

# On the Complexity of Group Isomorphism

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July 3, 2011

## Abstract

The group isomorphism problem consists in deciding whether two groups  $G$  and  $H$  given by their multiplication tables are isomorphic. An algorithm for group isomorphism attributed to Tarjan runs in time  $n^{\log n + O(1)}$ , c.f. [Mil78].

Miller and Monk showed in [Mil79] that group isomorphism can be many-one reduced to isomorphism testing for directed graphs. For groups with  $n$  elements, the graphs have valence at least  $n$ . We introduce a different reduction embedded in isomorphism tests, where the valence of the graphs, say  $X(G)$  and  $X(H)$ , and the complexity of the isomorphism test is closely related to the structure of the groups.

When given additionally to the input a composition series  $S$  for  $G$  and  $S'$  for  $H$ , then in the reduction we get graphs  $X(G, S)$ ,  $X(H, S')$  where an isomorphism  $\phi$  from  $X(G, S)$  onto  $X(H, S')$  gives an isomorphism  $\phi'$  from  $G$  onto  $H$  such that  $\phi'$  maps the subgroups in  $S$  blockwise onto those in  $S'$ . Then for  $p$ -groups, the reduction runs in time  $n^{cp}$  and for general groups in time  $n^{c \log n / \log \log n}$ , for a constant  $c$ .

Conversely, finding such composition series for the groups, i.e., such that there are isomorphisms that respect them, is as hard as the group isomorphism problem. We analyze the time complexity for three isomorphism testing algorithms with respect to two parameters which depend on the group structure.

## 1 Introduction

Two groups  $G, H$  with ground set  $\Omega = \{1, \dots, n\}$  are *isomorphic* if there is a mapping  $\phi : \Omega \rightarrow \Omega$  such that  $\phi(i) \cdot \phi(j) = \phi(i \circ j)$  where we assume that  $\circ$  is the group operation in  $G$  and  $\cdot$  in  $H$ . The *group isomorphism problem* is to decide whether two groups  $G$  and  $H$  are isomorphic. We consider finite groups given in *table representation*, i.e. a table of  $n \times n$  entries where the entry  $(i, j)$  contains the product  $ij$ .

The complexity of the group isomorphism problem has been studied for more than three decades. When finite groups are given by their generating sets then graph isomorphism is polynomial time Turing reducible to the permutation group isomorphism problem which is known to be in  $\text{NP} \cap \text{coAM}$  [Luk93].

In [Mil79], isomorphism testing of explicitly given structures is many-one reduced to directed graph isomorphism. Groups can also be represented in multiplication tables and equivalently as a ternary relation  $R$ , i.e.  $(a, b, c) \in R$  iff  $ab = c$ . As a corollary, group isomorphism can be reduced to graph isomorphism. In this reduction, the graphs have unbounded valence.

The following algorithm for isomorphism testing on two groups  $G, H$  with  $n$  elements given in table representation runs in time  $n^{\log n + O(1)}$ : compute a generating set  $A$  of size  $\log n$  for  $G$ . Then try all mappings from  $A$  bijectively onto each possible subset  $A'$  of  $H$ . There are  $\binom{n}{|A|} \cdot |A|!$  many such ordered sets. With the multiplication table of  $G$ , check for each such map whether it extends to an isomorphism from  $G$  onto  $H$ . We start with  $A$  and  $A'$  as partial ordered sets and recursively check whether  $\phi(i) \cdot \phi(j) = \phi(i \circ j)$  is consistent with  $A$  and  $A'$ . We extend the partial ordered sets whenever  $\phi(i \circ j)$  is a new element. This algorithm is attributed to Tarjan, c.f. [Mil78] and it is improved by Lipton, Snyder and Zalcstein [LSZ76] as a sharper  $O(\log^2 n)$  space algorithm.

Arvind and Torán [AT04] give a 2-round Arthur-Merlin protocol for the group non-isomorphism problem such that Arthur and Merlin use  $O(\log^6 n)$  random bits and  $O(\log^2 n)$  non-deterministic bits, respectively. They derandomize this protocol in the case of solvable groups.

For abelian groups when given as generating sets, isomorphism can be tested in linear time [Kav03], it is hard for ModL and contained in  $\text{ZPL}^{\text{ModL}}$  [AV04]. When given in multiplication tables then isomorphism testing is trivially in L, and also in  $\text{TC}^0(\text{FOLL})$  (see [CTW10]). On the one side, not far from abelian appear the hardest cases, namely nilpotent groups of class 2. For these groups, the center  $Z(G)$  and the quotient  $G/Z(G)$  are abelian. On the other side, Babai et al. [BCGQ11] consider groups without abelian normal subgroups. They prove that isomorphism testing for this class of groups is in time  $n^{O(1) + c \log \log n}$ . With parameter  $t(G)$ , defined to be the smallest  $t$  such that each minimal normal subgroup of  $G$  has at most  $t$  simple groups, isomorphism testing is in time  $n^{O(1) + c \log(t(G))}$  for a constant  $c$ . Quiao, Sarma and Tang [QMT11] present a framework to test isomorphism of groups with at least one normal Hall subgroup, when groups are given in multiplication tables.

**Group decomposition.** It is well known that a composition series even for permutation groups can be obtained in polynomial time [Luk87] and also in NC [BLS87]. For a group  $G$  it can be obtained as follows: First, compute the *socle*  $\text{Soc}(G)$  of  $G$ , this is a normal subgroup isomorphic to a direct product of simple groups. The factor group  $G/\text{Soc}(G)$  is then decomposed recursively. For groups with  $n$  elements, the composition series has length at most  $\log n$ .

Group decomposition is central to our isomorphism testing algorithms, also see Section 4. Initially, we do not decompose minimal normal subgroups and we guess an arrangement for those which are isomorphic within the socle. This gives so far a normal series  $S$  with semisimple factor groups. Let  $\text{seq}(S)$  denote the sequence of factor groups given by  $S$ . We consider two parameters which depend on the decomposition of the groups (also see Definition 4.7).

- The socle consists of a direct product of minimal normal subgroups. Let  $\beta$  be the maximum number of such isomorphic minimal normal subgroups in any socle of this decomposition process.
- The minimal normal subgroups are a direct product of isomorphic simple groups. Let  $\gamma$  be the maximum composition length of any minimal normal subgroup in any socle of this decomposition process. Hence, there is a minimal normal subgroup where we need at least  $\gamma$  group elements to generate it.

**Our contribution.** We show a reduction from group isomorphism onto graph isomorphism where the valence of the graphs depends on the group structure. We compare three isomorphism testing algorithms and analyze their time-complexity together with the parameters  $\beta$  and  $\gamma$ .

1. *Tarjan's algorithm when given a normal series as above.* Isomorphism testing for two groups  $G$  and  $H$  can be done in two steps.

First, we compute a normal series  $S$  for  $G$  and  $S'$  for  $H$  as above such that if  $G$  is isomorphic to  $H$  then also  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$  respecting the structure, i.e. mapping the subgroups in  $S$  of  $G$  blockwise onto those in  $S'$  of  $H$ .

Second, we fix coset representatives for each of the factor groups in  $S'$  as generators for  $H$ . Note, for each factor group we may need more than one element to generate it. For example, if a factor group is a direct product of  $k$  pairwise isomorphic cyclic groups then we have  $k$  generators (and  $2k$  generators if we have non-abelian simple groups instead), all these are taken as coset representatives.

Then we run through all possibilities to select coset representatives with respect to  $S$  as generators for  $G$ . If the mapping of the generators induces an isomorphism from  $G$  onto  $H$  then we accept, otherwise we reject. Let the factor groups in  $S$  be direct products of simple groups of size at most  $p$ . Then this algorithm runs in time  $n^{2\log_p n}$ .

2. *Reduction to graph isomorphism.* This algorithm also has two steps. The first step is as before, we run through all arrangements of  $\beta$  pairwise isomorphic minimal normal subgroups within a socle. In total, this can be done in time  $n^{\log \beta}$ .

The second step is different: We many-one reduce group isomorphism to graph isomorphism where the valence of the graphs is bounded by  $p+1$ . For this we run through all possibilities to select coset representatives for the factor groups as generators. This number is at most  $n^{c\gamma}$ , for a constant  $c$ . In the reduction, we first construct a complete tree where each node has  $p$  children. We have several copies of this tree and also of a further graph gadget to encode the group multiplications. Isomorphism testing for valence- $d$  graphs is in time  $n^{O(d)}$  [BL83]. Let the simple groups of the factor groups in  $S$  have size at most  $p$ . The group isomorphism algorithm runs then (for a constant  $c$ ) in time  $n^{c(\gamma+\log \beta+(p+1))}$ .

3. *Combination of both algorithms.* We distinguish between those minimal normal subgroups which are a direct product of simple groups of size  $> \alpha$  and the others. We take the first algorithm to find generators for the minimal normal subgroups in the case  $> \alpha$  and we take the second algorithm for the remaining minimal normal subgroups. For a constant  $c$ , the algorithm runs in time  $n^{c(\gamma+\log \beta)}(n^{c\log_\alpha n} + n^{c(\alpha+1)})$ .

The first algorithm runs faster if the composition length of  $G$  is small. The second algorithm is an improvement especially for groups where the decomposition has factor groups of small size (e.g. for  $p$ -groups where  $p$  is a small prime) *and* where also  $\gamma$  is small. The runtime of the third algorithm becomes minimal if we set  $\alpha = \log n / \log \log n$ . In Section 6 we prove the following theorem.

**Theorem 1.1** *Group isomorphism for groups with  $n$  elements given in table representation is in time (for a constant  $c$ ):*

$$n^{c(\gamma+\log \beta+\log n/\log \log n)}$$

In Section 3 we explain the reduction. In Section 5, we introduce the algorithm for  $p$ -groups.

**Theorem 1.2** *Group isomorphism for  $p$ -groups with  $n$  elements given in table representation is in time (for a constant  $c$ ):*

$$n^{c(p+\gamma+\log \beta)}$$

We conclude, when given composition series together with the groups then finding an isomorphism which respects them is a slightly easier task.

**Corollary 1.3** *Group isomorphism for groups  $G$  and  $H$  with  $n$  elements when given in table representation together with composition series  $S$  and  $S'$  which have to be respected is in time*

for groups (with constant  $c$ ):  $n^{c(\log n / \log \log n)}$   
and for  $p$ -groups:  $n^{cp}$ .

## 2 Preliminaries

**Groups.** A group  $G = (\Omega, \circ)$  is a set  $\Omega$  together with an operation  $\circ$ , i.e. a 2-ary function, which satisfy the axioms of closure and associativity,  $G$  has a unique identity element  $e$ , and unique inverse elements. We also write in short  $g \in G$  and mean that  $g \in \Omega$  and we write for a product  $g \circ h$  in short  $gh$ . We consider finite groups where  $\Omega$  consists of  $n$  elements, i.e. we say that  $G$  has *order* or *size*  $|G| = n$ .

For an integer  $i$ ,  $g^i$  is the element  $g \in \Omega$  multiplied  $i$  times with itself. If  $g^i = e$  for the smallest  $i \geq 1$ , then  $i$  is the *order* of  $g$  in the group  $G$ , in short we also write  $\text{ord}(g) = i$ . The set  $\{g, g^2, \dots, g^{i-1}\}$  is denoted the *powers* of  $g$ . The element  $g^{-1}$  is the *inverse* element of  $g$ , it satisfies the equation  $g^{-1}g = e$ .

We write  $H \leq G$  to denote that  $H$  is a *subgroup* of  $G$ . We use the following notion  $G \setminus H = \{g \in G \mid g \notin H\}$  which has a different meaning than that of factor groups  $G/H$  defined below.

