Electronic Colloquium on Computational Complexity, Report No. 132 (2011)

On Oscillation-free Chaitin h-random Sequences

Ludwig Staiger

Institut für Informatik, Martin-Luther-Universität Halle-Wittenberg
von-Seckendorff-Platz 1, D-06099 Halle, Germany
staiger@informatik.uni-halle.de

Abstract. The present paper generalises results by Tadaki [12] and Calude et al. [1] on oscillation-free partially random infinite strings. Moreover, it shows that oscillation-free partial Chaitin randomness can be separated from oscillation-free partial strong Martin-Löf randomness by Π_1^0-definable sets of infinite strings.

In the papers [11] and [2] several relaxations of randomness were defined. Subsequently, in [8] these were shown to be essentially different. The variants of partial randomness were characterised by different means such as Martin-Löf tests [11, 2], Solovay tests [8] and prefix [11] or a priori complexity [2]. Using description complexity partial randomness of an infinite string ξ was defined by linear lower bounds on the complexity of the n-length prefix $\xi \upharpoonright n$, that is, an infinite string was referred to as ε-random provided the complexity of $\xi \upharpoonright n$ was lower bounded by $\varepsilon \cdot n - O(1)$. In general, the mentioned papers did not require an upper bound on the complexity, except for [11] where an asymptotic upper bound was considered.

For the case of a priori complexity, the papers [9, 7] gave a description of infinite oscillation free ε-random strings where the upper complexity bound matches the lower bound up to an additive constant. For the case of prefix complexity the construction of similar infinite strings was accomplished in [12, 1]. The construction in [1] uses ε-universal prefix machines. Here it was observed in Theorem 6 that there are different (inequivalent) types of ε-universal machines.

In recent publications, based on Hausdorff's original paper [5] the concept of partial randomness was refined to functions of the logarithmic scale [6] or to more general gauge functions [10]. Here we showed that for a priori complexity and computable gauge functions $h : \mathbb{Q} \to \mathbb{R}$ there are oscillation-free h-random infinite strings.

The aim of the present paper is to show that, similarly to the results of [10], also in the case of prefix complexity one can refine ε-randomness to oscillation-free h-randomness. Moreover, our investigations reveal the reason of the paradox of [1, Theorem 6].

Cast into the language of gauge functions (cf. [4, 10]) the papers [12, 1] considered only the scale $h(t) = t^{\varepsilon}$, $\varepsilon \in (0, 1)$ computable, which results in complexity bounds of the form $\varepsilon \cdot n + O(1)$. The present paper refines this scale to a much larger class of gauge functions including also non-computable ones.

ISSN 1433-8092

The paper is organised as follows. First we introduce some notation and consider the concept of gauge functions. In the second section we investigate, for gauge functions h, h-universal machines as a generalisation of the ε-universal machines of [1]. In this section we also explain the paradox of [1, Theorem 6]. Then, in Section 3, we continue with further generalising results of [12, 1] to oscillation-free h-randomness for prefix complexity, and in the last part we show that oscillation-free h-random infinite sequences w.r.t. a priori complexity can be separated by Π_1^1-definable sets from oscillation-free h-random infinite sequences w.r.t. prefix complexity.

1 Notation and Preliminaries

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{Q}$ the set of rational numbers. Let $X = \{0, 1, \ldots, r-1\}$ be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as ω*-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.

We denote by $|w|$ the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$, and $\eta \upharpoonright n$ is the n-length prefix of η provided $|\eta| \geq n$. A language $W \subseteq X^*$ is referred to as *prefix-free* if $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$.

For a computable domain $\mathcal{D}$, such as $\mathbb{N}$, $\mathbb{Q}$ or X^*, we refer to a function $f : \mathcal{D} \to \mathbb{R}$ as *left computable* (or *approximable from below*) provided the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q < f(d)\}$ is computably enumerable. Accordingly, a function $f : \mathcal{D} \to \mathbb{R}$ is called *right computable* (or *approximable from above*) if the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q > f(d)\}$ is computably enumerable, and f is *computable* if f is right and left computable. Accordingly, a real number $\alpha \in \mathbb{R}$ as left computable, right computable or computable provided the constant function $f_\alpha(t) = \alpha$ is left computable, right computable or computable, respectively.

