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Flip-Pushdown Automata with k Pushdown Reversals and E0L Systems are Incomparable

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Abstract We prove that any propagating E0L system cannot generate the language $\{w\#w | w \in \{0,1\}^*\}$. This result, together with some known ones, enable us to conclude that the flip-pushdown automata with k pushdown reversals (i.e. the pushdown automata with the ability to flip its pushdown) and E0L systems are incomparable. This result solves an open problem stated in [3].

Keywords: Formal languages, E0L systems, pushdown automata

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1 Introduction

A flip-pushdown automaton, introduced by Sarkar [7], is an ordinary one-way pushdown automaton with the ability to flip its pushdown during the computation. It is known [7] that the flip-pushdown automata without any limit on the number of flips are equally powerful to Turing machines.

Holzer and Kutrib [3, 4] have shown that $k+1$ pushdown reversals are more powerful than k for deterministic and nondeterministic flip-pushdown automata, and, nondeterminism is more powerful than determinism for flip-pushdown automata with constant number of flips. Moreover, they considered in [3] some closure properties and as well as some computational problems of these language families. However, they left some problems considered in [3] open. Among others, what is the relationship between E0L (or ET0L) languages and the languages accepted by flip-pushdown automata with constant number of flips? (We define E0L systems below; for more information, see [6].) Although they have proved that the E0L language $\{a^n b^n c^n | n \geq 0\}$ cannot be recognized by such automata [3], they left open the second part of the problem.

In this paper we complete solution of the problem above by showing that the language $\{w\#w | w \in \{0,1\}^*\}$ cannot be derived by any (propagating) E0L system, but on the other hand, it can be accepted by a pushdown automaton with one flip. To show that $\{w\#w | w \in \{0,1\}^*\}$ is not E0L language, we use a proof technique (see the proof of Theorem 3 below) that is quite different from techniques based on combinatorial properties of languages (see [2, 6]).

Note that, in [1, 2, 5, 6], one can find several quite simple languages that are known to be not E0L languages, but it is not clear whether any of them is suitable for our purposes, (i.e. acceptable by a flip-pushdown automaton with constant number of flips).

2 Definitions

By $|M|$ we denote cardinality of a set M, by $|x|$ we denote the length of a word x and by λ we denote the empty word.

Definition 1. *An E0L system ia a quadruple $G = (\Sigma, P, \omega, \Delta)$, where Σ is a nonempty finite alphabet, $\omega \in \Sigma^*$, P is a finite set of productions of the* *form $\alpha \to \beta$, $\alpha \in \Sigma, \beta \in \Sigma^*$, and $\Delta \subseteq \Sigma$. If $\beta \neq \lambda$ for each production, then G is called propagating.*

Definition 2. *Let G be an E0L system $G = (\Sigma, P, \omega, \Delta)$. A derivation D in G is a triple (Θ, ν, p), where Θ is a finite set of ordered pairs of non-negative integers (the occurrence in D), ν is a function from Θ into Σ ($\nu(i,j)$ is the value of D at occurrence (i,j)), and p is a function form Θ into P ($p(i,j)$ is production of D at occurrence $\nu(i,j)$) satisfying the following conditions. There exist a sequence of words $\alpha_0, \alpha_1, \ldots, \alpha_t$ in Σ^* (called the trace of D) such that $t \geq 1$ and*

(i) $\Theta = \{(i,j) | 0 \leq i < t$ *and* $1 \leq j \leq |\alpha_i|\}$,

(ii) $\nu(i,j)$ *is the* j*th symbol in* α_i,

(iii) for $0 \leq i < t, \alpha_{i+1} = \delta_1 \delta_2 \ldots \delta_{|\alpha_i|}$, *where* $p(i,j)$ *is the production* $\nu(i,j) \to \delta_j$ *for* $1 \leq j \leq |\alpha_i|$.

In such a case D is said to be a *derivation* of α_t from α_0, and t is called the *length* of the derivation D, and we will write $\alpha_0 \Rightarrow_G^t \alpha_t$. Formally, $\alpha \Rightarrow_G^0 \alpha$ for each $\alpha \in \Sigma^*$. (We will omit the subscript G [the superscript t] if G is clear from the context [if $t = 1$].)

