Tight Approximation Bounds for Vertex Cover on Dense $k$-Partite Hypergraphs

Marek Karpinski∗ Richard Schmied† Claus Viehmann‡

Abstract
We establish almost tight upper and lower approximation bounds for the Vertex Cover problem on dense $k$-partite hypergraphs.

1 Introduction
A hypergraph $H = (V, E)$ consists of a vertex set $V$ and a collection of hyperedges $E$ where a hyperedge is a subset of $V$. $H$ is called $k$-uniform if every edge in $E$ contains exactly $k$ vertices. A subset $C$ of $V$ is a vertex cover of $H$ if every edge $e \in E$ contains at least a vertex of $C$.

The Vertex Cover problem in a $k$-uniform hypergraph $H$ is the problem of computing a minimum cardinality vertex cover in $H$. It is well known that the problem is $NP$-hard even for $k = 2$ (cf. [13]). On the other hand, the simple greedy heuristic which chooses a maximal set of nonintersecting edges, and then outputs all vertices in those edges, gives a $k$-approximation algorithm for the Vertex Cover problem restricted to $k$-uniform hypergraphs. The best known approximation algorithm achieves a slightly better approximation ratio of $(1 - o(1))k$ and is due to Halperin [11].

On the intractability side, Trevisan [22] provided one of the first inapproximability results for the $k$-uniform vertex cover problem and obtained an inapproximability factor of $k\pi$ assuming $P \neq NP$. In 2002, Holmerin [11] improved the factor to $k^{1-\epsilon}$. Dinur et al. [7, 8] gave consecutively two lower bounds, first $(k - 3 - \epsilon)$ and later on $(k - 1 - \epsilon)$. Moreover, assuming Khot’s Unique Games Conjecture (UGC) [17], Khot and Regev [18] proved an inapproximability factor of $k - \epsilon$ for the Vertex Cover problem on $k$-uniform hypergraphs. Therefore, it implies that the currently achieved ratios are the best possible.

∗Dept. of Computer Science and the Hausdorff Center for Mathematics, University of Bonn. Supported in part by DFG grants and the Hausdorff Center grant EXC59-1. Email: marek@cs.uni-bonn.de
†Dept. of Computer Science, University of Bonn. Work supported by Hausdorff Doctoral Fellowship. Email: schmied@cs.uni-bonn.de
‡Dept. of Computer Science, University of Bonn. Work partially supported by Hausdorff Center for Mathematics, Bonn. Email: viehmann@cs.uni-bonn.de
The Vertex Cover problem restricted to $k$-partite $k$-uniform hypergraphs, when the underlying partition is given, was studied by Lovász [20] who achieved a $\frac{k}{2}$-approximation. This approximation upper bound is obtained by rounding the natural LP relaxation of the problem. The above bound on the integrality gap was shown to be tight in [1]. As for the lower bounds, Guruswami and Saket [10] proved that it is NP-hard to approximate the Vertex Cover problem on $k$-partite $k$-uniform hypergraphs to within a factor of $\frac{k}{2} - \epsilon$ for $k \geq 5$. Assuming the Unique Games Conjecture, they also provided an inapproximability factor of $\frac{k}{2} - \epsilon$ for $k \geq 3$. More recently, Sachdeva and Saket [21] claimed a nearly optimal $NP$-hardness factor.

To gain better insights on lower bounds, dense instances of many optimization problems has been intensively studied [2, 15, 16, 14]. The Vertex Cover problem has been investigated in the case of dense graphs, where the number of edges is within a constant factor of $n^2$, by Karpinski and Zelikovsky [16], Eremeev [9], Clementi and Trevisan [6], later by Bar-Yehuda and Kehat [4] as well as Imamura and Iwama [12].

The Vertex Cover problem restricted to dense balanced $k$-partite $k$-uniform hypergraphs was introduced and studied in [5], where it was proved that this restricted version of the problem admits an approximation ratio better than $\frac{k}{2}$ if the given hypergraph is dense enough.

