

Expanding Generator Sets for Solvable Permutation Groups

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Abstract

Let $G = \langle S \rangle$ be a solvable permutation group given as input by the generating set S . I.e. G is a solvable subgroup of the symmetric group S_n . We give a deterministic polynomial-time algorithm that computes an *expanding generator set* for G . More precisely, given a constant $\lambda < 1$ we can compute an expanding generator set T of size $\tilde{O}(n^2)$ such that the undirected Cayley graph $\text{Cay}(G, T)$ is a λ -spectral expander, where the \tilde{O} notation suppresses $\log^{O(1)} n$ factors. In particular, this construction yields ε -bias spaces with improved size bounds for the groups \mathbb{Z}_d^n for any *constant* $\varepsilon > 0$.

1 Introduction

Let G be a finite group, and let $S = \langle g_1, g_2, \dots, g_k \rangle$ be a *generating set* for G . The *undirected Cayley graph* $\text{Cay}(G, S \cup S^{-1})$ is an undirected multigraph with vertex set G and edges of the form $\{x, xg_i\}$ for each $x \in G$ and $g_i \in S$. Since S is a generating set for G , $\text{Cay}(G, S \cup S^{-1})$ is a connected regular multigraph.

For a regular undirected graph $X = (V, E)$ of degree D on n vertices, its *normalized adjacency matrix* A_X is a symmetric matrix with largest eigenvalue 1. For $0 < \lambda < 1$, the graph X is an (n, D, λ) -*spectral expander* if the second largest eigenvalue of A_X , in absolute value, is bounded by λ .

Expander graphs are of great interest and importance in theoretical computer science, especially in the study of randomness in computation; the monograph by Hoory, Linial, and Wigderson [HLW06] is an excellent reference. A central problem is the explicit construction of expander graph families [HLW06, LPS88]. By explicit it is meant that the family of graphs has efficient deterministic constructions, where the notion of efficiency depends upon the application at hand, e.g. [Rei08]. Explicit constructions with the best known and

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near optimal expansion and degree parameters (the so-called Ramanujan graphs) are Cayley expander families [LPS88].

Alon and Roichman, in [AR94], show that every finite group has a logarithmic size expanding generator set using the probabilistic method. For any finite group G and $\lambda > 0$, they show that with high probability a random multiset S of size $O(\log |G|)$ picked uniformly at random from G is a λ -spectral expander. Algorithmically, if G is given as input by its multiplication table there is a randomized *Las Vegas* algorithm for computing S : we pick the multiset S of $O(\log |G|)$ many element from G and check in deterministic time $|G|^{O(1)}$ that $\text{Cay}(G, T)$ is a λ -spectral expander.

Wigderson and Xiao gave a derandomization of this algorithm in [WX08] (also see [AMN11] for an alternative proof of [WX08]). Given $\lambda > 0$ and a finite group G by a multiplication table, they show that in deterministic time $|G|^{O(1)}$ a multiset S of size $O(\log |G|)$ can be computed such that $\text{Cay}(G, T)$ is a λ -spectral expander.

This paper

Suppose the finite group G is a subgroup of the symmetric group S_n or the matrix group $\text{GL}_n(\mathbb{F}_p)$ and G is given as input by a *generating set* S , and not explicitly by a multiplication table. The question we address is whether we can compute an $O(\log |G|)$ size expanding generator set for G in deterministic polynomial time. Notice that if we can randomly (or nearly randomly) sample from the group G in polynomial time, then the Alon Roichman theorem implies that an $O(\log |G|)$ size sample will be an expanding generator set with high probability (though we do not know how to certify this in polynomial time).

This problem can be seen as a generalization of the construction of small bias spaces in, say \mathbb{F}_2^n [AGHP92]. It is easily proved (see e.g. [HLW06]), using some character theory of finite abelian groups, that ε -bias spaces are precisely expanding generator sets for \mathbb{F}_2^n (and this holds for any finite abelian group). Interestingly, the best known explicit construction of ε -bias spaces is of size $O(n^2/\varepsilon)$, whereas the Alon-Roichman theorem guarantees the existence of ε -bias spaces of size $O(n)$.

Subsequently, Azar, Motwani and Naor [AMN98] gave a construction of ε -bias spaces for finite abelian groups of the form \mathbb{Z}_d^n using Linnik's theorem and Weil's character sum bounds. The size of the ε -bias space they give is $O((d + n^2)^C)$ where the constant C comes from Linnik's theorem and the current best known bound for C is $11/2$.

In this paper we prove a more general result. Given any solvable subgroup G of S_n , where G is given by a generating set, we construct an expanding generator set T for G such that $\text{Cay}(G, T)$ is a λ -spectral expander for constant λ . Furthermore, $|T|$ is $\tilde{O}(n^2)$ which is close to some of the best known ε -bias space construction for \mathbb{F}_2^n [AGHP92, ABN⁺92].

An interesting question is whether we can find smaller expanding generator sets for a solvable group $G \leq S_n$. For an upper bound, by the Alon-Roichman theorem we know that there exist expanding generator sets for any G of size $O(\log |G|)$ which is bounded by $O(n \log n) = \tilde{O}(n)$. In general, given G an algorithmic question is to ask for a minimum size expanding generator set for G that makes the Cayley graph λ -spectral.

In this connection, it is interesting to note the following negative result that Lubotzky and Weiss in [LW93] have shown about solvable groups as expanders: Let $\{G_i\}$ be any infinite family of finite solvable groups $\{G_i\}$ such that each G_i has derived series of length bounded by some constant ℓ . Further, suppose that Σ_i is an arbitrary generating set for G_i such that its size $|\Sigma_i| \leq k$ for each i and some constant k . Then the Cayley graphs $\text{Cay}(G_i, \Sigma_i)$ do not form a family of expanders. In contrast, they also exhibit an infinite family of solvable groups in [LW93] that give rise to constant-degree Cayley expanders.

Coming back to our present paper, the main ingredients of our construction are the following:

- Let G be a finite group and N be a normal subgroup of G . Given expanding generator sets S_1 and S_2 for N and G/N respectively such that the corresponding Cayley graphs are λ -spectral expanders, we give a simple polynomial-time algorithm to construct an expanding generator set S for G such that $\text{Cay}(G, S)$ is also λ -spectral. Moreover, $|S|$ is bounded by a constant factor of $|S_1| + |S_2|$.
- We compute the derived series for the given solvable group $G \leq S_n$ in polynomial time using a standard algorithm [Luk93]. This series is of $O(\log n)$ length due to Dixon's theorem. Let the derived series for G be

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_k = \{1\}.$$

Assuming that we already have an expanding generator set for each quotient group G_i/G_{i+1} (which is abelian) of size $\tilde{O}(n^2)$, we apply the previous step repeatedly to obtain an expanding generator set for G of size $\tilde{O}(n^2)$. We can do this because the derived series is a normal series.