Let  $g \in G$ , then  $gH = \{gh \mid h \in H\}$  is a *left coset* of  $H$  in  $G$ , and  $Hg = \{hg \mid h \in H\}$  is a *right coset* of  $H$  in  $G$ . Any pair of left cosets (resp. right cosets) have the property that they contain either exactly the same set of elements or are disjoint. We write  $G$  as the union of its cosets

$$G = H + g_2H + \dots + g_rH$$

to indicate that the cosets  $H, g_2H, \dots, g_rH$  are disjoint and exhaust  $G$ . The elements  $g_2, \dots, g_r$  are the *coset representatives*, these are arbitrary elements from each coset. We write  $H$  in short for  $eH$ , and assume that the identity  $e$  is the representative for  $H$ . We use these notions also when given left cosets. A set of representatives of all the cosets is called a *transversal*.

A group given in *table representation*, also denoted *Cayley group*, consists of a multiplication table of size  $n$  by  $n$  filled with numbers in the range from 1 to  $n$ . The total size is  $n^2 \log n$  bits. A set of elements  $S = \{g_1, \dots, g_k\}$  of  $G$  is a *generating set* for a subgroup  $H \leq G$  if every  $g \in H$  can be expressed as a product of elements from  $S$ , we also write  $\langle S \rangle = H$ .

The *direct product* of two groups  $G = (\Omega, \cdot)$  and  $H = (\Omega', *)$ , denoted  $G \times H$ , is a group with element set  $\{(g, h) \mid g \in G, h \in H\}$  and an operation  $\circ$  defined elementwise  $(g, h) \circ (g', h') = (g \cdot g', h * h')$ .

A group is *commutative* or *abelian* if for all  $g, h \in G$ :  $gh = hg$  holds. An abelian group is isomorphic to the direct product of cyclic groups.

A subgroup  $H$  of  $G$  is said to be *normal* if for all  $g \in G$ ,  $gH = Hg$  and we write  $H \triangleleft G$ .  $H$  is a *minimal normal subgroup* of  $G$  if there is no other normal subgroup of  $G$  contained in  $H$ . A group is *simple* if it does not have non-trivial normal subgroups. A group is *semisimple* if it is the direct product of simple groups. The *socle* of  $G$  is a subgroup generated by all minimal normal subgroups, it is denoted  $\text{Soc}(G)$ . The socle is semisimple and it is a normal subgroup. We shall take the cosets  $Hg_i$  as the elements of a system  $K$ . We define the product in  $K$  as  $(Hg_i)(Hg_j) = Hg_k$  if  $g_i g_j \in Hg_k$  in  $G$ . If  $H$  is normal, then  $K$  is a group (c.f. [Hal99], p. 27). The product depends solely on the

cosets and not on the choice of the representatives.  $K$  is a group which we call the *factor group* or *quotient group* of  $G$  with respect to  $H$  and we write  $K = G/H$ .

A *normal series* of a group  $G$  is a finite sequence of subgroups  $G_1, \dots, G_k$  with

$$\{1\} = G_k \triangleleft G_{k-1} \triangleleft \dots \triangleleft G_1 = G.$$

A normal series is a *composition series* if for each  $i$ ,  $G_{i+1}$  is a proper normal subgroup of  $G_i$  and each factor group  $G_i/G_{i+1}$  is simple. The Jordan-Hölder Theorem ([Hal99], Theorem 8.4.4) states that if  $\{1\} = G_k \triangleleft G_{k-1} \triangleleft \dots \triangleleft G_1 = G$  and  $\{1\} = H_k \triangleleft H_{k-1} \triangleleft \dots \triangleleft H_1 = G$  are two composition series  $S_1$  and  $S_2$  for  $G$  respectively, then the factor groups  $G_i/G_{i+1}$  are isomorphic to  $H_{\pi(i)}/H_{\pi(i+1)}$  in some ordering  $\pi$  of the indices. Note, two non-isomorphic groups could have composition series with isomorphic composition factors.

**Definition 2.1** A complete set of coset representatives with respect to  $S_1$  is a sequence of tuples of group elements  $\vec{s} = (s_1, \dots, s_{k-1})$  with  $s_i = (a)$  (or  $s_i = (a, b)$ ) such that  $a$  (or  $a, b$ ) generate the simple factor group  $G_i/G_{i+1}$  that is cyclic (or non-abelian, respectively) and  $a, (b) \in G_i \setminus G_{i+1}$ , for each  $i \in \{1, \dots, k-1\}$ . All these group elements are a generating set for the group  $G$ .

We generalize these notions for normal series with semisimple factor groups. If  $G_i/G_{i+1}$  is semisimple then we define  $s_i$  to be a tuple of generators.

The *center* of a group  $G$  is the set of elements which commute with all elements of  $G$ , i.e.  $Z(G) = \{z \in G \mid \forall g \in G, gz = zg\}$ . This set forms a commutative subgroup of  $G$ . The *commutator* of two elements  $g, h \in G$  is the element  $[g, h] = g^{-1}h^{-1}gh$ . A *commutator subgroup* or *derived subgroup* is the group  $[G, G]$  generated by all the commutators  $\{[g, h] \mid g, h \in G\}$ . When iterating this, we get the *derived series* with  $G^{(0)} = G, G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ . It is a descending normal series  $G^{(r)} \triangleleft \dots \triangleleft G^{(1)} \triangleleft G^{(0)} = G$ . If  $G^{(i+1)} = G^{(i)}$  is non-trivial then this series terminates in a *perfect group*, i.e. it is equal to its own commutator subgroup. If  $G^{(i)} = \{1\}$  the trivial group, then the smallest such  $i$  is called the *derived length*. If all factor groups are commutative, then  $G$  is called *solvable*.

In contrast to the derived series, the *lower* or *descending central series* is defined  $G_r \triangleleft \dots \triangleleft G_2 \triangleleft G_1 = G$  where  $G_2 = [G, G]$  and  $G_{i+1} = [G_i, G]$ . Here  $G_{i+1}$  is a normal subgroup of  $G_i$  and the factor group  $G_i/G_{i+1}$  is cyclic, for all  $i$ . If the lower central series terminates in the trivial group  $G_i = \{1\}$  then  $G$  is *nilpotent*. Nilpotent groups are solvable, the converse does not hold. The smallest such  $i$  defines the *nilpotency class* of  $G$ .

The *conjugacy class* of a group element  $g \in G$  is the set  $cl(g) = \{h^{-1}gh \mid h \in G\}$ . The *normal closure* of a group element  $g \in G$  is the group  $ncl_G(g) = \langle cl(g) \rangle$ , i.e. a normal subgroup of  $G$  generated by all elements of the conjugacy class  $cl(g)$ . Note,  $ncl_G(g)$  is the smallest normal subgroup of  $G$  that contains  $g$ .

A *permutation* is a bijective mapping among elements in  $\Omega$ . The set of all permutations together with composition forms a group, the *symmetric group*  $Sym(\Omega)$ . An *automorphism*  $\phi \in Sym(G)$  is a permutation over group elements such that  $\phi(g)\phi(h) = \phi(gh)$ . The *automorphism group*  $Aut(G)$  of a group  $G$  is a group with automorphisms as elements.

**Graphs.** A *graph*  $G$  is a pair  $(V, E)$  with a set of *vertices*  $V = V(G)$  and *edges*  $E = E(G) \subseteq V \times V$ . We consider *simple* graphs, i.e. with undirected edges and without loops and multiedges. The *size* of a graph is the number of its vertices. The *distance* between two vertices  $u, v$  in a graph  $G$  is the length of the shortest path between  $u$  and  $v$ .

The *degree* or *valence* of a vertex  $v$  in a graph  $G$  is the number of edges which have  $v$  as end vertex. The *valence of a graph* is the maximum valence of its vertices.

A graph is *connected* if there is a path between any two vertices. A graph is a *tree* if it is connected and does not have a simple cycle. A *root* of a tree is a designated vertex. A *rooted tree* is a tree with a root. Let  $(u, v, \dots, r)$  be a simple path from  $u$  to the root  $r$  in a rooted tree. Then  $v$  is the *parent* of  $u$  and  $u$  is a *child* of  $v$ .

An *isomorphism* between graphs  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  is a bijective mapping  $\phi : V \rightarrow V$  such that  $\{u, v\} \in E_1$  if and only if  $\{\phi(u), \phi(v)\} \in E_2$ . Both graphs are *isomorphic* ( $G_1 \cong G_2$ ) if such an isomorphism exists. An *automorphism* of graph  $G$  is a permutation  $\phi : V(G) \rightarrow V(G)$  preserving the adjacency relation:  $\{u, v\} \in E(G) \Leftrightarrow \{\phi(u), \phi(v)\} \in E(G)$ . A *rigid* graph has no automorphisms except the identity.

**Complexity.** For the following and further complexity theoretic notions we refer to standard textbooks, for example [Pap94]. The class P (or NP) contains languages accepted by a (non-) deterministic Turing machine with polynomial time bound. The class L (or NL) contains the languages accepted by a (non-) deterministic Turing machine where the work-tape is restricted to  $O(\log n)$  bits. Non-determinism means, that the machines are allowed to *guess* bits while computing the solution and verify it within the restricted resource bounds. The class  $AC^i$  contains the languages accepted by a DLOGTIME uniform family of Boolean circuits of depth  $O(\log^i n)$  and size polynomial in  $n$ , with unbounded fanin *and-gates* and *or-gates*. The class  $NC^i$  is  $AC^i$  where *and-gates* and *or-gates* have bounded fanin (i.e. fanin two).  $NC = \bigcup_i NC^i$ . The following containments are known:

$$AC^0 \subset NC^1 \subseteq L \subseteq NL \subseteq AC^1 \subseteq NC^2 \subseteq \dots \subseteq NC \subseteq P \subseteq NP$$

A language  $L$  is  $AC^0$  *many-one reducible* (in short  $\leq_m^{AC^0}$ ) to a language  $L'$  if there is a total function  $f$  computable in  $AC^0$  so that for all words,  $w \in L_1$  if and only if  $w \in L_2$ . We consider reducibility with respect to DLOGTIME uniformity. The notions L, NC or P many-one reducibility are defined accordingly.

### 3 Reduction to Graph Isomorphism

In this section we reduce group isomorphism onto graph isomorphism. Following the reduction in [Mil79], for groups with  $n$  elements the valence of the graphs is at least  $n$ . We introduce a different reduction to graph isomorphism. If we break up the group into smaller pieces, i.e. characteristic subgroups, then in our construction the valence of the constructed graphs can be reduced.

**The construction.** Let  $G$  be a group with  $n$  elements in table representation. The reduction goes in two steps.

First, we define a graph  $T(G)$  as follows: For every group element  $g \in G$  there is an *element vertex*  $g$  in  $T(G)$ . There is a *root vertex*  $eG$ , it is connected to all element vertices in  $T(G)$ .

Second, from this tree we construct a graph  $X(G)$ .  $X(G)$  contains a main copy of  $T(G)$ . For every element vertex  $g$  in  $T(G)$ , we have a further copy  $T_g$  of  $T(G)$ . The root of  $T_g$  is identified with  $g$ . The leafs of  $T_g$  are connected to graph gadgets which encode the multiplication in  $G$ . Figure 1 shows the construction for a multiplication graph gadget  $M_{gh=k}$ . Hence, each node  $g^{(h)}$  is

connected to three multiplication graph gadgets, namely when  $g$  is multiplied with  $h$  in this order, i.e. to vertex  $x_{gh=k}$  in  $M_{gh=k}$ , and vice versa, i.e. to vertex  $y_{hg=i}$  in  $M_{hg=i}$ , and when  $g$  is the result of a multiplication with  $h$ , i.e. to vertex  $z_{jh=g}$  in  $M_{jh=g}$ , for corresponding group elements  $i, j \in G$ .