In this paper, we give a new approximation algorithm for the Vertex Cover problem restricted to dense $k$-partite $k$-uniform hypergraphs and prove that the achieved approximation ratio is almost tight assuming the Unique Games Conjecture.

## 2 Definitions and Notations

Given a natural number $i \in \mathbb{N}$, we introduce for notational simplicity the set $[i] = \{1, \ldots, i\}$ and set $[0] = \emptyset$. Let $S$ be a finite set with cardinality $s$ and $k \in [s]$. We will use the abbreviation $\binom{S}{k} = \{S' \subseteq S \mid |S'| = k\}$.

A $k$-uniform hypergraph $H = (V(H), E(H))$ consists of a set of vertices $V$ and a collection $E \subseteq \binom{V}{k}$ of edges. For a $k$-uniform hypergraph $H$ and a vertex $v \in V(H)$, we define the neighborhood $N_H(v)$ of $v$ by $(\bigcup_{e \in E(v)} e) \setminus \{v\}$ and the degree $d_H(v)$ of $v$ to be $|\{e \in E \mid v \in e\}|$. We extend this notion to subsets of $V(H)$, where $S \subseteq V(H)$ obtains the degree $d_H(S)$ by $|\{e \in E \mid S \subseteq e\}|$.

A $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ is a $k$-uniform hypergraph such that $V$ is a disjoint union of $V_1, \ldots, V_k$ with $|V_i \cap e| = 1$ for every $e \in E$ and $i \in [k]$. In the remainder, we assume that $|V_i| \geq |V_{i+1}|$ for all $i \in [k-1]$ and $k = O(1)$.

A balanced $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ is a $k$-partite $k$-uniform hypergraph with $|V_i| = \frac{|V|}{k}$ for all $i \in [k]$. We set $n = |V|$ and $m = |E|$ as usual.

For a $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ and $v \in V_i$, we introduce the $v$-induced hypergraph $H(v)$, where the edge set of $H(v)$ is defined by
{e \setminus \{v\} | v \in e \in E(H)}$ and the vertex set of $H(v)$ is partitioned into $V_i \cap N_H(v)$ with $i \in [k-1]$.

A vertex cover of a $k$-uniform hypergraph $H = (V(H), E(H))$ is a subset $C$ of $V(H)$ with the property that $e \cap C \neq \emptyset$ holds for all $e \in E(H)$. The Vertex Cover problem consists of finding a vertex cover of minimum size in a given $k$-uniform hypergraph. The Vertex Cover problem in $k$-partite $k$-uniform hypergraphs is the restricted problem, where a $k$-partite $k$-uniform hypergraph and its vertex partition is given as a part of the input.

We define a $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E(H))$ as $\epsilon$-dense for an $\epsilon \in [0, 1]$ if the following condition holds:

$$|E(H)| \geq \epsilon \prod_{i \in [k]} |V_i|$$

For $\ell \in [k-1]$, we introduce the notion of $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs. Given a $k$-partite $k$-uniform hypergraph $H$, if there exists an $I \in \binom{[k]}{\ell}$ and an $\epsilon \in [0, 1]$ such that for all $S$ with the property $|V_i \cap S| = 1$ for all $i \in I$ the condition

$$d_H(S) \geq \epsilon \prod_{i \in [k] \setminus I} |V_i|$$

holds, we define $H$ to be $\ell$-wise $\epsilon$-dense.

3 Our Results

In this paper, we give an improved approximation upper bound for the Vertex Cover problem restricted to $\epsilon$-dense $k$-partite $k$-uniform hypergraphs. The approximation algorithm in [5] yields an approximation ratio of

$$\frac{k}{k - (k-2)(1-\epsilon)^{k-\ell}}$$

for $\ell$-wise $\epsilon$-dense balanced $k$-partite $k$-uniform hypergraphs. Here, we design an algorithm with an approximation factor of

$$\frac{k}{2 + (k-2)\epsilon}$$

for the $\epsilon$-dense case which also improves on the $\ell$-wise $\epsilon$-dense balanced case for all $\ell \in [k-2]$ and matches their bound when $\ell = k-1$. A further advantage of this algorithm is that it applies to a larger class of hypergraphs since the considered hypergraph is not necessarily required to be balanced.

