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Rank Bounds for a Hierarchy of Lovász and Schrijver

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Abstract

Lovász and Schrijver [17] introduced several lift and project methods for 0-1 integer programs, now collectively known as Lovász-Schrijver (LS) hierarchies. Several lower bounds have since been proven for the rank of various linear programming relaxations in the LS and LS_+ hierarchies. In this paper we investigate rank bounds in the more general LS_* hierarchy, which allows lifts by any derived inequality as opposed to just $x \geq 0$ and $1 - x \geq 0$ in the LS hierarchy. Rank lower bounds for LS_* were obtained for the symmetric knapsack polytope by Grigoriev et al [14]. In this paper we show that LS_* rank is incomparable to other hierarchies like LS_+ and Sherali-Adams (SA) and show rank lower bounds for PHP_n^{n+1} and integrality gaps for optimization problems like MAX-CUT in LS_*. The rank lower bounds for LS_* follow from rank lower bounds for the SA_* hierarchy which is a generalization of the SA hierarchy in the same vein as LS_*. We show that the LS_* rank of PHP_n^{n+1} is $\sim \log_2 n$. We also extend the polynomial rank lower bounds and integrality gaps for MAX-CUT studied in Charikar et al. [5] for SA hierarchy to corresponding logarithmic rank lower bounds and integrality gaps in the LS_* hierarchy. The proof translates various known SA rank lower bounds [5] to weaker SA_* (and LS_*) rank lower bounds as long as the number of variables in the constraints of the initial linear program is small. In the reverse direction we give an example of a linear program with large number of variables in a constraint which has unit rank in SA_* (and LS_*) hierarchies but linear rank in SA (and LS_+) hierarchies.

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1 Introduction

Lift and project hierarchies like the Lovász-Schrijver (LS) hierarchy [17] and the Sherali-Adams (SA) hierarchy [24] have been extensively studied in the past few years both from an optimization perspective [7] and a proof complexity perspective [14]. Such hierarchies can be thought of as "lifting" a polytope to higher dimensions by multiplying the initial inequalities with inequalities of the form $x \geq 0$ and $1 - x \geq 0$ to obtain lifted inequalities. This raises the natural question of lifting the intial inequalities with more general inequalities, say the already known inequalities. Both [17] and [24] raise this question but leave it for future investigation. Grigoriev et al [14] formally introduced the LS_* proof system which allows for multiplication of any two derived inequalities i.e. the lifts are not restricted to $x \geq 0$ and $1 - x \geq 0$ and investigated rank lower bounds for LS_*. In this paper we continue the investigation of rank bounds for LS_* and its counterpart without projection - SA_*. We show upper bounds on LS_* and SA_* rank which demonstrate that even these simple generalizations of LS and SA can be relatively powerful and yet we can solve relevant optimization problems over polytopes obtained within a constant number of rounds in polynomial time. We also show that it is possible to generalize many rank lower bounds and integrality gaps for SA to SA_* and LS_* using current techniques. But first we give some background and motivation for studying such hierarchies.

A 0-1 integer program can be used to encode a combinatorial optimization problem (say MAX-SAT, MAX-CUT, Vertex Cover and so on). Essentially one is given a linear objective function which one wishes to minimize/maximize in presence of linear constraints such that the variables take 0-1 values. Since many of the optmization problems are known to be NP-complete one can attempt to relax the integer programming problem to a linear programming problem which can be solved tractably. The obtained fractional solution is rounded to give an integer solution which would ideally be provably close to the optimal [25]. Therefore the quality of the linear programming relaxation, usually measured by the worst case integrality gap of the problem instances, would be crucial in determining the quality of the final rounded solution. In the past two decades many linear programming hierarchies have been devised which on the one hand produce a sequence of linear programs (or even semidefinite programs) with non-increasing integrality gaps such that the sequence converges to the integer program eventually and on the other hand also ensure that it is easy to find the optimal (or approximately optimal) solution for any linear program, which occurs early in the sequence, relatively efficiently. See the surveys [7, 16].

Another motivation to study such hierarchies comes from proof complexity where they are treated as weak proof systems [20, 14]. The overall goal in proof complexity is to prove lower bounds on the size (and other natural parameters) for stronger and stronger proof systems starting from resolution all the way upto extended Frege. See [15, 23] for a survey. To summarize, from a proof systems perspective one ignores the objective function and starts with an infeasible integer program obtained by a suitable encoding of an unsatisfiable CNF. New inequalities are derived from the old using inference rules corresponding to the hierarchy under consideration. A valid proof i.e. a refutation is a succesful derivation of the empty polytope using the inference rules corresponding to the hierarchy in consideration. The aim is to prove bounds on the size and even the depth of the proof. For the purposes of this paper we will not make much of the differences between the above two viewpoints.

In this paper we will focus on the Lovász-Schrijver hierarchy [17] and the Sherali-Adams hierarchy [24] and their generalizations. Both these hierarchies can be described as "lift and project" hiearchies. In other words given the initial linear program one "lifts" it by introducing new "lifted" variables and "lifted" inequalities obtained from multiplying the original inequalities with expressions x_i and $1 - x_i$ (where x_i is a variable in the initial linear program). Next the variables $x_i x_i$ are identified with x_i for reasons of soundness and completeness that are formally explained in [7, 10, 17]. The difference between LS and SA lies in the next step i.e. the "project" step. In LS one must project back to the original variables before the next iteration of the lift step while in SA one needs to project back only at the end of all the lift steps. Both the seminal papers [17, 24] raise the prospect of lifts with respect to expressions other than just x_i and $1 - x_i$. However they do not investigate such general lifts any further and leave it for future investigation. Rank bounds for LS_* [14] and the closely related SA_* (formally defined later), which deal with such general lifts, are the main object of investigation in this paper.

Results and Techniques: First, we provide a formal definition of SA_* and relate it to LS_*. We also observe that it is possible to optimize over polytopes obtained within constant rounds of SA_* and LS_* in polynomial time. Next we investigate upper bounds for LS_* and SA_* on polytopes derived from matching problems in graphs. The reason for selecting such problems is that on the one hand the lifted inequalities have a simple structure making them easier to manipulate and on the other hand they are known to be hard for LS, LS_+, SA and even Lasserre hierarhies. The following table summarizes the known rank bounds for a linear encoding of the pigeon-hole principle on $K_{n+1,n}$ and the matching polytope on K_{2n+1}.

