotes a uniform element of X.

2.1 Distributional Complexity

A variant of the randomized complexity we use is *distributional* complexity. Let $\mathcal{D}_n$ be the set of distributions over $\{0,1\}^n$. The *cost* $\mathrm{C}(A, D)$ of a randomized decision tree algorithm A on n variables with respect to a distribution $D \in \mathcal{D}_n$ is the expected number of bits queried by A when x is sampled from D and over the random coins of A. The *distributional complexity* of a function f on n variables for δ two-sided error is $\Delta_\delta(f) = \max_{D \in \mathcal{D}_n} \min_{A \in \mathcal{P}^\delta_f} \mathrm{C}(A, D)$. The following observation is a well established route to proving lower bounds on worst case complexity.

Proposition 2.1. $\mathrm{R}_\delta(f) \geq \Delta_\delta(f)$.

2.2 The $\mathsf{3\text{-}MAJ}_h$ Function and the Hard Distribution

Let $\mathsf{MAJ}(x)$ denote the boolean majority function of its input bits. The ternary majority function $\mathsf{3\text{-}MAJ}_h$ is defined recursively on $n = 3^h$ variables, for every $h \geq 0$. We omit the height h when it is obvious from context. For $h = 0$ it is the identity function. For $h > 0$, let x be an input of length n and let $x^{(1)}, x^{(2)}, x^{(3)}$ be the first, second, and third $n/3$ variables of x. Then

$$\mathsf{3\text{-}MAJ}_h(x) = \mathsf{MAJ}(\mathsf{3\text{-}MAJ}_{h-1}(x^{(1)}),\ \mathsf{3\text{-}MAJ}_{h-1}(x^{(2)}),\ \mathsf{3\text{-}MAJ}_{h-1}(x^{(3)})).$$

In other terms, $\mathsf{3\text{-}MAJ}_h$ is defined by the read-once formula on the complete ternary tree T_h of height h in which every internal node is a majority gate. We identify the leaves of T_h from left to right with the integers $1, \ldots, 3^h$. For an input $x \in \{0,1\}^h$, the bit x_i defines the *value* of the leaf i, and then the values of the internal nodes are evaluated recursively. The value of the root is $\mathsf{3\text{-}MAJ}_h(x)$. For every node v in T_h different from the root, let $P(v)$ denote the parent of v. We say that v and w are siblings if $P(v) = P(w)$. For any node v in T_h, let $Z(v)$ denote the set of

variables associated with the leaves in the subtree rooted at v. We say that a node v is at depth d in T_h if the distance between v and the root is d. The root is therefore at depth 0, and the leaves are at depth h.

We now define recursively, for every $h \geq 0$, the set $\mathcal{H}_h$ of *hard inputs* of height h. In the base case $\mathcal{H}_0 = \{0, 1\}$. For $h > 0$, let

$$\mathcal{H}_h = \{(x, y, z) \in \mathcal{H}_{h-1} \times \mathcal{H}_{h-1} \times \mathcal{H}_{h-1} : \mathsf{3\text{-}MAJ}_{h-1}(x), \mathsf{3\text{-}MAJ}_{h-1}(y), \text{ and } \mathsf{3\text{-}MAJ}_{h-1}(z) \text{ are not all identical}\}.$$

The hard inputs consist of instances for which at each node v in the ternary tree, one child of v has value different from the value of v. The *hard distribution* on inputs of height h is defined to be the uniform distribution over $\mathcal{H}_h$. We call a hard input x *0-hard* or *1-hard* depending on whether $\mathsf{3\text{-}MAJ}_h(x) = 0$ or 1. We write $\mathcal{H}_h^0$ for the set of 0-hard inputs and $\mathcal{H}_h^1$ for the set of 1-hard inputs.

For an $x \in \mathcal{H}_h$, the *minority path* $M(x)$ is the path, starting at the root, obtained by following the child whose value disagrees with its parent. For $0 \leq d \leq h$, the node of $M(x)$ at depth d is called the depth d minority node, and is denoted by $M(x)_d$. We call the leaf $M(x)_h$ of the minority path the *absolute minority* of x, and denote it by $m(x)$.

2.3 The Jayram-Kumar-Sivakumar Lower Bound

For a deterministic decision tree algorithm B computing $\mathsf{3\text{-}MAJ}_h$, let $L_B(x)$ denote the set of variables queried by B on input x. Recall that $\mathcal{P}^{\delta}_{\mathsf{3\text{-}MAJ}_h}$ is the set of all randomized decision tree algorithms that compute $\mathsf{3\text{-}MAJ}_h$ with two-sided error at most δ. Jayram *et al.* define the function $I^{\delta}(h, d)$, for $d \leq h$:

$$I^{\delta}(h, d) = \min_{A \in \mathcal{P}^{\delta}_{\mathsf{3\text{-}MAJ}_h}} \mathbb{E}_{x \in_{\mathrm{R}} \mathcal{H}_h, B \in_{\mathrm{R}} A}[|Z(M(x)_d) \cap L_B(x)|].$$

In words, it is the minimum over algorithms computing $\mathsf{3\text{-}MAJ}_h$, of the expected number of queries below the dth level minority node, over inputs from the hard distribution. Note that $I^{\delta}(h, 0) = \min_{A \in \mathcal{P}^{\delta}_{\mathsf{3\text{-}MAJ}_h}} \mathrm{C}(A, \mathcal{H}_h)$, and therefore by Proposition 2.1, $\mathrm{R}_{\delta}(\mathsf{3\text{-}MAJ}_h) \geq I^{\delta}(h, 0)$.

We define $p_h^{\delta} = I^{\delta}(h, h)$, which is the minimal probability that a δ-error algorithm A queries the absolute minority of a random hard x of height h.

Jayram *et al.* prove a recursive lower bound for $I^{\delta}(h, d)$ using information theoretic arguments. A more elementary proof can be found in [LNPV06].

Theorem 2.2 (Jayram, Kumar, Sivakumar [JKS03])**.** *For all $0 \leq d < h$:*

$$I^{\delta}(h, d) \geq I^{\delta}(h, d + 1) + 2I^{\delta}(h - 1, d).$$

A simple computation gives then the following lower bound on $I^{\delta}(h, d)$, for all $0 \leq d \leq h$, expressed as a function of the p_i^{δ}'s:

$$I^{\delta}(h, d) \geq \sum_{i=d}^{h} \binom{h-d}{i-d} 2^{h-i} p_i^{\delta} \ .$$

When $d = 0$, this gives $I^{\delta}(h, 0) \geq \sum_{i=0}^{h} \binom{h}{i} 2^{h-i} p_i^{\delta}$. Putting this together with the fact that $\mathrm{R}_{\delta}(\mathsf{3\text{-}MAJ}_h) \geq I^{\delta}(h, 0)$, we get the following corollary:

Corollary 2.3. *Let $q, a > 0$ such that $p_i^\delta \geq a \cdot q^i$ for all $i \in \{0, 1, 2, \ldots, h\}$. Then $\mathrm{R}_\delta(\mathsf{3\text{-}MAJ}_h) \geq a(2+q)^h$.*

As mentioned in Section 1, Jayram *et al.* obtain the $(1-2\delta)(7/3)^h$ lower bound from this corollary by observing that $p_h^\delta \geq (1-2\delta)(1/3)^h$.

3 Improved Lower Bounds

3.1 First Improvement

Theorem 3.1. *For every error $\delta > 0$ and height $h \geq 0$, we have $p_h^\delta \geq (1-2\delta)2^{-h}$.*

Proof. We prove this theorem by induction. Clearly, $p_0^\delta \geq 1 - 2\delta$. It then suffices to show that $2p_h^\delta \geq p_{h-1}^\delta$ for $h \geq 1$. We do so by reduction as follows: let A be a randomized algorithm that achieves the minimal probability p_h^δ for height h formulae. We construct a randomized algorithm A' for height $h-1$ formulae such that the probability that A' errs is at most δ, and A' queries the absolute minority with probability at most $2p_h^\delta$. Since p_{h-1}^δ is the minimum probability of querying the absolute minority in the hard distribution, computed over all randomized algorithms on inputs of height $h-1$ with error at most δ, this implies that $2p_h^\delta \geq p_{h-1}^\delta$.

We now specify the reduction. For the sake of simplicity, we omit the error δ in the notation. We use the following definition:

Definition 3.2 (One level encoding scheme)**.** *A* one level encoding scheme *is a bijection $\psi : \mathcal{H}_{h-1} \times \{1,2,3\}^{3^{h-1}} \to \mathcal{H}_h$, such that for all (y, r) in the domain, $\mathsf{3\text{-}MAJ}_{h-1}(y) = \mathsf{3\text{-}MAJ}_h(\psi(y, r))$.*

Let $c : \{0,1\} \times \{1,2,3\} \to \mathcal{H}_1$ satisfying $b = \mathsf{MAJ}(c(b, s))$ for all inputs (b, s). Define the one level encoding scheme ψ induced *by c as follows: $\psi(y, r) = x \in \mathcal{H}_h$ such that for all $1 \leq i \leq 3^{h-1}$, $(x_{3i-2}, x_{3i-1}, x_{3i}) = c(y_i, r_i)$.*

To define A', we use the one level encoding scheme ψ induced by the following function: $c(y, 1) = y01$, $c(y, 2) = 1y0$, and $c(y, 3) = 01y$.