1.1 Gauge functions

A function $h : (0, \infty) \to (0, \infty)$ is referred to as a *gauge function* provided h is right continuous and non-decreasing.[1] If not stated otherwise, we will always assume that $\lim_{t \to 0} h(t) = 0$. As in [10] with a gauge function we associate a *modulus function* $g : \mathbb{N} \to \mathbb{N}$ which, roughly speaking, satisfies $h(r^{-g(n)}) \approx r^{-n}$ or, more precisely, $|-\log_r h(r^{-g(n)}) - n| = O(1)$.

We may define the modulus as follows

[1] In fact, since we are only interested in the values $h(r^{-n})$, $n \in \mathbb{N}$, the requirement of right continuity is just to conform with the usual meaning (cf. [4]).

Definition 1. $g(n) := \sup\{m : m \in \mathbb{N} \wedge r^{-n} < h(r^{-m})\}$

Here we use the convention $\sup \emptyset = 0$. Then we have

$$m \leq g(n) \iff h(r^{-g(n)}) > r^{-n} \text{ if } g(n) \neq 0\,. \tag{1}$$

Moreover, the following holds true.

Lemma 1. *If for all $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ satisfying $r^{-n} < h(r^{-m}) \leq r^{-n+1}$ then $h(r^{-g(n)}) \leq r^{-n+1}$.*

The assumption of Lemma 1 implies $h(r^{-n}) \geq r^{-n}$ and $h(r^{-(n+c)}) \geq h(r^{-n}) \cdot r^{-c}$. It is, in particular, satisfied if the function h is upwardly convex on $(0, 1)$ and $h(1) \geq 1$ (see [10, Lemma 3]).

For computable gauge functions $h : \mathbb{Q} \to \mathbb{R}$, relaxing Eq. (1) we obtain a corresponding computable modulus function.

Lemma 2 ([10, Lemma 4]). *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function satisfying the conditions that $1 < h(1) < r$ and for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $r^{-n} < h(r^{-m}) \leq r^{-n+1}$. Then there is a computable strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $r^{-n-1} < h(r^{-g(n)}) < r^{-n+1}$.*

2 Universal Machines

In this section we introduce and study the notion of h-universal machine.

A *machine* T is a partial computable function from X^* to X^*. We use machine and function synonymously.

A prefix-free machine is a machine whose domain is a prefix-free language. The prefix complexity of a word w induced by a prefix-free machine T, $H_T(w)$, is $H_T(w) = \inf\{|\pi| : T(\pi) = w\}$. From now on all Turing machines will be prefix-free and will be referred to simply as machines.

In analogy with [1] we say that a machine U is *h-universal* for a gauge function h if for all machines T there exists a constant $c_{U,T}$ such that for each program $\sigma \in X^*$ there exists a program $\pi \in X^*$ such that $U(\pi) = T(\sigma)$ and $-\log_r h(r^{-|\pi|}) \leq |\sigma| + c_{U,T}$. If $h(t) = t$ we get the classical notion of universal machine. Observe that, for gauge functions h, the function $\ell_h(n) := -\log_r h(r^{-n})$ is non-decreasing.

A machine U is *strictly h-universal* if U is h-universal but not h'-universal for any gauge function h' with $\lim_{n\to\infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$.

We fix a gauge function h and a universal machine T. We say that an ω-word ξ is *Chaitin h-random* if $H_T(\xi \upharpoonright n) \geq -\log_r h(r^{-n}) - O(n)$, and we say that ξ is *strictly Chaitin h-random* if ξ is Chaitin h-random and is not Chaitin h'-random for all gauge functions h' with $\lim_{n\to\infty} \frac{h'(r^{-n})}{h(r^{-n}}= 0$.

If T is universal and $h(t) = t^{\varepsilon}$, then we get Tadaki's definition of weak Chaitin ε-randomness (see [11, 2]), if $h(t) = t$, then we get the classical definition of randomness.