As a byproduct, we obtain a constructive proof that a vertex cover of an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E(H))$ is bounded from below by $\epsilon|V_k|$, which is shown to be sharp by constructing a family of tight examples.

On the other hand, we provide inapproximability results for the Vertex Cover problem restricted to $\ell$-wise $\epsilon$-dense balanced $k$-partite $k$-uniform hypergraphs under the Unique Games Conjecture. We also prove that this reduction yields a
matching lower bound if we use a conjecture on the Unique Games hardness of the Vertex Cover problem restricted to balanced $k$-partite $k$-uniform hypergraphs. This means that further restrictions such as $\ell$-wise density cannot lead to improved approximation ratios and our proposed approximation algorithm is best possible assuming this conjecture. In addition, we are able to prove an inapproximability factor under $P \neq NP$.

4 Approximation Algorithm

In this section, we give a polynomial time approximation algorithm with improved approximation factor for the Vertex Cover problem restricted to $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.

We state now our main result.

Theorem 1. There exists a polynomial time approximation algorithm with approximation ratio

$$\frac{k}{2 + (k - 2)\epsilon}$$

for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.

A crucial ingredient of the proof of Theorem 1 is Lemma 1, in which we show that we can extract efficiently a large part of an optimal vertex cover of a given $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$. More precisely, we obtain in this way a constructive proof that the size of a vertex cover of $H$ is bounded from below by $\epsilon|V_k|$. The procedure for the extraction of a part of an optimal vertex cover is given in Figure 1.

We now formulate Lemma 1:

Lemma 1. Let $H = (V_1, \ldots, V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph with $k \geq 1$. Then, the procedure $\text{Extract}(\cdot)$ computes in polynomial time a collection $R$ of subsets of $V(H)$ such that the size of $R$ is polynomial in $|V(H)|$ and $R$ contains a set $S$, which is a subset of an optimal vertex cover of $H$ and its cardinality is at least $\epsilon|V_k|$.

As a consequence, we obtain directly:

Corollary 1. Given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ with $k \geq 1$, the cardinality of an optimal vertex cover of $H$ is bounded from below by $\epsilon|V_k|$.
Procedure $\text{Extract}(\cdot)$

Input: $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E)$ with $k \geq 1$

1. IF $k = 1$ THEN
   (a) RETURN $\bigcup_{e \in E} e$

2. ELSE:
   (a) Let $(v_1, ..., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$
   (b) $R = \{v_1, ..., v_p\}$
   (c) FOR $i = 1, ..., p$ DO:
      i. $R_i = \{v_k \mid k \in [i-1]\}$
      ii. Invoke $\text{Extract}(H(v_i))$ with output $O$
      iii. $R = R \cup \{R_i \cup S \mid S \in O\}$

3. RETURN $R$

Figure 1: Procedure $\text{Extract}$

$\text{Extract}(\cdot)$ tries to obtain a large part of an optimal vertex cover of the $v_u$-induced hypergraph $H(v_u)$. Hence, we have to show that $H(v_u)$ must still be dense enough. We now give the proof of Lemma 1.

Proof. The proof of Lemma 1 will be split in several parts. In particular, we show that given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E(H))$, the procedure $\text{Extract}(\cdot)$ and its output $R$ possess the following properties:

1. $\text{Extract}(\cdot)$ constructs $R$ in polynomial time and the cardinality of $R$ is $O(n^k)$.
2. There is a $S \in R$ such that $S$ is a subset of an optimal vertex cover of $H$.
3. For every $S \in R$, the cardinality of $S$ is at least $|S| \geq \epsilon |V_k|$.