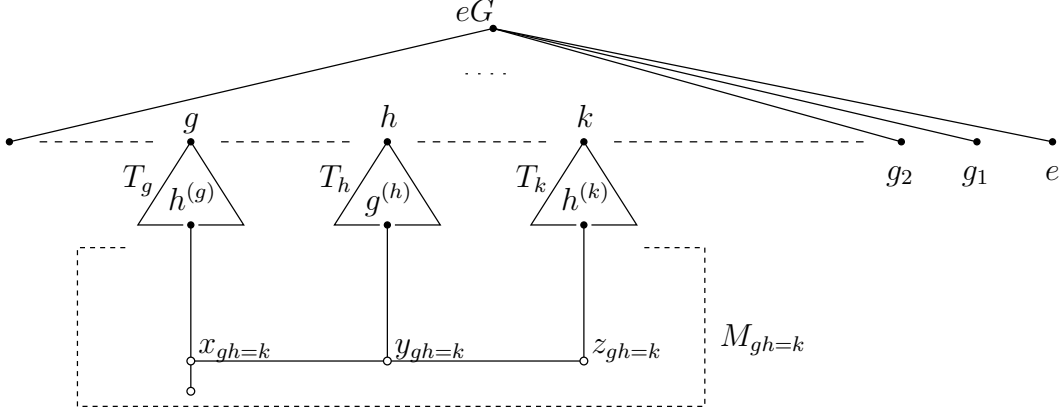


Figure 1: The graph  $X(G)$  with element vertices  $e, g_1, g_2, g, h, k$  is shown. A multiplication graph gadget  $M_{gh=k}$  is indicated by white nodes.

**Claim 3.1** *Every automorphism in  $G$  can be directly transformed into an automorphism in  $X(G)$ .*

**Proof.** To see this, the root node  $eG$  is the only node with valence  $n$  and so it is fixed. Hence, the vertices at each level in the construction are mapped onto each other. When the element vertices of the main copy  $T(G)$  in  $X(G)$  are fixed, then the whole graph is fixed. That is, because for every triple  $g, h, k \in G$  with  $gh = k$  there is a unique multiplication graph gadget and hence, there are unique paths from  $g$  to  $h$  and to  $k$ . Hence,  $g^{(h)}, h^{(g)}, k^{(h)}$  are fixed. Since every vertex at distance two to the root  $eG$  is connected to at least one multiplication graph gadget, all nodes of this level are fixed. The multiplication graph gadgets are rigid.  $\square$

**Refinement of groups.** The graph  $X(G)$  has vertices with valence at least  $n$ . We construct a new tree  $T'(G)$ . Let  $X'(G)$  be the graph  $X(G)$  where we replace each copy of  $T(G)$  by  $T'(G)$ . The goal is that in  $X'(G)$  the nodes have smaller valence but the automorphism properties of  $X'(G)$  are the same as in  $X(G)$ .

Let  $N$  be a characteristic subgroup in  $G$ . We get a normal series, namely  $\{1\} \triangleleft N \triangleleft G$ . Every group element  $g \in G$  can be written as a product  $h_1n$  where  $n \in N$  and  $h_1 \in h_1N$  is a coset representative. Let  $h_1, \dots, h_k$  be a complete set of coset representatives.

We define a new tree  $T'(G)$  as follows.

- For every group element  $g \in G$  with  $g = h_i n$  ( $i \in \{1, \dots, k\}$ ) there is an *element vertex*  $h_i n$  in  $T'(G)$ .
- For every coset representative  $h_1, \dots, h_k$  there is a vertex  $h_i N$  in  $T'(G)$ .
- There is a *root vertex*  $eG$  connected to the vertices  $h_1 N, \dots, h_k N$  in  $T'(G)$ .

The rest of the construction of  $X'(G)$  is the same as for  $X(G)$ . In particular, the tree copies connected to the leafs of  $T'(G)$  are copies of  $T'(G)$ .

**Remarks.** We use the property that  $N$  is characteristic and that the cosets  $h_1N, \dots, h_kN$  are mapped blockwise onto each other and that  $eN$  is blockwise fixed. The tree  $T'(G)$  respects these automorphisms. The valence of the graph is now the maximum of  $1+k$  and  $1+n/k$  where  $k$  is the number of cosets and  $|N| = n/k$ . Note, here we neglect that e.g.  $g^{(h)}$  has at least 4 neighbors. We remedy this later with minor changes to the construction. Figure 2 shows an example.

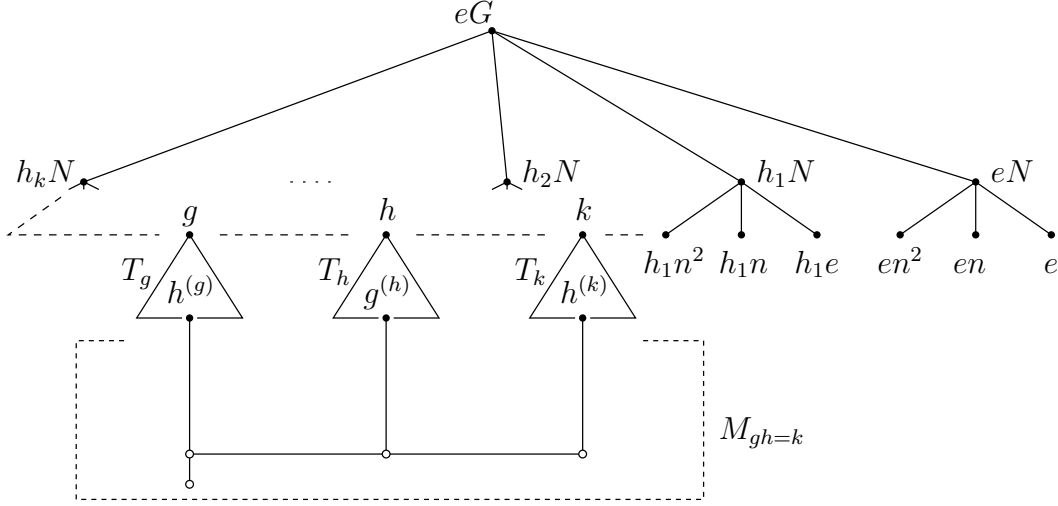


Figure 2: The graph  $X'(G)$  is shown with coset representatives  $e, h_1, \dots, h_k$  and with normal subgroup  $N$ , a cyclic group of three elements. The labels of the element vertices are written as products with an element in  $N$  and a coset representative.

Another important point is, that this construction does not depend on the selection of the coset representatives. For example, take  $h'_1 \in h_1N$  instead of  $h_1$  as coset representative. Since  $h'_1 \in h_1N$  there exists  $n \in N$  such that  $h'_1 = h_1n$  or equivalently  $h_1 = h'_1n^{-1}$ . Every element  $h_1n'$  can be written as  $(h'_1n^{-1})n' = h'_1(n^{-1}n')$ . Hence, taking a different element as coset representative implies a rearrangement among the children of the node  $h_1N$  in  $T'(G)$ . Also if  $h'_1 \in h_iN$  instead of  $h_1N$  then this means that we also permute the cosets. This implies a rearrangement among the nodes  $h_1N, \dots, h_kN$  in  $T'(G)$ .

We repeat the decomposition of  $N$  and the factor group  $G/N$  to further reduce the valence. That is, in the tree  $T'(G)$  we add new intermediate layers. Hence, we end up in a normal series where each of the subgroups is characteristic and the factor groups are semisimple. The valence depends on the size of the largest factor group in this series.

## 4 Decomposition of Groups

In this section we discuss some technical details for the decomposition of groups. The motivation here is that the group decomposition is central in the isomorphism algorithms, for example in the reduction part to graph isomorphism. It also has an influence in the complexity analysis.



It is well known that a composition series even for permutation groups can be obtained in polynomial time [Luk87] and also in NC [BLS87]. The recursive decomposition of groups into normal subgroups and factor groups ends up in simple groups. Here, we just distinguish between two types of simple groups, namely the cyclic groups and non-abelian simple groups. We use the following fact.

**Fact 4.1** *Finite simple non-abelian groups can be generated by two group elements.*

The proof of this depends on the classification of the finite simple groups. Clearly, a cyclic group of prime order can be generated by one element. The non-abelian simple groups can be generated by two elements where one is an involution (c.f. [MSW94]).

In the following we discuss two points. First, when decomposing groups we reduce the number of group elements we need to define a complete set of coset representatives. For example if we have  $h_1 \notin N$  as a representative for  $h_1N$ , then we can take  $h_1^2$  as a representative for the coset  $h_1^2N$ . The goal is to describe all coset representatives by a small set of generators, namely coset representatives for those elements that generate the factor group  $G/N$ . If the factor groups are simple, then we have at most two elements for this. We show now, that all group elements in a simple group can be arranged uniquely when one or two generators are fixed. Second, we show how the groups can be decomposed.

#### 4.1 Arrange group elements by generators

When given a set of generators for a group arranged in a unique order, then this induces a unique order for all group elements. This can be done for example by defining uniquely products to generate each group element and then arrange them lexicographically.

**Generator-representation for groups.** For a group  $G$  there is a normal series  $S$  where the factor groups  $G_i/G_{i+1}$  for  $i \in \{1, \dots, k-1\}$  are semisimple groups, i.e. direct products of simple groups which are cyclic of prime order for a prime, or simple and non-abelian and generated by two elements:

$$\{1\} = G_k \triangleleft G_{k-1} \triangleleft \dots \triangleleft G_1 = G$$

- If  $G_i/G_{i+1}$  for  $i \in \{1, \dots, k-1\}$  is cyclic, then we need one generator  $a_iG_{i+1}$  for  $G_i/G_{i+1}$ .
- If  $G_i/G_{i+1}$  is non-abelian and simple, then we need two generators  $a_iG_{i+1}, b_iG_{i+1}$  for  $G_i/G_{i+1}$ .
- If  $G_i/G_{i+1}$  is semisimple, a direct product of  $l$  simple groups, then we need at most  $2k$  generators  $a_{i,1}G_{i+1}, b_{i,1}G_{i+1}, \dots, a_{i,l}G_{i+1}, b_{i,l}G_{i+1}$  for  $G_i/G_{i+1}$ .

Let  $S$  be given by  $\vec{s} = (s_1, \dots, s_{k-1})$ , we say that an element  $g$  is given in *generator-representation*, if it can be expressed as a product  $w_1 \dots w_{k-1}$  where

- $w_i = a_i^{l_i}$  with  $s_i = (a_i)$  and  $l_i \in \{0, \dots, p_i - 1\}$  if  $G_i/G_{i+1}$  is cyclic,
- $w_i$  is a uniquely determined word with generators  $a_i, b_i$  with  $s_i = (a_i, b_i)$  if  $G_i/G_{i+1}$  is non-abelian and simple,
- $w_i$  is a unique determined word with generators  $a_{i,1}, b_{i,1}, \dots, a_{i,l}, b_{i,l}$  with  $s_i = (a_{i,1}, b_{i,1}, \dots, a_{i,l}, b_{i,l})$  if  $G_i/G_{i+1}$  is semisimple as above.

**Unique order for group elements.** The following lemma says how group elements in simple groups can be arranged uniquely according to their generator-representation. This unique order depends on a given composition series.

**Lemma 4.2** *Let  $G$  be a group and  $S$  a composition series given by  $\vec{s}$  as in Definition 2.1. There is a logspace computable function that brings every group element in  $G$  into a new order that is uniquely determined according to their generator-representation.*

**Proof.** Let  $\vec{s} = (s_1, \dots, s_{k-1})$ . The group elements are arranged in lexicographical order according to their generator representation, i.e. with highest priority sort group elements according to a word  $w_1$  of group elements in  $s_1$ , then those which are equal are sorted according to a word  $w_2$  of group elements in  $s_2$  and so on, until we sort according to a word  $w_{k-1}$  of group elements in  $s_{k-1}$ .

Clearly, if  $s_i = (a)$  (i.e.  $G_i/G_{i+1}$  is cyclic) then a word  $w_i$  is a power of  $a$ , these can be distinguished by their exponent  $a^0 < a^1 < \dots < a^{\text{ord}(a)-1}$ . If  $s_i = (a, b)$  (i.e.  $G_i/G_{i+1}$  is non-abelian and simple) and we have an order  $a < b$ , then we compare products of these generators as words  $w_i$  lexicographically. Hence, it remains to say how to get unique words for the elements in this factor group.