	LS	SA	LS_+	Lasserre	SA_*	LS_*
PHP on $K_{n+1,n}$	$n-1$ [14]	$n-1$ [9]	1 [14]	1	$\sim \frac{n}{2}$ (this)	$\sim \log_2 n$ (this)
Matching polytope on K_{2n+1}	$\geq 2n-1, \leq 2n^2-1$ [13]	$2n-1$ [18]	n [13]	$\Theta(n)$ [1]	$\leq n$ (this)	?

We also provide a simple pathological example where LS_* and SA_* require rank 1 but SA and LS_+ require rank n. In Section 5 we prove the following matching lower bound for the linear encoding of the PHP principle ($LPHP_n^{n+1}$) in SA_* and together with Observation 3.1 it implies a $\sim \log_2 n$ rank lower bound for LS_*.

Theorem 5.6. *Deriving the empty integer polytope for $LPHP_n^{n+1}$ requires rank greater than $\lfloor \frac{n}{2} \rfloor - 1$ in SA_*.*

A common feature of the above examples is that they all have some inequalities with many variables and the intuition behind this is discussed in Section 6 in context of the proof of Theorem 6.5 stated below.

Theorem 6.5. *The SA_* rank of MAX-CUT is $\Omega(n^{\gamma(\varepsilon)})$ and the LS_* rank of MAX-CUT is $\Omega(\log_2 n)$ and the integrality gap is $2 - \varepsilon$ in each case for a given $\varepsilon > 0$.*

The proof is based on ideas in [5] and also extends to other similar optimization problems. Our bounds proceed by translating the SA lower bound to SA_* lower bounds which then translate to LS_* lower bounds. We take such an indirect route because it is more difficult to characterize the facets or even verify that a given point belongs to the polytope obtained after even a few rounds of LS_* as opposed to SA or LS. The difficulty lies with understanding the projection operation which, not coincidentally, also seems to be a bottleneck for obtaining size lower bounds for LS and other similar proof systems [19].

A brief walkthrough of the paper follows. In Section 2 we give pointers to background material and other basic definitions. In Section 3 we give a survey of the couple of known rank bounds in LS_* and also discuss the relationship between SA_* and LS_* rank and algorithmic aspects related to these hierarchies. In Section 4 we give upper bounds on the LS_* and SA_* ranks of some polytopes with large LS, SA or LS_+ ranks. In Section 5 we use ideas from [9] to obtain a matching logarithmic lower bound (Theorem 5.6) on the LS_* rank of PHP_n^{n+1} via a $\frac{n}{2}$ lower bound on the SA_* rank. In Section 6 we use ideas in [5] to obtain LS_* and SA_* rank lower bounds and integrality gaps (Theorem 6.5) for MAX-CUT and discuss its extension for several other optimization problems as long as the initial linear program has constraints with only a small / constant number of variables. Finally, we present some open problems.

2 Preliminaries and definitions

We assume the reader is familiar with basics of combinatorial optimization and proof complexity. For details about polytopes, facets, extreme rays, linear programs and other definitions related to convex optimization the reader can refer to [3]. For details about linear programming hierarchies the reader can refer to [16, 7, 14] and for details about complexity theory and proof complexity the reader can refer to [15, 23, 22]. In this section we recollect a handfull of basic definitions needed for the purposes of this paper.

In this paper, the input is a polytope $K := \{x \in \mathbb{R}^n | Ax \geq b\}$ in Q_n i.e., the unit cube $[0,1]^n$ for some $n \in \mathbb{N}$. The constraints are assumed to be polynomially many in n and explicitly given. The definition of LS [17] uses the homogenized cone $\tilde{K}$ instead of the polytope K. The polytope K corresponding to the homogenized cone $\tilde{K}$ is simply obtained by intersecting it with the hyperplane $x_0 = 1$.

Definition 2.1 ([17])**.** *Given convex cones $\tilde{K_1}, \tilde{K_2}$ in $\mathbb{R}^{n+1}$ define the cone $M(\tilde{K_1}, \tilde{K_2})$ (the lifted LS cone) as the cone consisting of all $(n+1) \times (n+1)$ matrices Y in $\mathbb{R}$ satisfying the conditions:*

1. *Y is symmetric*
2. *$Y_{ii} = Y_{i0}$*
3. *$\tilde{K_1}^* Y \tilde{K_2}^* \geq 0$.*

Let $N(\tilde{K_1}, \tilde{K_2})$ denote the projection Ye_0 of $M(\tilde{K_1}, \tilde{K_2})$. Define $N(\tilde{K}, \tilde{Q}_n)$ (or simply $N(K)$) as the cone (polytope) obtained after a single LS lift and project step.

$N^r(K) = N(N^{r-1}(K))$ denotes the LS polytope of *rank* r. The SA hierarchy [24] is just like the LS hierarchy but with one difference: the project step is applied only at the end of all the lift steps. We do not give a moment matrix definition of SA hierarchy but it can be found in [16]. In this paper we will be mainly concerned with the characterization of LS and SA which is based on lifted constraints instead of lifted points (see [5, 18] or [16]) as in the above definition. Using essentially the Farkas Lemma and linear algebra the LS lift and project (2.1) can be alternatively characterized as follows.

Lemma 2.2 ([10, 17]). *Given an initial polytope K in $\mathbb{R}^n$, point $x \in N(K)$ iff it satisfies all linear inequalities of the form*

$$\sum_{i,j} \mu_i x_i h_j(x) + \sum_{i,j} \nu_i (1 - x_i) h_j(x) + \sum_i \gamma_i (x_i^2 - x_i)$$

where $h_i \geq 0$ are the constraints of K, μ_i, ν_i are non-negative reals and γ is any real.

Therefore one can formally define the LS proof system as follows.