Lemma 3. *The machine U is h-universal if and only if there exists a universal machine T and a constant $c_{U,T}$ such that $-\log_r h(r^{-H_U}) \leq H_T(w) + c_{U,T}$ for all $w \in X^*$.*

In [1] ε-universal machines were obtained from universal machines by padding the inputs. The next theorem shows that the same construction works also in the case of h-universal machines.

Theorem 1. *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function, and let $g : \mathbb{N} \to \mathbb{N}$ be a corresponding computable modulus function. If, for a universal machine T, we define $T_h(\pi \cdot 0^{g(|\pi|)-|\pi|}) := T(\pi)$ then $|{-\log_r h(r^{-H_{T_h}(w)})} - H(w)| = O(1)$ and T_h is a strictly h-universal machine.*

Proof. Let π be a minimal description of w, that is, $|\pi| = H(w)$. Then $\pi \cdot 0^{g(|\pi|)-|\pi|}$ is a minimal description of w w.r.t. T_h. Consequently, Lemma 2 proves $|{-\log_r h(r^{-H_{T_h}(w)})} - H(w)| = O(1)$. This also implies that T_h is h-universal.

Assume now T_h to be h'-universal for some h' tending faster to 0 than h. Then, on the one hand, $-\log_r h'(r^{-H_{T_h}(w)}) \leq H(w) + c$ for some constant c and, on the other hand, for every $i \in \mathbb{N}$ there is an n_i such that $-\log_r h'(r^{-H_{T_h}(w)}) \geq -\log_r h(r^{-H_{T_h}(w)}) + i$ for $H(w) \geq n_i$. This contradicts the relation $|{-\log_r h(r^{-H_{T_h}(w)})} - H(w)| = O(1)$. □

Next we give examples for Chaitin h-random ω-words. We follow the line of Theorem 3 of [1] and define for a machine U the ω-word $\Omega_U \in X^\omega$ as the r-ary expansion of the halting probability of a machine U, that is, $0.\,\Omega_U := \sum_{w \in \mathrm{dom}(U)} r^{-|w|}$.

Theorem 2. *Let h be a computable gauge function satisfying the hypothesis of Lemma 2 and let U be a h-universal machine. Then Ω_U is Chaitin h-random.*

Proof. As in the proof of Theorem 3 of [1] one defines a machine T for which $\mathbf{pref}(\Omega_U) \subseteq \mathrm{dom}(T)$ and $H_U(T(\Omega_U \upharpoonright m)) \geq m$. From $H_U(v) \geq m$, we obtain $-\log_r h(r^{-m}) \leq -\log_r h(r^{-H_U(T(v))}) \leq H(T(v)) \leq H(v) + c$ whenever $v \in \mathrm{dom}(T)$, and the assertion follows. □

We conclude this section by considering the paradox of Theorem 6 of [1]. Here inequivalent ε-universal machines $V_{\varepsilon,k}, k = 0, 1, \ldots$ were defined. The machines $V_{\varepsilon,k}$ had the property that $\lim_{|w|\to\infty} H_{V_{\varepsilon,k}}(w) - H_{V_{\varepsilon,k+1}}(w) = \infty$.

Recall the definition of $V_{\varepsilon,k}$. In terms of modulus function $g_k : \mathbb{N} \to \mathbb{N}$ they can be described as $V_{\varepsilon,k}(\pi 0^{g_k(|\pi|)-|\pi|}) := T(\pi)$ where T is a universal machine and $g_k(n) := \max\{n, \lfloor \frac{1}{\varepsilon} \cdot n - k \cdot \log_r n \rfloor\}$. In contrast to [1] where all machines $V_{\varepsilon,k}$ were ε-universal our Theorem 2 states that only $V_{\varepsilon,k}$ is strictly h_k-universal for gauge functions satisfying $h_k(r^{-g_k(n)}) = r^{-n}$. Since $g_k(n) - g_{k+1}(n)$ tends to infinity as n grows, the function h_{k+1} tends faster to 0 than h_k and, consequently, $V_{\varepsilon,k}$ is not h_{k+1}-universal.