On input y, algorithm A' picks a uniformly random string $r \in \{1,2,3\}^{3^{h-1}}$, and runs A on $x = \psi(y,r)$ and computes the same output. Notice that each bit of x_i of x is either determined by r alone or else it is $y_{\lceil i/3 \rceil}$. When A asks for a bit of x that is determined by r, then this value is "hard wired" in A' and A' makes no query. When A asks for a bit of x that is not determined by r, then and A' queries the corresponding bit of y. Observe that A' has error at most δ as $\mathsf{3\text{-}MAJ}_{h-1}(y) = \mathsf{3\text{-}MAJ}_h(\psi(y, r))$ for all r, and A has error at most δ. We claim now:

$$2 \Pr_{B \in_{\mathrm{R}} A,\, x \in_{\mathrm{R}} \mathcal{H}_h}[B(x) \text{ queries } x_{m(x)}] \quad \geq \quad \Pr_{B \in_{\mathrm{R}} A,\, (y,r) \in_{\mathrm{R}} \mathcal{H}'_h}[B'(y, r) \text{ queries } y_{m(y)}] \tag{1}$$

where $\mathcal{H}'_h$ is the uniform distribution over $\mathcal{H}_{h-1} \times \{1,2,3\}^{3^{h-1}}$ and B' is the algorithm that computes $x = \psi(y, r)$ and then evaluates $B(x)$. We prove this inequality by taking an appropriate partition of the probabilistic space of hard inputs $\mathcal{H}_h$, and prove Eq. 1 separately, on each set in the partition. For $h = 1$, the two classes of the partition are $\mathcal{H}_1^0$ and $\mathcal{H}_1^1$. For $h > 1$, the partition consists of the equivalence classes of the relation $\sim$ defined by $x \sim x'$ if $x_i = x'_i$ for all i such $P(i) \neq P(m(x))$ in the tree T.

Because ψ is a bijection, observe that this also induces a partition of (y, r), where $(y, r) \sim (y', r')$ if and only if $\psi(y, r) \sim \psi(y', r')$. Also observe that every equivalence class contains three elements.

Then Eq. 1 follows from the following stronger statement: for every equivalence class S, and for all B in the support of A, it holds that

$$\begin{aligned} 2 \Pr_{x \in_{\mathrm{R}} \mathcal{H}_h}[B(x) \text{ queries } x_{m(x)} \mid x \in S] \\ \geq \Pr_{(y,r) \in_{\mathrm{R}} \mathcal{H}'_h}[B'(y,r) \text{ queries } y_{m(y)} \mid \psi(y,r) \in S] \ . \end{aligned} \tag{2}$$

The same proof applies to all sets S, but to simplify the notation, we consider a set S that satisfies the following: for $x \in S$, we have $m(x) \in \{1,2,3\}$ and $x_{m(x)} = 1$. Observe that for each $j > 3$, the jth bits of all three elements in S coincide. Therefore, the restriction of B to the variables (x_1, x_2, x_3), when looking only at the three inputs in S, is a well-defined decision tree on three variables. We call this restriction C, and formally it is defined as follows: for each query x_j made by B for $j > 3$, C simply uses the value of x_j that is shared by all $x \in S$ and that we hard-wire into C; for each query x_j made by B where $j \in \{1,2,3\}$, C actually queries x_j. Note that the restriction C does not necessarily compute $\mathsf{3\text{-}MAJ}_1(x_1x_2x_3)$, for two reasons. Firstly, C is derived from B, which may err on particular inputs. But even if $B(x)$ correctly computes $\mathsf{3\text{-}MAJ}_h(x)$, it might happen that B never queries any of x_1, x_2, x_3, or it might query one and never query a second one, etc.

For any $x \in S$, recall that we write (y,r) the unique solution of $\psi(y,r) = x$. It holds for our choice of S that $m(y) = 1$ because we assumed $m(x) \in \{1,2,3\}$ and also $y_1 = y_{m(y)} = 0$ because we assumed $x_{m(x)} = 1$.

Observe that, for inputs $x \in S$, B queries $x_{m(x)}$ if and only if C queries the minority among x_1, x_2, x_3. Also, $B'(y,r)$ queries $y_{m(y)}$ if and only if $C(\psi(0,r_1))$ queries x_{r_1} (cf. definition of c). Furthermore, the distribution of $x_1x_2x_3$ when $x \in_{\mathrm{R}} S$ is uniform over $\mathcal{H}_1^0$. Similarly, the distribution of r_1 over uniform (y,r) conditioned on $\psi(y,r) \in S$ is identical to that of $(0,r_1) = \psi^{-1}(x_1x_2x_3)$ for $x_1x_2x_3 \in_{\mathrm{R}} \mathcal{H}_1^0$. Thus Eq. 2 is equivalent to:

$$\Pr_{x \in_{\mathrm{R}} \mathcal{H}_1^0}[C(x) \text{ queries } x_{r_1} \text{ where } \psi(0,r_1) = x] \leq 2 \Pr_{x \in_{\mathrm{R}} \mathcal{H}_1^0}[C(x) \text{ queries } x_{m(x)}] \ . \tag{3}$$

In principle, one can prove this inequality by considering all the (finitely many) decision trees C on three variables. We present here a somewhat more compact argument.

If C queries no bit, both sides of Eq. 3 are zero, so the inequality holds. We can therefore assume that C makes at least one query and, without loss of generality, we may also assume that the first query is x_1. We distinguish two cases.

If C makes a second query when the first query is evaluated to 0 then the right hand side of Eq. 3 is at least $4/3 = 2 \cdot (1/3 + 1/3)$ because there is a $1/3$ chance that the first query is $m(x)$ and $1/3$ chance that the second is $m(x)$. But the left hand side is at most 1, and therefore the inequality holds.

If C does not make a second query when the first query is evaluated to 0 then the left hand side is at most $2/3$ since for $x = 010$, we have $r_1 = 3$, but x_3 is not queried. With probability $1/3$ we have $m(x) = 1$, so the right hand side is at least $2/3$. We conclude that Eq. 3 holds for every C.

We remark that the decision tree algorithm making no queries is not the only one that makes Eq. 3 hold with equality. Another such algorithm is the following: first query x_1, if $x_1 = 0$, stop, else if $x_1 = 1$, query x_2 and stop.

To handle a general S, we replace $\{1,2,3\}$ with $m(x)$ and its two siblings. For S such that $x \in S$ satisfies $x_{m(x)} = 0$, the optimal algorithm C' is the same as the one described above, except that each 0 is changed to 1 and vice versa.

Therefore Eq. 3 holds for every C, which implies the theorem.

□

Combining Corollary 2.3 and Theorem 3.1, we obtain the following.

Corollary 3.3. $\mathrm{R}_\delta(3\text{-}\mathsf{MAJ}_h) \geq (1-2\delta)(5/2)^h$.

3.2 Further Improvements

Our proof from the previous section proceeds by proving a recurrence, using a one level encoding scheme, for the minimal probability that an algorithm queries the absolute minority bit. One can ask whether this is the best possible recurrence, and the following theorem states that it is possible to improve it by using higher level encoding schemes.

In the following, we omit δ from the notation when convenient.

Definition 3.4 (Uniform k-level encoding scheme)**.** *Let* $\mathcal{R} = (\{0,1\} \times \{1,2,3\})$ *and define* $c : \{0,1\} \times \mathcal{R} \to \mathcal{H}_1$ *by setting* $c(y,(b,1)) = yb(1-b)$, $c(y,(b,2)) = (1-b)yb$, *and* $c(y,(b,3)) = b(1-b)y$. *The* uniform k-level encoding scheme $\psi^{(k)}$ *is defined by the following recursion:*

1. *For* $h \geq 1$, $y \in \mathcal{H}_{h-1}$ *and* $r \in \mathcal{R}_h^{(1)} := \mathcal{R}^{3^{h-1}}$ *we set* $\psi^{(1)}(y,r) = x \in \mathcal{H}_h$ *such that* $(x_{3i-2}, x_{3i-1}, x_{3i}) = c(y_i, r_i)$, *for all* $1 \leq i \leq 3^{h-1}$;
2. *for* $h \geq k > 1$, $y \in \mathcal{H}_{h-k}$ *and* $(R,r) \in \mathcal{R}_h^{(k)} := \mathcal{R}_h^{(k-1)} \times \mathcal{R}^{3^{h-k}}$ *we set* $\psi^{(k)}(y,(R,r)) = \psi^{(k-1)}(\psi^{(1)}(y,r), R)$.