We will say that a language L is generated by G if

$$L = \{x | x \in \Delta^*, \omega \Rightarrow_G^t x \text{ for some } t \geq 0\}.$$

For some i, ($0 \leq i < t$), let $\alpha_i = \gamma_1 \gamma_2 \ldots \gamma_{|\alpha_i|}$, ($\gamma_j \in \Sigma$ for $1 \leq j \leq |\alpha_i|$), and let $\alpha_{i+1} = \delta_1 \delta_2 \ldots \delta_{|\alpha_i|}$ be from (iii) above. If $1 \leq d \leq h \leq |\alpha_i|$, hen we will say that the word $\delta_d \delta_{d+1} \ldots \delta_h$ with the position $(i+1, |\delta_1 \delta_2 \ldots \delta_{d-1}| + 1)$ is derived *under* D in one steps from the word $\gamma_d \gamma_{d+1} \ldots \gamma_h$ with the position (i, d).

Let $0 \leq j < m \leq t$ and let $\alpha_i = \alpha_i^I \alpha_i^{II} \alpha_i^{III}$ for some $\alpha_i^I, \alpha_i^{III} \in \Sigma^*, \alpha_i^{II} \in \Sigma^+$ for each i, ($j \leq i \leq m$). If the word α_{i+1}^{II} with the position $(i+1, |\alpha_{i+1}^I| + 1)$ is derived under D in one step from the word α_i^{II} with the position $(i, |\alpha_i^I| + 1)$ for each i, ($j \leq i < m$), then we will say that the word α_m^{II} with the position $(m, |\alpha_m^I| + 1)$ is derived under D in $m - j$ steps from the word α_j^{II} with the position $(j, |\alpha_j^I| + 1)$. (If the positions are clear from the context, then we will omit that information.)

3 Results

Now we will prove our main result: E0L systems and languages accepted by flip-pushdown automata with constant number of flips are incomparable. To do so, we will use the following two known Theorems.

Theorem 1 [3]. *Any flip-pushdown automaton with constant number of flips cannot accept the language* $L_1 = \{a^n b^n c^n | n \geq 0\}$.

Theorem 2 (A reformulated Theorem 2.1 from [6]).
There is an algorithm that given any E0L system generating a language without the empty word produces a propagating E0L systems generating the same language.

Since one can easily construct an E0L system generating L_1 and also construct a flip-pushdown automaton with one flip accepting $L_2 = \{w\#w | w \in \{0,1\}^*\}$, to derive our main result, now it is enough to prove the following

Theorem 3. *Any propagating E0L system cannot generate* L_2.

Before proving Theorem 3, we need the following Lemma 1 and its Corollary.

Lemma 1. *Let* $G = (\Sigma, P, \omega, \Delta)$ *be any propagating E0L system. Let* $\alpha \Rightarrow^s \beta$ *for some* $\alpha, \beta \in \Sigma$ *and some* $s > (|\Sigma|!) \cdot |\Sigma|^2$. *Then* $\alpha \Rightarrow^{s'} \beta$ *for* $s' = s - |\Sigma|!$

Proof: Since G is propagating one, there is a derivation D with a trace $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_s = \beta$, where $\alpha_i \in \Sigma$ for $0 \leq i \leq s$. For every $j = 0, 1, 2, \ldots, (|\Sigma|!) \cdot |\Sigma| - 1$, the sequence $\alpha_{t_j}, \alpha_{t_j+1}, \alpha_{t_j+2}, \ldots, \alpha_{t_j+|\Sigma|}$, where $t_j = j|\Sigma|$, must contain two elements $\alpha_{l_j}, \alpha_{m_j}$ with $t_j \leq l_j < m_j \leq t_{j+|\Sigma|}$ and $\alpha_{l_j} = \alpha_{m_j}$. For $i = 1, 2, \ldots, |\Sigma|$, let $B_i = \{j | 0 \leq j \leq (|\Sigma|!) \cdot |\Sigma| - 1, m_j - l_j = i\}$. Since $1 \leq m_j - l_j \leq |\Sigma|$ for each j, there is B_r with $|B_r| \geq (|\Sigma|!) \cdot |\Sigma| / |\Sigma| = |\Sigma|!$ If we modify D so that one segment $\alpha_{l_j+1}, \alpha_{l_j+2}, \ldots, \alpha_{m_j}$ with $j \in B_r$ may be deleted from the trace, (recall $\alpha_{l_j} = \alpha_{m_j}$), then we obtain a derivation of β from α of the length $s - r$. Thus, by deleting $|\Sigma|!/r$ such segments, we obtain a derivation of β from α of the length $s - |\Sigma|!$ □