Definition 2.3 ([14]). *Given a set C of linear inequalities on the variables $\{x_1, ..., x_n\}$ and add to that axioms $x_i^2 - x_i = 0$, we have the following inference rules for* **LS***:*

1. $\frac{p \geq 0}{p \cdot q \geq 0}$ *where $deg(pq) \leq 2$ and $q \in \{x_i, 1 - x_i : i \in [n]\}$.*
2. $\frac{p \geq 0 \quad q \geq 0}{\alpha p + \beta q \geq 0}$ *for $\alpha, \beta \in \mathbb{R}^+$.*

A valid refutation must obtain the contradiction $-1 \geq 0$.

The following is an alternative characterization of the LS rank from [14] in view of Definition 2.3. One can naturally visualize a LS derivation as a directed acyclic graph (DAG) and the LS rank of an inequality is its depth in the DAG where we only take into account edges corresponding to inference rule 1 from Definition 2.3 when measuring depth in the DAG. The LS rank of a polytope is the maximum of the LS ranks of the facet inequalities.

Let $S_n(\mathbb{R})$ denote the Smolensky ring $\mathbb{R}[\{x_i\}]/(\{x_i^2 - x_i\})$. The definition below is adapted from Section 3.2 of [16].

Definition 2.4. *Given a set C of linear inequalities on the variables $\{x_1, ..., x_n\}$ in $S_n(\mathbb{R})$, we have the following inference rule for r rounds of* **SA***:*

1. $p \cdot q \geq 0$ *where $p \in C, q = \Pi_{i \in I} q_i, q_i \in \{x_i, 1 - x_i : i \in [n]\}$ and $|I| \leq r$.*
2. $\frac{p \geq 0 \quad q \geq 0}{\alpha p + \beta q \geq 0}$ *for $\alpha, \beta \in \mathbb{R}^+$.*

Again a valid refutation must obtain $-1 \geq 0$.

Observe that if we wanted to derive all extreme inequalities obtained within r rounds of LS then we would have to first derive the extreme inequalities that can be obtained within $r - 1$ rounds of LS and then do one more lift and project step i.e. use *both* rules 1 and 2 in Definition 2.3. Unlike LS, the SA hierarchy is "static" in nature i.e. we might as well derive all our lifted inequalities, that we can derive in r rounds, at once via rule 1 in Definition 2.4 and then take relevant convex combinations with rule 2 as necessary. For more details about static (and dynamic) derivations see [14].

Proof systems vs. Linear program hierarchies: In general when the integer polytopes are not necessarily empty, the proof lines above can be equivalently represented as lifted linear inequalities (in higher dimensions) by mapping each $\Pi_{i \in I} x_i$ to lifted variables x_I. A more detailed discussion on lifted inequalities can be found in the papers [5, 18] and the surveys [7, 16]. Therefore we can always obtain the definition of a hierarchy in terms of lifted inequalities from the definition of the corresponding proof system, which is in terms of multinomials, and vice versa.

Finally, one can naturally generalize LS and SA by relaxing the restriction in the first inference rule in the definitions above. The LS_* proof system was defined by Grigoriev et al [14] as follows.

Definition 2.5 ([14]). *Given a set C of linear inequalities on the variables $\{x_1, ..., x_n\}$ and add to that axioms $x_i^2 - x_i = 0$, we have the following inference rules for* $\mathbf{LS}_*$*:*

1. $\frac{p \geq 0 \;\; q \geq 0}{p \cdot q \geq 0}$ *where* $deg(p \cdot q) \leq 2$.
2. $\frac{p \geq 0 \;\; q \geq 0}{\alpha p + \beta q \geq 0}$ *for* $\alpha, \beta \in \mathbb{R}^+$.

A valid refutation must obtain the contradiction $-1 \geq 0$.

Again only use of inference rule 1 counts towards the LS_* rank. Closely inspired by [14] and [24] one can formally define the intermediate generalization SA_* as follows:

Definition 2.6. *Given a set C of linear inequalities on the variables $\{x_1, ..., x_n\}$ in $S_n(\mathbb{R})$, we have the following inference rule for r rounds of* $\mathbf{SA}_*$*:*

1. $p \cdot q \geq 0$ *where* $p \in C, q = \Pi_{i \in I} q_i, q_i \in C$ *and* $|I| \leq r$.
2. $\frac{p \geq 0 \;\; q \geq 0}{\alpha p + \beta q \geq 0}$ *for* $\alpha, \beta \in \mathbb{R}^+$.

Again a valid refutation must obtain $-1 \geq 0$.

Observe that the difference between inequalities obtained within r rounds of SA and SA_* is that in the latter we can lift (or multiply) an initial constraint with respect to at most another r initial constraints as opposed to at most r constraints of the form x_i and $1 - x_i$ for $i \in [n]$. We refer the reader to [14] for definitions of other closely related geometric proof systems. Throughout this paper we are concerned with rank (and not size) so we can assume that, for r rounds of SA_* relaxation, projection to $\mathbb{R}^n$ is applied only at the end, if needed. Note that for brevity we always assume that the inequalities $x_i \geq 0$ and $1 - x_i \geq 0$ are implicitly present in C i.e. our polytopes are in Q_n. Finally, note that the polytope defined by rank r inequalities in LS_* (resp. SA_*) is a subset of the polytope defined by rank r inequalities in LS (resp. SA).

3 The LS_* hierarchy

3.0.1 Algorithmic aspects

Lovász and Schrijver [17] constructed a weak separation oracle for $N(K, Q)$ and $N(K, K)$ from a weak separation oracle for K assuming K is explicitly given by a polynomial number of constraints. The last assumption implies that one can not solve the separation problem over constant r, $r \geq 2$, rounds of LS_* in polynomial time by simple iteration over the facets of $N^{r-1}(K, K)$ $(= N(N^{r-2}(K, K), N^{r-2}(K, K)))$, since the intermediate polytopes may have exponentially many facets. It is possible that the situation for LS_* is similar to the case for Gomory-Chvátal cuts where the corresponding separation problem is known to be NP-complete [12]. However, it is possible to optimize over the projected polytope obtained after r rounds of SA_* relaxations in time $n^{O(r)}$ by simply solving the $n^{O(r)}$ sized lifted linear program. The following observation then allows for optimization over r rounds of LS_* by solving the lifted linear program for $2^r - 1$ rounds of SA_* instead.