We note that this encoding ψ is no longer a bijection. However, one can make essentially the same argument as Theorem 3.1. The advantage of this scheme over the one used in that theorem is the higher symmetry: while in the previous scheme a cyclic symmetry worked over any set of three siblings now the entire symmetric group acts on them. Because of this higher symmetry if one of three siblings has been queried, the remaining two are still playing symmetric roles.

We will need the following easy to prove observations that hold for all $h \geq k \geq 1$:

1. For all $y \in \mathcal{H}_{h-k}$ and $r \in \mathcal{R}_h^{(k)}$ we have $3\text{-}\mathsf{MAJ}_{h-k}(y) = 3\text{-}\mathsf{MAJ}_h(\psi^{(k)}(y,r))$.
2. For $(y,r) \in_{\mathrm{R}} \mathcal{H}_{h-k} \times \mathcal{R}_h^{(k)}$ the value $\psi(y,r)$ is distributed uniformly in $\mathcal{H}_h$.
3. For each $r \in \mathcal{R}_h^{(k)}$ and index i in the range $1 \leq i \leq 3^{h-k}$ there is a unique index $q_i(r)$ in the range $(i-1)3^k + 1 \leq q_i(r) \leq i3^k$ such that for all $y \in \mathcal{H}_{h-k}$ we have $x_{q_i(r)} = y_i$ for $x = \psi_h^{(k)}(y,r)$. If $1 \leq j \leq 3^h$ but j is not of the form $q_i(r)$ for any i, then x_j is independent of the choice of y. We call these bits of x the *fixed bits*.

We use the uniform k-level encoding schemes to obtain better bounds on p_h^δ. The argument is very similar to the argument in Theorem 3.1. We start with proving a lower bound on p_h^δ based on a parameter computable by considering all the (finitely many) decision trees acting on inputs from $\mathcal{H}_k^0$. Then we proceed to actually compute this parameter. The high symmetry helps reducing the cases to be considered, but as k grows the length of the calculation increases rather rapidly. We explain the basic structure of the calculation and also include a short Python code implementing it in Appendix A. For $k = 2$ we do the calculation without the use of a computer as an illustration,

for $k = 3, 4$ we include the results of running the code. A much more efficient method would be needed to make the calculation for $k = 5$ feasible.

Let us fix $k \geq 1$ and let C be a deterministic decision tree algorithm acting on inputs of length 3^k that queries at least one variable. We define

$$\alpha_C = \frac{\Pr_{x \in_R \mathcal{H}_k^0, (y,r) \in_R \psi^{-1}(x)}[C(x) \text{ queries } x_{q_1(r)}]}{\Pr_{x \in_R \mathcal{H}_k^0}[C(x) \text{ queries } x_{m(x)}]} \ ,$$

where $\psi = \psi_k^{(k)}$. Note that C queries at least one bits, and thus neither the numerator nor the denominator can be zero. This makes α_C is well defined and positive. Notice that α_C does not depend on the output of C, it depends only on what input bits C queries. We further define

$$\alpha_k = \max_C \alpha_C,$$

where the maximum extends over all deterministic decision trees on 3^k variables that query at least a single variable.

Theorem 3.5. *For every $k \geq 1$, $h \geq 0$ integers and $\delta \geq 0$ real, we have*

$$p_h^\delta \geq (1 - 2\delta)(\alpha_k / 2^k)\alpha_k^{-h/k}.$$

As a corollary we also have

$$\mathrm{R}_\delta(\textsf{3-MAJ}_h) \geq (1 - 2\delta)(\alpha_k / 2^k)(2 + \alpha_k^{-1/k})^h.$$

Proof. We concentrate on the proof of the first statement, then the second follows from Corollary 2.3.

The proof follows the same structure as that of Theorem 3.1 but using a depth-k recursion: we show that

$$\alpha_k p_h^\delta \geq p_{h-k}^\delta \tag{4}$$

if $h \geq k$. To bound p_h^δ in the base cases $h < k$ we use Theorem 3.1 stating $p_h^\delta \geq (1 - 2\delta)/2^h$, and the fact that $\alpha_k \leq 2^k$.

It remains to prove the Eq. 4. We proceed as in Theorem 3.1: we consider a randomized δ-error algorithm A for $\textsf{3-MAJ}_h$ that achieves the minimum defining p_h^δ and construct a randomized algorithm A' for $\textsf{3-MAJ}_{h-k}$ with the same error that queries the absolute minority of a uniform random element of $\mathcal{H}_{h-k}$ with probability at most $\alpha_k p_h^\delta$

To define A', we use the uniform k-level encoding scheme $\psi = \psi^{(k)}$ (see Figure 1 for an illustration of the $k = 2$ case). On input $y \in \mathcal{H}_{h-k}$ the algorithm A' picks a uniform random element r of $\mathcal{R}_h^{(k)}$ and applies the decision algorithm A to $x = \psi(y, r)$ with the convention that whenever a fixed bit of x is queried by A, then A' makes no query, and when a bit $x_{q_i(r)}$ is queried by A, then A' queries y_i. Define $\mathcal{H}'_h = \mathcal{H}_{h-k} \times \mathcal{R}_h^{(k)}$. Then $(y, r) \in \mathcal{H}'_h$ encodes $\psi(y, r) \in \mathcal{H}_h$.

We partition $\mathcal{H}_h$ again, this time into sets of size 3^l with $l = \sum_{i=0}^{k-1} 3^i$. For $h = k$, the two classes are $\mathcal{H}_k^0$ and $\mathcal{H}_k^1$. For $h > k$, the partition consists of the equivalence classes of the relation defined by $x \sim x'$ if $x_i = x'_i$ for all i such $P^{(k)}(i) \neq P^{(k)}(m(x)))$ in the tree T. Here $P^{(k)}(x)$ denotes the ancestor of the vertex x k levels above x (the k times iterated parent function). Namely, an equivalence class consists of inputs that are identical everywhere except the k-level subtree

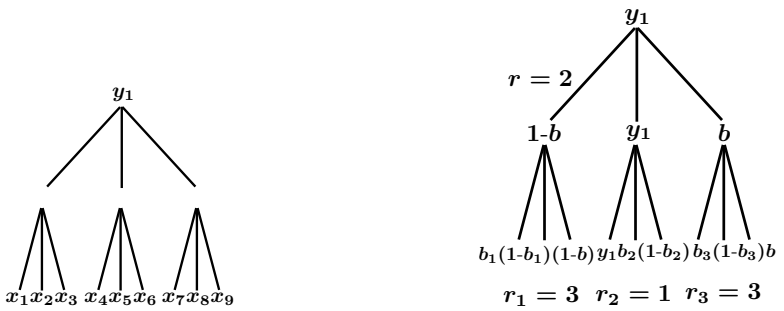


Figure 1: In a uniform 2-level encoding y_1 is encoded using 9 bits $x_1x_2\ldots x_9$. On the right hand side, an example with specific choices of r, r_1, r_2, r_3 for each level. In this example, when $y_1 = 0$, then b_3 encodes the minority bit if $b = b_3 = 0$.

containing their absolute minority. First observe that the uniformity of the encoding implies that for every equivalence class S, and all B in the support of A:

$$\begin{aligned}
\Pr_{x\in_{\mathrm{R}}\mathcal{H}_h}[B(x) \text{ queries } x_{m(x)} \mid x \in S] &= \Pr_{(y,r)\in_{\mathrm{R}}\mathcal{H}'_h, x=\psi(y,r)}[B(x) \text{ queries } x_{m(x)} \mid x \in S] \ , \\
\Pr_{(y,r)\in_{\mathrm{R}}\mathcal{H}'_h}[B'(y,r) \text{ queries } y_{m(y)} | \psi(y,r) \in S] &= \Pr_{x\in_{\mathrm{R}}\mathcal{H}_h, (y,r)\in_{\mathrm{R}}\psi^{-1}(x)}[B'(y,r) \text{ queries } y_{m(y)} | x \in S] \ .
\end{aligned}$$

We then prove that for every equivalence class S, and all B in the support of A, it holds that:

$$\alpha_k \Pr_{x\in_{\mathrm{R}}\mathcal{H}_h}[B(x) \text{ queries } x_{m(x)} \mid x \in S] \geq \Pr_{(y,r)\in_{\mathrm{R}}\mathcal{H}'_h}[B'(y,r) \text{ queries } y_{m(y)} \mid \psi(y,r) \in S], \qquad (5)$$

where we recall that B' is the algorithm that first computes $x = \psi(y,r)$ and then evaluates $B(x)$. Proving Eq. 5 for all B and S finishes the proof of the theorem.