- Finally, we consider the abelian quotient groups G_i/G_{i+1} and give a polynomial time algorithm to construct an expanding generator set for it of size $\tilde{O}(n^2)$. This construction applies a series decomposition of abelian groups as well as makes use of the Ajtai et al construction of expanding generator sets for \mathbb{Z}_t [AIK⁺90]. In particular, we note that our construction improves the Azar-Motwani-Naor construction of ε -bias spaces for \mathbb{Z}_d^n for any constant $\varepsilon > 0$ [AMN98].

We present the above three steps of the construction in the next three sections.

2 Combining Expanders for N and G/N

Let G be any finite group and N be a normal subgroup of G (i.e. $g^{-1}Ng = N$ for all $g \in G$). We denote this by $G \triangleright N \triangleright \{1\}$. Let $A \subset N$ be an expanding generator set for N and $\text{Cay}(N, A)$ be a λ -spectral expander. Similarly, suppose $B \subset G$ such that $\widehat{B} = \{Nx \mid x \in B\}$ is an expanding generator set for the quotient group G/N and $\text{Cay}(G/N, \widehat{B})$ is also a λ -spectral expander. Let $X = \{x_1, x_2, \dots, x_k\}$ denote a set of distinct coset representatives for

the normal subgroup N in G . In this section we show that $A \cup B$ is an expanding generator set for G . More precisely, we will show that $\text{Cay}(G, A \cup B)$ is a $\frac{1+\lambda}{2}$ -spectral expander.

In order to analyze the spectral expansion of the Cayley graph $\text{Cay}(G, A \cup B)$ it is useful to view vectors in $\mathbb{C}^{|G|}$ as elements of the group algebra $\mathbb{C}[G]$. The group algebra $\mathbb{C}[G]$ consists of linear combinations $\sum_{g \in G} \alpha_g g$ for $\alpha_g \in \mathbb{C}$. Addition in $\mathbb{C}[G]$ is component-wise, and clearly $\mathbb{C}[G]$ is a $|G|$ -dimensional vector space over \mathbb{C} . The product of $\sum_{g \in G} \alpha_g g$ and $\sum_{h \in G} \beta_h h$ is defined naturally as: $\sum_{g, h \in G} \alpha_g \beta_h gh$.

Let $S \subset G$ be any symmetric subset and let M_S denote the normalized adjacency matrix of the undirected Cayley graph $\text{Cay}(G, S)$. Now, each element $a \in G$ defines the linear map $M_a : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$ by $M_a(\sum_g \alpha_g g) = \sum_g \alpha_g ga$. Clearly, $M_S = \frac{1}{|S|} \sum_{a \in S} M_a$ and $M_S(\sum_g \alpha_g g) = \frac{1}{|S|} \sum_{a \in S} \sum_g \alpha_g ga$.

In order to analyze the spectral expansion of $\text{Cay}(G, A \cup B)$ we consider the basis $\{xn \mid x \in X, n \in N\}$ of $\mathbb{C}[G]$. The element $u_N = \frac{1}{|N|} \sum_{n \in N} n$ of $\mathbb{C}[G]$ corresponds to the uniform distribution supported on N . It has the following important properties:

1. For all $a \in N$ $M_a(u_N) = u_N$ because $Na = N$ for each $a \in N$.
2. For any $b \in G$ consider the linear map $\sigma_b : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$ defined by conjugation: $\sigma_b(\sum_g \alpha_g g) = \sum_g \alpha_g b^{-1}gb$. Since $N \triangleleft G$ the linear map σ_b is an automorphism of N . It follows that for all $b \in G$ $\sigma_b(u_N) = u_N$.

Now, consider the subspaces U and W of $\mathbb{C}[G]$ defined as follows:

$$U = \left\{ \left(\sum_{x \in X} \alpha_x x \right) u_N \right\}$$

$$W = \left\{ \sum_{x \in X} x \left(\sum_{n \in N} \beta_{n,x} n \right) \mid \sum_n \beta_{n,x} = 0, \forall x \in X \right\}$$

It is easy to see that U and W are indeed subspaces of $\mathbb{C}[G]$. Furthermore, we note that every vector in U is orthogonal to every vector in W , i.e. $U \perp W$. This follows easily from the fact that xu_N is orthogonal to $x \sum_{n \in N} \beta_{n,x} n$ whenever $\sum_{n \in N} \beta_{n,x} n$ is orthogonal to u_N . Note that $\sum_{n \in N} \beta_{n,x} n$ is indeed orthogonal to u_N when $\sum_{n \in N} \beta_{n,x} = 0$. We claim that $\mathbb{C}[G]$ is a direct sum of its subspaces U and W .

Proposition 2.1. *The group algebra $\mathbb{C}[G]$ has a direct sum decomposition $\mathbb{C}[G] = U + W$.*

Proof. Since $U \perp W$, it suffices to check that $\dim(U) + \dim(W) = |G|$. The set $\{xu_N \mid x \in X\}$ forms an orthogonal basis for U since for any $x \neq y \in X$, xu_N is orthogonal to yu_N . The cardinality of this basis is $|X|$.

Let $z_1, \dots, z_{|N|-1}$ be the $|N| - 1$ vectors orthogonal to the uniform distribution u_N in the eigenbasis for the Cayley graph $\text{Cay}(N, A)$. It is easy to see that the set $\{xz_j \mid x \in X, 1 \leq j \leq |N| - 1\}$ of size $|X|(|N| - 1)$ forms a basis for W . \square

We will now prove the main result of this section.

Lemma 2.2. *Let G be any finite group and N be a normal subgroup of G and $\lambda < 1/2$ be any constant. Suppose A is an expanding generator set for N so that $\text{Cay}(N, A)$ is a λ -spectral expander. Furthermore, suppose $B \subseteq G$ such that $\widehat{B} = \{Nx \mid x \in B\}$ is an expanding generator for the quotient group G/N and $\text{Cay}(G/N, \widehat{B})$ is also a λ -spectral expander. Then $A \cup B$ is an expanding generator set for G such that $\text{Cay}(G, A \cup B)$ is a $\frac{(1+\lambda)(\max\{|A|, |B|\})}{|A|+|B|}$ -spectral expander. In particular, if $|A| = |B|$ then $\text{Cay}(G, A \cup B)$ is a $\frac{(1+\lambda)}{2}$ -spectral expander.¹*

Proof. We will give the proof only for the case when $|A| = |B|$ (the general case is identical).