Corollary. *For* G *there is a constant* $c > 0$ *such that if* $\omega \Rightarrow^h x$ *for some* $x \in \Sigma^+$ *and for some* $h \geq 0$, *then* $\omega \Rightarrow^{h'} x$ *for some* $h' \leq c|x|$.

Proof: Let D be the shortest derivation of x from ω and let d denote its length. Since G is propagating one, there are words $\psi_1, \psi'_1, \psi_2, \psi'_2, \ldots, \psi_t, \psi'_t$ such that

$$\omega = \psi_1 \Rightarrow^{l_1} \psi'_1 \Rightarrow \psi_2 \Rightarrow^{l_2} \psi'_2 \Rightarrow \psi_3 \Rightarrow^{l_3} \psi'_3 \Rightarrow \cdots \Rightarrow \psi_t \Rightarrow^{l_t} \psi'_t = x,$$

where $t \geq 1$, $|\psi_i| = |\psi'_i|$ and $l_i \geq 0$ for $i = 1, 2, \ldots, t$, and $|\psi'_i| < |\psi_{i+1}|$ for $i = 1, 2, \ldots, t-1$, and $d = t - 1 + \sum_{i=1}^{t} l_i$. Suppose to the contrary that $d > c|x|$ for $c = (|\Sigma|!) \cdot |\Sigma|^2 + 2$. Thus, $l_j \geq c - 1$ for some j, since clearly $t \leq |x|$. Let $\psi_j = \delta_1\delta_2 \ldots \delta_m$ and $\psi'_j = \delta'_1\delta'_2 \ldots \delta'_m$, where $m = |\psi_j| = |\psi'_j|$ and $\delta_i, \delta'_i \in \Sigma$ for $i = 1, 2, \ldots, m$. Since G is propagating one, $\delta_i \Rightarrow^{l_j} \delta'_i$ for every i. By Lemma 1, $\delta_i \Rightarrow^{l'_j} \delta'_i$ for every i, where $l'_j = l_j - |\Sigma|!$ Hence $\psi_j \Rightarrow^{l'_j} \psi'_j$. Thus, by modifying D so that ψ_j may derive ψ'_j in l'_j steps (instead of l_j steps) we obtain a derivation of x with the length shorter than d - a contradiction! $\square$

Proof of Theorem 3. The structure of the proof is as follows. The main idea of the proof is formulated in Claim 3 and explained in its proof. To apply Claim 3 in the proof of Theorem 3, we firstly have to prove Claim 1 and Claim 2.

Suppose to the contrary that there is an E0L propagating system $G = (\Sigma, P, \omega, \Delta)$ generating L_2. Let us choose n large enough (it will be specified later how large it should be). For every nonnegative integer m, let

$$L_{n,m} = \{uvu\#uvu | u, v \in \{0,1\}^n, \ \omega \Rightarrow^m uvu\#uvu\}.$$

Note that, $L_{n,m} \subseteq L_2$. Clearly, $L_{n,m} = \emptyset$ for $m \leq 1$ and n large enough. Hence, by Corollary, there is l, $2 \leq l \leq c(6n+1)$, such that

$$|L_{n,l}| \geq 2^{2n}/(c(6n+1)), \tag{1}$$

for n large enough, where c is from Corollary for G. Fix such l and let

$$k = \max(\{|\delta| : \gamma \to \delta \text{ is in } P\} \cup \{|\omega|\}). \tag{2}$$

We will say that a word $x \in L_{n,l}$ is *wide*, if there is j, $(0 \leq j < l)$ and there are words $\delta_1, \delta_3 \in \Sigma^*$, a symbol $\delta_2 \in \Sigma$, and words $x_1, x_3 \in \Delta^*, x_2 \in \Delta^+$ such that $x = x_1x_2x_3$, $\omega \Rightarrow^j \delta_1\delta_2\delta_3$, $\delta_i \Rightarrow^{l-j} x_i$ for $i = 1, 2, 3$, $|x_2| \geq n/k$ and x_2 does not contain #.