(1.) Clearly, $R$ is upper bounded by $|V_1|^k = O(n^k)$ and therefore, the running time of $\text{Extract}(\cdot)$ is $O(n^k)$.
(2.) and (3.) We prove the remaining properties by induction. If we have $k = 1$, the set $\bigcup_{e \in E(H)} e$ is by definition an optimal vertex cover of $H = (V_1, E(H))$. Since $H$ is $\epsilon$-dense, the cardinality of $|E(H)|$ is lower bounded by $\epsilon |V_1|$. We assume that $k > 1$. Let $H = (V_1, ..., V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph and $OPT \subseteq V(H)$ an optimal vertex cover of $H$. Let $(v_1, ..., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E(H)|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$. If $\{v_1, ..., v_p\}$ is contained in $OPT$, we have constructed a subset of an
optimal vertex cover with cardinality

\[ p = \left\lceil \frac{|E(H)|}{\prod_{l \in [k-1]} |V_l|} \right\rceil \geq \frac{\epsilon \prod_{l \in [k]} |V_l|}{\prod_{l \in [k-1]} |V_l|} \geq \epsilon |V_k|. \]

Otherwise, there is an \( u \in [p] \) such that \( R_u \subseteq OPT \) and \( v_u \notin OPT \). But this means that an optimal vertex cover of \( H \) contains an optimal vertex cover of the \( v_u \)-induced \((k-1)\)-partite \((k-1)\)-uniform hypergraph \( H(v_u) \) in order to cover the edges \( e \in \{ e \in E \mid v_u \in e \} \). The situation is depicted in Figure 2.

\[ v_u \text{-induced Hypergraph } H(v_u) \]

![Figure 2: The \( v_u \)-induced \((k-1)\)-partite \((k-1)\)-uniform hypergraph \( H(v_u) \)](image)

By our induction hypothesis, \( Extract(H(v_u)) \) contains a set \( S_u \) which is a subset of a minimum vertex cover of \( H(v_u) \) and of \( OPT \). The only claim, which remains to be proven, is that the cardinality of \( S_u \) is large enough. More precisely, we show that \( |S_u| \) can be lower bounded by \( \epsilon |V_k| - |R_u| \). Therefore, we need to analyze the density of the \( v_u \)-induced hypergraph \( H(v_u) \). The edge set of \( H(v_u) \) is given by \( \{ e \setminus \{ v_u \} \mid v_u \in e \in E \} \). Thus, we have to obtain a lower bound on the degree of \( v_u \). Since \( |\{ e \in E \mid e \cap R_u \neq \emptyset \}| \) is upper bounded by \( |R_u| \prod_{l \in [k-1]} |V_l| \), the vertices in \( V_k \setminus R_u \) possess the average degree of at least

\[
\sum_{v \in V_k \setminus R_u} \deg_H(v) \geq \frac{\epsilon \prod_{l \in [k]} |V_l| - |\{ e \in E \mid e \cap R_u \neq \emptyset \}|}{|V_k \setminus R_u|} \geq \frac{\epsilon \prod_{l \in [k]} |V_l| - |R_u| \prod_{l \in [k-1]} |V_l|}{|V_k \setminus R_u|} \geq \frac{(\epsilon |V_k| - |R_u|) \prod_{l \in [k-1]} |V_l|}{|V_k \setminus R_u|} \]

(1) (2) (3)
Notice that hypergraph is depicted in Figure 3. Let \( u \) with \( i \) all \( k \)-partite \( H \). Then, \( \left| V_{k-1} \right| \) lower bounded by

\[
\left| E_u \right| \geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|}
\]