Let $v \in \mathbb{C}[G]$ be any vector such that $v \perp \mathbf{1}$ and M denote the adjacency matrix of the Cayley graph $\text{Cay}(G, A \cup B)$. Our goal is to show that $\|Mv\| \leq \frac{1+\lambda}{2}\|v\|$. Notice that the adjacency matrix M can be written as $\frac{1}{2}(M_A + M_B)$ where $M_A = \frac{1}{|A|} \sum_{a \in A} M_a$ and $M_B = \frac{1}{|B|} \sum_{b \in B} M_b$.²

Claim. For any two vectors $u \in U$ and $w \in W$, we have $M_A u \in U$, $M_A w \in W$, $M_B u \in U$, $M_B w \in W$, i.e. U and W are invariant under the transformations M_A and M_B .

Proof. Consider vectors of the form $u = xu_N \in U$ and $w = x \sum_{n \in N} \beta_{n,x} n$, where $x \in X$ is arbitrary. By linearity, it suffices to prove for each $a \in A$ and $b \in B$ that $M_a u \in U$, $M_b u \in U$, $M_a w \in W$, and $M_b w \in W$. Notice that $M_a u = xu_N a = xu_N = u$ since $u_N a = u_N$. Furthermore, we can write $M_a w = x \sum_{n \in N} \beta_{n,x} n a = x \sum_{n' \in N} \gamma_{n',x} n'$, where $\gamma_{n',x} = \beta_{n,x}$ and $n' = na$. Since $\sum_{n' \in N} \gamma_{n',x} = \sum_{n \in N} \beta_{n,x} = 0$ it follows that $M_a w \in W$. Now, consider $M_b u = ub$. For $x \in X$ and $b \in B$ the element xb can be *uniquely* written as $x_b n_{x,b}$, where $x_b \in X$ and $n_{x,b} \in N$.

$$\begin{aligned} M_b u &= xu_N b = xb(b^{-1}u_N b) \\ &= x_b n_{x,b} \sigma_b(u_N) = x_b n_{x,b} u_N = x_b u_N \in U. \end{aligned}$$

Finally,

$$\begin{aligned} M_b w &= x \left(\sum_{n \in N} \beta_{n,x} n \right) b = xb \left(\sum_{n \in N} \beta_{n,x} b^{-1} n b \right) \\ &= x_b n_{x,b} \sum_{n \in N} \beta_{b n b^{-1}, x} n \\ &= x_b \sum_{n \in N} \gamma_{n,x} n \in W. \end{aligned}$$

¹The sizes of A and B is not a serious issue for us. Since we consider multisets as expanding generator sets, notice that we always ensure $|A|$ and $|B|$ are within a factor of 2 of each other by scaling the smaller multiset appropriately. Indeed, in our construction we can even ensure when we apply this lemma that the multisets A and B are of the same cardinality which is a power of 2.

²In the case when $|A| \neq |B|$, the adjacency matrix M will be $\frac{|A|}{|A|+|B|} M_A + \frac{|B|}{|A|+|B|} M_B$.

Here, we note that $\gamma_{n,x} = \beta_{n',x}$ and $n' = b(n_{x,b}^{-1}n)b^{-1}$. Hence $\sum_{n \in N} \gamma_{n,x} = 0$, which puts $M_b w$ in the subspace W as claimed. \square

Claim. Let $u \in U$ such that $u \perp \mathbf{1}$ and $w \in W$. Then:

1. $\|M_A u\| \leq \|u\|$.
2. $\|M_B w\| \leq \|w\|$.
3. $\|M_B u\| \leq \lambda \|u\|$.
4. $\|M_A w\| \leq \lambda \|w\|$.

Proof. Since M_A is the normalized adjacency matrix of the Cayley graph $\text{Cay}(G, A)$ and M_B is the normalized adjacency matrix of the Cayley graph $\text{Cay}(G, B)$, it follows that for any vectors u and w we have the bounds $\|M_A u\| \leq \|u\|$ and $\|M_B w\| \leq \|w\|$.

Now we prove the third part. Let $u = (\sum_x \alpha_x x)u_N$ be any vector in U such that $u \perp \mathbf{1}$. Then $\sum_{x \in X} \alpha_x = 0$. Now consider the vector $\hat{u} = \sum_{x \in X} \alpha_x Nx$ in the group algebra $\mathbb{C}[G/N]$. Notice that $\hat{u} \perp \mathbf{1}$. Let $M_{\hat{B}}$ denote the normalized adjacency matrix of $\text{Cay}(G/N, \hat{B})$. Since it is a λ -spectral expander it follows that $\|M_{\hat{B}} \hat{u}\| \leq \lambda \|\hat{u}\|$. Writing out $M_{\hat{B}} \hat{u}$ we get $M_{\hat{B}} \hat{u} = \frac{1}{|B|} \sum_{b \in B} \sum_{x \in X} \alpha_x Nxb = \frac{1}{|B|} \sum_{b \in B} \sum_{x \in X} \alpha_x Nx_b$, because $xb = x_b n_{x,b}$ and $Nxb = Nx_b$ (as N is a normal subgroup). Hence the norm of the vector $\frac{1}{|B|} \sum_{b \in B} \sum_{x \in X} \alpha_x Nx_b$ is bounded by $\lambda \|\hat{u}\|$. Equivalently, the norm of the vector $\frac{1}{|B|} \sum_{b \in B} \sum_{x \in X} \alpha_x x_b$ is bounded by $\lambda \|\hat{u}\|$. On the other hand, we have

$$\begin{aligned} M_B u &= \frac{1}{|B|} \sum_b \left(\sum_x \alpha_x x \right) u_N b = \frac{1}{|B|} \sum_b \left(\sum_x \alpha_x x b \right) b^{-1} u_N b \\ &= \frac{1}{|B|} \left(\sum_b \sum_x \alpha_x x_b n_{x,b} \right) u_N = \frac{1}{|B|} \left(\sum_b \sum_x \alpha_x x_b \right) u_N \end{aligned}$$

For any vector $(\sum_{x \in X} \gamma_x x)u_N \in U$ it is easy to see that the norm $\|(\sum_{x \in X} \gamma_x x)u_N\| = \|\sum_{x \in X} \gamma_x x\| \|u_N\|$. Therefore,

$$\begin{aligned} \|M_B u\| &= \left\| \frac{1}{|B|} \sum_b \sum_x \alpha_x x_b \right\| \|u_N\| \\ &\leq \lambda \left\| \sum_{x \in X} \alpha_x x \right\| \|u_N\| \\ &= \lambda \|u\|. \end{aligned}$$

We now show the fourth part. For each $x \in X$ it is useful to consider the following subspaces of $\mathbb{C}[G]$

$$\mathbb{C}[xN] = \left\{ x \sum_{n \in N} \theta_n n \mid \theta_n \in \mathbb{C} \right\}.$$

For any distinct $x \neq x' \in X$, since $xN \cap x'N = \emptyset$, vectors in $\mathbb{C}[xN]$ have support disjoint from vectors in $\mathbb{C}[x'N]$. Hence $\mathbb{C}[xN] \perp \mathbb{C}[x'N]$ which implies that the subspaces $\mathbb{C}[xN], x \in X$ are pairwise mutually orthogonal. Furthermore, the matrix M_A maps $\mathbb{C}[xN]$ to $\mathbb{C}[xN]$ for each $x \in X$.