The paradox of Theorem 6 of [1] occurs because the family of gauge functions $\left(t^\varepsilon\right)_{\varepsilon \in (0,1)}$ admits intermediate computable functions, e.g. functions of the logarithmic scale like $h(t) = t^\varepsilon \cdot \left(\log_r \frac{1}{t}\right)^k$ (see [5]) but these functions were not taken into consideration in the definition of ε-universality.

3 Oscillation-freeness

The aim of this section is to show that for a large class of gauge function there exist oscillation-free Chaitin h-random ω-words, that is, $\xi \in X^\omega$ such that $|H(\xi \upharpoonright n) + \log_r h(r^{-n})| = O(1)$.

We start with a generalisation of [1, Proposition 9].

Proposition 1. *Let $h : \mathbb{Q} \to \mathbb{N}$ be a gauge function such that for every $d \in \mathbb{N}$ there is an ℓ_d such that the inequality*

$$H(n) + d - 1 \leq -\log_r \frac{h(r^{-(n+\ell)})}{h(r^{-\ell})} \leq n - \big(H(n) + d - 1\big) \tag{2}$$

holds for all $\ell \geq \ell_d$ and, depending on the value of d, for all sufficiently large $n \in \mathbb{N}$.

Then there are $c, \ell' \in \mathbb{N}$ such that for all words $w \in X^*$, $|w| \geq \ell'$, exist words $v, u \in X^c$ such that

$$H(w) - c < H(wu) + \log_r h(r^{-(|w|+c)}) \leq H(w) + \log_r h(r^{-|w|}) - 1, \text{ and} \tag{3}$$

$$H(wv) + \log_r h(r^{-(|w|+c)}) \geq H(w) + \log_r h(r^{-|w|}) + 1\,. \tag{4}$$

Proof. As in the proof of Proposition 9 of [1], given $w \in X^*$ and $c \in \mathbb{N}$, one finds strings v and $u = 0^c$ such that

$$H(wv) \geq H(w) + c - H(c) - d \text{ and } |H(w0^c) - H(w)| \leq H(c) + d \tag{5}$$

where the constant d is independent of w and c. Thus $H(w) - c < H(w0^c)$ if c is large enough.

Now, depending on d, choose a sufficiently large ℓ' and the remaining inequalities follow from Eqs. (5) and (2). $\square$

Remark 1. The assumption of Proposition 1 is a little bit involved. Due to the fact that $H(n)$ is a slowly growing function one easily observes that Eq. (2) is satisfied whenever there are real numbers $\gamma, \overline{\gamma} \in (r^{-1}, 1)$ such that $\gamma^n \leq \frac{h(r^{-(n+\ell)})}{h(r^{-\ell})} \leq \overline{\gamma}^n$ for all $\ell, n \in \mathbb{N}$. The latter is satisfied, in particular, for all length-invariant unbounded (p, q)-premeasures in the sense of [8].

The next theorem is an existence theorem for oscillation-free Chaitin h-random ω-words where h is a gauge functions fulfilling Eq. (2). This, in particular, guarantees that for arbitrary $\varepsilon \in (0, 1)$ oscillation-free Chaitin ε-random ω-words exist. The subsequent theorem will then consider the constructive case.

Theorem 3. *Let $h : \mathbb{Q} \to \mathbb{R}$ be a gauge function satisfying Eq. (2) and the assumption of Lemma 1. Then there is an ω-word $\xi \in X^\omega$ and a constant c_h such that $|H(\xi \upharpoonright n) - (-\log_r h(r^{-n}))| \leq c_h$.*

Proof. We proceed as in the proof of Theorem 10 of [1]. In view of $h(r^{-n}) \geq r^{-n}$ we choose sufficiently large constants c and ℓ' from Proposition 1 such that

$-\log_r h(r^{-\ell'}) < H(w)$ for some $w, |w| = \ell'$, and we define $W \subseteq X^{\ell'} \cdot (X^c)^*$ as follows.