Consider the Cayley graph of  $G_i/G_{i+1}$  where we have directed edges labeled with  $a_i, a_i^{-1}$  and  $b_i, b_i^{-1}$ . Since  $a_i, b_i$  are generators, this graph is connected. We define an order  $a_i < a_i^{-1} < b_i < b_i^{-1}$ .

**Claim 4.3** *In a Cayley graph, if all generators are arranged in a unique order then there is a logspace computable function which arranges all group elements in a unique order.*

**Proof.** The proof is adapted from [DLN08], where planar 3-connected graphs embedded in a plane are canonized. This procedure also works in logspace for general graphs when given a cyclic arrangement for the edges going around each vertex, c.f. *oriented graphs* in [Wag10].

Let  $a_1 < a_1^{-1} < \dots < a_{k-1} < a_{k-1}^{-1}$  be a unique order for all generators. We complete this to a cyclic arrangement  $\rho$  with  $a_{k-1}^{-1} < a_1$ . Hence, the Cayley graph is an oriented graph now. We follow a path starting at node  $e$  in direction of the edge labeled with  $a_1$  with a *universal exploration sequence* [Rei08]. That is, a logspace machine traverses the whole graph and reaches in polynomial number of steps every vertex at least once. Let  $p$  be this path. A second logspace machine goes through this path  $p$  and computes all the positions when each vertex is reached for the first time. These paths up to a certain position can be seen as a product with generators that evaluates to a group element. We can sort all group elements according to the position of their first occurrence in this path.  $\square$

This completes the proof of Lemma 4.2.  $\square$

Since the computations are deterministic and independent from the table representation of the group elements, we immediately get the following corollary.

**Corollary 4.4** *Let  $G, H$  be two groups and  $S, S'$  be composition series given by complete sets of coset representatives. If these induce an isomorphism from  $G$  onto  $H$  then Lemma 4.2 gives that isomorphism from  $G$  onto  $H$ .*

## 4.2 Computation of a composition series.

Let  $G$  be a group with  $n$  elements. We decompose  $G$  into a characteristic subgroup  $N$  (the socle of  $G$ ) and a factor group  $G/N$ . We decompose the factor group recursively until we end up in a *normal series* where each factor group is semisimple (i.e. a direct product of simple groups).

For this we compute first the normal closure  $\text{ncl}(x)$  for all elements  $x \in G$ . Note,  $\text{ncl}(x) = G$  for all  $x \in G$  if and only if  $G$  is already simple. The algorithm will give us generators. If  $G_i/G_{i+1}$  is cyclic, then we get one coset representative, if it is non-abelian and simple, then we get two and if  $G_i/G_{i+1}$  is semisimple with composition length  $l$ , then we get up to  $2l$  coset representatives.

Note,  $\text{ncl}(x)$  is the smallest normal subgroup which contains  $x$ . These are minimal normal subgroups, i.e. a direct product of isomorphic simple groups. We run through all elements  $x \in G$  and select those, where  $\text{ncl}(x)$  is not contained in another subgroup  $\text{ncl}(y)$ . It is also useful to have the following (c.f. Proposition 1.5.1 in [Faw09]).

**Fact 4.5** *Any two distinct minimal normal subgroups of a group  $G$  must intersect trivially.*

The socle  $\text{Soc}(G)$  is a subgroup generated by all minimal normal subgroups. The socle is a normal subgroup of  $G$ , it is a direct product of simple groups. For these and more facts, see [Faw09].

There is one task, namely to break up the socle into its simple groups. For this we use Fact 4.5, i.e., that the simple groups from the socle come from all the minimal normal subgroups. Hence, it suffices to break up each minimal normal subgroup  $N$  into its simple groups:

- If  $N$  is abelian, then consider all elements of prime order. Go through them from left to right and select an element as generator if it is not generated by the group elements of prime order to the left.

To see this, we refer to Lemma 3.2.1 and 3.2.2 in [Hal99], namely that every group element can be written uniquely as a product of prime power elements, and prime power elements can be generated by prime elements.

- If  $N$  is not abelian, then the simple groups in  $N$  are not abelian. We need two generators. We search a pair of elements in  $N$  which generates a simple group  $K_1$  which is normal in  $N$ . For this, we compute  $\text{ncl}_N(g)$  for  $g \in G$  until we find a non-trivial normal subgroup  $K_1$  of  $N$ . We do this recursively for  $N/K_1$ . Hence, we get generators for all simple groups  $K_1, \dots, K_i$  in  $N = K_1 \times \dots \times K_i$ .

**Arrange factor groups in composition series.** The socle of a group is semisimple, it is generated by the minimal normal subgroups. These are pairwise disjoint, i.e. their intersection contains  $\{e\}$  only. Minimal normal subgroups are a direct product of isomorphic simple groups. Hence, the recursive decomposition of groups into normal subgroups and factor groups is not unique. For example, the cyclic group  $C_6$  has two different composition series:

$$\{1\} \triangleleft C_2 \triangleleft C_6 \text{ and } \{1\} \triangleleft C_3 \triangleleft C_6.$$

Here, the factor groups are  $C_2$  and  $C_3$  which appear in different order. The following theorem says, that the factor groups are isomorphic. But this does not mean that groups are isomorphic if they have isomorphic factor groups.

**Theorem 4.6** (*Jordan-Hölder*) *Two composition series of a group have isomorphic composition factors.*

For an isomorphism test, the first task is to arrange the simple groups of the socle by their isomorphism type. This can be done as follows.

In the case of abelian groups isomorphism testing can be done in linear time [Kav03]. In the algorithm, the orders of all group elements are simply compared. Hence, when sorting these sequences they can be compared lexicographically to define an order  $\prec$  on them.

We define an order  $G \prec H$  on two simple non-abelian groups if one of the following holds:

- $|G| < |H|$  or
- $|G| = |H|$  but  $T_G < T_H$  which is defined as follows. Let  $(a', b')$  be a pair of elements which generate  $H$ . Since we have two generators only for simple groups, we run through all pairs  $(a, b)$  of group elements in  $G$ . As in Claim 4.3 we compute Cayley graphs with  $a < b$  in  $G$  and  $a' < b'$  in  $H$ . The claim says, that the group elements can be arranged in a unique order in logspace. Now, compare the multiplication tables  $T_G$  and  $T_H$  with group elements arranged in this order line by line and bit by bit.

We define  $G = H$  if the groups are isomorphic, i.e. if neither  $G \prec H$  nor  $H \prec G$  holds. We define  $G \preceq H$  if  $G \prec H$  or  $G = H$  holds.

With this order we define an order on the composition factors in a composition series of a group.

**Definition 4.7** *A sorted sequence of composition factors is a sequence of factor groups in a composition series where factor groups are sorted as follows.*

1. *According to the recursive decomposition into normal subgroups and factor groups, i.e.  $\{1\} \triangleleft \text{Soc}(G) \triangleleft G$ , where we refine  $G/\text{Soc}(G)$  recursively as we do for  $G$ .*
2. *The socles in the decomposition process are direct products of minimal normal subgroups, i.e.  $\text{Soc}(G) = K_1 \times \dots \times K_i$ , where (for  $j \in \{1, \dots, i\}$ ) each  $K_j = K_{j,1} \times \dots \times K_{j,\gamma_j}$  is a direct product of isomorphic simple groups. Arrange  $K_{1,1}, \dots, K_{i,1}$  according to  $\prec$ , for example if  $K_{1,1} \preceq \dots \preceq K_{i,1}$  then  $\{1\} \triangleleft \dots \triangleleft \text{Soc}(G)$  where all the factor groups are arranged  $K_{1,1}, \dots, K_{1,\gamma_1}, K_{2,1}, \dots, K_{i,\gamma_i}$  in this order from left to right.*

*Let  $\beta$  be the maximum number of isomorphic minimal normal subgroups in  $K_1, \dots, K_i$  in all levels of recursion. Let  $\gamma = \max_{j=1}^i \gamma_j$  be the maximum number of isomorphic simple groups in  $K_1, \dots, K_i$  in all levels of recursion.*

For a composition series  $S$ , let  $\text{seq}(S)$  be the sequence of composition factors in  $S$ . The order of the composition factors induces an order on their generators, i.e. the corresponding coset representatives in  $\vec{s}$ . We assume, that the members in  $\vec{s}$  are arranged in this way.

Let  $G$  and  $H$  be two groups. For an isomorphism test, we compute a composition series for  $G$ , i.e. by recursively computing socles, and then we have to arrange in  $H$  within each of the socles in the decomposition process:

- $K_1, \dots, K_i$  if these are isomorphic (i.e. a direct product of the same number of isomorphic simple groups) and
- within each  $K_j$  ( $1 \leq j \leq i$ ) all the simple groups.

We will see that in our isomorphism testing algorithms we will do these final arrangements with non-determinism.

In arbitrary two composition series the composition factors are ordered with respect to Step 1 in Definition 4.7. That is, because the socle of a group is a *characteristic* subgroup. According to Step 2, we arrange the factor groups with respect to the order  $\triangleleft$ . This can be done, because the groups  $K_1, \dots, K_i$  are all normal subgroups in  $G$ .

**Lemma 4.8** *For two isomorphic groups  $G$  and  $H$ , there are composition series  $S$  for  $G$  and  $S'$  for  $H$  as in Definition 4.7 with  $(G, \text{seq}(S))$  isomorphic to  $(H, \text{seq}(S'))$  such that this isomorphism maps the subgroups in this series  $S$  blockwise onto those in  $S'$ .*

**The algorithm.** In Algorithm 1 we show how a composition series for a group can be computed where the factor groups are arranged as in Definition 4.7. We give some comments to the three main parts of this algorithm.

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**Algorithm 1** CompSeries: Compute composition series for groups.

---

**input:** group  $G$  with  $n$  elements

**output:** composition series for  $G$  given by a complete set of coset representatives

**initialize:** set of groups  $NCL = \{\}$

{ 1. compute minimal normal subgroups }

1: **for each**  $g \in G$  **do**

2:    $\text{ncl}_G(g) = \langle g_1, \dots, g_k \rangle$  with  $\{g_1, \dots, g_k\} = \{h^{-1}gh \mid h \in G\}$

3:   **if**  $\exists N \in NCL : \text{ncl}_G(g) < N$  **then**  $NCL = \{\text{ncl}_G(g)\}$    **else**  $NCL \leftarrow \text{ncl}_G(g)$

4: **end for**

5:  $\text{Soc}(G) = \langle N_1, \dots, N_k \rangle$  with  $\{N_1, \dots, N_k\} = NCL$

6: **if**  $\forall g \in G : \text{ncl}_G(g) = G$  **then**  $G$  is simple, return  $\{G\}$

{ 2. compute simple groups }

7: **for each**  $N \in NCL$  **do**

8:   **if**  $N$  is abelian **then**

9:     **for each**  $\{g \in N \mid \text{ord}(g) \text{ prime}, g \notin \langle S(N) \rangle\}$  **do**  $S(N) \leftarrow (g)$

10:   **if**  $N$  is not abelian **then**

11:     **for each**  $\{(g, h) \in N \times N \mid g \neq h, \exists n \in N : \langle g, h \rangle \triangleleft \text{ncl}_N(n)\}$  **do**

12:       **if**  $\langle g, h \rangle \cap \langle S(N) \rangle = \{e\}$  **then**  $S(N) \leftarrow (g, h)$

13:   **end for**

{ 3. compute all composition series }

14:  $\vec{s}_1 = \text{CompSeries}(G/\text{Soc}(G))$  where  $G/\text{Soc}(G) = \{g\text{Soc}(G) \mid g \in (G \setminus \text{Soc}(G)) \cup \{e\}\}$

15: choose arbitrarily  $(N_1, \dots, N_k) \in \text{Sym}(NCL)$  with  $s_i \preceq s_j$  for all  $i < j, s_i \in N_i, s_j \in N_j$

16: choose arbitrarily  $(s_{1,1}, \dots, s_{1,l_1}, \dots, s_{k,1}, \dots, s_{k,l_k}) \in \text{Sym}(S(N_1)) \times \dots \times \text{Sym}(S(N_k))$

17:  $\vec{s}_2 = (s_{1,1}, \dots, s_{1,l_1}, \dots, s_{k,1}, \dots, s_{k,l_k})$  represents the socle, i.e.

$$\text{Soc}(G) = G_{1,1} \triangleleft G_{1,2} \triangleleft \dots \triangleleft G_{k,l_k} \triangleleft G_{k,l_k+1} = \{1\}$$

$$\text{with } G_{i,j}/G_{i,j+1} = \langle aG_{i,j+1} \rangle \text{ if } s_{i,j} = (a) \text{ and}$$

$$\text{with } G_{i,j}/G_{i,j+1} = \langle aG_{i,j+1}, bG_{i,j+1} \rangle \text{ if } s_{i,j} = (a, b)$$

18:  $\vec{s} = (s_1, \dots, s_i, s_{i+1}, \dots, s_j)$  with  $(s_1, \dots, s_i) = \vec{s}_1, (s_{i+1}, \dots, s_j) = \vec{s}_2$

19: return  $\vec{s}$

---

In the first part, we compute the minimal normal subgroups and put them into the set  $NCL$ . The socle  $Soc(G)$  is then generated by all members in  $NCL$ . If  $\text{ncl}(g) = G$  for all  $g \in G$  then  $G$  is simple, the composition series is trivially  $\{1\} \triangleleft G$ .