Let us fix S and let z be the variable part of the input: the 3^k variables in the k-level subtree of the absolute minority. Note that the set of possible values of z is either $\mathcal{H}^0_k$ or $\mathcal{H}^1_k$, depending on S. Now a deterministic decision tree B on inputs from S can be considered a deterministic decision tree C for z. Indeed, the queries B asks outside z have a deterministic answer in S that can be hard wired in C. In case C asks no queries at all, then Eq. 5 is satisfied with zero on both sides of the inequality. Otherwise, if the possible values of z are coming from $\mathcal{H}^0_k$ Eq. 5 follows from $\alpha_C \leq \alpha_k$ (which, in turn, comes from the definition of α_k as a maximum). Finally if the possible values of z are the 1-hard inputs, then Eq. 5 is satisfied by symmetry. □

To apply Theorem 3.5 we need to compute (or estimate) α_k. For any fixed k this is a finite computation, but even to do it for small values of k one needs to do better than going through all the deterministic decision trees C on 3^k variables. For a fixed k, a decision tree C and $\alpha \geq 0$ we introduce

$$\rho_\alpha(C) = \Pr_{x\in_{\mathrm{R}}\mathcal{H}^0_k, (y,r)\in_{\mathrm{R}}\psi^{-1}(x)}[C(x) \text{ queries } x_{q_1(r)}] - \alpha \Pr_{x\in_{\mathrm{R}}\mathcal{H}^0_k}[C(x) \text{ queries } x_{m(x)}] \ . \qquad (6)$$

For the decision tree C_0 not querying any variables we have $\rho_\alpha(C) = 0$, for other decision trees C we have $\rho_\alpha(C) > 0$ if and only if $\alpha_C > \alpha$. Thus, we have $\alpha_k > \alpha$ if and only if there exists C with $\rho_\alpha(C) > 0$. Finding the maximum $\max_C \rho_\alpha(C)$ therefore answers the question whether $\alpha_k > \alpha$.

The advantage of this approach lies in the linearity of ρ_α, it makes it easier to maximize $\rho_\alpha(C)$ than α_C itself.

Let us call a bit of the hard input $x \in \mathcal{H}_k$ *absolute majority* if the flipping of this input flips the value of $\mathsf{3\text{-}MAJ}_k(x)$. Note that there are exactly 2^k such bits for all hard inputs, these are the ones where all vertices on the root to leaf path of the ternary tree evaluate to the same value.

Notice that for a fixed $x \in \mathcal{H}_k^0$ and $(y, r) \in_{\mathrm{R}} \psi^{-1}(x)$ the position $q_1(r)$ (where the k-level encoding $\psi = \psi^{(k)}$ "hides" the input variable y) is uniformly distributed over the 2^k absolute majority positions. Thus, we can simplify Eq. 6 defining ρ_α as follows:

$$\rho_\alpha(C) = 2^{-k} p_q - \alpha p_m, \tag{7}$$

where p_q is the expected number of absolute majority bits queried by $C(x)$ for $x \in_{\mathrm{R}} \mathcal{H}_k^0$ and p_m is the probability that the absolute minority bit is queried by C for $x \in_{\mathrm{R}} \mathcal{H}_k^0$.

We call a *configuration* a situation during the execution of a decision tree when some of the variables are already queried and we know their values, while the others are not known. The next action of the decision tree is either to stop (and produce an output that is irrelevant for us now) or to pick a variable not queried yet and query it. In the latter case the next configuration is determined by the value of the chosen variable.

A decision tree is determined by the actions it takes in the possible configurations. Given configuration γ, the optimal decision tree takes the action that maximizes the linear combination in Eq. 7 *conditioned on reaching this configuration*. Namely it maximizes:

$$\rho_\alpha(C, \gamma) = 2^{-k} P_q(\gamma) - \alpha P_m(\gamma), \tag{8}$$

where $P_q(\gamma)$ is the expected number of absolute majority bits queried by $C(x)$, when x is a uniform random 0-hard x consistent with γ, while $P_m(\gamma)$ is the probability that $C(x)$ queries the absolute minority bit for a uniform random 0-hard x consistent with γ. The point is that the optimal action in a configuration γ can be found independently of the actions taken at configurations inconsistent with γ. (A similar statement is false for the maximization of α_C.)

Note that $\rho_\alpha(C, \gamma)$ is easy to compute if C stops at γ, while if C queries a new bit at γ, then $\rho_\alpha(C, \gamma)$ can be easily computed from $\rho_\alpha(C, \gamma')$ and $\rho_\alpha(C, \gamma'')$, where γ' and γ'' are the two configurations the new query may bring γ.

This leads to the following dynamic programming algorithm: consider all configurations in the order where evaluating further variables yields configurations considered earlier. For each configuration γ we find the optimal action and store the value of $\rho_\alpha(C, \gamma)$ for the optimal C. Clearly, $\rho_\alpha(C) = \rho(\alpha(C, \gamma_0)$, where γ_0 is the initial configuration (no variable is asked yet).

The number of configurations is 3^{3^k}. This makes the above algorithm infeasible even for $k = 3$. We make the number of configurations considered shrink considerably by the following simple tricks.

1. We use the symmetries. All the configurations that can be transformed into each other using the automorphisms of the ternary tree are equivalent. We consider only one in each equivalence class.

2. We identify two types of configurations when the optimal action is clear without any computation. First, if the root is evaluated, then we have necessarily queried all absolute majority bits, so the optimal strategy is to stop. Second, if an unqueried variable cannot be the absolute minority variable, we may as well query it right away. We know that the vertices of

the path to the absolute minority evaluate alternatively to 0 and 1. If a vertex in odd depth evaluates to 0 or in even depth evaluates to 1, then it is not on the absolute minority path and all variables under it can be queried. But if a vertex in odd depth evaluates to 1, or in even depth evaluates to 0, then its *siblings* are not on the absolute minority path and the variables under the siblings should be queried.

We call a configuration *stable* if neither rule in point 2 applies. It is enough to store $\rho_\alpha(C,\gamma)$ for stable γ. If $\rho_\alpha(C,\gamma)$ is needed for some unstable configuration γ we apply the above rules (possible repeatedly) to find what stable configurations they lead to and use the corresponding stored values to compute $\rho_\alpha(C,\gamma)$.

Note that if N_k denotes the number of equivalence classes of stable configurations in depth k, then we have the recursion $N_0 = 1$, $N_k = \binom{N_{k-1}+1}{2} + \binom{N_{k-1}+2}{3}$. Indeed, the stable configuration for depth 0 is the one in which the only variable is not known. The recursion comes from noticing that the children of the root in a stable configuration of depth k are duals of stable configurations of depth $k-1$ or are fully evaluated. But no child of the root can be evaluated to 1 in a stable configuration and at most one of them can be evaluated to 0, so the stable configuration of depth k is determined by an unordered pair or triple of stable configurations of depth $k-1$. (Note that to avoid dealing with duals the code Python code in the appendix considers recursive NOT-3-MAJ functions to avoid dealing with dulas. This is the function computed by the a negated majority gate in every node of a ternary tree.) The recursion gives the following values.

$$\begin{aligned} N_1 &= 2 \\ N_2 &= 7 \\ N_3 &= 112 \\ N_4 &= 246,792 \\ N_5 &= 2,505,258,478,767,772 \end{aligned}$$

This makes the computation feasible for $k \leq 4$. We can even go through the seven stable configurations in depth 2 without the use of a machine. We do just that in the next section as an illustration.

As explained earlier, having an algorithm to maximize $\rho_\alpha(C)$ we can answer questions whether $\alpha_k > \alpha$. Instead of a binary search we find the exact value of α_k as follows. With little modification our algorithm gives not just the maximum of $\rho_\alpha(C)$ but also α_C for the decision tree C maximizing $\rho_\alpha(C)$. We can start with an arbitrary $\alpha \leq \alpha_k$ and until we find that $max_C \rho_\alpha(C) = 0$ we can repeatedly increase α to this value α_C. Clearly, this finds the maximum α_k in a finite number of iterations. Instead of bounding the number of iterations in general we mention that starting from the initial value $\alpha = 0$ we arrived to α_k in at most four iterations in the $k = 2, 3, 4$ cases. Our computations show:

$$\begin{aligned} \alpha_1 &= 2 \\ \alpha_2 &= \frac{24}{7} \\ \alpha_3 &= \frac{12231}{2203} \\ \alpha_4 &= \frac{2027349}{216164} \end{aligned}$$

Using the value of α_4 Theorem 3.5 yields the following bound.

Corollary 3.6.

$$\mathrm{R}_\delta(3\text{-}\mathsf{MAJ}_h) \geq (1/2-\delta)\left(2+\left(\frac{216164}{2027349}\right)^{1/4}\right)^h > (1/2-\delta)2.57143^h.$$

We also state the consequence of the value α_2. This bound is somewhat weaker, but is obtained without the help of a computer.

