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Constructive dimension and Hausdorff dimension: the case of exact dimension

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Abstract The present paper generalises results by Lutz and Ryabko. We prove a martingale characterisation of exact Hausdorff dimension. On this base we introduce the notion of exact constructive dimension of (sets of) infinite strings.
Furthermore, we generalise Ryabko's result on the Hausdorff dimension of the set of strings having asymptotic Kolmogorov complexity $\leq \alpha$ to the case of exact dimension.

The papers [13,14,16,17,7,8] show a close connection between Hausdorff dimension and constructive dimension or, equivalently, asymptotic Kolmogorov complexity of (sets of) infinite strings. In all these papers,the Hausdorff dimension of a set is defined as usual (cf. [3,4]) to be a real number. It is interesting to observe that already Hausdorff in his paper [6] defined the (fractal) dimension of a set to be a real function of a special shape. To distinguish it from the "usual" Hausdorff dimension Hausdorff's original definition is referred to as exact Hausdorff dimension [9,5,10].

The aim of the present paper is to generalise results by Lutz [7,8] and Ryabko [13] to the case of this exact dimension. First we deal with the martingale characterisation of Hausdorff dimension [7,8]. This leads in a natural way to a definition of exact constructive dimension. From this we derive the particularly interesting fact that the exact dimension of an infinite string ξ can be identified with Levin's [22] universal left computable continuous semi-measure restricted to the set of finite prefixes of ξ. As a further consequence we obtain a connection to the a priori complexity (cf. [20,21]) of finite strings yielding just another proof that constructive dimension equals Kolmogorov complexity (cf. [18]).

Having a priori complexity in mind we generalise Ryabko's result that the set of infinite strings having asymptotic Kolmogorov complexity $\leq \alpha$ has Hausdorff dimension α to the case of exact dimensions. Finally we apply our results to the family of functions of the logarithmic scale, as considered by Hausdorff [6]. Here we show that, unlike the case of asymptotic

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Kolmogorov complexity, the results involving exact dimensions depend on the kind of complexity (cf. [20,21]) of finite strings.

1 Notation and Preliminaries

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{Q}$ the set of rational numbers. Let X be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.

$|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$, and $\eta[0..n]$ is the n-length prefix of η provided $|\eta| \geq n$. A language $W \subseteq X^*$ is referred to as *prefix-free* if $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$. If $W \subseteq X^*$ then $\mathrm{Min}_{\sqsubseteq} W := \{w : w \in W \wedge \forall v(v \in W \rightarrow v \not\sqsubset w)\}$ is the (prefix-free) set of minimal w.r.t. $\sqsubseteq$ elements of W.

A *super-martingale* is a function $\mathcal{V} : X^* \rightarrow [0, \infty)$ which satisfies $\mathcal{V}(e) \leq 1$ and the super-martingale inequality

$$r \cdot \mathcal{V}(w) \leq \textstyle\sum_{x \in X} \mathcal{V}(wx) \text{ for all } w \in X^* . \tag{1}$$

If Eq. (1) is satisfied with equality $\mathcal{V}$ is called a martingale. Closely realted with (super-)martingales are continuous (or cylindrical) (semi-)measures $\mu : X^* \rightarrow [0, 1]$ where $\mu(e) \leq 1$ and $\mu(w) \leq \sum_{x \in X} \mu(wx)$ for all $w \in X^*$.

Indeed, if $\mathcal{V}$ is a super-martingale then $\mu(w) := r^{-|w|} \cdot \mathcal{V}(w)$ is a continuous (semi-)measure, and vice versa. It should be mentioned that for any continuous semi-measure μ and every prefix-free subset $w \subseteq X^*$ the inequality $\sum_{w \in W} \mu(w) \leq 1$ holds. This proves also the corresponding super-martingale inequality for prefix-free sets $W \subseteq X^*$.

$$\mathcal{V}(e) \geq \textstyle\sum_{w \in W} r^{-|w|} \cdot \mathcal{V}(w) \tag{2}$$

2 Hausdorff's approach

A function $h : (0, \infty) \rightarrow (0, \infty)$ is referred to as a *gauge function* provided h is positive, right continuous and non-decreasing. The h-dimensional outer measure of F on the space X^ω is given by

$$\mathcal{H}^h(F) := \lim_{n \to \infty} \inf\{ \sum_{v \in V} h(r^{-|v|}) : F \subseteq V \cdot X^\omega \wedge \min_{v \in V} |v| \geq n\} . \tag{3}$$

If $\lim_{t\to 0} h(t) > 0$ then $\mathcal{H}^h(F) < \infty$ if and only if F is finite.

The usual α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ is defined by the family of gauge functions $h_\alpha(t) = t^\alpha$, that is, $\mathcal{H}^\alpha = \mathcal{H}^{h_\alpha}$. Here $h_0(t) = t^0$ defines the counting measure on X^ω.