In the second part, we compute for each minimal normal subgroup  $N$  its simple groups. We distinguish the situation whether  $N$  is abelian or not. Then we need one or two generators for a simple group, respectively.

In Line 9 and 12, we write in short  $\langle S(N) \rangle$  and mean the group generated by all group elements put into  $S(N)$ .

In Lines 10 to 12, we run through all pairs  $g, h \in N$  until we find one which generates  $N$ . Recall, that a direct product  $GH$  of simple groups  $G$  and  $H$  has normal subgroups, namely  $G$  and  $H$ , and  $G \cap H = \{e\}$ .

In the third part, in  $\vec{s}_1$  we encode a composition series for the factor group  $G/Soc(G)$  recursively. In  $\vec{s}_2$  we encode a composition series for  $Soc(G)$ . We take the generators (i.e. in the tuples  $s_{i,j}$  for  $i \in \{1, \dots, k\}, j \in \{1, \dots, l_i\}$ ) of all the composition factors as coset representatives and get a complete set of coset representatives for  $G$ . The factor groups are arranged according to the recursive decomposition and then according to  $\prec$  as in Definition 4.7. In Line 18,  $\vec{s}_2$  is the same as in Line 17, the elements are just relabeled.

**Composition series for an isomorphism test.** For an isomorphism test we modify Algorithm 1 to get a composition series for the second group  $H$ . This is done as follows.

- Line 9: guess a generating set for  $N$ .  
If  $N$  is a direct product of  $\gamma_N$  cyclic groups, then we need  $\gamma_N \log |N|$  non-deterministic bits.
- Line 11 and 12: guess a generating set for  $N$ .  
If  $N$  is a direct product of  $\gamma_N$  simple non-abelian groups, then we need  $2\gamma_N \log |N|$  non-deterministic bits.
- Line 15: guess an arrangement  $(N_1, \dots, N_k) \in Sym(NCL)$  with  $s_i \preceq s_j$  for all  $i < j, s_i \in N_i, s_j \in N_j$ . For this we need  $k \log k$  bits. Since  $k \leq \beta \leq \log n$  we need at most  $\beta \log \beta \leq \log n \log \log n$  non-deterministic bits.
- Line 16: guess an arrangement  $(s_{1,1}, \dots, s_{1,l_1}, \dots, s_{k,1}, \dots, s_{k,l_k}) \in Sym(S(N_1)) \times \dots \times Sym(S(N_k))$ . At most  $l_i \log l_i$  non-deterministic bits for each  $l_i$  are required. Suppose that  $\gamma_N = l_1 = \dots = l_k$ . Since  $k\gamma_N \leq \log n$  we need at most  $\sum_{i=1}^k l_i \log l_i \leq \sum_{i=1}^k l_i \log(\gamma_N) \leq k\gamma_N \log(\gamma_N)$  non-deterministic bits. Hence, we need at most  $\log n \log \log n$  non-deterministic bits.

The total demand on non-deterministic bits which corresponds to the Lines 9, 11 and 12 is computed as follows. The worst-case appears if we have minimal normal subgroups of large size. Let the largest minimal normal subgroup in the decomposition process be  $N^*$  which is a direct product of  $\gamma$  simple groups. Then  $|N^*| = p^\gamma$  if  $N^*$  is a direct product of  $\gamma$  simple groups of size at most  $p$ . We need at most  $2\gamma \log p^\gamma = 2\gamma^2 \log p$  non-deterministic bits. The number of such minimal normal subgroups is at most  $\log n / \log p^\gamma = \log n / (\gamma \log p)$ . Hence, the total demand is at most  $\log n / (\gamma \log p) \cdot 2\gamma^2 \log p = 2\gamma \log n$  bits.

The total demand on non-deterministic bits which corresponds to Line 15 is computed as follows. Recall that  $\beta$  is the largest number of isomorphic minimal normal subgroups among all the socles in the decomposition process. We get the worst-case if each socle consists of exactly  $\beta$  isomorphic minimal normal subgroups. Let  $d$  be the total depth of recursion. Then the total demand is  $d\beta \log \beta$

bits. Note, that  $d\beta \leq \log n$  since we have at most  $\log n$  factor groups. Hence, we get the worst-case if  $d = 1$ . It follows, that  $d \leq \log n/\beta$  and the total demand is  $\log n/\beta \cdot \beta \log \beta = \log n \log \beta \leq \log n \log \log n$  bits.

The total demand on non-deterministic bits which corresponds to Line 16 is computed as follows. Recall, that at recursion level  $i$ ,  $l_1, \dots, l_k \leq \gamma_i$  and we said that we need at most  $k\gamma_i \log(\gamma_i)$  non-deterministic bits. Let  $d$  be the total depth of recursion. We get the worst-case if  $l_1 = \dots = l_k = \gamma_i$  for all  $i \in \{1, \dots, d\}$  and if  $\gamma = \gamma_1 = \dots = \gamma_i = \dots = \gamma_d$ . It follows, that  $dk\gamma \leq \log n$  since we have at most  $\log n$  factor groups. Thus,  $dk \leq \log n/\gamma$  and the total demand is at most  $(\log n/\gamma) \cdot \gamma \log \gamma \leq \log n \log \gamma$ .

**Theorem 4.9** *Let  $G, H$  be two isomorphic groups. A composition series  $S$  for  $G$  and  $S'$  for  $H$  such that  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$  can be computed in NP with access to at most  $2\gamma \log n + \log n \log \beta + \log n \log \gamma \leq O(\log^2 n)$  non-deterministic bits.*

However, in general this is no improvement to the known upper bounds for group isomorphism. In the following sections we analyze algorithms with respect to the parameters  $\beta$  and  $\gamma$  from Definition 4.7.

## 5 Isomorphism for $p$ -groups

To get a composition series  $S$  for  $G$  we decompose the group  $G$  as in Algorithm 1 and guess the composition series  $S'$  for the other group  $H$  as in Theorem 4.9.

For a prime number  $p$ , let  $(G, S)$  be a  $p$ -group  $G$  over  $n$  elements with a composition series  $S$ . Let the factor groups in  $S$  be cyclic of order  $p$ .

$$\{1\} = G_k \triangleleft G_{k-1} \triangleleft \dots \triangleleft G_1 = G$$

Let  $\vec{g} = ((g_1), \dots, (g_{k-1}))$  be a complete set of coset representatives with respect to  $S$  with generator  $g_1 G_{i+1} \in G_i/G_{i+1}$ . Here,  $G_i/G_{i+1}$  is a cyclic group of order  $p$ .

Let  $\text{seq}(S)$  be the sequence of factor groups of  $S$ . Recall, any element in a coset could be taken as a representative. So an isomorphism from  $(G, \text{seq}(S))$  onto  $(H, \text{seq}(S'))$  just maps cosets blockwise onto each other.

**Theorem 5.1** *There is an  $\text{AC}^0$ -computable function that on input of groups  $G, H$  and  $S, S'$  computes a graph  $X_p(G, S)$  and  $X_p(H, S')$  with at most  $11n^2 + 1$  vertices which have valence at most  $p + 1$  such that  $X_p(G, S)$  is isomorphic to  $X_p(H, S')$  if and only if  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$ , i.e. there exist composition series such that their subgroups  $G_i$  and  $H_i$  are isomorphic.*

**Proof.** We construct a graph  $X_p(G, S)$  for  $G$  and  $S$  in two steps. The construction for  $X_p(H, S')$  is done accordingly.

In the first step we define a tree  $T_p(G, S)$ . Intuitively speaking, this tree is based on the structure of the composition series  $S$ .

1. For every group element  $g \in G$  we have an *element vertex* in  $T_p(G, S)$ .
2. For all  $i \in \{1, \dots, k-1\}$  and each coset  $gG_i$  and subgroup  $eG_i$  of the factor group  $G/G_i$  we have a *coset vertex*  $gG_i$  and  $eG_i$  in  $T_p(G, S)$ . Note, for  $i = 1$  we call  $eG_1 = eG$  the *root* of  $T_p(G, S)$ .

3. For each element vertex  $g$  and coset vertex  $gG_{k-1}$  we have an edge  $\{g, gG_{k-1}\}$  in  $T_p(G, S)$ .
4. For each pair of coset vertices  $gG_i, gg'G_{i+1}$  with  $g \in gG_i$  and  $g' \in G_{i+1}$  we have an edge  $\{gG_i, gg'G_{i+1}\}$  in  $T_p(G, S)$ .

Now we construct  $X_p(G, S)$ . We use  $T_p(G, S)$  and define a further graph gadget to simulate the multiplication rule.

1. We have a main copy of  $T_p(G, S)$  in  $X_p(G, S)$ . The root node of the main copy is connected to a *color graph gadget*, namely a path of length two to distinguish this vertex from the others. For each leaf node  $v$  in  $T_p(G, S)$  we have a copy  $T_v$  of  $T_p(G, S)$ . We identify the root node of  $T_v$  with  $v$ .
2. For each node  $v$  in  $T_p(G, S)$ , we have for each leaf node  $w$  of  $T_v$  five nodes  $w_{\leftarrow}, w_{\rightarrow}, w_l, w_r, w_{=}$  in  $X_p(G, S)$ . We have edges  $(w, w_l), (w_l, w_{\leftarrow}), (w, w_r), (w_r, w_{\rightarrow}), (w_r, w_{=})$  in  $X_p(G, S)$ .
3. For each pair of group elements  $g, h \in G$  we simulate the multiplication  $gh = k$  as follows. We define a *multiplication graph gadget* that is connected to the vertices  $h_{\leftarrow}^{(g)}$  in  $T_g$ ,  $g_{\rightarrow}^{(h)}$  in  $T_h$ , and  $h_{=}^{(k)}$  in  $T_k$ . One gadget  $M_{gh=k}$  is shown in Figure 3.