$$W := \left\{w : w \in X^{\ell'} \cdot (X^c)^* \wedge \forall v\big(v \in W \wedge v \sqsubseteq w \to H(v) > -\log_r h(r^{-|v|})\big)\right\}.$$

Since there is a $w \in X^{\ell'}$ with $-\log_r h(r^{-\ell'}) < H(w)$, we have $W \neq \emptyset$. By induction, Eq. (4) shows that every $w \in W$ has an extension $wv \in W$ where $|v| = c$. Moreover, since h is non-decreasing, $H(wv) + \log_r h(r^{-|wv|}) \leq H(w) + \log_r h(r^{-|w|}) + 2c$ for $|v| = c$.

Let $H(w) > -\log_r h(r^{-|w|})+c+1$. In view of Lemma 1 we have $-\log_r h(r^{-|w|+c}) \leq -\log_r h(r^{-|w|}) + c$. Then the first part of Eq. (3) shows $-\log_r h(r^{-|w|+c}) + 1 < H(w0^c)$, that is, $w0^c \in W$. Finally, the second part of Eq. (4) shows that then $H(w0^c) + \log_r h(r^{-|w|+c}) < H(w) + \log_r h(r^{-|w|})$.

Thus there is an infinite sequence $w_0 \sqsubset w_1 \sqsubset \cdots \sqsubset w_i \sqsubset$ of words in W such that $|w_{i+1}| - |w_i| = c$ and $|H(w_i) + \log_r h(r^{-|w_i|})|$ remains bounded. □

Now consider the language W defined in the preceding proof. If h is a computable function, W is the complement of an computably enumerable language. Hence the infinite paths through W build a Π_1^0-definable ω-language $F \subseteq X^\omega$. Then the leftmost w.r.t. the lexicographical ordering ω-word ξ_{left} in F defines a left computable real $0\,.\xi_{\mathrm{left}}$.

We show that ξ_{left} is oscillation-free h-random. Since $H(w) > -\log_r h(r^{-|w|})$ for $w \in W$, it suffices to verify that $H(\xi_{\mathrm{left}} \restriction n) + \log_r h(r^{-n}) \leq c_h$ for some constant c_h. We use the parameters c and ℓ' from the proof of Theorem 3.

Let $\mathbf{pref}(\xi_{\mathrm{left}}) \cap W = \{w_i : i \in \mathbb{N} \wedge |w_i| = \ell' + i \cdot c\}$ where w_0 is the leftmost word in W. Choose a constant $c_h > \max\{H(w_0) + \log_r h(r^{-\ell'}), 4c\}$. Then $H(w_0) + \log_r h(r^{-|w_0|}) \leq c_h$. Assume that this relation holds for $j = 0, \ldots, i$. If $H(w_i) + \log_r h(r^{-|w_i|}) \leq 2c$ then $H(w_i v) + \log_r h(r^{-|w_i v|}) \leq 4c$ for all $v \in X^c$. Thus $H(w_{i+1}) + \log_r h(r^{-|w_{i+1}|}) \leq c_h$. If $2c < H(w_i) + \log_r h(r^{-|w_i|}) \leq c_h$ then $w_i 0^c$ is the leftmost successor of w_i in W and $0 < H(w_i) + \log_r h(r^{-|w_i|}) - 2c \leq H(w_i 0^c) + \log_r h(r^{-|w_i|+c}) < H(w_i) + \log_r h(r^{-|w_i|}) \leq c_h$.

This proves the following constructive version of Theorem 3.

Theorem 4. *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function which satisfies Eq. (2) and the hypothesis of Lemma 2. Then there exists an oscillation-free Chaitin h-random ω-word ξ such that $0\,.\xi$ is a left computable real.*

4 A Separation Theorem

In the preceding section we showed the existence of oscillation-free Chaitin h-random ω-words. For the gauge functions fulfilling the assumption of Lemma 2 we proved the existence of Π_1^0-definable ω-languages containing such ω-words as leftmost ones.