In this case it is possible to define the (usual) Hausdorff dimension of a set $F \subseteq X^\omega$ as

$$\dim_{\mathrm{H}} F := \sup\{\alpha : \alpha = 0 \vee \mathcal{H}^\alpha(F) = \infty\} = \inf\{\alpha : \alpha \geq 0 \wedge \mathcal{H}^\alpha(F) = 0\}\,.$$

As we see from Eq. (3) for our purposes the behaviour of gauge function is of interest only in a small vicinity of 0. Moreover, in many cases we are not interested in the exact value of $\mathcal{H}^h(F)$ when $0 < \mathcal{H}^h(F) < \infty$. Thus we can often make use of scaling a gauge function and altering it in a range $(\varepsilon, 1]$ apart from 0.

The following properties of gauge functions h and the related measure $\mathcal{H}^h$ are proved in the standard way.

Property 1. Let h, h' be gauge functions.

1. If $c_1 \cdot h(r^{-n}) \leq h'(r^{-n}) \leq c_2 \cdot h(r^{-n})$ for some $c_1, c_2,\ 0 < c_1 \leq c_2$, then $c_1 \cdot \mathcal{H}^h(F) \leq \mathcal{H}^{h'}(F) \leq c_2 \cdot \mathcal{H}^h(F)$.
2. If $\lim\limits_{n\to\infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0$ then $\mathcal{H}^{h'}(F) < \infty$ implies $\mathcal{H}^h(F) = 0$, and $\mathcal{H}^h(F) > 0$ implies $\mathcal{H}^{h'}(F) = \infty$.

Here the first property could be called equivalence of gauge functions. In fact, if h and h' are equivalent in the sense of Property 1.1 then for all $F \subseteq X^\omega$ the measures $\mathcal{H}^h(F)$ and $\mathcal{H}^{h'}(F)$ are both zero, finite or infinite. In the same way the second property gives an ordering of gauge functions. The ordering is denoted by $\prec$ where $h' \prec h$ is an abbreviation for $\lim\limits_{n\to\infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0$, that is, $h(r^{-n})$ tends faster to 0 than $h'(r^{-n})$ as n tends to infinity.

By analogy to the change-over-point $\dim_{\mathrm{H}} F$ for $\mathcal{H}^\alpha(F)$ the partial ordering $\prec$ yields a suitable notion of Hausdorff dimension in the range of arbitrary gauge functions.

Definition 1. We refer to a gauge function h as *exact Hausdorff dimension function* for $F \subseteq X^\omega$ provided

$$\mathcal{H}^{h'}(F) = \begin{cases} \infty\,, & \text{if } h' \prec h\ \text{, and} \\ 0\,, & \text{if } h \prec h'\,. \end{cases}$$

Remark that, since $\prec$ is not a total ordering, nothing is said about the measure $\mathcal{H}^{h'}(F)$ for functions h' which are equivalent or not comparable to h. Hausdorff called a function h *dimension* of F provided $0 < \mathcal{H}^h(F) < \infty$. This case is covered by our definition and Property 1.

One easily observes that $h_0(t) := t$ yields $\mathcal{H}^{h_0}(F) \leq 1$, thus $\mathcal{H}^{h'}(F) = 0$ for all $h',\ h_0 \prec h'$. Therefore, we can always assume that a gauge function satisfies $h(t) > t^2,\ t \in (0,1)$.

2.1 Exact Hausdorff dimension and martingales

In this section we show a generalisation of Lutz's theorem to arbitrary gauge functions. To obtain a transparent notation we do not use Lutz's s-gale notation but instead we follow Schnorr's approach of combining martingales with order functions. For a discussion of both approaches see Section 13.2 of [2].

Let, for a super-martingale $\mathcal{V} : X^* \to [0,\infty)$, a gauge function h and a value $c \in (0,\infty]$ be $S_{c,h}[\mathcal{V}] := \{\xi : \xi \in X^\omega \wedge \limsup_{n\to\infty} \frac{\mathcal{V}(\xi[0..n])}{r^n \cdot h(r^{-n})} \geq c\}$. In particular, $S_{\infty,h}[\mathcal{V}]$ is the set of all ω-words on which the super-martingale $\mathcal{V}$ is successful w.r.t. the order function $f(n) = r^n \cdot h(r^{-n})$ in the sense of Schnorr [15].

Now we can prove the analogue to Lutz's theorem. In view of Property 1 we split the assertion into two parts.

Lemma 1. *Let $F \subseteq X^\omega$ and h, h' be gauge functions such that $h \prec h'$ and $\mathcal{H}^h(F) < \infty$. Then $F \subseteq S_{\infty,h'}[\mathcal{V}]$ for some martingale $\mathcal{V}$.*

Proof. First we follow the lines of the proof of Theorem 13.2.3 in [2] and show the assertion for $\mathcal{H}^h(F) = 0$. Thus there are prefix-free subsets $U_i \subseteq X^*$ such that $F \subseteq \bigcap_{i\in\mathbb{N}} U_i \cdot X^\omega$ and $\sum_{u\in U_i} h(r^{-|u|}) \leq 2^{-i}$.