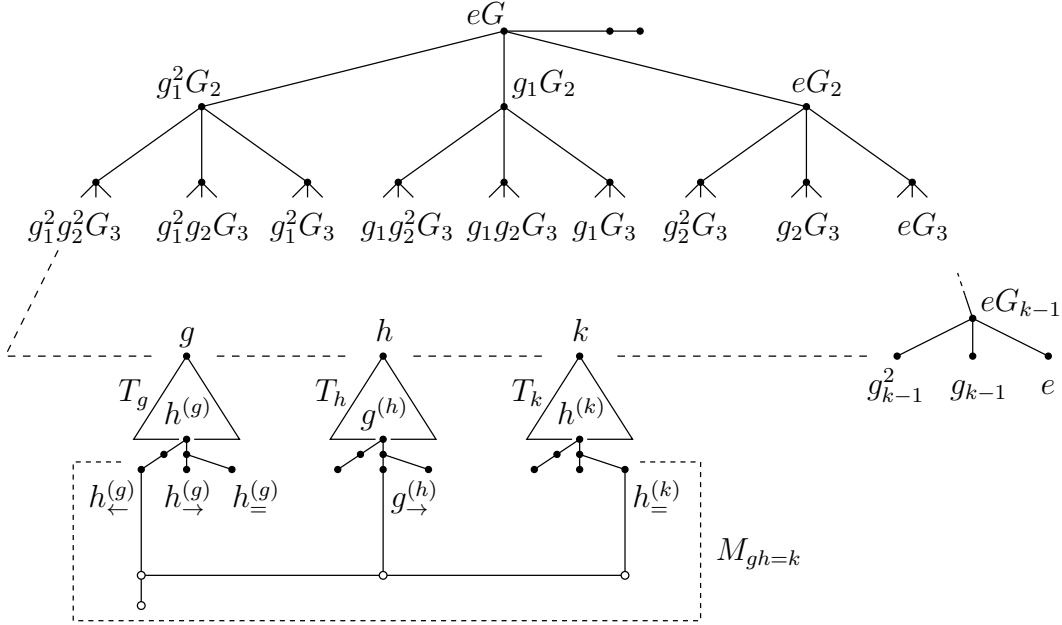


Figure 3: The graph  $X_p(G, S)$  with  $\gamma = 1$  and a multiplication graph gadget  $M_{gh=k}$  indicated by white nodes.

We prove now that  $X_p(G, S)$  has all the properties stated in Theorem 5.1.



**Claim 5.2** *The graph  $X_p(G, S)$  has at most  $11n^2 + 1$  vertices.*

**Proof.**

- $T_p(G, S)$  is a complete tree with  $n$  leaves where each inner node at the same depth  $i$  has the same number  $\max\{p, \gamma_i\}$  of children, it has at most  $\sum_{i=1}^{\log_p(n)-1} p^i \leq 2n - 1$  vertices.
- The color graph gadget is a path of length two connected to the root node of  $T_p(G, S)$ .
- For each of the  $n$  leaf nodes of  $T_p(G, S)$  we have a copy of  $T_p(G, S)$ . Since the root nodes of these copies are identified with the leaf nodes of  $T_p(G, S)$  we do not count these nodes twice. We get at most  $n((2n - 1) - 1)$  vertices.
- Every leaf node  $v$  of these copies is connected to a subtree with five nodes  $v_l, v_r, v_{\leftarrow}, v_{\rightarrow}$  and  $v_{=}$ . There are  $n^2$  such leaf nodes. Hence, we get further  $5n^2$  vertices.
- Every multiplication gate has 4 vertices, when not counting the vertices  $v_{\leftarrow}, v_{\rightarrow}$  and  $v_{=}$  twice. We have  $n^2$  multiplication gates. We get  $4n^2$  vertices.

We have in total at most  $2n - 1 + 2 + n((2n - 1) - 1) + 5n^2 + 4n^2 = 11n^2 + 1$  vertices.  $\square$

**Claim 5.3** *There is a logspace-computable function that computes the graph  $X_p(G, S)$ .*

**Proof.** Since we have coset representatives  $(g_1, \dots, g_{k-1})$ , every element  $gG_i \in G_1/G_i$  can be obtained uniquely by following a path

$$(eG, w_1G_2, w_1w_2G_3, \dots, w_1 \cdots w_{i-1}G_i),$$

where  $w_jG_{j+1}$  is a coset vertex, for each  $j \in \{1, \dots, i-1\}$ . This is related to the following product:

$$gG_i = w_1 \cdot w_2 \cdots w_{i-1}G_i$$

The logspace machine goes through all products. The construction of the tree is done as in a depth first traversal through the resulting tree  $T_p(G, S)$ . We evaluate the products for all group elements  $g = w_1 \cdot w_{k-1}$  for all  $w_i$  in a preprocessing step. This can be done in logspace. A further logspace machine relabels the group elements by such products, i.e. we rewrite the whole group table.

It is easy to see, that with access to the new group table each step of the graph construction can be done in  $AC^0$ .  $\square$

**Claim 5.4** *The graph  $X_p(G, S)$  has valence at most  $p + 1$  for all  $p \geq 2$ .*

**Proof.** The nodes in the tree  $T_p(G, S)$  have one parent and  $p$  children, because in Step 4 of the construction, there are at most  $p$  cosets  $gg'G_{i+1}$  in  $gG_i$  with  $g \in gG_i$  and  $g' \in G_{i+1}$ . The root node is connected to a color graph gadget and has also valence  $p + 1$ . The leafs are identified with the root node of a copy of  $T_p(G, S)$  and have also valence  $p + 1$ . Each leaf node of the copies of  $T_p(G, S)$  is connected to at most one multiplication graph gadget and has valence two. To see this we argue, that for every pair of variables in  $gh = k$  the third variable is uniquely determined in a group. The vertices of the multiplication graph gadgets and the color graph gadget have valence at most 3, this is not greater than  $p + 1$  for all  $p \geq 2$ .  $\square$

**Claim 5.5** *The graph  $X_p(G, S)$  is isomorphic to  $X_p(H, S')$  if and only if  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$ .*

**Proof.** Consider an isomorphism  $\phi$  between  $(G, \text{seq}(S))$  and  $(H, \text{seq}(S'))$ , we argue now that we get an isomorphism between  $X_p(G, S)$  and  $X_p(H, S')$ . That is, for every pair of elements  $g, h$  in  $G$  we show that the property  $\phi(g)\phi(h) = \phi(gh)$  can be transformed into an isomorphism between the graphs.

Clearly, an isomorphism from  $(G, \text{seq}(S))$  onto  $(H, \text{seq}(S'))$  is also an isomorphism when just considering  $T_p(G, S)$  and  $T_p(H, S')$ . The subgroups in  $S$  and  $S'$  correspond to characteristic normal subgroups, hence, these have to be mapped onto each other. Now, consider  $X_p(G, S)$  and  $X_p(H, S')$ . Let  $gh = k$ , between three leaf nodes  $g, h, k$  of  $T_p(G, S)$  in  $X_p(G, S)$  there are unique simple paths going through a single multiplication graph gadget  $M_{gh=k}$  such that:

- except  $g, h, k$  there is no other vertex visited in  $T_p(G, S)$ ,
- there is a unique simple path from  $g$  of  $T_p(G, S)$  to  $h \stackrel{(g)}{\leftarrow}$  in  $T_g$ ,
- there is a unique simple path from  $h \stackrel{(g)}{\leftarrow}$  of  $T_g$  to  $g \stackrel{(h)}{\rightarrow}$  of  $T_h$  in  $M$ ,
- there is a unique simple path from  $g \stackrel{(h)}{\rightarrow}$  of  $T_h$  to  $h$  of  $T_p(G, S)$  in  $T_h$ ,
- there are unique simple paths from  $h \stackrel{(g)}{\leftarrow}$  of  $T_g$  and  $g \stackrel{(h)}{\rightarrow}$  of  $T_h$  to  $h \stackrel{(k)}{=}$  in  $T_k$  in  $M$ ,
- there is a unique simple path from  $h \stackrel{(k)}{=}$  of  $T_k$  to  $k$  of  $T_p(G, S)$  in  $T_k$ .

Hence, if  $\phi$  is an isomorphism of  $(G, \text{seq}(S))$  onto  $(H, \text{seq}(S'))$  then in  $X_p(H, S')$  there is also a multiplication graph gadget  $M_{\phi(g)\phi(h)=\phi(k)}$  such that these unique simple paths exist. This isomorphism mimics the permutation from  $T_p(G, S)$  onto  $T_p(H, S')$  also at each copy of the tree, e.g.  $T_g$  in  $X_p(G, S)$  is mapped onto  $T_{\phi(g)}$  in  $X_p(H, S')$ , and for every leaf vertex  $v$  with  $\phi(v) = w$ , the vertices  $v \leftarrow, v \rightarrow, v =$  in  $T_g$  are mapped via  $\phi$  onto  $w \leftarrow, w \rightarrow, w =$  in  $T_{\phi(g)}$ .

Now to the other direction. Since the root node of  $T_p(G, S)$  is distinguished from the others, any isomorphism maps this root node onto the root node of  $T_p(H, S')$ . Vertices at the same distance are mapped onto each other, hence  $T_p(G, S)$  is mapped onto  $T_p(H, S')$ . This also holds for the copies of the tree rooted at the children and the multiplication graph gadgets.

Any isomorphism respects the multiplication rules of the groups: There are multiplication graph gadgets just for the multiplication rules, i.e. if  $gh = k$  in  $G$  then there is no gadget  $M_{gh=k'}$  for any  $k' \neq k$ . Since the multiplication graph gadget is rigid, there is no isomorphism that maps a vertex  $v \leftarrow$  in  $X_p(G, S)$  onto any vertex  $w \rightarrow$  or  $w =$  in  $X_p(H, S')$  and vice versa. We conclude, every isomorphism  $\phi$  from  $(G, \text{seq}(S))$  onto  $(H, \text{seq}(S'))$  maps  $M_{gh=k}$  in  $X_p(G, S)$  onto  $M_{\phi(g)\phi(h)=\phi(k)}$  in  $X_p(H, S')$ . Hence, if  $(G, \text{seq}(S))$  is not isomorphic to  $(H, \text{seq}(S'))$  then we cannot get an isomorphism from  $X_p(G, S)$  onto  $X_p(H, S')$ .

There is a one-to-one correspondence between automorphisms of  $X_p(G, S)$  and automorphisms of  $(G, \text{seq}(S))$ . Assume, the leaf nodes of  $T_p(G, S)$  are fixed. Since any three leaf vertices of  $T_p(G, S)$  have at most one rigid multiplication graph gadget in common, all of them are fixed. Since every vertex  $w \stackrel{(v)}{\leftarrow}, w \stackrel{(v)}{\rightarrow}, w \stackrel{(v)}{=}$  of every tree  $T_v$  in  $X_p(G, S)$  is connected to a multiplication graph gadget, all these vertices are fixed. We conclude, that every automorphism of  $(G, \text{seq}(S))$  induces a unique automorphism of  $X_p(G, S)$ . Hence, this also holds for isomorphisms from  $X_p(G, S)$  onto  $X_p(H, S')$ . □

This completes the proof of Theorem 5.1. □

Note, the graph  $X_p(G, S)$  has the property that it is a *cone-graph* with logarithmic depth bound. That is, from every vertex, there is a unique path to the root node of  $T_p(G, S)$ .

**Complexity.** For an isomorphism test, we compute first a composition series  $S$  and guess then a corresponding composition series  $S'$ , such that we get a sorted sequences of factor groups  $\text{seq}(S)$  and  $\text{seq}(S')$ , as in Definition 4.7. Recall, that  $\beta$  is the largest number of isomorphic minimal normal subgroups in a socle along the decomposition process and that  $\gamma$  is the size of the largest minimal normal subgroup in any socle of the decomposition process.

By Theorem 4.9 we need  $2\gamma \log n + \log n \log \beta + \log n \log \gamma \leq O(\log^2 n)$  non-deterministic bits or deterministic time  $n^{2\gamma + \log \beta + \log \gamma}$ . The complexity of bounded valence GI is in polynomial time:

**Theorem 5.6** ([BL83]) *Isomorphism on graphs of valence at most  $d$  can be tested in deterministic time  $n^{O(d)}$ .*

We put this together and get for a constant  $c$  the running time  $n^{c(\gamma + \log \beta)} \cdot n^{c(p+1)}$ . This completes the proof of Theorem 1.2.