Claim 1. *$L_{n,l}$ contains at most $|L_{n,l}|/2$ wide words.*

Proof: Let x be any wide word with the values $j, \delta_1, \delta_2, \delta_3, x_1, x_2, x_3$ as above. First, let us prove that there is no any other word $x_2' \neq x_2$, $x_2' \in \Delta^*$, with $\delta_2 \Rightarrow^{l-j} x_2'$, since otherwise $\omega \Rightarrow^j \delta_1\delta_2\delta_3 \Rightarrow^{l-j} x_1x_2'x_3$. But $x_1x_2'x_3$ cannot belong into $L_2 = \{w\#w | w \in \{0,1\}^*\}$, because for any two words $y_1, y_3 \in \Delta^*$, from which exactly one of them contains #, there is at most one word $y_2 \in \Delta^*$ such that $y_1y_2y_3 \in L_2$. Hence, the word x_2 is uniquely determined by the couple $(\delta_2, l-j)$.

Since $x_1x_2x_3 = x = uvu\#uvu$ for some $u, v \in \{0,1\}^n$, and since x_2 does not contain # (see the definition of wide words above), there are words $z_0, z_1, z_3, z_4 \in \{0,1\}^*$ such that $z_0z_1x_2z_3z_4 = uvu$, where $|z_1x_2z_3| = 2n$ if $|x_2| < 2n$, and $z_1 = z_3 = \lambda$ if $|x_2| \geq 2n$. Hence the word uvu is uniquely determined by the quadruple $(|z_0|, z_1, x_2, z_3)$, since uvu is a periodic word with the period of the length $2n$, and hence the word uvu is uniquely determined by its subword $z_1x_2z_3$ of the length at least $2n$ and by the position of the subword $z_1x_2z_3$ in the word uvu determined by $|z_0|$. Consequently, also the wide word $x = uvu\#uvu$ is uniquely determined by the same quadruple.

Since x_2 is uniquely determined by the couple $(\delta_2, l-j)$, (see above), the number of all the wide words cannot exceed the number of all 5-tuples of the form $(|z_0|, z_1, \delta_2, l-j, z_3)$, where $0 \leq |z_0| \leq n$, $\delta_2 \in \Sigma$, $1 \leq l-j \leq c(6n+1)$, $|z_1z_3| \leq 2n - n/k$, $z_1, z_3 \in \{0,1\}^*$. But the number of such 5-tuples is at most

$$(n+1) \cdot |\Sigma| \cdot (c(6n+1)) \cdot (2n - n/k + 1) \cdot 2^{2n-n/k+1} \leq 2^{2n-1}/(c(6n+1))$$

for n large enough. By (1), Claim 1 follows. □

Let x be any word in $L_{n,l}$. Let D be any derivation of x from ω of the length l with a trace $\omega = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_l = x$. Since x contains exactly one #, the following values d_i's, β_i's and $y_i\#z_i$'s are uniquely determined by D. For $i = 0, 1, 2, \ldots, l-1$, there is an integer d_i such that a subword of x containing # (denote this subword by $y_i\#z_i$) is derived under D in $l-i$ steps from the d_ith symbol of α_i; let β_i denote this symbol. Thus, (a) hold.

(a) The d_ith symbol of α_i, i.e. β_i derives $y_i\#z_i$ under D in $l-i$ steps for $0 \leq i \leq l-1$.

Again, the fact that x contains exactly one # yields that for $i = 0, 1, 2, \ldots, l-2$, the production that has been applied according to D to the d_ith symbol of α_i has to have a form

$$\beta_i \to \xi_{i+1}\beta_{i+1}\varphi_{i+1}$$

for some $\xi_{i+1}, \varphi_{i+1} \in \Sigma^*$, where $\alpha_{i+1} = \varrho_{i+1}\xi_{i+1}\beta_{i+1}\varphi_{i+1}\tau_{i+1}$ for some $\varrho_{i+1}, \tau_{i+1} \in \Sigma^*$, where $|\varrho_{i+1}\xi_{i+1}\beta_{i+1}| = d_{i+1}$. But it means, together with (a), that for $i = 0, 1, 2, \ldots, l-2$ there are $r_{i+1}, s_{i+1} \in \Delta^*$ such that (b) and (c) hold.