Define $\mathcal{V}_i(w) := \begin{cases} r^{|w|} \cdot \sum_{wu\in U_i} h(r^{-|wu|}), & \text{if } w \in \mathbf{pref}(U_i) \setminus U_i \text{ , and} \\ \sup\{r^{|v|} \cdot h(r^{-|v|}) : v \sqsubseteq w \wedge v \in U_i\}, & \text{otherwise}^1. \end{cases}$

In order to prove that $\mathcal{V}_i$ is a martingale we consider three cases:

$w \in \mathbf{pref}(U_i) \setminus U_i$: Since then $U_i \cap w \cdot X^* = \bigcup_{x\in X} U_i \cap wx \cdot X^*$, we have $\mathcal{V}_i(w) = r^{|w|} \cdot \sum_{wu\in U_i} h(r^{-|wu|}) = r^{-1} \cdot \sum_{x\in X} r^{|wx|} \sum_{wxu\in U_i} h(r^{-|wxu|}) = r^{-1} \cdot \sum_{x\in X} \mathcal{V}_i(wx)$.

$w \in U_i \cdot X^*$: Let $w \in v \cdot X^*$ where $v \in U_i$. Then $\mathcal{V}_i(w) = \mathcal{V}_i(wx) = r^{|v|} \cdot h(r^{-|v|})$ whence $\mathcal{V}_i(w) = r^{-1} \cdot \sum_{x\in X} \mathcal{V}_i(wx)$.

$w \notin \mathbf{pref}(U_i) \cup U_i \cdot X^*$: Here $\mathcal{V}_i(w) = \mathcal{V}_i(wx) = 0$.

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[1] This yields $\mathcal{V}_i(w) = 0$ for $w \notin \mathbf{pref}(U_i) \cup U_i \cdot X^*$.

Now, set $\mathcal{V}(w) := \sum_{i\in\mathbb{N}} \mathcal{V}_i(w)$.

Then, for $\xi \in \bigcap_{i\in\mathbb{N}} U_i \cdot X^\omega$ there are $n_i \in \mathbb{N}$ such that $\xi[0..n_i] \in U_i$ and we obtain $\frac{\mathcal{V}(\xi[0..n_i])}{r^{n_i}\cdot h'(r^{-n_i})} \geq \frac{\mathcal{V}_i(\xi[0..n_i])}{r^{n_i}\cdot h'(r^{-n_i})} = \frac{h(r^{-n_i})}{h'(r^{-n_i})}$ which tends to infinity as i tends to infinity.

Now let $\mathcal{H}^h(F) < \infty$. Then $h \prec \sqrt{h\cdot h'} \prec h'$. Thus $\mathcal{H}^{\sqrt{h\cdot h'}}(F) = 0$ and we can apply the first part of the proof to the functions $\sqrt{h\cdot h'}$ and h'. □

The next lemma is in some sense a converse to Lemma 1

Lemma 2. *Let h be a gauge function, $c \in (0,\infty]$ and $\mathcal{V}$ be a super-martingale. Then $\mathcal{H}^h(S_{c,h}[\mathcal{V}]) \leq \frac{\mathcal{V}(e)}{c}$.*

Proof. It suffices to prove the assertion for $c < \infty$.

Define $V_k := \{w : w \in X^* \wedge |w| \geq k \wedge \frac{\mathcal{V}(w)}{r^{|w|}\cdot h(r^{-|w|})} \geq c - 2^{-k}\}$ and set $U_k := \mathrm{Min}_\sqsubseteq V_k$. Then $S_{c,h}[\mathcal{V}] \subseteq \bigcap_{k\in\mathbb{N}} U_k \cdot X^\omega$.

Now $\sum\limits_{w\in U_k} h(r^{-|w|}) \leq \sum\limits_{w\in U_k} h(r^{-|w|}) \cdot \frac{\mathcal{V}(w)}{r^{|w|}\cdot h(r^{-|w|})} \cdot \frac{1}{c-2^{-k}} = \frac{1}{c-2^{-k}} \cdot \sum\limits_{w\in U_k} \frac{\mathcal{V}(w)}{r^{|w|}} \leq \frac{\mathcal{V}(e)}{c-2^{-k}}$ (cf. Eq. (2)). Thus $\mathcal{H}^h(\bigcap_{k\in\mathbb{N}} U_k \cdot X^\omega) \leq \frac{\mathcal{V}(e)}{c}$. □

Lemmata 1 and 2 yield the following martingale characterisation of exact Hausdorff dimension functions.