Corollary 3.7. $\mathrm{R}_\delta(3\text{-}\mathsf{MAJ}_h) \geq (1/2-\delta)(2+\sqrt{7/24})^h > (1/2-\delta)2.54006^h$.

3.3 Computing $\alpha_2 = 24/7$

Here we fix $k = 2$ and consider the deterministic decision trees C on 9 variables. We run these decision trees on inputs from $\mathcal{H}_2^0$.

Observe that we already know that $\alpha_C \leq 4$ for all C from the proof of Theorem 3.1. It is easy to check that $\alpha_{C_0} = 3$ for the decision tree C_0 with the following strategy: first query x_1, if $x_1 = 1$, stop, else if $x_1 = 0$, query x_2 and x_3; then if $\mathsf{MAJ}(x_1x_2x_3) = 0$, stop, else query all remaining bits and stop.These bounds show that $3 \leq \alpha_2 \leq 4$, so it is enough to consider ρ_α for the values of α in the range $[3, 4]$.

It turns out that for these values the same decision tree maximizes $\rho_\alpha(C)$ among the deterministic decision trees querying at least one variable. It is the decision tree C' given in Figure 2. In the figure, "Stop" means stop and "All" means to completely evaluate all variables, except in the case when not asking a variable is explicitly indicated. We state the optimality of C' in the following lemma.

Lemma 3.8. *Let C be any deterministic decision tree on 9-bit inputs asking at least a single query and let C' be the decision tree depicted in Figure 2. Then for all $\alpha \in [3, 4]$, $\rho_\alpha(C) \leq \rho_\alpha(C')$.*

Proof. Recall that the action in a configuration γ of the decision tree C that maximizes $\rho_\alpha(C)$ is the one that maximizes $\rho_\alpha(C, \gamma) = 2^{-2}P_q(\gamma) - \alpha P_m(\gamma)$. To simplify notation, we simply write ρ, P_q and P_m for these values if the configuration γ considered is clear from the context.

We call any set of 3 sibling nodes a *clause*, that is $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$ are clauses. We say a clause is *evaluated* if its majority is known.

We will argue what an algorithm maximizing $\rho_\alpha(C)$ should do. We begin with three simple rules that are the special cases of the general rules we used to reduce the number of configurations considered.

1. If a bit 0 is evaluated, then evaluate all remaining bits in its clause.
2. If two bits in a clause are evaluated to 1 (this is the minority clause), evaluate all remaining bits in other clauses.
3. if two clauses have been evaluated to 0, then stop.

In what follows we systematically consider all configurations, where the three rules above do not apply (the stable configurations) and decide what an optimal decision should do next in those situations to maximize ρ_α.

4. A single majority (0) clause is evaluated and either no variables are evaluated in either of the other clauses or a single 1 is evaluated in both the other clauses. In this case, stopping is the best strategy. Indeed, $m(x)$ has not been queried yet. Therefore if we stop, then $P_m = 0$ and $P_q = 2$. This gives $\rho = 1/2$. But if C continues by querying at least one more bit, then $P_m \geq 1/6$ or $P_m \geq 1/4$ (since there are either 6 or 4 remaining unqueried variables, and they are symmetric) and $P_q \leq 4$. Therefore, $\rho = 2^{-2}P_q - \alpha P_m \leq 1 - \alpha/6 \leq 1/2$ since $\alpha \geq 3$.

5. A single majority clause is evaluated and one more bit is evaluated to 1, but nothing more. We argue that stopping is best in this case just as in the previous case. The argument is more involved because there is no symmetry between all unqueried variables, we have to compare stopping separately to querying a variable inside or outside the untouched clause. There are 9 inputs consistent with this partial evaluation. If we stop, then $P_q = 2$ and $P_m = 0$, so we have $\rho = 1/2$.

   If we query a variable in the clause containing the single 1 bit, then there are 3 consistent input in which the next queried bit is $m(x)$, so we have $P_m \geq 1/3$ and $P_q \leq 4$, thus $\rho \leq 0$ since $\alpha \geq 3$.

   If we query a variable in the untouched clause, then there is 1 out of the 9 consistent inputs for which this next queried variable is $m(x)$, making $P_m \geq 1/9$. There are 4 more consistent inputs for which this variable evaluates to 1. In this case we arrive in the configuration covered by item 4 above, and using that rule we should stop, leaving 2 out the 4 absolute majority bits unqueried. Thus, we have $P_q \leq 4 - 4/9 \cdot 2 = 28/9$ and $\rho \leq 4/9$, which is still less than the $1/2$ obtained if we stop.

6. A single 1 has been evaluated in each of the three clauses and no other bit has been queried. In this case reading another bit is the best strategy (the choice of which bit is unimportant because of symmetry). Observe that no bit 0 has been evaluated yet. Therefore if C stops, we have $\rho = P_q = P_m = 0$. If C continues to query another bit, then which bit to query does not matter by symmetry and rest of the algorithm is determined by the earlier rules yielding $P_q = 8/3$, $P_m = 1/6$, and $\rho = 2/3 - \alpha/6 \geq 0$.

7. A single bit 1 has been evaluated in each of two different clauses and the third clause is untouched. Then the best is to evaluate a bit of the third clause. Again, if C stops we have $\rho = P_m = P_q = 0$ and ρ_α are 0.

   If C reads another bit in one of the clauses containing a single 1 bit, then the rest of the decision tree algorithm is determined by the earlier rules and we get $P_q = 14/5$ and $P_m = 1/5$ with $\rho = 7/10 - \alpha/5$. Note that whether this option is better or stopping depends on the value of α.

   If C reads a bit in the untouched clause, then by using the rules already presented in the previous cases, we calculate $P_q = 12/5$ and $P_m = 2/15$ yielding $\rho = 3/5 - 2\alpha/15$. This happens to be more than either 0 or $7/10 - \alpha/5$ in the entire range of α considered.

8. A single bit 1 has been evaluated in one clause and no other clauses are touched. Then the best is to evaluate a bit of another clause. Again, if C stops, then we have $\rho = P_m = P_q = 0$.

   If C evaluates another bit in the clause containing 1, then the rest of C is determined by earlier rules and we have $P_q = 3$, $P_m = 1/4$ and $\rho = 3/4 - \alpha/4 \leq 0$.

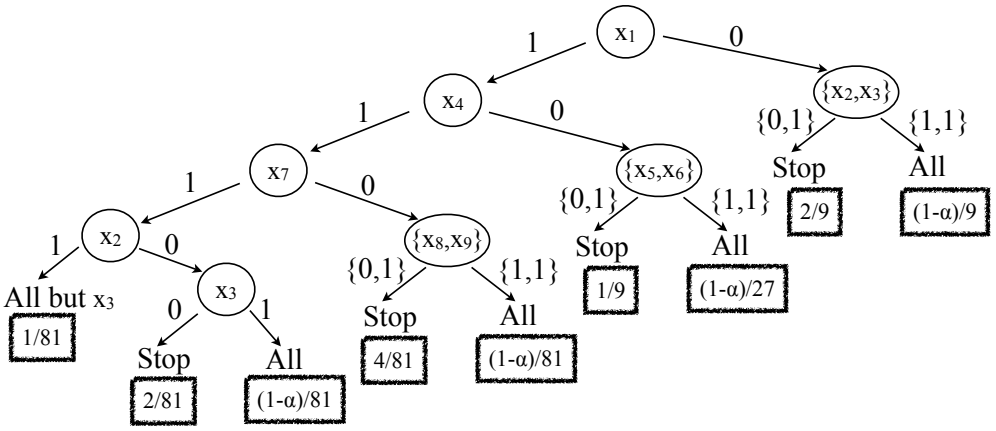


Figure 2: Picture of C' with the contribution to $\rho_{C'}$ (in boxes) for each branch of C'

But if C queries a bit in an untouched clause, then similar calculations yield $P_q = 7/3$, $P_m = 5/36$ and $\rho = 7/12 - 5\alpha/36 > 0$, making this the best choice.

Following all the above rules (and always choosing the smallest index where symmetry allows us to choose) we arrive to a well defined decision tree, namely to C'. This finishes the proof of the lemma. □

Theorem 3.9. $\alpha_2 = 24/7$

Proof. As observed in the paragraph before Lemma 3.8 we have $3 \leq \alpha_2 \leq 4$. By the lemma we know that $\rho_\alpha(C)$ in this range is maximized by either C' or the decision tree not querying any variable (the latter giving $\rho_\alpha = 0$). We have $\alpha \geq \alpha_2$ if and only if this maximum is 0, so we are done if we calculate $\rho_\alpha(C')$. The contribution of each branch of the algorithm is calculated on Figure 2 summing to $(48-14\alpha)/81$. This is positive for $\alpha < 24/7$, so we have $\alpha_2 = \alpha_{C'} = 24/7$. □

4 Improved Depth-Two Algorithm

In this section, we present a new zero-error algorithm for computing 3-MAJ$_h$. For the key ideas behind it, we refer the reader to Section 1.