Now, consider any vector $w = \sum_{x \in X} x (\sum_{n \in N} \beta_{n,x} n)$ in W . Letting $w_x = x (\sum_{n \in N} \beta_{n,x} n) \in \mathbb{C}[xN]$ for each $x \in X$ we note that $M_A w_x \in \mathbb{C}[xN]$ for each $x \in X$. Hence, by Pythagoras theorem we have $\|w\|^2 = \sum_{x \in X} \|w_x\|^2$ and $\|M_A w\|^2 = \sum_{x \in X} \|M_A w_x\|^2$. Since $M_A w_x = x M_A (\sum_{n \in N} \beta_{n,x} n)$, it follows that $\|M_A w_x\| = \|M_A (\sum_{n \in N} \beta_{n,x} n)\| \leq \lambda \|\sum_{n \in N} \beta_{n,x} n\| = \lambda \|w_x\|$.

Putting it together, it follows that $\|M_A w\|^2 \leq \lambda^2 (\sum_{x \in X} \|w_x\|^2) = \lambda^2 \|w\|^2$. \square

We now complete the proof of the lemma. Consider any vector $v \in \mathbb{C}[G]$ such that $v \perp \mathbf{1}$. Let $v = u + w$ where $u \in U$ and $w \in W$. Let \langle, \rangle denote the inner product in $\mathbb{C}[G]$. Then we have

$$\begin{aligned} \|Mv\|^2 &= \frac{1}{4} \|(M_A + M_B)v\|^2 \\ &= \frac{1}{4} \langle (M_A + M_B)v, (M_A + M_B)v \rangle \\ &= \frac{1}{4} \langle M_A v, M_A v \rangle + \frac{1}{4} \langle M_B v, M_B v \rangle + \frac{1}{2} \langle M_A v, M_B v \rangle \end{aligned}$$

We consider each of the three summands in the above expression.

$$\begin{aligned} \langle M_A v, M_A v \rangle &= \langle M_A(u + w), M_A(u + w) \rangle \\ &= \langle M_A u, M_A u \rangle + \langle M_A w, M_A w \rangle + 2 \langle M_A u, M_A w \rangle. \end{aligned}$$

By Claim 2 and the fact that $U \perp W$, $\langle M_A u, M_A w \rangle = 0$. Thus we get

$$\begin{aligned} \langle M_A v, M_A v \rangle &= \langle M_A u, M_A u \rangle + \langle M_A w, M_A w \rangle \\ &\leq \|u\|^2 + \lambda^2 \|w\|^2, \text{ from Claim 2.} \end{aligned}$$

By an identical argument Claims 2 and 2 imply $\langle M_B v, M_B v \rangle \leq \lambda^2 \|u\|^2 + \|w\|^2$. Finally

$$\begin{aligned} \langle M_A v, M_B v \rangle &= \langle M_A(u + w), M_B(u + w) \rangle \\ &= \langle M_A u, M_B u \rangle + \langle M_A w, M_B w \rangle + \langle M_A u, M_B w \rangle + \langle M_A w, M_B u \rangle \\ &= \langle M_A u, M_B u \rangle + \langle M_A w, M_B w \rangle \\ &\leq \|M_A u\| \|M_B u\| + \|M_A w\| \|M_B w\| \text{ (by Cauchy-Schwarz inequality)} \\ &\leq \lambda \|u\|^2 + \lambda \|w\|^2, \text{ which follows from Claim 2} \end{aligned}$$

Combining all the inequalities, we get

$$\|Mv\|^2 \leq \frac{1}{4} (1 + 2\lambda + \lambda^2) (\|u\|^2 + \|w\|^2) = \frac{(1 + \lambda)^2}{4} \|v\|^2.$$

Hence, it follows that $\|Mv\| \leq \frac{1+\lambda}{2} \|v\|$. \square

2.1 A Derandomized Squaring Step

Notice that $\text{Cay}(G, A \cup B)$ is only a $\frac{1+\lambda}{2}$ -spectral expander. We can compute another expanding generator set S for G from $A \cup B$, using *derandomized squaring* [RV05], such that $\text{Cay}(G, S)$ is a λ -spectral expander. We recall a result in [RV05, Observation 4.3, Theorem 4.4] about derandomized squaring applied to Cayley graphs which we recall in some detail.

Theorem 2.3 ([RV05]). *Let G be a finite group and U be an expanding generator set such that $\text{Cay}(G, U)$ is a λ' -spectral expander and H be a consistently labeled d -regular graph with vertex set $\{1, 2, \dots, |U|\}$ for a constant d such that H is a μ -spectral expander. Then $\text{Cay}(G, U) \otimes H$ is a directed Cayley graph for the same group G and with generator set $S = \{u_i u_j \mid (i, j) \in E(H)\}$. Furthermore, if A is the normalized adjacency matrix for $\text{Cay}(G, U) \otimes H$ then for any vector $v \in \mathbb{C}^{|G|}$ such that $v \perp \mathbf{1}$:*

$$\|Av\| \leq (\lambda'^2 + \mu)\|v\|.$$

Observe that in the definition of the directed Cayley graph $\text{Cay}(G, U) \otimes H$ (in the statement above) there is an identification of the vertex set $\{1, 2, \dots, |U|\}$ of H with the generator multiset U indexed as $U = \{u_1, u_2, \dots, u_{|U|}\}$.

Alternatively, we can also identify the vertex set $\{1, 2, \dots, |U|\}$ of H with the generator multiset U indexed as $U = \{u_1^{-1}, u_2^{-1}, \dots, u_{|U|}^{-1}\}$, since U is closed under inverses and, as a multiset, we assume for each $u \in U$ both u and u^{-1} occur with same multiplicity. Let us denote this directed Cayley graph by $\text{Cay}(G, U^{-1}) \otimes H$. Clearly, by the above result of [RV05] the graph $\text{Cay}(G, U^{-1}) \otimes H$ also has the same expansion property. I.e. if A' denotes its normalized adjacency matrix for $\text{Cay}(G, U^{-1}) \otimes H$ then for any vector $v \in \mathbb{C}^{|G|}$ such that $v \perp \mathbf{1}$:

$$\|A'v\| \leq (\lambda'^2 + \mu)\|v\|.$$

We summarize the above discussion in the following lemma.