(b) $y_i\#z_i = r_{i+1}y_{i+1}\#z_{i+1}s_{i+1}$,

(c) r_{i+1} $[s_{i+1}]$ is derived from ξ_{i+1} [from φ_{i+1}] under D in $l-i-1$ steps.

Claim 2. *Let x be any word in $L_{n,l}$ that is not wide. Let D be any derivation of x from ω of the length l with the corresponding values $\alpha_i, d_i, y_i, z_i, \beta_i$ for $0 \le i \le l-1$, and $\varphi_i, \xi_i, r_i, s_i$ for $1 \le i \le l-1$, as above. Then there are indices $0 \le p, q \le l-1$ such that $n \le |y_p| \le 2n$ and $n \le |z_q| \le 2n$. Moreover, $|y_i| \ge |y_j|$ and $|z_i| \ge |z_j|$ for $i < j$.*

Proof: One can easily observe that to prove Claim 2, it is enough to show that (3), (4) and (5) hold.

$$0 \le |y_i| - |y_{i+1}| \le n \text{ and } 0 \le |z_i| - |z_{i+1}| \le n \text{ for } 0 \le i \le l-2, \tag{3}$$

$$2n \le |y_0| \text{ and } 2n \le |z_0|, \tag{4}$$

$$|y_{l-1}| \le n \text{ and } |z_{l-1}| \le n. \tag{5}$$

We will prove (3) - (5) only for y_i's. The proof for z_i's is similar.

Choose any i, $0 \le i \le l-2$. Since x is not wide and r_{i+1} does not contain #, (because the subword $y_i\#z_i$ of x contains the unique symbol # and $y_i = r_{i+1}y_{i+1}$, by (b)), then any symbol of ξ_{i+1} cannot derive (during the derivation of r_{i+1} from ξ_{i+1} under D, recall (c)) any subword of r_{i+1} of the length at least n/k. Thus, $|r_{i+1}| \le (n/k)|\xi_{i+1}| \le (n/k)|\xi_{i+1}\beta_{i+1}\varphi_{i+1}| \le (n/k) \cdot k = n$, by (2). This yields (3), since $y_i = r_{i+1}y_{i+1}$, by (b).

Clearly, there are $\xi_0, \varphi_0 \in \Sigma^*$ such that $\omega = \alpha_0 = \xi_0\beta_0\varphi_0$, where $|\xi_0\beta_0| = d_0$, see the selection of d_0, β_0 and $y_0\#z_0$ above. Since $y_0\#z_0$ is a subword of x, there are $r_0, s_0 \in \Delta^*$ with $x = r_0y_0\#z_0s_0$. Similarly, as in (c) above, r_0 $[s_0]$ is derived from ξ_0 [from φ_0] under D in l steps, since D derives $x = r_0y_0\#z_0s_0$ from $\omega = \alpha_0 = \xi_0\beta_0\varphi_0$ in l steps (see above), and $y_0\#z_0$ is derived from β_0 under D in l steps, by (a). Since x is not wide and r_0 evidently does not contain #, then any symbol of ξ_0 cannot derive (during the derivation of r_0 from ξ_0 under D) any subword of r_0 of the length at least n/k. Thus, $|r_0| \le (n/k)|\xi_0| \le (n/k)|\omega| \le (n/k) \cdot k = n$, by (2). Since $x = r_0y_0\#z_0s_0$ is in $L_{n,l}$, then $|r_0y_0| = 3n$. Consequently, $|y_0| \ge 3n - n = 2n$. This yields (4).

By (a), the d_{l-1}th symbol of α_{l-1}, i.e. β_{l-1} derives $y_{l-1}\#z_{l-1}$ under D in one step. It means that $\beta_{l-1} \to y_{l-1}\#z_{l-1}$ is a production of G. Thus, (5) holds, since $|y_{l-1}| \leq |y_{l-1}\#z_{l-1}| \leq k \leq n$ for n large enough, by (2).