Theorem 1. *Let $F \subseteq X^\omega$. Then a gauge function h is an exact Hausdorff dimension function for F if and only if*

1. *for all gauge functions h' with $h \prec h'$ there is a super-martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$, and*
2. *for all gauge functions h'' with $h'' \prec h$ and all super-martingales $\mathcal{V}$ it holds $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$.*

Lemmata 1 and 2 also show that we can likewise formulate Theorem 1 for martingales instead of super-martingales.

3 Constructive dimension: the exact case

The constructive dimension is a variant of dimension defined analogously to Theorem 1 using only left computable super-martingales. For the usual family of gauge functions $h_\alpha(t) = t^\alpha$ it was introduced by Lutz [7] and resulted, similarly to $\dim_{\mathrm{H}}$ in a real number assigned to a subset $F \subseteq X^\omega$. In the case of left computable super-martingales the situation turned out to be simpler because the results of Levin [22] and Schnorr [15] show that there is an optimal left computable super-martingale $\mathcal{U}$, that is, every other left computable super-martingale $\mathcal{V}$ satisfies $\mathcal{V}(w) \leq c_{\mathcal{V}} \cdot \mathcal{U}(w)$ for all $w \in X^*$ and some constant $c_{\mathcal{V}} > 0$ not depending on w. Thus we may define

Definition 2. *Let $F \subseteq X^\omega$. We refer to $h : \mathbb{R} \to \mathbb{R}$ as an* exact constructive dimension function *for F provided $F \subseteq S_{\infty,h'}[\mathcal{U}]$ for all $h', h \prec h'$ and $F \not\subseteq S_{\infty,h''}[\mathcal{U}]$ for all $h'', h'' \prec h$.*

Originally, Levin showed that there is an optimal left computable continuous semi-measure $\mathbf{M}$ on X^*. As usual, we call a function $\mu : X^* \to [0, \infty)$ a *continuous* (or *cylindrical*) *semi-measure* on X^* provided $\mu(e) \leq 1$ and $\mu(w) \geq \sum_{x \in X} \mu(wx)$ for all $w \in X^*$. One easily verifies that μ is a continuous semi-measure if and only if $\mathcal{V}(w) := r^{|w|} \cdot \mu(w)$ is a super-martingale.

Thus we might use $\mathcal{U}_{\mathbf{M}}$ with $\mathcal{U}_{\mathbf{M}}(w) := r^{|w|} \cdot \mathbf{M}(w)$ as our optimal left computable super-martingale. The proof of the next theorem makes use of this fact and of the inequality $\mathbf{M}(w) \geq \mathbf{M}(w \cdot v)$.

Theorem 2. *The function h_ξ defined by $h_\xi(r^{-n}) := \mathbf{M}(\xi[0..n])$ is an exact constructive dimension function for the set $\{\xi\}$.*

Closely related to Levin's optimal left computable semi-measure is the *a priori entropy* (or *complexity*) $\mathrm{KA} : X^* \to \mathbb{N}$ defined by

$$\mathrm{KA}(w) := \lfloor -\log_r \mathbf{M}(w) \rfloor \tag{4}$$

First we mention the following bound from [11].

Theorem 3. *Let $F \subseteq X^\omega$, h be a gauge function and $\mathcal{H}^h(F) > 0$.*

Then for every $c > 0$ with $\mathcal{H}^h(F) > c \cdot \mathbf{M}(e)$ there is a $\xi \in F$ such that $\mathrm{KA}(\xi[0..n]) \geq_{\mathrm{a.e.}} -\log_r h(r^{-n}) - \log_r c$.

This lower bound on the maximum complexity of an infinite string in F yields a set-theoretic lower bound on the success sets $S_{c,h}[\mathcal{U}]$ of $\mathcal{U}$.

Theorem 4. *Let $-\infty < c < \infty$ and let h be a gauge function. Then there is a $c' > 0$ such that*

$$\{\xi : \exists^\infty n(\mathrm{KA}(\xi[0..n]) \leq \log_r h(r^{-n}) + c)\} \subseteq S_{c',h}[\mathcal{U}].$$

Corollary 1. *Let h, h' be gauge functions such that $h \prec h'$. Then*

1. $\{\xi : \exists c \exists^\infty n(\mathrm{KA}(\xi[0..n]) \leq \log_r h(r^{-n}) + c)\} \subseteq S_{\infty,h'}[\mathcal{U}]$, *and*
2. $\mathcal{H}^{h'}(\{\xi : \exists c \exists^\infty n(\mathrm{KA}(\xi[0..n]) \leq -\log_r h(r^{-n}) + c)\}) = 0$.

4 Complexity

In this section we are going to show that, analogously to Ryabko's and Lutz's results for the "usual" dimension the bound given in Corollary 1 is tight for a large class of (computable) gauge functions. To this end we prove that certain sets of infinite strings diluted according to a gauge function h have positive Hausdorff measure $\mathcal{H}^h$.