As before, we identify the formula 3-MAJ$_h$ with a complete ternary tree of height h. In the description of the algorithm we adopt the following convention. Once the algorithm has determined the value b of the subformula rooted at a node v of the formula 3-MAJ$_h$, we also use v to denote this bit value b.

The algorithm is a combination of two depth-2 recursive algorithms. The first one, EVALUATE (see Algorithm 1), takes a node v of height $h(v)$, and evaluates the subformula rooted at v. The interesting case, when $h(v) > 1$, is depicted in Figure 3. The first step, permuting the input, means applying a random permutation to the children y_1, y_2, y_3 of v and independent random permutations to each of the three sets of grandchildren.

The second algorithm, COMPLETE (see Algorithm 2), is depicted in Figure 4. It takes two arguments v, y_1, and completes the evaluation of the subformula 3-MAJ$_h$ rooted at node v,

Algorithm 1 EVALUATE(v): evaluate a node v.

Input: Node v with subtree of height $h(v)$.
Output: the bit value $\mathsf{3\text{-}MAJ}_h(Z(v))$ of the subformula rooted at v

Let $h = h(v)$

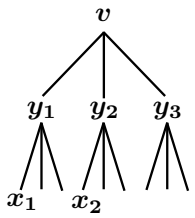
if $h = 0$ **then** ▷ First base case: $h = 0$ (v is a leaf)
  Query $Z(v)$ to get its value a **return** a
end if

Let y_1, y_2, y_3 be a uniformly random permutation of the children of v

if $h = 1$ **then** ▷ Second base case: $h = 1$
  EVALUATE(y_1) and EVALUATE(y_2)
  if $y_1 = y_2$ **then return** y_1
  elsereturn EVALUATE(y_3)
  end if
end if

Let x_1 and x_2 be chosen uniformly at random from the children of y_1 and y_2, resp. ▷ Recursive case
▷ use the attached figure as a guide

EVALUATE(x_1) and EVALUATE(x_2)



if $x_1 \neq x_2$ **then**
  EVALUATE(y_3)
  Let $b \in \{1, 2\}$ be such that $x_b = y_3$
  COMPLETE(y_b, x_b)
  if $y_b = y_3$ **then return** y_b
  elsereturn COMPLETE(y_{3-b}, x_{3-b})
  end if
else[$x_1 = x_2$]
  COMPLETE(y_1, x_1)
  if $y_1 = x_1$ **then**
    COMPLETE(y_2, x_2)
    if [**then**$y_2 = y_1$]$y_2 = x_2$ **return** y_1
    else[$y_2 \neq y_1$] **return** EVALUATE(y_3)
    end if
  else[$y_1 \neq x_1$]
    EVALUATE(y_3)
    if $y_3 = y_1$ **then return** y_1
    elsereturn COMPLETE(y_2, x_2)
    end if
  end if
end if

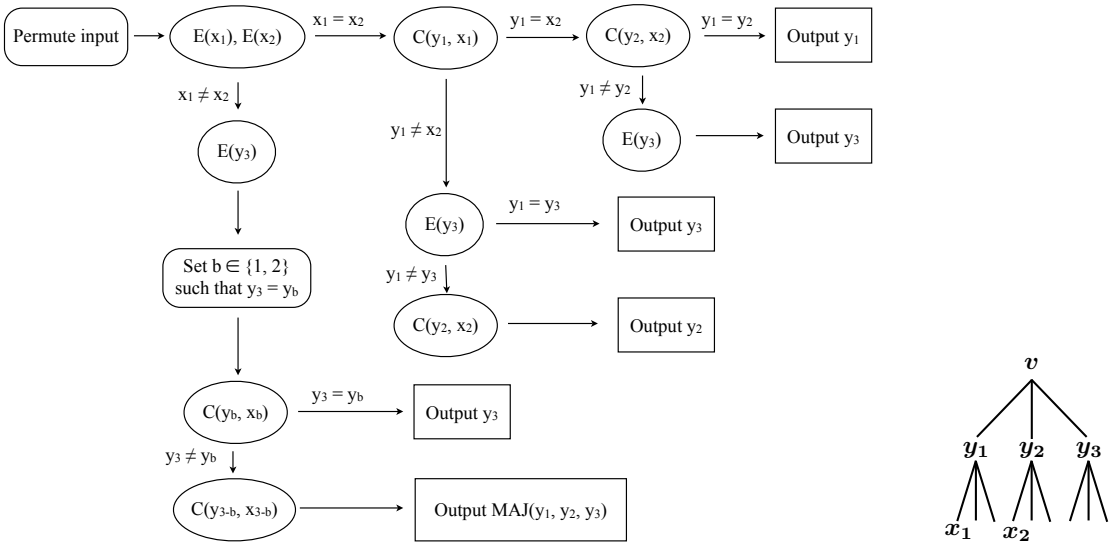


Figure 3: Pictorial representation of algorithm EVALUATE on a subformula of height $h(v) \geq 2$ rooted at v. It is abbreviated by the letter 'E' when called recursively on descendants of v. The letter 'C' abbreviates the second algorithm COMPLETE.

where $h(v) \geq 1$, and y_1 is a child of v whose value has already been evaluated. The first step, permuting the input, means applying a random permutation to the children y_2, y_3 of v and independent random permutations to each of the two sets of grandchildren of y_2, y_3. Note that this is similar in form to the depth 2 algorithm of [JKS03].

To evaluate an input of height h, we invoke EVALUATE(r), where r is the root. The correctness of the two algorithms follows by inspection—they determine the values of as many children of the node v as is required to compute the value of v.

For the complexity analysis, we study the expected number of queries they make for a worst-case input of fixed height h. (*A priori*, we do not know if such an input is a hard input as defined in Section 2.2.) Let $T(h)$ be the worst-case complexity of EVALUATE(v) for v of height h. For COMPLETE(v, y_1), we distinguish between two cases. Let y_1 be the child of node v that has already been evaluated. The complexity given that y_1 is the minority child of v is denoted by S^{m}, and the complexity given that it is a majority child is denoted by S^{M}.

The heart of our analysis is the following set of recurrences that relate T, S^{M} and S^{m} to each other.

Lemma 4.1. *It holds that* $S^{\mathrm{m}}(1) = 2$, $S^{\mathrm{M}}(1) = \frac{3}{2}$, $T(0) = 1$, *and* $T(1) = \frac{8}{3}$.

For all $h \geq 1$, *it holds that*

$$S^{\mathrm{M}}(h) \leq S^{\mathrm{m}}(h) \qquad \textit{and} \qquad S^{\mathrm{M}}(h) \leq T(h) \ . \tag{9}$$

Algorithm 2 COMPLETE(v, y_1): finish the evaluation of the subformula rooted at node v

Input: Node v of height $h(v)$; child y_1 of v which has already been evaluated
Output: the bit value 3-MAJ$_h(Z(v))$

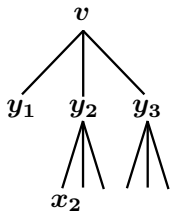
Let $h = h(v)$

Let y_2, y_3 be a uniformly random permutation of the two children of v other than y_1

if $h = 1$ **then** ▷ Base case
  EVALUATE(y_2)
  if $y_2 = y_1$ **then return** y_1
  elsereturn EVALUATE(y_3)
  end if
end if

Let x_2 be chosen uniformly at random from the children of y_2 ▷ Recursive case
▷ use the attached figure as a guide

EVALUATE(x_2)



if $y_1 \neq x_2$ **then**
  EVALUATE(y_3)
  if $y_1 = y_3$ **then return** y_1
  elsereturn COMPLETE(y_2, x_2)
  end if
else$[y_1 = x_2]$
  EVALUATE(y_2, x_2)
  if $y_1 = y_2$ **then return** y_1
  elsereturn EVALUATE(y_3)
  end if
end if

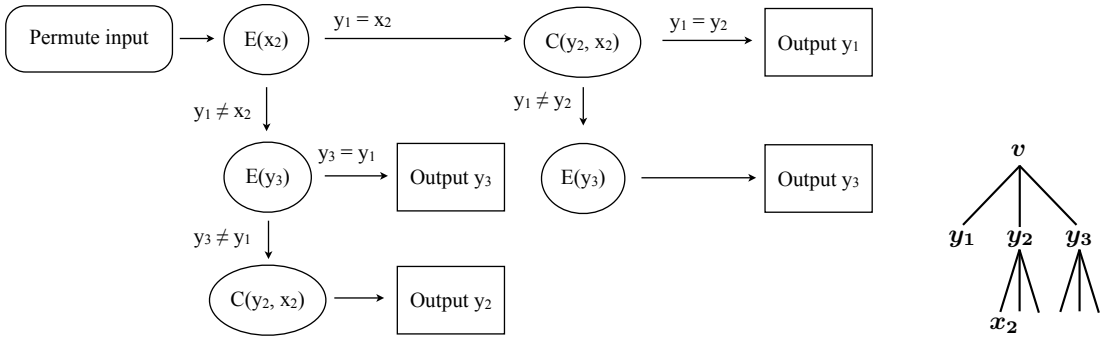


Figure 4: Pictorial representation of algorithm COMPLETE on a subformula of height $h \geq 1$ rooted at v one child y_1 of which has already been evaluated. It is abbreviated by the letter 'C' when called recursively on descendants of v. Calls to EVALUATE are denoted 'E'.