Lemma 2.4. *Let G be a finite group and U be a generator multiset for G such that for each $u \in U$ both u and u^{-1} occur with the same multiplicity (i.e. U is symmetric and preserves multiplicities). Suppose $\text{Cay}(G, U)$ is a λ' -spectral expander. Let H be a consistently labeled d -regular graph with vertex set $\{1, 2, \dots, |U|\}$ for a constant d such that H is a μ -spectral expander. Then $\text{Cay}(G, S)$ is an undirected Cayley graph for the same group G and with generator set $S = \{u_i u_j \mid (i, j) \in E(H)\} \cup \{u_i^{-1} u_j^{-1} \mid (i, j) \in E(H)\}$. Furthermore, $\text{Cay}(G, S)$ is a $(\lambda'^2 + \mu)$ -spectral expander of degree $2d|U|$.*

We can, for instance, use the graphs given by the following lemma for H in the above construction.

Lemma 2.5 ([RV05]). *For some constant $Q = 4^q$, there exists a sequence of consistently labelled Q -regular graphs on Q^m vertices whose second largest eigenvalue is bounded by $1/100$ such that given a vertex $v \in [Q^m]$ and an edge label $x \in [Q]$, we can compute the x^{th} neighbor of v in time polynomial in m .*

Suppose $\text{Cay}(G, U)$ is a $3/4$ -spectral expander and we take H given by the above lemma for derandomized squaring, then it is easy to see that with a constant number of squaring operations we will obtain a generator set S for G such that $|S| = O(|U|)$ and $\text{Cay}(G, S)$ is a $1/4$ -spectral expander. Putting this together with Lemma 2.2 we obtain the following consequence which we will use repeatedly in the rest of the paper.

Lemma 2.6. *Let G be a finite group and N be a normal subgroup of G such that $N = \langle A \rangle$ and $\text{Cay}(N, A)$ is a $1/4$ -spectral expander. Further, let $B \subseteq G$ and $\widehat{B} = \{Nx \mid x \in B\}$ such that $G/N = \langle \widehat{B} \rangle$ and $\text{Cay}(G/N, \widehat{B})$ is a $1/4$ -spectral expander. Then in time polynomial³ in $|A| + |B|$, we can construct an expanding generator set S for G , such that $|S| = O(|A| + |B|)$ and $\text{Cay}(G, S)$ is a $1/4$ -spectral expander.*

3 Normal Series and Solvable Permutation Groups

Let $G \leq S_n$ such that

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_r = \{1\}$$

is a *normal series* for G . I.e. G_i is a normal subgroup of G for each i and hence G_i is a normal subgroup of G_j for each $j < i$.

Lemma 3.1. *Let $G \leq S_n$ with normal series $\{G_i\}_{i=0}^r$ as above. Further, for each i let B_i be a generator set for G_i/G_{i+1} such that $\text{Cay}(G_i/G_{i+1}, B_i)$ is a $1/4$ -spectral expander. Let $s = r \cdot \max_i \{|B_i|\}$. Then in deterministic time polynomial in n and s we can compute a generating set B for G such that $\text{Cay}(G, B)$ is a $1/4$ -spectral expander and $|B| = c^{\log r} s$ for some constant $c > 0$.*

Proof. The proof is an easy application of Lemma 2.6. First suppose we have three indices k, ℓ, m such that $G_k \triangleright G_\ell \triangleright G_m$ and $\text{Cay}(G_k/G_\ell, S)$ and $\text{Cay}(G_\ell/G_m, T)$ both are $1/4$ -spectral expanders. Then notice that we have the groups $G_k/G_m \triangleright G_\ell/G_m \triangleright \{1\}$ and the group $\frac{G_k}{G_\ell}$ is isomorphic to $\frac{G_k/G_m}{G_\ell/G_m}$ via a natural isomorphism. Hence $\text{Cay}(\frac{G_k/G_m}{G_\ell/G_m}, \widehat{S})$ is also a $1/4$ -spectral expander, where \widehat{S} is the image of S under the said natural isomorphism. Therefore, we can apply Lemma 2.6 by setting G to G_k/G_m and N to G_ℓ/G_m to get a generator set U for G_k/G_m such that $\text{Cay}(G_k/G_m, U)$ is $1/4$ -spectral and $|U| \leq c(|S| + |T|)$.

To apply this inductively to the entire normal series, assume wlog its length $r = 2^t$. Inductively assume that in the normal series

$$G = G_0 \triangleright G_{2^i} \triangleright G_{2 \cdot 2^i} \triangleright G_{3 \cdot 2^i} \cdots \triangleright G_r = \{1\},$$

for each quotient group $G_{j2^i}/G_{(j+1)2^i}$ we have an expanding generator set of size $c^i s$ that makes $G_{j2^i}/G_{(j+1)2^i}$ $1/4$ -spectral. Now, consider the three groups $G_{(2j)2^i} \triangleright G_{(2j+1)2^i} \triangleright G_{(2j+2)2^i}$

³Though the lemma holds for any finite group G , the caveat is that the group operations in G should be polynomial-time computable. Since we focus on permutation groups in this paper we will require it only for quotient groups $G = H/N$ where H and N are subgroups of S_n .

and setting $k = 2j2^i$, $\ell = (2j + 1)2^i$ and $m = (2j + 2)2^i$ in the above argument we get expanding generator sets for $G_{2j2^i}/G_{(2j+2)2^i}$ of size $c^{i+1}s$ that makes it $1/4$ -spectral. The lemma follows by induction. \square

3.1 Solvable permutation groups

Now we apply the above lemma to solvable permutation groups. Let G be any finite solvable group. The *derived series* for G is the following chain of subgroups of G :

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_k = \{1\}$$

where, for each i , G_{i+1} is the *commutator subgroup* of G_i . I.e. G_{i+1} is the normal subgroup of G_i generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G_i$. It turns out that G_{i+1} is the minimal normal subgroup of G_i such that G_i/G_{i+1} is abelian. Furthermore, the derived series is also a *normal series*. I.e. each G_i is in fact a normal subgroup of G itself. It also implies that G_i is a normal subgroup of G_j for each $j < i$.

Our algorithm will crucially exploit a property of the derived series of solvable groups $G \leq S_n$. This is a theorem of Dixon [Dix68] which states that the length k of the derived series of a solvable subgroup of S_n is bounded by $5 \log_3 n$.