4.1 A generalised dilution principle

We are going to show that for a large family of gauge functions set of finite positive measures can be constructed. Our construction is a generalisation of Hausdorff's 1918 construction. Instead of his method of cutting out middle thirds in the unit interval we use the idea of dilution functions as presented in [19]. In fact dilution appears much earlier (see e.g. [1,16,8])

We consider *prefix-monotone* mappings, that is, mappings $\varphi : X^* \to X^*$ satisfying $\varphi(w) \sqsubseteq \varphi(v)$ whenever $w \sqsubseteq v$. We call a function $g : \mathbb{N} \to \mathbb{N}$ a *modulus function* for φ provided $|\varphi(w)| = g(|w|)$ for all $w \in X^*$. This, in particular, implies that $|\varphi(w)| = |\varphi(v)|$ for $|w| = |v|$ when φ has a modulus function.

Every prefix-monotone mapping $\varphi : X^* \to X^*$ defines as a limit a partial mapping $\overline{\varphi} :\subseteq X^\omega \to X^\omega$ in the following way: $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$ whenever $\varphi(\mathbf{pref}(\xi))$ is an infinite set, and $\overline{\varphi}(\xi)$ is undefined when $\varphi(\mathbf{pref}(\xi))$ is finite.

If, for some strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, the mapping φ satisfies the conditions $|\varphi(w)| = g(|w|)$ and for every $v \in \mathbf{pref}(\varphi(X^*))$ there are $w_v \in X^*$ and $x_v \in X$ such that

$$\varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v) \wedge \forall y(y \in X \wedge y \neq x_v \to v \not\sqsubseteq \varphi(w_v \cdot y)) \qquad (5)$$

then we call φ a *dilution function* with modulus g. If φ is a dilution function then $\overline{\varphi}$ is a one-to-one mapping. For the image $\overline{\varphi}(X^\omega)$ we obtain the following bounds on its Hausdorff measure.

Theorem 5. *Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, φ a corresponding dilution function and $h : (0, \infty) \to (0, \infty)$ be a gauge function. Then*

1. $\mathcal{H}^h(\overline{\varphi}(X^\omega)) \leq \liminf_{n\to\infty} \frac{h(r^{-g(n)})}{r^{-n}}$
2. *If* $c \cdot r^{-n} \leq_{\mathrm{ae}} h(r^{-g(n)})$ *then* $c \leq \mathcal{H}^h(\overline{\varphi}(X^\omega))$.

Proof. The first assertion follows from $\overline{\varphi}(X^\omega) \subseteq \bigcup_{|w|=n} \varphi(w) \cdot X^\omega$ and $|\varphi(w)| = g(|w|)$.

The second assertion is obvious for $\mathcal{H}^h(\overline{\varphi}(X^\omega)) = \infty$. Let $\mathcal{H}^h(\overline{\varphi}(X^\omega)) < \infty$, $\varepsilon > 0$, and $V \cdot X^\omega \supseteq \overline{\varphi}(X^\omega)$ such that $\sum_{v\in V} h(r^{-|v|}) \leq \mathcal{H}^h(\overline{\varphi}(X^\omega)) + \varepsilon$. The set $W_V := \{w_v \cdot x_v : v \in V \wedge \varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v)\}$ (see Eq. (5)) is prefix-free and it holds $W_V \cdot X^\omega \supseteq X^\omega$. Thus W_V is finite and $\sum_{w\in W_V} r^{-|w|} = 1$.

Assume now $\min\{|v| : v \in V\}$ large enough such that $c \cdot r^{-|v|} \leq_{\mathrm{ae}} h(r^{-|v|})$ for all $v \in V$.

$$\text{Then } \sum_{v\in V} h(r^{-|v|}) \geq \sum_{wx\in W_V} h(r^{-|\varphi(wx)|}) = \sum_{wx\in W_V} h(r^{-g(|wx|)}) \geq \sum_{wx\in W_V} c \cdot r^{-|wx|} = c\,.$$

As $\varepsilon > 0$ is arbitrary, the assertion follows. □

Corollary 2. *If* $c \cdot r^{-n} \leq_{\mathrm{ae}} h(r^{-g(n)}) \leq c' \cdot r^{-n}$ *then* $c \leq \mathcal{H}^h(\overline{\varphi}(X^\omega)) \leq c'$.

In connection with Theorem 5 and Corollary 2 it is of interest which gauge functions allow for a construction of a set of positive finite measure via dilution. Hausdorff's cutting out was demonstrated for upwardly convex[2] gauge functions. We consider the slightly more general case of functions fulfilling the following.

Lemma 3. *If a gauge function h is upwardly convex on some interval $(0, \varepsilon)$ and $\lim_{t\to 0} h(t) = 0$ then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is an $m \in \mathbb{N}$ satisfying*

$$r^{-n} < h(r^{-m}) \leq r^{-n+1}\,. \tag{6}$$

In particular, Eq. (6) implies that the gauge function h does not tend faster to 0 than the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$.