Observation 3.1. *Given a polytope $K \in \mathbb{R}^n$, the projection (to $\mathbb{R}^n$) of the polytope obtained after at most $2^k - 1$ rounds of SA_* is a subset of the polytope obtained in k rounds of LS_*.*

Proof. The proof is by induction on k. The base case $k = 0$ (we start with the same initial constraints for both) and even $k = 1$ (when $LS_* = SA_*$) are clear. Assume that any LS_* inequality derived within $k-1$ rounds can be expressed as a positive linear combination of SA_* inequalities obtained after at most $2^{k-1}-1$ rounds. We now derive all the rank k LS_* inequalities in at most $2^k - 1$ rounds of SA_*. Since all lifted inequalities in round k of LS_* are generated by multiplying at most two inequalities obtained after round $k-1$, induction hypothesis and the defintion of SA_* (2.6) implies that the resulting lifted inequality can be generated by some positive linear combination of inequalties obtained in $2.(2^{k-1} - 1) + 1$ $(= 2^k - 1)$ rounds of SA_*. Hence the proof follows. ☐

3.0.2 Known rank bounds

Grigoriev et al. [14] prove the following lower bound for symmetric knapsack inequalities (i.e. $\sum_{i=1}^{n} x_i = \alpha$ for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$).

Theorem 3.2 ([14]). *Let SK_n denote the symmetric knapsack inequality with n odd and $\alpha = \frac{n}{2}$ then*

1. *Any LS_+ refutation of SK_n has rank at least $\frac{n}{4}$.*
2. *Any $LS_{+,*}$ refutation of SK_n has rank at least $\log_2 n - 1$.*

The only known super-logarithmic rank lower bound in LS_* is due to Beame et al. [2]. For brevity we restate their main result without a detailed explanation but simply note that LS_* proofs can be simulated by $R^{cc}(k)$ proofs with only $O(\log n)$ factor increase in rank and polynomial increase in size.

Lemma 3.3 ([2]). *There is a family of bipartitie graphs $\mathcal{G}$ and a family of polysize CNF formulae $G := Lift_{k-1}(\mathcal{G}PHP)$ on n variables that require refutation rank $n^{\Omega(1/k)}$ and tree-like size $exp(n^{\Omega(1/k)})$ in any $R^{cc}(k)$ system for any $k \leq (1-\epsilon)\log\log n$ for some positive absolute constant ϵ.*

The author is unaware of any systematic investigation of upper bounds for LS_* or SA_* rank.

4 Upper bounds on LS_* and SA_* rank

In this section we show that some well known inequalities have small LS_* and SA_* rank when compared with other hierarchies like SA and LS_+. The lower bounds in the following sections imply that LS_* and SA_* do not provide any algorithmic advantage when compared with SA as long as the intial linear program has constraints with few variables. However, this still leaves out various versions of the travelling salesman problem and the bounded degree spanning tree problem which occur in practical scenarios. In this paper we will only deal with upper bounds on LS_* and SA_* rank of much simpler linear programs which have an inequality with many variables. We start with the pigeon hole principle (PHP_n^{n+1}).

The motivation behind studying PHP comes from the fact that it is one of the cornerstone problems in proof complexity as witnessed by the survey [21]. The PHP^{n+1}_n (falsely) claims that there exists a (possibly) multivalued everywhere defined injection between the two partitions of the bigraph $K_{n+1,n}$. A refutation of this claim in a proof system successfully derives a contradiction using the proof rules starting from the suitably encoded axioms for PHP. It is known that PHP has polynomial size and logarithmic rank in the Gomory-Chvátal cutting planes (CP) proof system [20]. Pudlak [20] shows that PHP has a polynomial size refutation in LS. Grigoriev et al. [14] prove a rank lower bound of $n-1$ for PHP^{n+1}_n in LS and Dantchev et al [9] prove a rank lower bound of $n-1$ for PHP^{n+1}_n in SA. PHP^{n+1}_n can be encoded as a linear programming relaxation as follows.

Definition 4.1 ([4])**.** *For $m > n$, define the $\mathbf{LPHP^m_n}$ polytope as a Linear encoding of the PHP^m_n principle by the set of linear inequalities*

$$Q_i := \sum_{j\in[n]} x_{ij} - 1 \geq 0, \ \ (\forall i \in [m]). \tag{4.1}$$

$$Q_{jk,i} := 1 - x_{ji} - x_{ki} \geq 0, \ \ (\forall j \neq k, \ \ j,k \in [m], \ \ i \in [n]) \tag{4.2}$$

In the integer solution variable x_{ij} is 1 if $i \mapsto j$ and 0 otherwise and so the integer polytope for $LPHP^m_n$ is empty. The L in $LPHP$ stands for Linear (encoding).

Theorem 4.2. *The LS_* rank of $LPHP^{n+1}_n$ (Definition 4.1) is at most $\lceil \log_2(n+1)\rceil - 1$.*

Proof. We give only a brief proof sketch. Given $Q_{jk,i} \geq 0$ one round of LS implies $x_{ji}x_{ki} = 0$ for $j \neq k$. Using this fact to cancel out the quadratic terms and with some algebraic simplification one can rewrite $Q_{lm,i}Q_{pq,i} \geq 0$ as $1 - x_{li} - x_{mi} - x_{pi} - x_{qi} \geq 0$, for distinct l, m, p and q, after one round of LS_*. Now in the second round of the LS_* refutation we can multiply two such different linear inequalities derived in first round and cancel out the quadratic terms to obtain an inequality of the form $1 - \sum_{k=1}^{8} x_{j_k i} \geq 0$ for distinct j_k. Iterating for $\lceil \log_2(n+1)\rceil - 1$ rounds of LS_* we can derive $R_i := 1 - \sum_{j\in[n+1]} x_{ji} \geq 0$ for $i \in [n]$. Next we observe that

$$\sum_{i=1}^{n+1} Q_i + \sum_{i=1}^{n} R_i = -1 \geq 0$$

which gives the empty polytope as required. □

Using similar ideas we can show it is possible to derive inequalities $R_i \geq 0$, and therefore refute $LPHP^{n+1}_n$, in $\lceil \frac{n+1}{2} \rceil$ rounds of SA_*.