**Theorem 1.2** *Group Isomorphism for  $p$ -groups with  $n$  elements given in table representation is in time (for a constant  $c$ ):*

$$n^{c(p + \gamma + \log \beta)}$$

## 6 Isomorphism test for groups

In this section we give three isomorphism tests. The first is the classical isomorphism test, it can be seen as a variant of the algorithm attributed to Tarjan, c.f. [Mil78]. The second algorithm is an extension of what we introduced for  $p$ -groups in Section 5. The third is a combination of both algorithms.

For all these algorithms we assume, that for a group  $G$  the composition factors of a composition series  $S$  are given together with a complete set of coset representatives  $\vec{s}$ . We address this sequence of composition factors of  $S$  by  $\text{seq}(S)$ . Note, any element in a coset could be taken as a representative. So an isomorphism that respects two composition series  $S$  of  $G$  and  $S'$  of another group  $H$  maps subgroups in  $S$  blockwise onto those in  $S'$ .

### 6.1 Classical Isomorphism Test

**The isomorphism test.** Let  $G$  and  $H$  be two groups. By Theorem 4.9 we conclude that a composition series  $S$  for  $G$  and  $S'$  for  $H$  can be computed in time  $n^{2\gamma + \log \beta + \log \gamma + c}$  for a constant  $c$ , such that if  $G$  is isomorphic to  $H$  then also  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$ .

Let  $(H, S')$  be a group with normal series  $S'$  and a complete set of coset representatives  $\vec{h} = (h_{1,1}, \dots, h_{1,\gamma_1}, \dots, h_{k-1,\gamma_{k-1}})$ . For an isomorphism test, we guess a complete set of coset representatives  $\vec{g} = (g_{1,1}, \dots, g_{1,\gamma_1}, \dots, g_{k-1,\gamma_{k-1}})$  in  $G$  which are mapped onto  $(h_{1,1}, \dots, h_{1,\gamma_1}, \dots, h_{k-1,\gamma_{k-1}})$  in this order. Note, if  $G_i/G_{i+1}$  is cyclic, then  $g_{i,1} = (a_{i,1})$  and if  $G_i/G_{i+1}$  is non-abelian and simple, then  $g_{i,1} = (a_{i,1}, b_{i,1})$ . If  $G_i/G_{i+1}$  is semisimple then we have  $g_{i,1}, \dots, g_{i,\gamma_i}$  of the same type. We assume that  $g_{i,j} = (a_{i,j}, b_{i,j})$  is mapped onto  $h_{i,j} = (a'_{i,j}, b'_{i,j})$  by mapping  $a_{i,j}$  onto  $a'_{i,j}$  and  $b_{i,j}$  onto  $b'_{i,j}$ .

We write the group elements in generator-representation and arrange them in increasing lexicographical order according to their representation. We relabel the elements according to their new

order as in Lemma 4.2. We write the multiplication tables for  $G$  and  $H$  where the elements are sorted and compare them line by line and bit by bit.

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**Algorithm 2** Isomorphism testing for Cayley Groups

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**Input:** multiplication tables of two groups  $G, H$  with  $n$  elements

**Computation:** accept if  $G$  is isomorphic to  $H$  and reject otherwise

- 1: compute  $S'$  for  $H$  a normal series together with complete sets of coset representatives  $\vec{h} = (h_{1,1}, \dots, h_{k-1, \gamma_{k-1}})$  according to Algorithm 1
  - 2: guess a complete set of coset representatives  $\vec{g} = (g_{1,1}, \dots, g_{k-1, \gamma_{k-1}})$  with respect to  $(G, S)$
  - 3: **for each**  $g \in G$  (or  $H$ ) **do** compute  $repr(g)$  a word with  $g_{1,1}, \dots, g_{k-1, \gamma_{k-1}}$  in  $G$  with respect to  $\vec{g}$  or a word with  $h_{1,1}, \dots, h_{k-1, \gamma_{k-1}}$  in  $H$  with respect to  $\vec{h}$
  - 4: relabel generators according to their order in  $\vec{g}$  (and  $\vec{h}$ )
  - 5: let  $T_G$  be  $G$  (and  $T_H$  be  $H$ ) in table representation where elements are sorted according to their new labels in increasing lexicographical order
  - 6: compare  $T_G, T_H$  lexicographically line by line and bit by bit
  - 7: **if**  $T_G = T_H$  **then** accept and halt
  - 8: reject and halt
- 

We give some notes to Algorithm 2.

**Step 1:** *compute a normal series for  $G$  and  $H$  according to Algorithm 1.* For the normal series we have coset representatives, i.e. one or two generators if a factor group is simple and  $k$  or  $2k$  generators if a factor group is semisimple, a direct product of  $k$  simple isomorphic groups.

**Step 2:** *guess a complete set of coset representatives  $S$  for  $G$ .* For each factor group we guess the same number of generators as we have for the corresponding factor group in  $S'$  for  $H$ .

**Step 3:** *For each group element  $g$ , compute its generator representation.* The representation depends on the generators in  $\vec{g}$  and  $\vec{h}$ . The representation for a group element  $g \in G$  is a product of generators with unique words according to Lemma 4.2. For example, if  $G_i/G_{i+1}$  is a direct product of  $\gamma_i$  cyclic isomorphic groups, then  $g_i = (a_{i,1}, \dots, a_{i,\gamma_i})$  and the  $i$ -th term in this product is  $a_{i,1}^{l_{i,1}} \dots a_{i,\gamma_i}^{l_{i,\gamma_i}}$  with  $l_{i,j} \in \{0, \dots, \text{ord}(a_{i,j}) - 1\}$  and  $j \in \{1, \dots, \gamma_i\}$ . Whereas if  $G_i/G_{i+1}$  is non-abelian and simple, then instead of  $a_{i,j}$  we have  $a_{i,j}, b_{i,j}$ , correspondingly, and the  $i$ -th term is a word with  $a_i, b_i$ .

**Step 4:** *Relabel generators according to their order in  $\vec{g}$  (and  $\vec{h}$ ).* The new labels are taken from the set  $\{1, \dots, n\}$  in increasing order.

**Step 5:** *Compute the multiplication table where elements are sorted by their new labels.* We relabel the group elements as in Lemma 4.2 and sort them in increasing lexicographical order.

**Step 6 to 8:** *Accept iff the tables  $T_G$  and  $T_H$  are equal.* The comparison is done lexicographically line by line and for each line element by element. The elements are compared bit by bit. If  $T_G = T_H$  then we accept, else we reject.

**Complexity.** Every step can be done in polynomial time by an NP-machine. We calculate now the demand on non-deterministic bits in Algorithm 2.

In Step 3, we guess generators, these are different to the coset representatives from Step 2. An isomorphism then maps generators onto each other. For  $i \in \{1, \dots, k-1\}$  and  $j \in \{1, \dots, \gamma_i\}$  we

guess  $a_{i,j}, b_{i,j} \in G_i \setminus G_{i+1}$ , i.e. we need at most  $2 \log(|G_i| - |G_{i+1}|) \leq 2 \log n$  bits. If  $G_i/G_{i+1}$  has order  $p^{\gamma_i}$ , then we have  $\gamma_i$  generators, each of order  $p$ . For an upper bound, let every generator have order at least  $p$ . Hence, in total we have at most  $\log_p n$  generators and we need at most  $O(\log n) \cdot \log_p n$  non-deterministic bits to guess all generators. We get the following Theorem.

**Theorem 6.1** *Let the factor groups in  $S$  of  $G$  have size at least  $p$ . Group Isomorphism on groups with  $n$  elements given in table representation can be tested by an NP-machine with access to at most  $O(\log n \log_p n)$  non-deterministic bits.*

**Remarks.** Instead of non-determinism, the computations can be done in deterministic time  $n^c \cdot 2^{\log n \log_p n} = n^{c+\log_p n}$  for a constant  $c$ .

In the worst case we have  $p = 2$  and then we reach the upper bound from Tarjan's algorithm. Later, for the complexity analysis of the third isomorphism test it is important to consider this bound with parameter  $p$ .

## 6.2 Isomorphism test: Reduction to Bounded Valence Graph Isomorphism.

In the second isomorphism testing algorithm we compute a composition series for  $G$  and guess a composition series for  $H$  where each subgroup is characteristic. Then we reduce the isomorphism problem onto graph isomorphism. The valence of the resulting graph depends on the size of the largest factor group.

**The reduction.** We generalize the reduction of Theorem 5.1 from  $p$ -groups to arbitrary groups.

Let  $(G, S)$  be a group  $G$  over  $n$  elements with a composition series  $S$  as in Algorithm 1 where each factor group  $G_i/G_{i+1}$  has order  $p_i$ :

$$\{1\} = G_k \triangleleft G_{k-1} \triangleleft \cdots \triangleleft G_1 = G$$

and let  $\vec{s} = (s_1, \dots, s_{k-1})$  be a complete set of coset representatives for  $S$ . Correspondingly, we guess  $(H, S')$  a group  $H$  with a complete set of coset representatives  $\vec{s}' = (s'_1, \dots, s'_{k-1})$  and composition series  $S'$ :

$$\{1\} = H_k \triangleleft H_{k-1} \triangleleft \cdots \triangleleft H_1 = H$$

If the factor groups do not have the same order, or are not of the same type, then  $(G, S)$  is not isomorphic to  $(H, S')$ . We prove the following theorem.

**Theorem 6.2** *Let  $p = \max\{p_1, \dots, p_{k-1}\}$ . There is an  $AC^0$ -computable function that computes a graph  $X(G, S)$  and  $X(H, S')$  with at most  $11n^2 + 1$  vertices which have valence at most  $p + 1$  such that  $X(G, S)$  is isomorphic to  $X(H, S')$  if and only if  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$ .*

**Proof.** First, we generalize the construction of  $X_p(G, S)$  in the proof of Theorem 5.1 from  $p$ -groups to groups and define a new graph  $X(G, S)$ .

First, we construct a tree  $T(G, S)$  the same way as  $T_p(G, S)$  in the proof of Theorem 5.1. If the factor group  $G_i/G_{i+1}$  is a non-abelian simple group, e.g.  $A_5$  the alternating group on 5 elements, then we have  $p_i = |G_i/G_{i+1}| = 60$  since  $|A_5| = 60$ .

The rest of the construction of  $X(G, S)$  is identical to the construction of  $X_p(G, S)$  in the proof of Theorem 5.1.

We prove now that  $X(G, S)$  has all the properties stated in Theorem 6.2.

**Claim 6.3** *The graph  $X(G, S)$  has at most  $11n^2 + 1$  vertices.*

The proof goes the same lines as in the proof of Claim 5.2. In  $X(G, S)$ , the inner nodes do not have the same valence, but every node has at least two children. Hence, we get at most  $11n^2 + 1$  vertices.

**Claim 6.4** *There is a logspace-computable function that computes the graph  $X(G, S)$ .*

The proof goes the same lines as the proof of Claim 5.3.

**Claim 6.5** *The graph  $X(G, S)$  has valence at most  $p + 1$  for any  $p \geq 2$ .*

Recall, that there is no factor group of order greater than  $p$ . The proof follows the lines of the proof of Claim 5.4.

**Claim 6.6** *The graph  $X(G, S)$  is isomorphic to  $X(H, S')$  if and only if  $(G, \text{seq}(S))$  is isomorphic to  $(H, \text{seq}(S'))$ .*

The proof is similar to the proof of Claim 5.5. In the proof we have a different tree structure, namely  $T(G, S)$  instead of  $T_p(G, S)$ . Both trees are rooted and complete. In  $T(G, S)$  nodes at the same distance to the root have the same valence. Hence, there are automorphisms that map a vertex onto every other vertex which has the same distance to the root node  $eG_1$ .