Lemma 3.2. *Suppose $G \leq S_n$ is a solvable group with derived series*

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_k = \{1\}$$

such that for each i we have an expanding generator set B_i for the abelian quotient group G_i/G_{i+1} such that $\text{Cay}(G_i/G_{i+1}, B_i)$ is a $1/4$ -spectral expander. Let $s = r \max_i \{|B_i|\}$. Then in deterministic time polynomial in n and s we can compute a generating set B for G such that $\text{Cay}(G, B)$ is a $1/4$ -spectral expander and $|B| = 2^{O(\log k)} s = (\log n)^{O(1)} s$.

Proof. Follows by a direct application of Lemma 3.1. \square

Given a solvable permutation group $G \leq S_n$ by a generator set the polynomial-time algorithm for computing an expanding generator set will proceed as follows: in deterministic polynomial-time we first compute [Luk93] generator sets for each subgroup $\{G_i\}_{1 \leq i \leq k}$ in the derived series for G . In order to apply the above lemma it suffices to compute an expanding generator set B_i for G_i/G_{i+1} such that $\text{Cay}(G_i/G_{i+1}, B_i)$ is $1/4$ -spectral. We deal with this problem in the next section.

4 Abelian Quotient Groups

As explained above we are now left with the problem of computing expanding generator sets for the abelian quotient groups G_i/G_{i+1} . We prove a couple of easy lemmas that will allow us to further simplify the problem.

Lemma 4.1. *Let H and N be subgroups of S_n such that N is a normal subgroup of H and H/N is abelian. Let $p_1 < p_2 < \dots < p_k$ be the set of all primes bounded by n and $e = \lceil \log n \rceil$. There is an onto homomorphism ϕ from the product group $\mathbb{Z}_{p_1^e}^n \times \mathbb{Z}_{p_2^e}^n \times \dots \times \mathbb{Z}_{p_k^e}^n$ onto the abelian quotient group H/N .*

Proof. Since H is a subgroup of S_n it has a generator set of size at most $n - 1$. Let $\{x_1, x_2, \dots, x_n\}$ be a generator (multi)set for H . Each permutation x_i can be written as a product of disjoint cycles and the order, r_i , of x_i is the lcm of the lengths of these disjoint cycles. Thus we can write for each i

$$r_i = p_1^{e_{i1}} p_2^{e_{i2}} \dots p_k^{e_{ik}},$$

where the key point to note is that $p_j^{e_{ij}} \leq n$ for each i and j because r_i is the lcm of the disjoint cycles of permutation x_i . Clearly, $e_{ij} \leq e = \lceil \log n \rceil$.

Now, define the elements $y_{ij} = x_i^{r_i/p_j^{e_{ij}}}$. Notice that the order, $o(y_{ij})$, of y_{ij} is $p_j^{e_{ij}}$.

Let $(a_{11}, \dots, a_{n1}, \dots, a_{1k}, \dots, a_{nk})$ be an element of the product group $\mathbb{Z}_{p_1^e}^n \times \mathbb{Z}_{p_2^e}^n \times \dots \times \mathbb{Z}_{p_k^e}^n$, where for each i we have $(a_{i1}, \dots, a_{ni}) \in \mathbb{Z}_{p_i^e}^n$. Let $b_{ij} = p_j^{e-e_{ij}} a_{ij}$ for each i and j . Now define the mapping ϕ as

$$\phi(a_{11}, \dots, a_{n1}, \dots, a_{1k}, \dots, a_{nk}) = N \left(\prod_{j=1}^k \prod_{i=1}^n y_{ij}^{a_{ij}} \right).$$

Since H/N is abelian, it is easy to see that ϕ is a homomorphism. To see that ϕ is onto, consider $Nx_1^{f_1} \dots x_n^{f_n} \in H/N$. Clearly, the cyclic subgroup generated by x_i is the direct product of its p_j -Sylow subgroups generated by y_{ij} for $1 \leq j \leq k$. Hence $x_i^{f_i} = y_{i1}^{a_{i1}} \dots y_{ik}^{a_{ik}}$ for some $(a_{i1}, \dots, a_{ik}) \in \mathbb{Z}_{p_1^{e_{i1}}} \times \dots \times \mathbb{Z}_{p_k^{e_{ik}}}$. This vector (a_{11}, \dots, a_{nk}) is a preimage of $Nx_1^{f_1} \dots x_n^{f_n}$, implying that ϕ is onto. \square

Suppose H_1 and H_2 are two finite groups such that $\phi : H_1 \rightarrow H_2$ is an onto homomorphism. In the next lemma we show that the ϕ -image of an expanding generator set for H_1 , is an expanding generator set for H_2 .

Lemma 4.2. *Suppose H_1 and H_2 are two finite groups such that $\phi : H_1 \rightarrow H_2$ is an onto homomorphism. Furthermore, suppose $\text{Cay}(H_1, S)$ is a λ -spectral expander. Then $\text{Cay}(H_2, \phi(S))$ is also a λ -spectral expander.*

Proof. Let $N = \text{Ker}(\phi)$ be the kernel of the onto homomorphism ϕ . Then H_1/N is isomorphic to H_2 and the lemma is equivalent to the claim that $\text{Cay}(H_1/N, \widehat{S})$ is a λ -spectral expander, where $\widehat{S} = \{Ns \mid s \in S\}$ is the corresponding generator set for H_1/N . We can check by a direct calculation that all eigenvalues of the normalized adjacency matrix of $\text{Cay}(H_1/N, \widehat{S})$ are also eigenvalues of $\text{Cay}(H_1, S)$. This claim also follows from well-known results in the ‘‘expanders monograph’’ [HLW06, Lemma 11.15, Proposition 11.17]. In order to apply these results we note that H_1 naturally defines a permutation action on the quotient

group H_1/N by $h : Nx \mapsto Nxh$ for each $h \in H_1$ and $Nx \in H_1/N$. Then the Cayley graph $\text{Cay}(H_1/N, \widehat{S})$ is just the Schreier graph for this action and the generator set S of H_1 and, by [HLW06, Proposition 11.17], all eigenvalues of $\text{Cay}(H_1/N, \widehat{S})$ are eigenvalues of $\text{Cay}(H_1, S)$ and the lemma follows. \square

Now, suppose $H, N \leq S_n$ are groups given by generator sets where $N \triangleleft H$ and H/N is abelian. By Lemmas 4.1 and 4.2, it suffices to describe a polynomial (in n) time algorithm for computing an expanding generator set of size $\widetilde{O}(n^2)$ for the product group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$. In the following section we solve this problem. Our solution improves the Azar-Motwani-Naor construction of ε -bias spaces for \mathbb{Z}_d^n [AMN98] for constant $\varepsilon > 0$, that we describe in Section 5.