Proof. If h is monotone, upwardly convex on $(0, \varepsilon)$ and $h(0) = 0$ then, in particular, $h(\gamma) \geq \gamma \cdot h(\gamma')/\gamma'$ whenever $0 \leq \gamma \leq \gamma' \leq \varepsilon$. Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$ be the largest number such that $r^{-n} < h(r^{-m})$. Then $h(r^{-m-1}) \leq r^{-n} < h(r^{-m}) \leq r \cdot h(r^{-m-1}) \leq r^{-n+1}$. □

Remark 1. Using the scaling factor $c = r^{n_0}$, that is, considering $c \cdot h$ instead of h and taking $h'(t) = \min\{c \cdot h(t), r\}$ one can always assume that $n_0 = 0$ and $h'(1) > 1$. Defining then $g(n) := \max\{m : m \in \mathbb{N} \wedge r^{-n} < h(r^{-m})\}$ we obtain via Property 1 and Corollary 2 that for every gauge function h fulfilling Eq. (6) there is a subset F_h of X^ω having finite and positive $\mathcal{H}^h$-measure.

[2] A function $f : \mathbb{R} \to \mathbb{R}$ is called *upwardly convex* if $f(a+t(b-a)) \geq f(a)+t(f(b)-f(a))$ for all $t \in [0, 1]$.

4.2 Computable gauge functions

The aim of this section is to show that the modulus function g and thus the dilution function φ can be chosen computable if only the gauge function h fulfilling Eq. (6) is computable.

For a computable domain $\mathcal{D}$, such as $\mathbb{N}$, $\mathbb{Q}$ or X^*, we refer to a function $f : \mathcal{D} \to \mathbb{R}$ as *left computable* (or *approximable from below*) provided the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q < f(d)\}$ is computably enumerable. Accordingly, a function $f : \mathcal{D} \to \mathbb{R}$ is called *right computable* (or *approximable from above*) if the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q > f(d)\}$ is computably enumerable, and f is *computable* if f is right and left computable.

If we refer to a function $f : \mathcal{D} \to \mathbb{Q}$ as computable we usually mean that it maps the domain $\mathcal{D}$ to the domain $\mathbb{Q}$, that is, it returns the exact value $f(d) \in \mathbb{Q}$.

Lemma 4. *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function satisfying the conditions that $1 < h(1) < r$ and for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $r^{-n} < h(r^{-m}) \le r^{-n+1}$. Then there is a computable strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $r^{-n-1} < h(r^{-g(n)}) < r^{-n+1}$.*

Proof. We define g inductively. To this end we compute for every $n \ge 1$ a closed interval I_n such that $h(r^{-g(n)}) \in I_n \subset (r^{-n}, \min I_{n-1})$

We start with $g(0) := 0$ and $I_{-1} = [r, r+1]$ and estimate I_0 as an sufficiently small approximating interval of $h(r^{-g(0)}) > 1$ satisfying $I_0 \subseteq (1, r)$.

Assume now that for n the value $g(n)$ and the interval I_n satisfying $h(r^{-g(n)}) \in I_n \subset (r^{-n}, \min I_{n-1})$ are computed.

We search for an m and an approximating interval $I(m)$, $h(r^{-m}) \in I(m)$, such that $I(m) \subset (r^{-n-1}, \min I_n)$. Since $\liminf_{m\to\infty} h(r^{-m}) = 0$ and $\exists m(r^{-n-1} < h(r^{-m}) \le r^{-n}) < \min I_n$ this search will eventually be successful. Define $g(n+1)$ as the first such m found by our procedure and set $I_n := I(m)$.

Finally, the monotonicity of h implies $g(n+1) > g(n)$. □

With Corollary 2 we obtain the following.

Corollary 3. *Under the hypotheses of Lemma 4 there is a computable dilution function $\varphi : X^* \to X^*$ such that $r^{-1} \le \mathcal{H}^h(\overline{\varphi}(X^\omega)) \le r$.*

4.3 Complexity of diluted infinite strings

In the final part of this Section 4 we show that for a large class of computable gauge functions h is an exact dimension function for the set

$\{\xi : \exists c \exists^\infty n (\mathrm{KA}(\xi[0..n]) \leq -\log_r h(r^{-n}) + c)\}$. This proves a converse to Corollary 1.2. To this end we use the following lower bound on the maximum a priori complexity of a diluted string from [19].

Theorem 6. *Let $\varphi : X^* \to X^*$ be a one-to-one prefix-monotone recursive function satisfying Eq (5) with strictly increasing modulus function g. Then*
$|\mathrm{KA}(\overline{\varphi}(\xi)[0..g(n)]) - \mathrm{KA}(\xi[0..n])| \leq O(1)$ *for all $\xi \in X^\omega$ and all $n \in \mathbb{N}$.*

We obtain our result.