In the next example we move on to the complete graph K_{2n+1}. The fractional matching polytope M_F [13] of K_{2n+1} is given by the following constraints

$$Q_v := 1 - \sum_{u\in N(v)} x_{uv} \geq 0, \quad \forall v \in [2n+1].$$

Let $E(S)$ denote the set of edges induced by the set of vertices S. Then the corresponding integer polytope M_I is given by the following constraints

$$R_S := \frac{|S|-1}{2} - \sum_{e\in E(S)} x_e \geq 0, \quad \forall S \subseteq [2n+1], \ |S| \ odd.$$

The inequalities in M_I are known as "blossom" inequalities. They share similarities with the comb inequalities in travelling salesman polytopes. The SA rank of M_I is $2n-1$ [18] while the LS rank lies between $2n-1$ and $2n^2-1$ [13]. M_I has LS_+ rank n [13] and Lasserre rank $\Theta(n)$ [1]. We now show that the SA_* rank of M_I is n i.e. equal to the LS_+ rank as opposed to the SA rank. We will use the following characterization of SA relaxations of M_F from Proposition 3.1 in [18].

For $I, J \subseteq E(K_{2n+1})$, let $\Pi_{e\in I}x_e\Pi_{e\in J}(1-x_e)$ denote the "standard multiplier" $S_{I,J}$ where *I is a matching, J is a star, and the vertex sets $V(I)$ and $V(J)$ induced by I and J are disjoint.* Given polynomial $p(\bar{x})$ on x_is we multilinearize it to $m(\bar{x})$. For a multinomial $m(\bar{x})$, let $\phi(m)$ denote the linear combination over variables z_i $i \in \mathbb{N}$ obtained by replacing each monomial $\Pi_{e\in L}x_e$ by the variable $z_{|L|}$ provided that L is a matching and 0 otherwise.

Lemma 4.3 ([18])**.** *The value of the k-lifted SA linear program for maximum matching in K_{2n+1} is equal to that of the following modified linear program:*

$$\max_{z_1,z_2..} \binom{2n+1}{2} z_1$$

such that

1. *$z_{|I|} = 0$ if $|I| \geq n+1$, $|I| \leq k$.*
2. *All constraints of the form $\phi(Q_v S_{I,J}) \geq 0$ where $S_{I,J}$ is a standard multiplier with $|I|+|J| \leq k$ and $v \notin V(I) \cup V(J)$.*
3. *All constraints of the form $\phi(S_{I,J}) \geq 0$ where $S_{I,J}$ is a standard multiplier with $|I|+|J| \leq k+1$.*

Mathieu and Sinclair [18] prove that for $k = 2n-1$ the projection of the above SA polytope coincides with M_I. Based on their result we have the following result.

Theorem 4.4. *The SA_* rank of the matching polytope for K_{2n+1} is at most n.*

Proof. Let $S'_{I,J} := (1-\sum_{e\in J} x_e)\Pi_{e\in I}x_e$ denote the "modified" standard multiplier obtained from $S_{I,J}$. Let vertex r_J be the root of star J. Observe that $1-\sum_{e\in J} x_e = Q_{r_J} + \sum_{\{e:r_J\in e\}\setminus J} x_e$ is a valid inequality for M_F. Therefore the lifted constraint corresponding to $Q_v S'_{I,J} \geq 0$ can be derived in at most $\min\{n, |I|+1\}$ rounds of SA_*. We need at most $|I|+1$ rounds if J is non-empty and $|I| < n$ as $V(I)$ and $V(J)$ are disjoint by definition, and we need at most $\min\{n, |I|\}$ rounds if J is empty as the edges in I form a matching. Next we cancel out all lifted variables x_L, where L is not a matching, by using the following useful fact: If edges e_1, e_2 share a vertex then the lifted variable $x_{e_1,e_2} = 0$ after one round of SA and therefore any lifted variable $x_I = 0$ for $e_1, e_2 \in I$ after $|I|-1$ rounds of SA. Note that all necessary $x_L = 0$ above can also be derived in n rounds of SA. The following result is essentially a restatement of Lemma 3.3 in [18] which does not really depend on the properties of SA hierarchy. We repeat the proof in the appendix.

Lemma 4.5 ([18])**.** *There exists an optimal symmetric solution y^s to the maximum matching linear program for K_{2n+1} obtained after r rounds of SA_* such that y^s_L is the same for every set of $|L| \leq r+1$ edges which form a matching i.e. for matchings L and L', $|L| = |L'| \implies y^s_L = y^s_{L'}$.*

Hence given an optimal point y for the linear program obtained after n rounds of SA_* there exists a symmetric point y^s which is also feasible and optimal for the same linear program. Therefore if the point y^s satisfies the lifted constraint obtained from $Q_v S'_{I,J} \geq 0$ after removing all lifted variables corresponding to non-matchings, then y^s satisfies the constraint $\phi(Q_v S'_{I,J}) \geq 0$ where $z_{|L|} \mapsto y^s_L$ for some matching L. Similarly if y^s satisfies the lifted constraint from $S'_{I,J} \geq 0$ then it satisfies $\phi(S'_{I,J}) \geq 0$ where $z_{|L|} \mapsto y^s_L$ for some matching L. But by definition of $S_{I,J}$, $S'_{I,J}$ and ϕ we have $\phi(Q_v S_{I,J}) = \phi(Q_v S'_{I,J})$ and $\phi(S_{I,J}) = \phi(S'_{I,J})$. So any extreme point in the n-lifted SA_* polytope obtained as above can be extended to an optimal point in the linear program in Lemma 4.3 for $k = 2n-1$, by appending zeros in the coordinates for the variables $z_{|I|}$ when $|I| \geq n+1$. Since in the case of the matching polytope proving an integrality gap of 1 with respect to the natural objective function ($\max \sum_{e \in E(K_{2n+1})} x_e$) suffices to show that we have converged to the integer polytope, the preceeding discussion together with the SA rank bound in [18] implies Theorem 4.4. □

Next we move on to a third example which is a bit removed from graph matchings. It demonstrates that LS_* and SA_* rank are not comparable to SA and LS_+ rank when the intial linear program has inequalities with many variables.