Let \( H(v_u) \) be defined by \( (V_1^u, \ldots, V_{k-1}^u, E_u) \) with \( |V_i^u| \leq |V_i| \) for all \( i \in [k-1] \). By our induction hypothesis, the size of every set contained in \( \text{Extract}(\cdot) \) is at least

\[
\frac{|E_u|}{\prod_{i \in [k-1]} |V_i^u|} |V_{k-1}| \geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u| \prod_{i \in [k-1]} |V_i^u|} |V_{k-1}|
\]

\[
|E_u| \geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u| \prod_{i \in [k-1]} |V_i^u|} |V_{k-1}|
\]

\[
|E_u| \geq \frac{(\epsilon|V_k| - |R_u|) |V_k|}{|V_k \setminus R_u|} |V_{k-1}|
\]

\[
|E_u| \geq \frac{(\epsilon|V_k| - |R_u|) |V_k|}{|V_k \setminus R_u|} |V_{k-1}|
\]

In (4), we used the fact that \( |V_i^u| \leq |V_i| \) for all \( i \in [k-1] \). Whereas in (5), we used our assumption \( |V_k| \leq |V_{k-1}| \). All in all, we obtain

\[
|R_u \cup S_u| \geq |R_u| + (\epsilon|V_k| - |R_u|) = \epsilon|V_k|.
\]

Clearly, this argumentation on the size of \( R_u \cup S_u \) holds for every \( u \in [p] \) and the proof of Lemma 1 follows.

Before we state our approximation algorithm and prove Theorem 1, we show that the bound in Lemma 1 is tight. In particular, we define a family of \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs \( H(k, l, \epsilon) = (V_1, \ldots, V_k, E(H)) \) with \( |V_i| = \frac{|V|}{l} \) for all \( i \in [k] \), \( k \geq 1 \), \( \epsilon \in \left\{ \frac{\epsilon}{l} | u \in [l] \right\} \) and \( l \geq 1 \) such that \( \text{Extract}(\cdot) \) returns a subset of an optimal vertex cover with cardinality of exactly \( \epsilon|V_k| \).

**Lemma 2.** The bound of Lemma 1 is tight.

**Proof.** Let us define \( H(k, p, \epsilon) = (V_1, \ldots, V_k, E) \). For a fixed \( p \geq 1 \) and \( k \geq 1 \), every partition \( V_i \) with \( i \in [k] \) consists of a set of \( l \) vertices. Let us fix a \( \epsilon = \frac{u}{l} \) with \( u \in [l] \). Then, \( H(k, l, \epsilon) \) contains the set \( V_k^u \subseteq V_k \) of \( u \) vertices such that \( E = \{ \{v_1, v_2, \ldots, v_k\} \mid v_1 \in V_{k-1}^u, v_2 \in V_2, \ldots, v_k \in V_k \} \). An example of such a hypergraph is depicted in Figure 3.

Notice that \( H(k, l, \epsilon) = (V_1, \ldots, V_k, E) \) is \( \epsilon \)-dense, since

\[
\frac{|E|}{\prod_{j \in [k]} |V_j|} = \frac{|V_k^u|}{|V_k|} = \frac{u}{l} = \epsilon.
\]
Figure 3: An example of a hypergraph $H(k, l, \epsilon)$

The procedure $Extract(\cdot)$ returns a set $R$, in which $V_k^u$ is contained, since $V_k^u$ is the set of the $p$ heaviest vertices of $V_k$. Hence, we obtain $|V_k^u| = \frac{|V_k|}{V_k^u}|V_k| = \epsilon|V_k|$. On the other hand, the remaining hypergraph $H' = (V_1, ..., V_k \setminus V_k^u, E(H'))$ with edge set $E(H') = \{e \in E \mid e \cap V_k^u = \emptyset\}$ is already covered, since $E(H')$ is by definition of $H(k, p, \epsilon)$ the empty set. Therefore, $V_k^u$ is a vertex cover of $H(k, p, \epsilon)$ and since, according to Corollary 1, every vertex cover is bounded from below by $\epsilon|V_k|$, $V_k^u$ must be an optimal vertex cover.