Theorem 7. *If $h : \mathbb{Q} \to \mathbb{R}$ is a computable gauge function satisfying Eq. (6) then there is a $c \in \mathbb{N}$ such that*

$$\mathcal{H}^h(\{\zeta : \mathrm{KA}(\zeta[0..\ell]) \leq_{\text{a.e.}} -\log_r h(r^{-\ell}) + c\}) > 0.$$

Proof. From the gauge function h we construct a computable dilution function φ with modulus function g such that $r^{-(l+k+1)} < g(r^{-g(l)}) < r^{-(l+k-1)}$ for a suitable constant k (cf. Lemma 4 and Remark 1). Then, according to Corollary 3, $\mathcal{H}^h(\overline{\varphi}(X^\omega)) > 0$.

Using Theorem 6 we obtain $\mathrm{KA}(\overline{\varphi}(\xi)[0..g(l)]) \leq \mathrm{KA}(\xi[0..l]) + c_1 \leq l + c_2$ for suitable constants $c_1, c_2 \in \mathbb{N}$. Let $n \in \mathbb{N}$ satisfy $g(l) < n \leq g(n+1)$. Then $\mathrm{KA}(\overline{\varphi}(\xi)[0..n] \leq \mathrm{KA}(\overline{\varphi}(\xi)[0..g(l+1)] \leq l + 1 + c_2$.

Now from $l + k - 1 < -\log_r h(r^{-g(l)}) \leq -\log_r h(r^{-n})$ we obtain the assertion $\mathrm{KA}(\overline{\varphi}(\xi)[0..n] \leq -\log_r h(r^{-n}) + k + c_2$. □

5 Functions of the logarithmic scale

The final part of this paper is devoted to a generalisation of the "usual" dimensions using Hausdorff's family of functions of the logarithmic scale. This family is, similarly to the family $h_\alpha(t) = t^\alpha$, also linearly ordered and, thus, allows for more specific versions of Corollary 1.2 and Theorem 7.

A function of the form where the first non-zero exponent satisfies $p_i > 0$

$$h_{(p_0,\dots,p_k)}(t) = t^{p_0} \cdot \prod_{i=1}^{k} (\log^i \tfrac{1}{t})^{p_i} \qquad (7)$$

is referred to as a *function of the logarithmic scale* (see [6]). Here we have the convention that $\log^i t = \max\{\underbrace{\log_r \dots \log_r}_{i \text{ times}} t \ , 1\}$.

One observes that the lexicographic order on the tuples $(p_0, \dots, p_k)$ yields an order of the functions $h_{(p_0,\dots,p_k)}$ in sense that $(p_0, \dots, p_k) >_{\text{lex}} (q_0, \dots, q_k)$ if and only if $h_{(q_0,\dots,q_k)}(t) \prec h_{(p_0,\dots,p_k)}(t)$.

This gives rise to a generalisation of the "usual" Hausdorff dimension as follows.

$$\begin{aligned}\dim_{\mathrm{H}}^{(k)} F &:= \sup\{(p_0,\dots,p_k) : \mathcal{H}^{h_{(p_0,\dots,p_k)}}(F) = \infty\}\\ &= \inf\{(p_0,\dots,p_k) : \mathcal{H}^{h_{(p_0,\dots,p_k)}}(F) = 0\}\end{aligned} \tag{8}$$

When taking supremum or infimum we admit also values $-\infty$ and ∞ although we did not define the corresponding functions of the logarithmic scale. E.g. $\dim_{\mathrm{H}}^{(1)} F = (0,\infty)$ means that $\mathcal{H}^{h_{(0,\gamma)}}(F) = \infty$ but $\mathcal{H}^{h_{(\alpha,-\gamma)}}(F) = 0$ for all $\gamma \in (0,\infty)$ and all $\alpha > 0$.

The following theorems generalise Ryabko's [13] result on the "usual" Hausdorff dimension (case $k = 0$) of the set of strings having asymptotic Kolmogorov complexity $\leq p_0$.

Let $h_{(p_0,\dots,p_k)}$ be a function of the logarithmic scale. We define its first logarithmic truncation as $\beta_h(t) := -\log_r h_{(p_0,\dots,p_{k-1})}$. Observe that $\beta_h(r^{-n}) = p_0 \cdot n + \sum_{i=1}^{k-1} p_i \cdot \log^i n$ and $-\log h_{(p_0,\dots,p_k)}(r^{-n}) = \beta_h(r^{-n}) + p_k \cdot \log^k n$, for sufficiently small $t > 0$.

Then from Corollary 1.2 we obtain the following result analogous to Ryabko's theorem.

Theorem 8 ([12]). *Let $k > 0$, $(p_0,\dots,p_k)$ be a $(k+1)$-tuple and $h_{(p_0,\dots,p_k)}$ be a function of the logarithmic scale. Then*
$\dim_{\mathrm{H}}^{(k)}\left\{\xi : \xi \in X^\omega \wedge \liminf_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k\right\} \leq (p_0,\dots,p_k)\,.$

Using Theorem 7 we obtain a partial converse to Theorem 8 slightly refining Satz 4.11 of [12].