Let $\mathbf{K}^{\mathbf{n}}_\alpha$ denote the polytope in $\mathbb{R}^n$ defined by the following two constraints

$$g^n_\alpha := \sum_{i=1}^{n} x_i - \alpha \geq 0,\ \alpha \in (0,1);\ f := 1 - \sum_{i=1}^{n} x_i \geq 0 \tag{4.3}$$

over variables $x_i \in [0,1]$. Note that $(K^n_\alpha)_I := \{x \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$ denotes the required integer polytope. Cheung [6] (also Cook and Dash [8]) prove that the LS_+ and SA rank of the polytope $g^n_\alpha \geq 0$ (for $\alpha \in (0,1)$) is n. The following theorem gives an upper bound on the LS_* and SA_* rank of K^n_α.

Theorem 4.6. *$\forall \alpha \in (0,1)$ the LS_* and SA_* rank of K^n_α is 1.*

Proof. Observe that using the multiplication and summation rules of LS_* give

$$(\sum_{i=1}^{n} x_i - \alpha)(1 - \sum_{i=1}^{n} x_i) + \sum_{i=1}^{n}(x_i^2 - x_i) + \sum_{i \neq j} x_i x_j = \alpha \sum_{i=1}^{n} x_i - \alpha \geq 0$$

which immediately gives $(K^n_\alpha)_I$. □

It remains to show that rank lower bounds given in [6] from $g^n_\alpha \geq 0$ in SA and LS_+ hierarchies extend to K^n_α.

In the case of SA, the proof given in [6] just verifies that the point

$$i \in [n] \Rightarrow y_{\{i\}} = \frac{\alpha}{\alpha n + 1 - \alpha},\quad |I| \geq 2 \Rightarrow y_I = 0 \tag{4.4}$$

lies in the polytope obtained after $n-1$ SA-lifts of $g^n_\alpha \geq 0$ for $\alpha \in (0,1)$. It is also easy to verify that y is contained in $n-1$ SA-lifts of $1 - \sum_{i=1}^n x_i \geq 0$. This is because any SA lift other than with $\Pi_{i \in I}(1 - x_i)$ leads to $0 \geq 0$ on instantiation with coordinates of y. Moreover even in case of lifts with $\Pi_{i \in I}(1 - x_i)$ we are left with $1 - \sum_{i \in [n]} y_i = \frac{1-\alpha}{\alpha n + 1 - \alpha} > 0$ for $\alpha \in (0,1)$. Hence we get the following result.

Observation 4.7. *The SA rank of K^n_α is n.*

The rank lower bound proof for $g^n_\alpha \geq 0$ in LS_+ is by induction on n and shows that $y \in N_+^{n-1}(g^n_\alpha \geq 0)$. The proof also works in prescence of $f \geq 0$ and is deferred to the appendix.

5 PHP

In this section we give the proof of a linear lower bound on the SA_* rank of $LPHP$ (Theorem 5.6) and therefore a logarithmic lower bound on the LS_* rank of $LPHP$. The strategy of the proof will be to show that the SA_* lifted polytope is non-empty i.e. it contains a given point. This point is the same as that in [9] but defined using different notation for consistency with the usual definition of SA.

Definition 5.1 ([9]). *A* **partial bijection** *in P^n_{n-1} is a bijection $[2..n+1]\backslash\{i\} \to [2..n]$ for $i \in [2..n+1]$. P^n_{n-1} denotes the set of partial bijections from $[2..n+1] \to [2..n]$.*

Definition 5.2 ([9]). *Given $I \subseteq \{(p,q)|p \in [2..n+1], q \in [2..n]\}$, I is* **self-inconsistent** *if $(i,j),(i,k) \in I$ and $j \neq k$, or $(j,i),(k,i) \in I$ and $j \neq k$.*

If I is not self-inconsistent then I is self-consistent. Intuitively a self-consistent set I naturally corresponds to a bijection of two sets of size $|I|$.

Definition 5.3 ([9]). *Given $I \subseteq \{(p,q)|p \in [2..n+1], q \in [2..n]\}$, I is* **inconsistent** *with $\pi \in P^n_{n-1}$ if either I is self-inconsistent or*

- *$(i,j) \in I$ and $\pi(k) = j$ for $i \neq k$*
- *$(i,j) \in I$ and $\pi(i) = l$ for $j \neq l$.*

If I is not inconsistent with π then I and π are consistent with each other. Intuitively a restriction of π would correspond to the bijection represented by I. For brevity let $(*)$ denote the wildcard character (for example $(r,*)$ stands for the set $\{(r,a)|a \in [2..n]\}$) and let $N = n^2 + n$ i.e. the number of variables in $LPHP^{n+1}_n$. Now we define the evaluation function i.e. the lifted point.

Definition 5.4 ([9]). *An* **evaluation** *$V : x_I \to \mathbb{R}$ is a function defined on all lifted variables obtained from monomials of degree at most $n-1$ and linearly extended to the lifted inequalities. For $I \subseteq \{(p,q)|p \in [2..n+1], q \in [2..n]\}, |I| \leq n-1$ define $V(x_I)$ as the fraction of all $n!$ partial bijections P^n_{n-1} consistent with I.*

If $(1,i) \in I$ (resp. $(i,1) \in I$) then $V(x_I) := V(x_{(r,i):I})$ (resp. $V(x_I) := V(x_{(i,r):I})$), where $(r,i) : I$ (resp. $(i,r) : I$) denotes (r,i) (resp. (i,r)) substituted for all instances of the form $(1,i)$ (resp. $(i,1)$) in I and $(r,*) \notin I$ (resp. $(*,r) \notin I$). Note that such an r exists since $|I| \leq n-1$.