This completes the proof of Theorem 6.2. □

**Complexity.** Recall, that  $\beta$  is the maximum number of isomorphic minimal normal subgroups in a socle along the decomposition process and that  $\gamma$  is the maximum composition length of a minimal normal subgroup in any socle of the decomposition process. By Theorem 4.9 we need  $2\gamma \log n + \log n \log \beta + \log n \log \gamma \leq O(\log^2 n)$  non-deterministic bits or deterministic time  $n^{2\gamma + \log \beta + \log \gamma + c}$  for a constant  $c$ .

Let the factor groups in  $S$  and  $S'$  have size at least  $p$ . The isomorphism test runs for a constant  $c$  in time  $n^c$  plus the time complexity  $n^{c(p+1)}$  for bounded valence GI (see Theorem 5.6, [BL83]). We get the following theorem.

**Theorem 6.7** *Group Isomorphism for groups with  $n$  elements given in table representation can be tested in deterministic time  $O(n^{c(\gamma + \log \beta)} \cdot n^{c(p+1)})$  for a constant  $c$ .*

### 6.3 Combine both Isomorphism Tests

Algorithm 2 has its weak point when considering groups when decomposed into a large number of factor groups of small size, whereas the complexity of the second algorithm is bad if there are any factor groups of large size.

The idea is that we use both algorithms as a subroutine in the new algorithm. The algorithm gets additionally to the input a number  $\alpha$  as parameter. For factor groups of size larger than  $\alpha$  we guess the generators as in Algorithm 2. Since the number of corresponding factor groups is small, this also keeps the runtime of Algorithm 2 low in the new algorithm. Then we modify the construction of the graph  $X(G, S)$ , such that it has valence at most  $\alpha + 1$ . With parameter  $\alpha$  we improve the runtime of the new isomorphism testing algorithm.

**Changes to the graphs.** Since we said that we guess some of the generators, we make changes to the graphs from the reduction. We modify  $X(G, S)$  for a group  $G$  and a composition series  $S$  that is given by a complete set of coset representatives  $\vec{s} = (s_1, \dots, s_{k-1})$ .

**Lemma 6.8** *Let  $A = s_{i_1} \cup \dots \cup s_{i_j}$  be a set of generators, a subset of the generators in  $s_1, \dots, s_{k-1}$ . Let  $\alpha$  be the size of the factor group with largest size among those in  $S$  which do not have generators in  $A$ . Then we can compute a graph  $X(G, S, A)$  of valence at most  $\alpha + 1$  that behaves like  $X(G, S)$  but where any automorphism fixes the generators in  $A$  elementwise.*

**Proof.** Let  $a_i \in A$  be a generator, such that  $a_i G_{i+1}$  is a coset representative for a cyclic factor group  $G_i/G_{i+1}$  in  $(G, S)$ . Let  $a_i, b_i \in A$  be generators with respect to a non-abelian simple factor group  $G_i/G_{i+1}$  in  $(G, S)$ . Let  $D_i$  be the set of nodes at distance  $i$  to the root  $eG$ . That is,  $gG_i \in D_i$  with  $g$  any product of generators in  $w_1, \dots, w_i$ .

We do the following changes for every node  $gG_i \in D_i$  in the graph  $X(G, S)$ :

- *Remove edges to the children of  $gG_i$ .* For example, in the cyclic case remove  $\{gG_i, gg_{i+1}^{l_i+1}G_{i+1}\}$  where  $g_{i+1}$  is a fixed coset representative. In the non-abelian simple case, remove  $gw_jG_i$  where  $w_j$  is a word with  $a_i, b_i$  as in the proof of Lemma 4.2.
- *Arrange edges according to generators  $a_i$  (or  $a_i, b_i$ ).* Let  $l = \text{ord}(a_i) - 1$  (or let  $l$  be the order of the group generated by  $a_i, b_i$ ). Arrange the children from right to left:

$$(ga_i^l G_{i+1}, ga_i^{l-1} G_{i+1}, \dots, ga_i^1 G_{i+1}, gG_{i+1}).$$

This is done according to Lemma 4.2.

- *Connect the children from right to left to the leafs of a binary tree with root  $gG_i$  such that children have the same distance to  $gG_i$ .* We connect pairwise leafs or subtrees from right to left to form larger subtrees, inductively. The tree contains nodes with one or two children. See Figure 4 for an example.
- *Color the leaf connected to  $ga_i G_{i+1}$  (or the leafs connected to  $ga_i G_{i+1}$  and  $gb_i G_{i+1}$ ).* In the cyclic case, we connect  $ga_i G_{i+1}$  to an extra vertex. In the non-abelian simple case, we connect  $ga_i G_{i+1}$  to an extra vertex and  $gb_i G_{i+1}$  to a path of length three. These can be distinguished from all other vertices, because there is no vertex with the same distance to the root  $eG$  connected to a single vertex or a path of length three (we ignore nodes that come from other paths which serve as a coloring of nodes).

The colored nodes  $ga_i G_{i+1}$  (or  $ga_i G_{i+1}$  and  $gb_i G_{i+1}$ ) are fixed. If  $a_i G_{i+1}$  (or  $a_i G_{i+1}$  and  $b_i G_{i+1}$ ) are fixed in  $G_i/G_{i+1}$ , then every automorphism of the group fixes all cosets of the factor group  $G_i/G_{i+1}$ . Hence, all children of the node  $gG_i$  are fixed in any automorphism of the group.

The tree structure guarantees that the distances to the root  $eG$  remain unchanged for all vertices. The graph  $X(G, S)$  is a cone graph, it follows that this also holds for  $X(G, S, A)$ .

□

**The algorithm.** We give some notes to Algorithm 3.

In Lines 1 and 2, we compute a composition series  $S'$  for  $H$  and guess a composition series  $S$  for  $G$  together with coset representatives that we obtain from Algorithm 1.

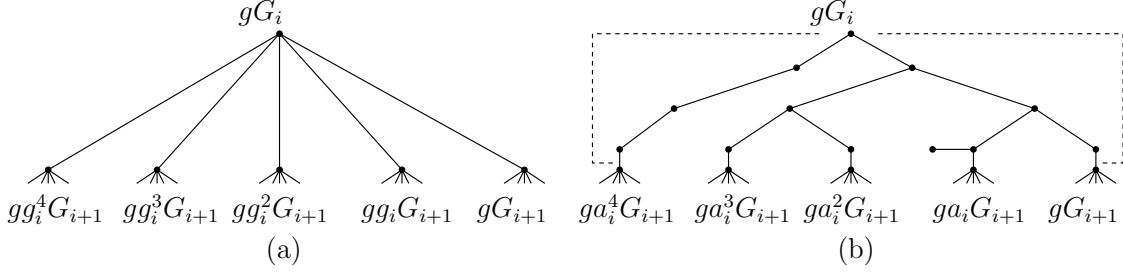


Figure 4: The situation is shown where  $G_i/G_{i+1}$  is a cyclic factor group.

(a) The node  $gG_i$  and its children  $gg_i^j G_{i+1}$  for all  $j \in \{0, \dots, l\}$  in  $X(G, S)$ , with  $l = 4$ .

(b) The graph gadget connected to these nodes in  $X(G, S, A)$  enclosed within the dashed box. Note,  $a_i$  can be any element in  $G_i \setminus G_{i+1}$ .

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**Algorithm 3** Isomorphism testing for Cayley Groups

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**Input:** multiplication tables  $G, H$  of two groups with  $n$  elements, parameter  $\alpha$

**Computation:** accept if  $G$  is isomorphic to  $H$  and reject otherwise

- 1: let  $S'$  be  $\vec{s}' = (s'_1, \dots, s'_{k-1}) = \text{CompSeries}(H)$  a composition series for  $H$  from Algorithm 1
  - 2: guess  $S$  with  $\vec{s} = (s_1, \dots, s_{k-1})$  from  $\text{CompSeries}(G)$  a composition series for  $G$  from Algorithm 1 as in Theorem 4.9
  - 3: **for**  $i \in \{1, \dots, k-1\}$  **do**
  - 4:   **if**  $G_i/G_{i+1}$  is of order  $> \alpha$  **then**
  - 5:     guess  $t = (a)$  (or  $t = (a, b)$ ) with  $\langle aG_{i+1}, bG_{i+1} \rangle \cong G_i/G_{i+1}$
  - 6:      $t_i = t, A \leftarrow a, b, A' \leftarrow a', b'$
  - 7:   **end if**
  - 8: **end for**
  - 9: **if**  $X(G, S, A) \cong X(H, S', A')$  then accept and halt
  - 10: reject and halt
- 

In Lines 3 and 4, we run through the factor groups of size greater than  $\alpha$ .

In Line 5, for each such factor group  $G_i/G_{i+1}$  we guess one or two generators for  $t_i$ , depending on whether the factor groups are cyclic or non-abelian simple groups, respectively.

In Line 6, we put the guessed generators in  $t_i$  also in a set  $A$  and correspondingly those in  $s'_i$  in a set  $A'$ .

In Line 9, we construct the graphs where we treat generators in the sets  $A$  and  $A'$  specially. If the graphs are isomorphic, then we accept and halt.

In Line 10, we reject and halt.

**The complexity of group isomorphism.** We calculate the runtime of Algorithm 3.

In Line 1 the composition series for  $H$  can be computed in polynomial time [Luk87, BLS87], also see Algorithm 1.

In Line 2, to guess the composition series for  $G$  we need

$$2\gamma \log n + \log n \log \beta + \log n \log \gamma \leq O(\log^2 n)$$



non-deterministic bits by Theorem 4.9. For a deterministic algorithm, we could run Algorithm 3 several times, trying all possibilities, i.e. it requires at most  $n^{c(\gamma+\log \beta)}$  time.

In Line 5, we say that we guess generators for composition factors of size  $> \alpha$ . For this we need  $\log n \log_\alpha n$  non-deterministic bits only. For a deterministic algorithm, we could run Algorithm 3 several times, trying all possibilities as generators. Hence, the running time is multiplied with  $2^{\log n \log_\alpha n} = n^{\log_\alpha n}$ .

In Line 9, we invoke an isomorphism testing algorithm for graphs of valence at most  $\alpha + 1$ . This algorithm runs in time  $n^{O(\alpha+1)}$ .

Hence, the total running time (for a constant  $c$ ) is:

$$n^{c(\gamma+\log \beta)}(n^{c \log_\alpha n} + n^{c\alpha})$$

Note, that  $\beta$  and  $\gamma$  depend on the input whereas  $\alpha$  can be chosen. We minimize the runtime if  $\log_\alpha n = \alpha$ , that is

$$\log n / \log \alpha = \alpha.$$

Now, we substitute  $\alpha$  by  $\log n / \log \alpha$  on the left side and get the following two equations which are equivalent:

$$\log n / \log(\log n / \log \alpha) = \alpha \tag{1}$$

$$\log n / (\log \log n - \log \log \alpha) = \alpha \tag{2}$$

If we substitute  $\alpha$  again by  $\log n / \log \alpha$  on the left side, and since  $\log \log \alpha \leq \log \log \log n$  we proved the main theorem.

**Theorem 1.1** *Group isomorphism for groups with  $n$  elements given in table representation is in time (for a constant  $c$ ):*

$$n^{c(\gamma+\log \beta+\log n / \log \log n)}$$

## Conclusion

We consider algorithms which reduce group isomorphism onto graph isomorphism. The complexity depends on the number of composition series which have to be computed for the groups. When composition series for the groups are fixed, then an isomorphism test which respects them can be done in time less than  $n^{\log n}$ , in the case of  $p$ -groups in polynomial time.

It would be interesting whether the number of composition series considered in the isomorphism tests can be reduced to improve the complexity, or whether techniques from graph theory or algorithms used for graph isomorphism can be modified to work with groups.

**Acknowledgments.** We thank V. Arvind, László Babai, Paolo Codenotti, Johannes Köbler, Youming Qiao and Jacobo Torán for helpful discussions and finding an error in a previous version.

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