Finally, for all $h \geq 2$, it holds that

$$S^{\mathrm{m}}(h) = T(h-2) + T(h-1) + \frac{2}{3} S^{\mathrm{M}}(h-1) + \frac{1}{3} S^{\mathrm{m}}(h-1) \ , \tag{10}$$

$$S^{\mathrm{M}}(h) = T(h-2) + \frac{2}{3} T(h-1) + \frac{1}{3} S^{\mathrm{M}}(h-1) + \frac{1}{3} S^{\mathrm{m}}(h-1) \ , \quad \textit{and} \tag{11}$$

$$T(h) = 2\,T(h-2) + \frac{23}{27} T(h-1) + \frac{26}{27} S^{\mathrm{M}}(h-1) + \frac{18}{27} S^{\mathrm{m}}(h-1) \ . \tag{12}$$

Proof. We prove these relations by induction. The bounds for $h \in \{0, 1\}$ follow immediately by inspection of the algorithms. To prove the statement for $h \geq 2$, we assume the recurrences hold for all $l < h$. Observe that it suffices to prove Equations (10), (11), (12) for height h, since the values of the coefficients immediately imply that Inequalities (9) holds for h as well.

Equation (10). Since COMPLETE(v, y_1) always starts by computing the value of a grandchild x_2 of v, we get the first term $T(h-2)$ in Eq. (10). It remains to show that the worst-case complexity of the remaining queries is $T(h-1) + (2/3)S^{\mathrm{M}}(h-1) + (1/3)S^{\mathrm{m}}(h-1)$.

Since y_1 is the minority child of v, we have that $y_1 \neq y_2 = y_3$. The complexity of the remaining steps is summarized in the next table in the case that the three children of node y_2 are not all equal. In each line of the table, the worst case complexity is computed given the event in the first cell of the line. The second cell in the line is the probability of the event in the first cell over the random permutation of the children of y_2. This gives a contribution of $T(h-1) + (2/3)S^{\mathrm{M}}(h-1) + (1/3)S^{\mathrm{m}}(h-1)$.

$S^{\mathrm{m}}(h)$ (we have $y_1 \neq y_2 = y_3$)		
event	probability	complexity
$y_2 = x_2$	$2/3$	$T(h-1) + S^{\mathrm{M}}(h-1)$
$y_2 \neq x_2$	$1/3$	$T(h-1) + S^{\mathrm{m}}(h-1)$

This table corresponds to the worst case, as the only other case is when all children of y_2 are equal, in which the cost is $T(h-1) + S^{\mathrm{M}}(h-1)$. Applying Inequality (9) for $h-1$, this is a smaller contribution than the case where the children are not all equal.

Therefore the worst case complexity for S^{m} is given by Eq. (10). We follow the same convention and appeal to this kind of argument also while deriving the other two recurrence relations.

Equation (11). Since COMPLETE(v, y_1) always starts by computing the value of a grandchild x_2 of v, we get the first term $T(h-2)$ in Eq. (11). There are then two possible patterns, depending on whether the three children y_1, y_2, y_3 of v are all equal. If $y_1 = y_2 = y_3$, we have in the case that all children of y_2 are not equal that:

$S^{\mathrm{M}}(h)$ if $y_1 = y_2 = y_3$		
event	probability	complexity
$y_2 = x_2$	$2/3$	$S^{\mathrm{M}}(h-1)$
$y_2 \neq x_2$	$1/3$	$T(h-1)$

As in the above analysis of Eq. (10), applying Inequalities (9) for height $h-1$ implies that the complexity in the case when all children of y_2 are equal can only be smaller, therefore the above table describes the worst-case complexity for the case when $y_1 = y_2 = y_3$.

If y_1, y_2, y_3 are not all equal, we have two events $y_1 = y_2 \neq y_3$ or $y_1 = y_3 \neq y_2$ of equal probability as y_1 is a majority child of v. This leads to the following tables for the case where the

children of y_2 are not all equal

$S^{\mathrm{M}}(h)$ given $y_1 = y_2 \neq y_3$		
event	prob.	complexity
$y_2 = x_2$	2/3	$S^{\mathrm{M}}(h-1)$
$y_2 \neq x_2$	1/3	$T(h-1) + S^{\mathrm{m}}(h-1)$

$S^{\mathrm{M}}(h)$ given $y_1 = y_3 \neq y_2$		
event	prob.	complexity
$y_2 = x_2$	2/3	$T(h-1)$
$y_2 \neq x_2$	1/3	$T(h-1) + S^{\mathrm{m}}(h-1)$

As before, one can apply Inequalities (9) for height $h-1$ to see that the worst case occurs when the children of y_2 are not all equal.

From the above tables, we deduce that the worst-case complexity occurs on inputs where y_1, y_2, y_3 are not all equal. This is because one can apply Inequalities (9) for height $h-1$ to see that, line by line, the complexities in the table for the case $y_1 = y_2 = y_3$ are upper bounded by the corresponding entries in each of the latter two tables. To conclude Eq. (11), recall that the two events $y_1 = y_2 \neq y_3$ and $y_1 = y_3 \neq y_2$ occur with probability 1/2 each:

$$\begin{aligned} S^{\mathrm{M}}(h) &= T(h-2) + \frac{1}{2}\left[\frac{2}{3}\,S^{\mathrm{M}}(h-1) + \frac{1}{3}\left(T(h-1) + S^{\mathrm{m}}(h-1)\right)\right] \\ &\quad + \frac{1}{2}\left[\frac{2}{3}\,T(h-1) + \frac{1}{3}\left(T(h-1) + S^{\mathrm{m}}(h-1)\right)\right] \ . \end{aligned}$$

Equation (12). Since EVALUATE(v) starts with two calls to itself to compute x_1, x_2, we get the first term $2\,T(h-2)$ on the right hand side. For the remaining complexity, we consider two possible cases, depending on whether the three children y_1, y_2, y_3 of v are equal. If $y_1 = y_2 = y_3$, assuming that the children of y_1 are not all equal, and the same for the children of y_2, we have

$T(h)$ given $y_1 = y_2 = y_3$		
event	probability	complexity
$y_1 = x_1, y_2 = x_2$	4/9	$2S^{\mathrm{M}}(h-1)$
$y_1 = x_1, y_2 \neq x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1)$
$y_1 \neq x_1, y_2 = x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1)$
$y_1 \neq x_1, y_2 \neq x_2$	1/9	$T(h-1) + S^{\mathrm{m}}(h-1)$

As before, the complexities are in non-decreasing order, and we observe that Inequalities (9) for height $h-1$ implies that in a worst case input the children of y_1 are not all equal, and the same for the children of y_2.

If y_1, y_2, y_3 are not all equal, we have three events $y_1 = y_2 \neq y_3$, $y_1 \neq y_2 = y_3$ and $y_3 = y_1 \neq y_2$ each of which occurs with probability 1/3. This leads to the following analyses

$T(h)$ given $y_1 = y_2 \neq y_3$		
event	probability	complexity
$y_1 = x_1, y_2 = x_2$	4/9	$2S^{\mathrm{M}}(h-1)$
$y_1 = x_1, y_2 \neq x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 = x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 \neq x_2$	1/9	$T(h-1) + 2S^{\mathrm{m}}(h-1)$

$T(h)$ given $y_1 \neq y_2 = y_3$		
event	probability	complexity
$y_1 = x_1, y_2 = x_2$	4/9	$T(h-1) + S^{\mathrm{M}}(h-1)$
$y_1 = x_1, y_2 \neq x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 = x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 \neq x_2$	1/9	$T(h-1) + 2S^{\mathrm{m}}(h-1)$

$T(h)$ given $y_3 = y_1 \neq y_2$		
event	probability	complexity
$y_1 = x_1, y_2 = x_2$	4/9	$T(h-1) + S^{\mathrm{M}}(h-1)$
$y_1 = x_1, y_2 \neq x_2$	2/9	$T(h-1) + S^{\mathrm{M}}(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 = x_2$	2/9	$T(h-1) + S^{\mathrm{m}}(h-1)$
$y_1 \neq x_1, y_2 \neq x_2$	1/9	$T(h-1) + 2S^{\mathrm{m}}(h-1)$

In all three events, we observe that Inequalities (9) for height $h-1$ implies that in a worst case input, the children of y_1 are not all equal, and the same for the children of y_2.