In a recent paper [10] we proved that, for a different kind of h-randomness (strong Martin-Löf randomness in the sense of [2]), there are Π_1^0-definable ω-languages containing oscillation-free h-random ω-words. We obtained these ω-languages by diluting ω-words. The concept of strong Martin-Löf randomness can be defined using the a priori complexity of words KA. For a definition of KA see [13, 9, 10] or [3][2]. We mention here only the following properties of KA.

Property 1. 1. An ω-word ξ is random if and only if $|\mathrm{KA}(\xi \upharpoonright n) - n| = O(1)$,
2. $\mathrm{KA}(wv) \geq \mathrm{KA}(w) - O(1)$, for $w, v \in X^*$, and
3. $H(w) \geq \mathrm{KA}(w) - O(1)$ where the difference is unbounded.

For dilution we use prefix monotone mappings. Every prefix-monotone mapping $\varphi : X^* \to X^*$ defines as a limit a partial mapping $\overline{\varphi} :\subseteq X^\omega \to X^\omega$ in the following way: $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$ whenever $\varphi(\mathbf{pref}(\xi))$ is an infinite set, and $\overline{\varphi}(\xi)$ is undefined when $\varphi(\mathbf{pref}(\xi))$ is finite.

If a (modulus) function $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing we define a dilution function $\varphi : X^* \to X^*$ as follows.

$$\begin{aligned} \varphi(e) &:= 0^{g(0)} \text{ and} \\ \varphi(wx) &:= \varphi(w) \cdot x \cdot 0^{g(n+1)-g(n)-1} \text{ for } w \in X^* \text{ and } x \in X \end{aligned} \tag{6}$$

If φ is a dilution function then φ and also $\overline{\varphi}$ are one-to-one mappings. If, moreover, g is computable then φ is also computable and $\overline{\varphi}(X^\omega)$ is a Π_1^0-definable ω-language.

It holds the following estimate on the a priori complexity of a diluted string (see [9, Theorem 3.1]).

Lemma 4. *Let g be a computable strictly increasing modulus function and let φ be defined via Eq. (6). Then*
$$\left|\mathrm{KA}\big(\overline{\varphi}(\xi \upharpoonright g(n))\big) - \mathrm{KA}\big(\xi \upharpoonright n\big)\right| \leq O(1) \textit{ for all } \xi \in X^\omega .$$

From Lemmata 4 and 2 and the above Property 1.2 we obtain immediately the following (cf. also [9, Theorem 3.3]).

Proposition 2. *Let h be a computable gauge function, g a corresponding computable modulus function and let φ be defined via Eq. (6). Then $|\mathrm{KA}(\xi \upharpoonright (n+1)) - \mathrm{KA}(\xi \upharpoonright n)| = O(1)$ implies $|\mathrm{KA}(\overline{\varphi}(\xi) \upharpoonright g(n)) + \log_r h(r^{-n})| = O(1)$.*

Property 1.1 shows that Proposition 2 holds for random ω-words ξ. In that case $\overline{\varphi}(\xi)$ is strongly Martin-Löf h-random. Next we consider the situation for prefix complexity. Here we have $|H(w) - H(\varphi(w))| = O(1)$ whenever φ is a partial computable one-to-one function. Thus we obtain a theorem analogous to Lemma 4 for prefix complexity H.

Lemma 5. *Let g be a computable strictly increasing modulus function and let φ be defined via Eq. (6). Then*
$$\left|H\big(\overline{\varphi}(\xi \upharpoonright g(n))\big) - H\big(\xi \upharpoonright n\big)\right| \leq O(1) \textit{ for all } \xi \in X^\omega .$$

This much preparation allows us to prove our separation theorem.

Theorem 5. *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function which satisfies Eq. (2) and the hypothesis of Lemma 2. Then there exists a Π_1^0-definable ω-language which contains an oscillation-free strongly Martin-Löf h-random ω-word ξ but no oscillation-free Chaitin h-random ω-word.*

[2] In [3] a priori complexity is denoted by KM.