Observe (by symmetry arguments) that V is well defined for all monomials in $S_N(\mathbb{R})$ of degree at most $n-1$. Note the similarities with the definition in Section 4 of [9]. Using ideas in the proof of Proposition 11 in [9] we can prove the following Lemma.

Lemma 5.5. *Let Q_i be defined as in equation (4.1) then $V(x_I Q_i) = 0$ for any monomial x_I and $|I| \leq n-2$.*

Proof. First suppose $i \neq 1$ and $(1,*),(*,1) \notin I$. Observe that I is self-consistent otherwise the statement of the Lemma follows immediately. Let P'_i denote the set of $\pi \in P^n_{n-1}$ consistent with I such that i remains unmatched. By Definition 5.4

$$\sum_{j=2}^{n} V(x_{I\cup(i,j)}) + \frac{|P'_i|}{n!} = V(x_I). \tag{5.1}$$

Equation 5.1 is true since either

1. $(\exists a \in [2..n])((i,a) \in I)$ then observe that by the definition of evaluation: $(\forall b \neq a)(V(x_{I\cup(i,b)}) = 0)$, $P'_i = 0$ (since $\pi' \in P'_i \Rightarrow ((\exists l \neq i)(\pi'(l) = a))$, and $V(x_{I\cup(i,a)}) = V(x_I)$. Note that in this case the statement of the Lemma follows so we may assume $(i,*) \notin I$ from now.
2. Otherwise, equation (5.1) follows from definition of V.

Observe that $|I| \leq n-2 \Rightarrow ((\exists l \in [2..n])((*,l) \notin I))$ so that $V(x_{I\cup(i,1)}) := V(x_{I\cup(i,l)})$. Note that I self-consistent and $(i,*) \notin I$ implies that $I \cup (i,l)$ is self-consistent. It now suffices to show that there is a *bijection* between P'_i and the set of partial bijections consistent with $I \cup (i,l)$ (denoted by P_{il}).

To see $|P_{il}| \geq |P'_i|$, observe that for $\pi' \in P'_i \Rightarrow (\exists! i' \in [2..n+1])(i' \mapsto l)$ one replaces i' by i to obtain a unique $\pi \in P_{il}$. The consistency of π with $I \cup (i,l)$ follows from consistency of π' with I and the construction of π.

To see $|P_{il}| \leq |P'_i|$, given $\pi \in P_{il} \Rightarrow ((\exists! i' \in [2..n+1] \setminus \{i\})(i' \notin Dom(\pi))$ replace i by i' to obtain a unique $\pi' \in P'_i$. The consistency of π' with I follows from consistency of π with $I \cup (i,l)$.

Therefore $\frac{|P'_i|}{n!} = V(x_{I\cup(i,l)})$ so the statement is true when $i \neq 1$ and $(1,*), (*,1) \notin I$.

Otherwise, if $i = 1$ or $(1,a) \in I$ or $(a,1) \in I$ then we can reduce this case to the previous case by substituting $i' \in [2..n+1], j' \in [2..n]$ such that $(i',*), (*,j') \notin I \cup (i,j)$, in place of 1. After the substitutions note if I is not self-consistent or $(i,*) \in I$ then $V(x_I Q_i) = 0$ immediately follows. Otherwise, if $i = 1$ or $(1,a) \in I$ then substituting 1 with i' reduces this case to when $i \neq 1$ and $(1,*) \notin I$ respectively. If $(a,1) \in I$ then observe that $\sum_{j=1}^{n} V(x_{I\cup(i,j)}) = V(x_{(a,j''):I\cup(i,j')}) + \sum_{j=1, j\neq j'}^{n} V(x_{(a,j'):I\cup(i,j)})$ where $j'' \neq j'$ and $(*,j'') \notin I$. Let $I' := (a,j') : I$. By definition of V, $V(x_{(a,j''):I\cup(i,j')}) = V(x_{I'\cup(i,j'')}) = V(x_{I'\cup(i,1)})$. So by essentially interchanging the roles of 1 and j' in the right coordinates of I while keeping $V(x_I Q_i)$ invariant we now need to prove $V(x_{I'} Q_i) = 0$. Therefore if $i = 1$ or $(1,*) \in I$ or $(*,1) \in I$ then we are able to reduce all such cases to the previous one. Hence the proof follows. □

Theorem 5.6. *Deriving the empty integer polytope for $LPHP_n^{n+1}$ (Definition 4.1) requires rank greater than $\lfloor \frac{n}{2} \rfloor - 1$ in SA_*.*

Proof. A rank k_* SA_* form is derived from lifting an expression of the form:

$$F := \Pi_{i\in S_1} Q_i \Pi_{(jk,l)\in S_2} Q_{jk,l} \Pi_{p\in S_3}(1 - x_p) \Pi_{q\in S_4} x_q$$

where the meaning of the sets S_i is intuitive, $\Sigma_{i=1}^{4} |S_i| \leq k_* + 1$, and $F \in S_N(\mathbb{R})$. Observe that,

$$Q_{jk,l} = 1 - x_{jl} - x_{kl} = (1 - x_{jl})(1 - x_{kl}) - x_{jl}x_{kl}.$$

Any evaluation of F which is defined by a linear combination of its value on lifted monomials will be invariant under the above rewrite. Therefore we can rewrite F as a linear combination of forms with at most one $Q_{jk,l}$ without changing $V(F)$ as below.

$$\begin{aligned} F' \quad := \quad & \sum_i Q_i g_i + \sum_{j\neq k,l} x_{jl}x_{kl}h_{jk,l} \\ & + \sum_{j\neq k,l} \alpha^{+}_{j,k,l} Q_{jk,l} \Pi_{p\in S}(1 - x_p) \Pi_{q\in T} x_q \\ & + \sum_{p,q} \beta^{+}_{p,q} \Pi_{p\in S'}(1 - x_p) \Pi_{q\in T'} x_q \end{aligned}$$

such that $(\forall g_i \in S_N(\mathbb{R}))(deg(g_i) \leq 2k_*)$, $(\forall h_{jk,l} \in S_N(\mathbb{R}))(deg(h_{jk,l}) \leq 2k_*)$, $|S|+|T| \leq 2k_*$, $|S'|+|T'| \leq k_*+1$ and $\alpha^+, \beta^+ \in \mathbb{R}^+$.