Applying Inequalities (9) for height $h-1$, it follows that line by line the complexities in the last three tables are at least the complexities in the table for the case $y_1 = y_2 = y_3$. Therefore the worst case also corresponds to an input in which y_1, y_2, y_3 are not all equal. We conclude Eq. (12) as before, by taking the expectation of the complexities in the last three tables.

Theorem 4.2. *$T(h), S^{\mathrm{M}}(h)$, and $S^{\mathrm{m}}(h)$ are all in $\mathrm{O}(\alpha^h)$, where $\alpha \leq 2.64944$.*

Proof. We make an ansatz $T(h) \leq a\,\alpha^h$, $S^{\mathrm{M}}(h) \leq b\,\alpha^h$, and $S^{\mathrm{m}}(h) \leq c\,\alpha^h$, and find constants a, b, c, α for which we may prove these inequalities by induction.

The base cases tell us that $2 \leq c\alpha$, $\frac{3}{2} \leq b\alpha$, $1 \leq a$, and $\frac{8}{3} \leq a\alpha$.

Assuming we have constants that satisfy these conditions, and that the inequalities hold for all appropriate $l < h$, for some $h \geq 2$, we derive sufficient conditions for the inductive step to go through.

By the induction hypothesis, Lemma 4.1, and our ansatz, it suffices to show

$$a + \tfrac{3a+2b+c}{3}\alpha \leq c\,\alpha^2 \qquad a + \tfrac{2a+b+c}{3}\alpha \leq b\,\alpha^2 \qquad 2a + \tfrac{23a+26b+18c}{27}\alpha \leq a\,\alpha^2 \tag{13}$$

The choice $\alpha = 2.64944$, $a = 1.02$, $b = 0.559576 \times a$, and $c = 0.755791 \times a$ satisfies the base case as well as all the Inequalities (13), so the induction holds. □

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A Python code

```
def prep(dep):
  global depth,g,d
  # depth = depth of recursive NOT-3-MAJ circuits considered
  # g[i] = list of depth i stable configurations for 0 <= i <= depth
  # for a record r representing a configuration of depth i
  # r[0] = # of evaluations giving 0
  # r[1] = # of evaluations giving 1
  # r[2] = # of queried absolute majority bits in all evaluations with correct
      output
  # r[3:6] = indices of the three children, sorted (where applicable)
  # d[i] = dictionary that tells the index of a stable configuration of depth
      i from the indices of its 3 depth i-1 children
  # prep sets the values of these global variables (will not be changed)
  depth=dep
  g=[[[0,1,depth%2],[1,1,0]]]
  # two stable configurations in g[0]: a variable set to 1 and a not queried
      variable
  d=[{}]
  # empty dictionary in d[0]
  for i in range(1,depth+1):
    # building g[i] and d[i] under the names gg and dd
    gl=g[-1]
    gg=[[0,1,((depth-i)%2)*2**i]]
    # the first record represents a fully queried subtree of depth i
        evaluating to 1
    dd={}
    for a in range(len(gl)):
      for b in range(max(a,1),len(gl)):
        for c in range(b,len(gl)):
          # enforcing a <= b <= c and no two fully evaluated siblings in a
              stable configuration
          dd[(a,b,c)]=len(gg)
          A,B,C=gl[a],gl[b],gl[c]
          gg.append([A[0]*B[1]*C[1]+A[1]*B[0]*C[1]+A[1]*B[1]*C[0],A[1]*B[0]*C
              [0]+A[0]*B[1]*C[0]+A[0]*B[0]*C[1],A[0]*B[1]*C[2]+A[0]*B[2]*C[1]+
              A[1]*B[0]*C[2]+A[1]*B[2]*C[0]+A[2]*B[0]*C[1]+A[2]*B[1]*C[0],a,b,
              c])
          # inserting the current record in gg and dd - formula is long but
              simple
    g.append(gg)
    d.append(dd)
    # inserting the final gg and dd as g[i] and d[i]

def check(a0,a1):
  global alpha,ot,opt
  # alpha is considered as the rational a0/a1 but the integers are stored
  # ot is a table dinamically built for the strategy C maximizing rho_alpha(C)
  # ot[a][0] = # of inputs in which the optimal strategy C applied after
      configuration g[depth][a] queries abs. minority
```

```
  # ot[a][1] = # total # of absolute majority bits revealed for C applied
      after g[depth][a]
  alpha=[a0,a1]
  ot=[[0,0]]
  # g[depth][0] is impossible
  for a in range(1,len(g[-1])):
    opt=[0,g[-1][a][2]]
    # opt is the best current guess for ot[a]
    # here it is set to value corresponding to stopping at the configuration g
        [depth][a]
    # opt[0] = 0 as the absolute minority is not queried in any stable
        configuration
    adj(depth,a,[],0)
    # here adj is applied to the root of the current configuration g[depth][a]
    # it craws through the entire tree and updates opt if querying a variable
        is better than the current optimum
    ot.append(opt)
    # opt is now the correct optimum - it is inserted in the list
  return ot[-1]
  # here max_C rho_alpha(C) = ot[-1][1]/(2**depth*N)-alpha*ot[-1]/N, where N
      is the total # of 0-hard inputs
  # also: alpha(C) = ot[-1][1]/(ot[-1][0]*2**depth) (unless ot[-1]=[0,0] and C
       queries no input bit)

def adj(i,a,s,t):
  global ot,opt
  # here we consider what happens if a vertex W in a stable configuration is
      evaluated to 0 or 1
  # g[i][a] = the configuration below W
  # s is a list of depth-i pairs of siblings to add to g[i][a] to arrive to
      the stable depth d configuration considered
  # first goal compute tt as follows
  # tt[0] # of inputs in which the optimal strategy C applied after W is set
      to 0 (instable) queries abs. minority
  # tt[1] total # of absolute majorit bits revealed in same situation
  # tt[2] # of consistent inputs with W on the root to absolute minority path
  # t = (same as tt but for the parent of W)
  if s==[]:
    tt=[0,2**i,1,0,0]
    # no input considered gives 0 at the root
  else:
    s1,s2=g[i][s[0][0]],g[i][s[0][1]]
    # the siblings of W
    nn,ne=s1[1]*s2[1],s1[0]*s2[1]+s2[0]*s1[1]
    tt=[nn*t[0]+ne*t[3],nn*t[1]+ne*t[4],nn*t[2]]
    # based on the rule to evaluate the siblings of W if W evaluates to 0
    if s[0][0]==0:
      tt.extend([s2[0]*t[0],s2[0]*t[1]])
      # based on the fact that if W evaluates to 1 and a so does a sibling,
          then the parent evaluates to 0
    else:
```

```
      # we have a stable configurationa and use d to find its index in g[depth
          ]
      w,j=0,i
      for si in s:
        j=j+1
        if w<=si[0]:
          triple=(w,si[0],si[1])
        elif w<=si[1]:
          triple=(si[0],w,si[1])
        else:
          triple=(si[0],si[1],w)
        # here we sorted the three siblings w, si[0] and si[1]
        w=d[j][triple]
      # by now w is index of the full stable configuration and ot[w] contains
          the numbers we seek
      tt.extend(ot[w])
  if i>0:
    # if W is not a variable we recursively call adj on its non-evaluated
        children
    A=g[i][a]
    if A[3]>0:
      adj(i-1,A[3],[A[4:6]]+s,tt)
    adj(i-1,A[4],[[A[3],A[5]]]+s,tt)
    adj(i-1,A[5],[A[3:5]]+s,tt)
  else:
    # if W is a variable we compute the effects of querying it, compare to opt
        and update opt if needed
    pair=[tt[0]+tt[2]+tt[3],tt[1]+tt[4]]
    if (pair[0]-opt[0])*alpha[0]*2**depth<(pair[1]-opt[1])*alpha[1]:
      opt=pair

import fractions
def alpha(dep):
  # finds minimal alpha with rho_alpha(C) <= 0 for all C = max alpha(C) for
      non-empty C
  # by recursively applying check(alpha) that either gives a higher alpha or
      confirms that alpha is optimal
  prep(dep)
  p=0
  q=1
  x=check(p,q)
  while (x != [0,0]):
    f=fractions.Fraction(x[1],x[0]*(2**dep))
    p=f.numerator
    q=f.denominator
    x=check(p,q)
  print("For depth ",dep,", optimal alpha is ",f)
  print("leading to a lower bound of (",2+(p/q)**(-1/dep),")^h")
```


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