Next, we state our approximation algorithm for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs defined in Figure 4. The approximation algorithm combines the procedure $Extract(\cdot)$ to generate a large enough subset of an optimal vertex cover together with the $\frac{k}{2}$-approximation algorithm due to Lovász [20] applied to the remaining instance.

**Algorithm** $Approx(\cdot)$

**Input:** $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E)$ with $k \geq 3$

1. $T = \{V_k\}$
2. invoke procedure $Extract(H)$ with output $R$
3. for all $S \in R$ do :
   (a) $H_S = (V(H) \setminus S, \{e \in E(H) \mid e \cap S = \emptyset\})$
   (b) obtain a $\left(\frac{5}{2}\right)$-approximate solution $S_k$ for $H_S$
   (c) $T = T \cup \{S_k \cup S\}$
4. Return the smallest set in $T$

Figure 4: Algorithm $Approx(\cdot)$
We now prove Theorem 1.

Proof. Let $H = (V_1, \ldots, V_k, E)$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph. From Lemma 1, we know that the procedure $\text{Extract}(\cdot)$ returns in polynomial time a collection $C$ of subsets of $V(H)$ such that there is a set $S$ in $C$, which is contained in an optimal vertex cover of $H$. Moreover, we know that the size of $S$ is lower bounded by $\epsilon |V_k|$.

Next, we analyze the approximation ratio of our approximation algorithm $\text{Approx}(\cdot)$. Clearly, the size of an optimal vertex cover of $H$ is upper bounded by $|V_k|$. Let us denote by $\text{OPT}'$ the size of an optimal vertex cover of the remaining hypergraph $H'$ defined by removing all edges $e$ of $H$ with $e \cap S \neq \emptyset$. Furthermore, let $S'$ be the solution of the $\frac{k}{2}$-approximation algorithm applied to $H'$. The approximation ratio of $\text{Approx}(\cdot)$ is bounded by

$$\frac{|S| + |S'|}{|S| + |\text{OPT}'|} \leq \frac{k}{k |S| + \frac{k}{2} |\text{OPT}'|} \leq \frac{2 + (k-2) |S| + \frac{k}{2} |\text{OPT}'|}{|S| + \frac{k}{2} |\text{OPT}'|} \leq \frac{k}{2 + (k-2) |S| + \frac{k}{2} |\text{OPT}'|} \leq \frac{k}{2 + (k-2) |V_k|} \leq \frac{k}{2 + (k-2) \epsilon}$$

In (11), we used the fact that the size of the output of $\text{Approx}(\cdot)$ is upper bounded by $|V_k|$. Therefore, we have $|S| + \frac{k}{2} |\text{OPT}'| \leq |V_k|$. In (12), we know from Lemma 1 that $|S| \geq \epsilon |V_k|$.

5 Inapproximability Results

In this section, we prove hardness results for the Vertex Cover problem restricted to $\ell$-wise $\epsilon$-dense balanced $k$-uniform $k$-partite hypergraphs under the Unique Games Conjecture [17] as well as under the assumption $P \neq NP$.

5.1 UGC-Hardness

The Unique Games-hardness result of [10] was obtained by applying the result of Kumar et al. [19], with a modification to the LP integrality gap due to Ahorani et al. [1]. More precisely, they proved the following inapproximability result:
Theorem 2. [10] For every $\delta > 0$ and $k \geq 3$, there exist a $n_\delta$ such that given $H = (V_1, \ldots, V_k, E(H))$ as an instance of the Vertex Cover problem in balanced $k$-partite $k$-uniform hypergraphs with $|V(H)| \geq n_\delta$, the following is UGC-hard to decide:

- The size of a vertex cover of $H$ is at least $|V| \left( \frac{1}{2(k-1)} - \delta \right)$.
- The size of an optimal vertex cover of $H$ is at most $|V| \left( \frac{1}{k(k-1)} + \delta \right)$.