Proof. From Lemma 2 we obtain a computable strictly increasing modulus function g such that $|-\log_r h(r^{-g(n)}) - n| \leq 1$. Define φ according to Eq.(6) and choose an arbitrary random ω-word $\zeta \in X^\omega$. Then Proposition 2 shows that $\overline{\varphi}(\zeta)$ is oscillation-free strongly Martin-Löf h-random.

Next we show that the Π_1^0-definable ω-language $\overline{\varphi}(X^\omega)$ does not contain any oscillation-free Chaitin h-random ω-word.

Assume that, for some $\xi \in X^\omega$, the ω-word $\overline{\varphi}(\xi)$ is oscillation-free Chaitin h-random. Then $|H(\overline{\varphi}(\xi) \upharpoonright g(n)) + \log_r H(r^{-g(n)})| = O(1)$, and, consequently, $|H(\xi \upharpoonright n) - n| = O(1)$. But this is impossible as $H(\xi \upharpoonright n) \geq n - c$, for all $n \in \mathbb{N}$, implies $\lim_{n\to\infty} H(\xi \upharpoonright n) - n = \infty$. □

References

1. Calude, C.S., Hay, N.J., Stephan, F.: Representation of left-computable ε-random reals. J. Comput. System Sci. 77(4), 812–819 (2011), http://dx.doi.org/10.1016/j.jcss.2010.08.001
2. Calude, C.S., Staiger, L., Terwijn, S.A.: On partial randomness. Ann. Pure Appl. Logic 138(1-3), 20–30 (2006), http://dx.doi.org/10.1016/j.apal.2005.06.004
3. Downey, R.G., Hirschfeldt, D.R.: Algorithmic Randomness and Complexity. Theory and Applications of Computability, Springer-Verlag, New York (2010)
4. Graf, S., Mauldin, R.D., Williams, S.C.: The exact Hausdorff dimension in random recursive constructions. Mem. Amer. Math. Soc. 71(381), x+121 (1988)
5. Hausdorff, F.: Dimension und äußeres Maß. Math. Ann. 79(1-2), 157–179 (1918), http://dx.doi.org/10.1007/BF01457179
6. Mielke, J.: Verfeinerung der Hausdorff-Dimension und Komplexität von ω-Sprachen. Ph.D. thesis, Martin-Luther-Universität Halle-Wittenberg (2010), http://nbn-resolving.de/urn:nbn:de:gbv:3:4-2816, [in German]
7. Mielke, J., Staiger, L.: On oscillation-free ε-random sequences II. In: Bauer, A., Hertling, P., Ko, K.I. (eds.) Computability and Complexity in Analysis. Dagstuhl Seminar Proceedings, vol. 09003. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, Germany (2009), http://drops.dagstuhl.de/opus/volltexte/2009/2269
8. Reimann, J., Stephan, F.: Hierarchies of randomness tests. In: Goncharov, S.S., Downey, R., Ono, H. (eds.) Mathematical logic in Asia, pp. 215–232. World Scientific, Hackensack, NJ (2006)
9. Staiger, L.: On oscillation-free ε-random sequences. Electr. Notes Theor. Comput. Sci. 221, 287–297 (2008)
10. Staiger, L.: Constructive dimension and hausdorff dimension: The case of exact dimension. In: Owe, O., Steffen, M., Telle, J.A. (eds.) Fundamentals of Computation Theory. Lecture Notes in Computer Science, vol. 6914, pp. 252–263. Springer (2011)
11. Tadaki, K.: A generalization of Chaitin's halting probability Ω and halting self-similar sets. Hokkaido Math. J. 31(1), 219–253 (2002)
12. Tadaki, K.: A new representation of Chaitin Ω number based on compressible strings. In: Calude, C.S., Hagiya, M., Morita, K., Rozenberg, G., Timmis, J. (eds.) Unconventional Computation. Lecture Notes in Computer Science, vol. 6079, pp. 127–139. Springer (2010)
13. Uspensky, V.A., Shen, A.: Relations between varieties of Kolmogorov complexities. Math. Systems Theory 29(3), 271–292 (1996), http://dx.doi.org/10.1007/BF01201280

ECCC ISSN 1433-8092

http://eccc.hpi-web.de