4.1 Improved small-bias spaces for abelian groups

In this section we give a deterministic polynomial (in n) time construction of an $\widetilde{O}(n^2)$ size expanding generating set for the product group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$.

Consider the following *normal series* for this product group given by the subgroups $K_i = \mathbb{Z}_{p_1}^{n_{e-i}} \times \mathbb{Z}_{p_2}^{n_{e-i}} \times \dots \times \mathbb{Z}_{p_k}^{n_{e-i}}$ for $0 \leq i \leq e$. Clearly,

$$K_0 \triangleright K_1 \triangleright \dots \triangleright K_e = \{1\}$$

This is obviously a normal series since $K_0 = \mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$ is abelian. Furthermore, $K_i/K_{i+1} = \mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$.

Since the length of this series is $e = \lceil \log n \rceil$ we can apply Lemma 3.1 to construct an expanding generator set of size $\widetilde{O}(n^2)$ for K_0 in polynomial time assuming that we can compute an expanding generator set of size $\widetilde{O}(n^2)$ for $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$ in deterministic polynomial time.

Thus, it suffices to efficiently compute an $\widetilde{O}(n^2)$ -size expanding generating set for the product group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$.

In [AIK⁺90], Ajtai et al, using some number theory, gave a deterministic polynomial time expanding generator set construction for the cyclic group \mathbb{Z}_t , where t is given in *binary*.

Theorem 4.3. *Let t be a positive integer given in binary as an input. Then there is a deterministic polynomial-time (i.e. in $\text{poly}(\log t)$ time) algorithm that computes an expanding generating set T for \mathbb{Z}_t of size $O(\log^* t \log t)$, where $\log^* t$ is the least positive integer such that a tower of k 2's bounds t . Furthermore, $\text{Cay}(\mathbb{Z}_t, T)$ is λ -spectral for each constant λ .*

Now, consider the group $\mathbb{Z}_{p_1 p_2 \dots p_k}$. Since $p_1 p_2 \dots p_k$ can be represented by $O(n \log n)$ bits in binary, we apply the above theorem to compute an expanding generating set of size $\widetilde{O}(n)$ for $\mathbb{Z}_{p_1 p_2 \dots p_k}$ in $\text{poly}(n)$ time. Let $m = O(\log n)$ be a positive integer to be fixed in the analysis later. Consider the product group $M_0 = \mathbb{Z}_{p_1}^m \times \mathbb{Z}_{p_2}^m \times \dots \times \mathbb{Z}_{p_k}^m$ and for $1 \leq i \leq m$ let $M_i = \mathbb{Z}_{p_1}^{m-i} \times \mathbb{Z}_{p_2}^{m-i} \times \dots \times \mathbb{Z}_{p_k}^{m-i}$. Clearly, the groups M_i form a *normal series* for M_0 :

$$M_0 \triangleright M_1 \triangleright \dots \triangleright M_m = \{1\}.$$

and the quotient groups are $M_i/M_{i+1} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1 p_2 \dots p_k}$. Since we can compute in $\text{poly}(n)$ time an expanding generating set for $\mathbb{Z}_{p_1 p_2 \dots p_k}$ of size $\tilde{O}(n)$ by Theorem 4.3, we can again apply Lemma 3.1 to this normal series and, given $\lambda > 0$, compute in polynomial (in n) time an expanding generator set of size $\tilde{O}(n)$ for the product group M_0 such that the corresponding Cayley graph is λ -spectral.

Now we are ready to describe the expanding generator set construction for $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$.

4.1.1 The final construction

For $1 \leq i \leq k$ let m_i be the least positive integer such that $p_i^{m_i} > cn$ (where c is a suitably large constant). Thus, $p_i^{m_i} \leq cn^2$ for each i . For each i , $\mathbb{F}_{p_i}^{m_i}$ be the finite field of $p_i^{m_i}$ elements which can be deterministically constructed in polynomial time since it is polynomial sized. Clearly, there is an onto homomorphism ψ from the group $\mathbb{Z}_{p_1}^m \times \mathbb{Z}_{p_2}^m \times \dots \times \mathbb{Z}_{p_k}^m$ to the additive group of $\mathbb{F}_{p_1}^{m_1} \times \mathbb{F}_{p_2}^{m_2} \times \dots \times \mathbb{F}_{p_k}^{m_k}$. Thus, if S is the expanding generator set of size $\tilde{O}(n)$ constructed above for $\mathbb{Z}_{p_1}^m \times \mathbb{Z}_{p_2}^m \times \dots \times \mathbb{Z}_{p_k}^m$ it follows that $\psi(S)$ is an expanding generator multiset of size $\tilde{O}(n)$ for the additive group $\mathbb{F}_{p_1}^{m_1} \times \mathbb{F}_{p_2}^{m_2} \times \dots \times \mathbb{F}_{p_k}^{m_k}$. Define $T \subset \mathbb{F}_{p_1}^{m_1} \times \mathbb{F}_{p_2}^{m_2} \times \dots \times \mathbb{F}_{p_k}^{m_k}$ to be any (say, the lexicographically first) set of cn many k -tuples such that for any two tuples (x_1, x_2, \dots, x_k) and $(x'_1, x'_2, \dots, x'_k)$ in T are distinct in all coordinates. I.e. $x_j \neq x'_j$ for all $j \in [k]$. It is obvious that we can construct T by picking the first cn such tuples in lexicographic order.

Now we will define the expanding generator set R . Let $x = (x_1, x_2, \dots, x_k) \in T$ and $y = (y_1, y_2, \dots, y_k) \in \psi(S)$. Define $v_i = (y_i, \langle x_i, y_i \rangle, \langle x_i^2, y_i \rangle, \dots, \langle x_i^{n-1}, y_i \rangle)$ where $x_i^j \in \mathbb{F}_{p_i}^{m_i}$ and $\langle x_i^j, y_i \rangle$ is the inner product modulo p_i of the elements x_i^j and y_i seen as p_i -tuples in $\mathbb{Z}_{p_i}^{m_i} \cong \mathbb{F}_{p_i}^{m_i}$. Hence, v_i is an n -tuple and $v_i \in \mathbb{Z}_{p_i}^n$. Now define

$$R = \{(v_1, v_2, \dots, v_k) \mid x \in T, y \in \psi(S)\}.$$

Notice that $|R| = \tilde{O}(n^2)$. We claim that R is an expanding generator set for the product group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$. Let $(\chi_1, \chi_2, \dots, \chi_k)$ be a nontrivial character of the product group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$, i.e. there is at least one j such that χ_j is nontrivial. Let ω_i be a primitive p_i^{th} root of unity. Recall that, since χ_i is a character there is a corresponding vector $\beta_i \in \mathbb{Z}_{p_i}^n$, i.e. $\chi_i : \mathbb{Z}_{p_i}^n \rightarrow \mathbb{C}$ and $\chi_i(u) = \omega_i^{\langle \beta_i, u \rangle}$ for $u \in \mathbb{Z}_{p_i}^n$ and the inner product in the exponent is a modulo p_i inner product. The character χ_i is nontrivial if and only if β_i is a nonzero element of $\mathbb{Z}_{p_i}^n$.