As the starting point of our reduction, we use Theorem 2 and prove the following:

Theorem 3. For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio

$$\frac{k}{2 + \frac{2(1-k)(k-2)}{k(k-2)}\epsilon} - \delta$$

for the Vertex Cover problem in $\ell$-wise $\epsilon$-dense $k$-partite $k$-uniform hypergraphs assuming the Unique Games Conjecture.

Proof. First, we concentrate on the $\epsilon$-dense case and afterwards, we extend the range of $\ell$. As a starting point of the reduction, we use the $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ from Theorem 2 and construct an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H' = (V'_1, \ldots, V'_k, E')$.

Let us start with the description of $H'$. First, we join the set $C_i$ of $\frac{\epsilon}{1-\epsilon} \frac{n}{k}$ vertices to $V_i$ for every $i \in [k]$ and add all possible edges $e$ of $H'$ to $E'$ with the restriction $C_i \cap e \neq \emptyset$. Thus, we obtain $|V'_i| = \frac{n}{k} + \frac{\epsilon}{1-\epsilon} \frac{n}{k}$ for all $i \in [k]$.

Now, let us analyze how the size of the optimal solution of $H'$ transforms. We denote by $OPT'$ an optimal vertex cover of $H'$. The UGC-hard decision question from Theorem 2 transforms into the following:

$$n \left( \frac{1}{2(k-1)} - \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n \left( \frac{1}{k(k-1)} + \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in $\epsilon$-dense hypergraphs for every $\delta' > 0$ to within:

$$n \left( \frac{1}{2(k-1)} - \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n \left( \frac{1}{k(k-1)} + \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k}$$

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in $\epsilon$-dense hypergraphs for every $\delta' > 0$ to within:

$$n \left( \frac{1}{2(k-1)} - \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k} = \frac{1-\epsilon}{2(k-1)} - \delta(1 - \epsilon) + \frac{\epsilon}{k}$$

$$n \left( \frac{1}{k(k-1)} + \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k} = \frac{1-\epsilon}{k(k-1)} + \delta(1 - \epsilon) + \frac{\epsilon}{k}$$

$$\frac{(1-\epsilon)k}{2(k-1)} + \frac{2\epsilon(k-1)}{k(k-1)} - \delta'$$

$$\frac{1-\epsilon}{(k-1)k} + \frac{\epsilon(k-1)}{k(k-1)} - \delta'$$

(15)
Finally, we have to verify that the constructed hypergraph $H'$ is indeed $\epsilon$-dense. Notice that $H'$ can have at most $(|V_1|) = (\frac{n}{k} + \frac{\epsilon}{1-\epsilon})^k$ edges. Therefore, we obtain the following:

\[
\left(\frac{\epsilon}{1-\epsilon} \frac{n}{k}\right) \left(\frac{n}{k} + \frac{\epsilon}{1-\epsilon} \frac{n}{k}\right)^{k-1} = \frac{n}{k} \left(1 + \frac{\epsilon}{1-\epsilon}\right) = \frac{\epsilon}{1-\epsilon} = \epsilon
\]

Notice that the constructed hypergraph is also $\ell$-wise $\epsilon$-dense balanced. Hence, we obtain the same inapproximability factor in this case as well.

Next, we combine the former construction with a conjecture about Unique Games hardness of the Vertex Cover problem in balanced $k$-partite $k$-uniform hypergraphs. In particular, we postulate the following:

**Conjecture 1.** Given a balanced $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ with $k \geq 3$, let $OPT$ denote an optimal vertex cover of $H$. For every $\delta > 0$, the following is UGC-hard to decide:

\[
|V| \left(\frac{1}{k} - \delta\right) \leq |OPT| \quad \text{or} \quad |OPT| \leq |V| \left(\frac{2}{k^2} + \delta\right)
\]

Combining Conjecture 1 with the construction in Theorem 3, it yields the following inapproximability result which matches precisely the approximation upper bound achieved by our approximation algorithm described in Section 4:

**Theorem 4.** For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k - 1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio

\[
\frac{k}{2 + (k-2)\epsilon} - \delta
\]
for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming Conjecture 1.