This completes proof of Claim 2 (for y's). $\square$

Claim 3. *Let $x = uvu\#uvu$ be any word in $L_{n,l}$ with $|u| = n$ that is not wide. Let $D, \alpha_i, d_i, y_i, z_i, \beta_i$ for $0 \leq i \leq l-1$, and r_i, s_i for $1 \leq i \leq l-1$, and p, q be the values for x from Claim 2. Let $x' = u'v'u'\#u'v'u'$ be any other word in $L_{n,l}$ with $|u'| = n$ that is not wide and let $D', \alpha'_i, d'_i, y'_i, z'_i, \beta'_i$ for $0 \leq i \leq l-1$, and r'_i, s'_i for $1 \leq i \leq l-1$, and p', q' be analogous values for x', (i.e. (a) - (c) above and Claim 2 hold for these values). Let $u \neq u', p = p', q = q', \beta_p = \beta'_{p'}$ and $\beta_q = \beta'_{q'}$. Then there is a word x'' that does not belong into L_2 but that can be derived in G*

Proof: Assume $p \geq q$. (The proof is similar for $q > p$.) Since $y_0\#z_0$ is a subword of x, there are words $r_0, s_0 \in \Sigma^*$ with $x = r_0y_0\#z_0s_0$. Thus, by (b), $r_0r_1 \ldots r_{p-1}r_py_p\#z_ps_ps_{p-1} \ldots s_1s_0 = x = uvu\#uvu$. Hence the suffix of the length n of y_p is u, since $|u| = n$ and $n \leq |y_p|$, by Claim 2. Similarly, $r'_0r'_1 \ldots r'_{p-1}r'_py'_p\#z'_ps'_ps'_{p-1} \ldots s'_1s'_0 = x' = u'v'u'\#u'v'u'$ for some $r'_0, s'_0 \in \Sigma^*$. Claim 2 and the assumptions $p \geq q = q'$ yield $|z'_p| \leq |z'_{q'}| \leq 2n$. Hence, the suffix of the length n of the word $s'_ps'_{p-1} \ldots s'_1s'_0$ is u', since $|s'_ps'_{p-1} \ldots s'_1s'_0| = |u'v'u'| - |z'_p| \geq 3n - 2n = n = |u'|$. By the assumption of Claim 3, $u \neq u'$. Hence, the word $x'' = r'_0r'_1 \ldots r'_{p-1}r'_py_p\#z_ps'_ps'_{p-1} \ldots s'_1s'_0$ cannot belong into L_2, since the words $r'_0r'_1 \ldots r'_{p-1}r'_py_p$ and $z_ps'_ps'_{p-1} \ldots s'_1s'_0$ have different suffixes u and u' of the length n, (see above). But, on the other hand, the word x'' can be derived from ω simply by modifying the derivation D' of x' so that the d'_pth symbol of α'_p, i.e. β'_p, may derive $y_p\#z_p$ (like under D, see (a)) instead of $y'_p\#z'_p$, (recall the assumptions $p = p'$ and $\beta'_{p'} = \beta_p$ of Claim 3). $\square$

In order to complete the proof of Theorem 3, now it is enough to show existence of words x and x' satisfying Claim 3. To do so, we proceed as follows. For each $u \in \{0,1\}^n$, let

$$M_u = \{x | x = uvu\#uvu, v \in \{0,1\}^n, x \in L_{n,l}, x \text{ is not wide}\}.$$

Note that $|M_u| \leq 2^n$ for each u and $M_u \cap M_{u'} = \emptyset$ for $u \neq u'$. Hence, by Claim 1 and by (1), the number of nonempty sets M_u is at least $(|L_{n,l}| - |L_{n,l}|/2)/2^n \geq 2^n/(2c(6n+1))$. For each nonempty set M_u, mark arbitrary one word x in M_u and assign to x the values p, q, β_p, β_q from Claim 2. There are two marked words $x \in M_u$ and $x' \in M_{u'}$ for some $u \neq u'$ with the same values p, q, β_p, β_q, since the number of marked words (i.e. the number of nonempty sets M_u) is greater than the number of all possible different quadruples (p, q, β_p, β_q), which is at most $l^2|\Sigma|^2 \leq c^2(6n+1)^2|\Sigma|^2 < 2^n/(2c(6n+1))$ for n large enough.

This completes the proof of Theorem 3. $\square$

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