Theorem 9. *Let $k > 0$, $(p_0,\dots,p_k)$ be a $(k+1)$-tuple where $p_0 > 0$ and $p_0,\dots,p_{k-1}$ are computable numbers. Then for the function $h_{(p_0,\dots,p_k)}$ it holds*

$$\dim_{\mathrm{H}}^{(k)}\left\{\xi : \xi \in X^\omega \wedge \limsup_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} \leq p_k\right\} = (p_0,\dots,p_k)\,.$$

Ryabko's [13] theorem is independent of the kind of complexity we use. The following example shows that, already in case $k = 1$, Theorem 8 does not hold for plain Kolmogorov complexity KS (cf. [20,21,2]).

Example 1. It is known that $\mathrm{KS}(\xi[0..n]) \leq n - \log_r n + O(1)$ for all $\xi \in X^\omega$ (cf. [2, Corollary 3.11.3]). Thus every $\xi \in X^\omega$ satisfies $\liminf\limits_{n\to\infty} \frac{\mathrm{KS}(\xi[0..n]) - n}{\log_r n} < -\frac{1}{2}$. Consequently,

$\dim_{\mathrm{H}}^{(1)}\{\xi : \xi \in X^\omega \wedge \liminf\limits_{n\to\infty} \frac{\mathrm{KS}(\xi[0..n]) - n}{\log_{|X|} n} < -\frac{1}{2}\} = (1,0) >_{\mathrm{lex}} (1,-\frac{1}{2})$.

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A Proofs

A.1 Proof of Theorem 1

Proof. Assume h to be exact for F and $h \prec h'$. Then $h \prec \sqrt{h \cdot h'} \prec h'$. Thus $\mathcal{H}^{\sqrt{h \cdot h'}}(F) = 0$ and applying Lemma 1 to $\sqrt{h \cdot h'}$ and h' yields a super-martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$.

If $h'' \prec h$ then $\mathcal{H}^{h''}(F) = \infty$ and according to Lemma 2 $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$ for all super-martingales $\mathcal{V}$.

Conversely, let Conditions 1 and 2 be satisfied. Let $h \prec h'$, and let $\mathcal{V}$ be a super-martingale such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$. Now Lemma 2 shows $\mathcal{H}^{h'}(F) \leq \mathcal{H}^{h'}(S_{\infty,h'}[\mathcal{V}]) = 0$.

Finally, suppose $h'' \prec h$ and $\mathcal{H}^{h''}(F) < \infty$. Then $\mathcal{H}^{\sqrt{h \cdot h''}}(F) = 0$ and Lemma 1 shows that there is a super-martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,\sqrt{h \cdot h''}}[\mathcal{V}]$. This contradicts Condition 2. □

A.2 Proof of Theorem 4

Proof. If ξ has infinitely many prefixes such that $\mathbf{M}(\xi[0..n]) = h_\xi(r^{-n}) \geq c \cdot h(r^{-n})$ then, since $\mathcal{U}(w) \geq c'' \cdot r^n \cdot \mathbf{M}(w)$ for a suitable $c'' > 0$, we obtain $\limsup_{n\to\infty} \frac{\mathcal{U}(\xi[0..n])}{r^n \cdot h(r^{-n})} \geq \limsup_{n\to\infty} \frac{c'' \cdot r^n \cdot \mathbf{M}(\xi[0..n])}{r^n \cdot h(r^{-n})} \geq \frac{c''}{c}$ □

A.3 Proof of Theorem 8

Proof. From $\liminf_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k$ follows $\mathrm{KA}(\xi[0..n]) \leq \beta_h(2^{-n}) + p'_k \cdot \log^k n + O(1)$ for some $p'_k < p_k$. Thus $h_{(p_0,\dots,p'_k)} \prec h_{(p_0,\dots,p_k)}$ and the assertion follows from Corollary 1.2. □

A.4 Proof of Theorem 9

Proof. Let $p'_k < p_k$ be a computable number. Then $h_{(p_0,\dots,p'_k)}$ is a computable gauge function, $h_{(p_0,\dots,p'_k)} \prec h_{(p_0,\dots,p_k)}$ and $\mathcal{H}^h(\{\xi : \mathrm{KA}(\xi[0..n]) \leq -\log_r h(r^{-n}) + c_h\}) > 0$ for $h = h_{(p_0,\dots,p'_k)}$ and some constant c_h. Moreover $\mathrm{KA}(\xi[0..n]) \leq -\log_r h(r^{-n}) + c_h$ implies $\limsup\limits_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} \leq p_k$. Thus $\dim_{\mathrm{H}}^{(k)} \left\{ \xi : \xi \in X^\omega \wedge \limsup\limits_{n\to\infty} \frac{\mathrm{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} \leq p_k \right\} \geq (p_0, \dots, p'_k)$.

As p'_k can be made arbitrarily close to p_k the assertion follows. □

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