Proof. The UGC-hard decision question from Conjecture 1 transforms into the following:

\[ n \left( \frac{1}{k} - \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n \left( \frac{2}{k^2} + \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k} \]

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense k-partite k-uniform hypergraphs for every \( \delta' > 0 \) to within:

\[ \frac{n \left( \frac{1}{k} - \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k}}{n \left( \frac{2}{k^2} + \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k}} = \frac{n \left( \frac{1}{k} - \delta \right) (1 - \epsilon) + \frac{\epsilon n}{k}}{n \left( \frac{2}{k^2} + \delta \right) (1 - \epsilon) + \frac{\epsilon n}{k}} \]  

(23)

\[ = \frac{n \frac{n}{k}}{n \frac{2}{k^2} (1 - \epsilon) + \frac{k \epsilon n}{k^2}} - \delta' \]  

(24)

\[ = \frac{k}{2(1 - \epsilon) + k \epsilon} - \delta' \]  

(25)

\[ = \frac{k}{2 + (k - 2) \epsilon} - \delta' \]  

(26)

5.2 NP-Hardness

Recently, Sachdeva and Saket proved in [21] a nearly optimal NP-hardness of the Vertex Cover problem on balanced k-uniform k-partite hypergraphs. More precisely, they obtained the following inapproximability result:

Theorem 5. [21] Given a balanced k-partite k-uniform hypergraph \( H = (V, E) \) with \( k \geq 4 \), let \( OPT \) denote an optimal vertex cover of \( H \). For every \( \delta > 0 \), the following is NP-hard to decide:

\[ |V| \left( \frac{k}{2(k + 1)(2(k + 1) + 1)} - \delta \right) \leq |OPT| \]

or

\[ |V| \left( \frac{1}{k(2(k + 1) + 1)} + \delta \right) \geq |OPT| \]

Combining our reduction from Theorem 2 with Theorem 5, we prove the following inapproximability result under the assumption \( P \neq NP \):

Theorem 6. For every \( \delta > 0 \), \( \epsilon \in (0, 1) \), \( \ell \in [k - 1] \), and \( k \geq 4 \), there is no polynomial time approximation algorithm with an approximation ratio

\[ \frac{k^2(1 - \epsilon) + \epsilon 2(k + 1)(2(k + 1) + 1)}{2(k + 1)[1 - \epsilon + \epsilon (2(k + 1) + 1)]} - \delta \]
for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming P ≠ NP.

Proof. The NP-hard decision question from Theorem 5 transforms into the following:

\[ n \left( \frac{k}{2(k+1)(2(k+1)+1)} - \delta \right) + \frac{\epsilon}{1-\epsilon k} \frac{n}{k} \leq |OPT'| \]

or

\[ n \left( \frac{1}{k(2(k+1)+1)} + \delta \right) + \frac{\epsilon}{1-\epsilon k} \frac{n}{k} \geq |OPT'| \]

Assuming NP ≠ P, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense hypergraphs for every \( \delta' > 0 \) to within:

\[ \frac{k(1-\epsilon)}{2(k+1)(2(k+1)+1)} + \frac{\epsilon}{k} - \delta' = \frac{k^2(1-\epsilon) + \epsilon 2(k+1)(2(k+1)+1)}{k(2(k+1)+1)} - \delta' \]

\[ = \frac{k^2(1-\epsilon) + \epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1 - \epsilon + \epsilon(2(k+1)+1)]} - \delta' \]  

(27)

(28)

\[ \Box \]

6 Further Research

An interesting question remains about even tighter lower approximation bounds for our problem, perhaps connecting it more closely to the integrality gap issue of the LP of Lovász [20].

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References


[7] I. Dinur, V. Guruswami, and S. Khot *Vertex Cover on \( k \)-Uniform Hypergraphs is Hard to Approximate within Factor \( (k - 3 - \epsilon) \)*, ECCC TR02-027, 2002.