Since the characters $(\chi_1, \chi_2, \dots, \chi_k)$ of the abelian group $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$ are also the eigenvectors for the group (for any generator set for it). In particular, for the set R as well the characters are the eigenvectors, and the nontrivial characters are orthogonal to $\mathbf{1}$. Thus, in order to prove that R is an expanding generating set for $\mathbb{Z}_{p_1}^n \times \mathbb{Z}_{p_2}^n \times \dots \times \mathbb{Z}_{p_k}^n$, it is enough to bound the following exponential sum estimate for the nontrivial characters $(\chi_1, \chi_2, \dots, \chi_k)$ since that directly bounds the second largest eigenvalue in absolute value.

$$\begin{aligned}
\left| \mathbb{E}_{x \in T, y \in \psi(S)} [\chi_1(v_1) \chi_2(v_2) \dots \chi_k(v)] \right| &= \left| \mathbb{E}_{x \in T, y \in \psi(S)} [\omega_1^{\langle \beta_1, v_1 \rangle} \dots \omega_k^{\langle \beta_k, v_k \rangle}] \right| \\
&= \left| \mathbb{E}_{x \in T, y \in \psi(S)} [\omega_1^{\langle p_1(x_1), y_1 \rangle} \dots \omega_k^{\langle p_k(x_k), y_k \rangle}] \right| \\
&\leq \mathbb{E}_{x \in T} \left| \mathbb{E}_{y \in \psi(S)} [\omega_1^{\langle p_1(x_1), y_1 \rangle} \dots \omega_k^{\langle p_k(x_k), y_k \rangle}] \right|,
\end{aligned}$$

where $p_i(x) = \sum_{\ell=0}^{n-1} \beta_{i,\ell} x^\ell \in \mathbb{F}_{p_i^m}[x]$ for $\beta_i = (\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,n})$. Since the character is nontrivial suppose $\beta_j \neq 0$, which means p_j is a nonzero polynomial of degree at most $n-1$. Hence the probability that $p_j(x_j) = 0$, when x is picked from T is bounded by $\frac{n}{cn}$.

On the other hand, when $p_j(x_j) \neq 0$ the tuple $(p_1(x_1), \dots, p_k(x_k))$ defines a nontrivial character of the group $\mathbb{Z}_{p_1}^m \times \dots \times \mathbb{Z}_{p_k}^m$. Since S is an expanding generating set for the abelian group $\mathbb{Z}_{p_1}^m \times \dots \times \mathbb{Z}_{p_k}^m$, the character defined by $(p_1(x_1), \dots, p_k(x_k))$ is also an eigenvector for $\mathbb{Z}_{p_1}^m \times \dots \times \mathbb{Z}_{p_k}^m$, in particular w.r.t. generator set S . Hence, we have that $\left| \mathbb{E}_{y \in S} [\omega_1^{\langle p_1(x_1), y_1 \rangle} \dots \omega_k^{\langle p_k(x_k), y_k \rangle}] \right| \leq \varepsilon$, where the parameter ε can be fixed to an arbitrary small constant by Theorem 4.3. Hence the above estimate is bounded by $\frac{n}{cn} + \varepsilon = \frac{1}{c} + \varepsilon$ which can be made an arbitrarily small constant by choosing c suitably. To summarize, the above discussion along with Lemmas 4.1 and 4.2 directly yields the following theorem.

Theorem 4.4. *Let p_1, p_2, \dots, p_k be all primes bounded by n and $\lambda > 0$ any constant. In deterministic polynomial (in n) time we can construct an expanding generator set of size $\tilde{O}(n^2)$ for the product group $\mathbb{Z}_{p_1}^n \times \dots \times \mathbb{Z}_{p_k}^n$ that makes it λ spectral. Consequently, if H and N are subgroups of S_n given by generator sets and H/N is abelian then in deterministic polynomial time we can compute an expanding generator set of size $\tilde{O}(n^2)$ for H/N that makes it λ spectral.*

Finally, we state the main theorem which follows directly from the above theorem and Lemma 3.2.

Theorem 4.5. *Let $G \leq S_n$ be a solvable permutation group given by a generator set and $\lambda > 0$ any constant. Then in deterministic polynomial time we can compute an expanding generator set S of size $\tilde{O}(n^2)$ such that the Cayley graph $\text{Cay}(G, S)$ is a λ -spectral expander.*

5 Comparison with known results

In [AMN98] Azar, Motwani, and Naor first considered the construction of ε -bias spaces for abelian groups, specifically for the group \mathbb{Z}_d^n . For arbitrary d and any $\varepsilon > 0$ they construct ε -bias spaces of size $O((d + n^2/\varepsilon^2)^C)$, where C is the constant in Linnik's Theorem. The construction involves finding a suitable prime (or prime power) promised by Linnik's theorem which can take time upto $O((d + n^2)^C)$. The current best known bound for C is $\leq 11/2$ (and assuming ERH it is 2). Their construction yields a polynomial-size ε -bias space for $d = n^{O(1)}$. In fact when $d = O(\log n)^{O(1)}$, their construction is of size $O(n^2)$.

It is interesting to compare with our results in Section 4.1: Let d be any positive integer with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ such that each p_i is $O(\log n)$ bit sized and each e_i is bounded by $O(\log n)$. Then note that we can efficiently find the prime factorization of d . Now, it follows from the construction described in Section 4.1 that in polynomial time we can compute an $\tilde{O}(n^2)$ size ε -bias space for \mathbb{Z}_d^n for any constant $\varepsilon > 0$. Notice that for *constant* ε this is a significant improvement upon the construction in [AMN98] for such d , in particular for $d = n^{O(1)}$. Also, we note that for $d = O(\log n)^{O(1)}$ and constant ε , our construction yields an $\tilde{O}(n)$ size ε -bias space. The reason is that we get an $O(\log^* n \log \log n)$ size expanding generating set for $\mathbb{Z}_{p_1 p_2 \dots p_k}$ (where p_1, p_2, \dots, p_k are the distinct prime factors of d) using [AIK⁺90].

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