Now suppose one could refute $LPHP$ by a rank k_* (for $2k_* \leq n-2$) SA_* proof i.e., one could derive -1 by some positive linear combination of lifted inequalities derived from forms in equation 5.2. At this point it suffices to prove that V (from definition 5.4) has $V(F') \geq 0$, since it would give an immediate contradiction.

By definition 5.4 V is linear and $(\forall j,k,l)(V(x_{jl}x_{kl}h_{jk,l}) = 0)$. Furthermore Lemma 5.5 implies that $(\forall i)(V(Q_i g_i) = 0)$ hence only the last two types of expressions in equation 5.2 i.e., $Q_{jk,l}\Pi_{p\in S}(1-x_p)\Pi_{q\in T}x_q$ and $\Pi_{p\in S'}(1-x_p)\Pi_{q\in T'}x_q$, remain to be taken care of. However, these two lifted expressions can also be derived by $2k_*$ rounds of SA alone so the fact that their valuations are non-negative follows simply from the proof of Dantchev et al. [9] for SA ([9] proves a lower bound of $n-1$ on the rank of $LPHP_n^{n+1}$ in SA). Note that this can also be proved directly with a little more work. Therefore the statement follows. □

Corollary 5.7. *The LS_* rank of $LPHP_n^{n+1}$ is at least $\log_2\lfloor \frac{n}{2} \rfloor$.*

Proof. The result follows from Observation 3.1 and Theorem 5.6. □

Note that the bound above extends to the weaker functional-PHP but not to onto-PHP (see [21] for definitions) which can be shown to have LS_* rank 1. In the latter case, the natural LS_+ refutation is also an LS_* refutation.

6 MAX-CUT and related bounds

The proof in Section 5 used some of the structure of the PHP inequalities in order to translate the SA bounds to SA_*. In this section we will make minimal use of such problem structure. We will extend SA rank lower bounds for MAX-CUT in the SA_* hierarchy (and LS_* hierarchy via Observation 3.1) using ideas from [5]. The proof techniques extend to other optimization problems as long as the initial linear program has small number of variables in each constraint. Throughout this section $G = (V_G, E_G)$ will represent a graph on n vertices.

Definition 6.1 ([11])**.** *Given graph G and distributions (i.e. discrete probability measure) μ_T on cuts (i.e. subsets) of $T \subseteq V_G$ for every T such that $|T| \leq k$, the distributions μ_T are k-**locally consistent** if for any A, Q, T, $A \subseteq Q \subseteq T$ implies $\mu_T(\{B | B \subseteq T, B \cap Q = A\}) = \mu_Q(A)$.*

Charikar et al [5] (also [11]) deduce that given a linear program it is sufficient to prove the existence of "locally consistent" probability distributions on subsets of size k to show $\Omega(k)$ rank lower bound (and also integrality gaps) in the SA hierarchy for the original linear program.

Lemma 6.2 ([11, 5])**.** *Given k-locally consistent probability distribution of cuts on graph G, the vector $x_{ij} = \mu_{\{i,j\}}(\{\{i\},\{j\}\})$ (i.e. equivalently interpretable as the probability that i, j lie on different sides of the cut) lies in the SA cut polytope obtained after $\frac{k}{2} - \frac{3}{2}$ rounds.*

Charikar et al. [5] also show that the existence of certain discrete metric spaces leads to $\Omega(n^{\gamma(\varepsilon)})$-locally consistent distributions over cuts such that $x_{ij} = \mu_{\{i,j\}}(\{\{i\},\{j\}\})$ (the probability that i and j are separated by the cut), with integrality gap $2-\varepsilon$.

Definition 6.3 ([5])**.** *Let $x_{ij} \in [0,1]$ denote variables corresponding to vertex pair i, j ($i \neq j$) in an undirected graph G.*

$$x_{ij} \geq 0, 1 - x_{ij} \geq 0, x_{ij} = x_{ji}, \tag{6.1}$$

$$x_{ij} + x_{jk} - x_{ik} \geq 0 \quad \forall i, j, k \in V(G), \tag{6.2}$$

$$2 - x_{ij} - x_{jk} - x_{ik} \geq 0 \quad \forall i, j, k \in V(G). \tag{6.3}$$

The above polytope is referred to as the **cut polytope**. *The MAX-CUT linear programming relaxation optimizes $\sum_{e \in E_G} x_e$ over the cut polytope.*

In order to generalize the above mentioned two step approach in [5] to the SA_* hierarchy one just needs to generalize Lemma 6.2 to SA_*. The proof is based on ideas in [5, 11] and relies mainly on the linearity of expectation.

Lemma 6.4. *Given k-locally consistent distribution of cuts on graph G, the vector $x_{ij} = \mu_{\{i,j\}}(\{\{i\},\{j\}\})$ (i.e. equivalently probability that i, j lie on different sides of a cut) lies in SA_* cut polytope obtained after $\frac{k}{3} - 1$ rounds.*

Proof. Consider a non-empty set $I = \{(i_1, j_1), ..., (i_p, j_p)\}$ of size at most $\frac{k}{2}$. Let $Q_I \subseteq V_G$ denote the set $\{i_1, j_1, ..., i_p, j_p\}$ of size at most k. Let $1^I_{ij}(X)$ denote an indicator variable that is 1 if the cut X on Q_I seprates vertices i and j for $i, j \in Q_I$ and 0 otherwise. Using the convention in Section 5, the coordinates of the SA_* solution are again given by a linear evaluation function $V : x_I \to \mathbb{R}$, where $I \subseteq V_G \times V_G, |Q_I| \leq k$. V is defined as follows.

$$V(x_I) := E_{\mu_{Q_I}}(\Pi_{(i,j)\in I} 1^I_{ij}). \tag{6.4}$$

In other words $V(x_I)$ is the probability that all pairs of vertices in I are separated by a cut chosen according to μ_{Q_I}.

Let $V(R_i(\{x_{ab}, x_{cd}, x_{ef}\}))$ denote the evaluation of an intial constraint R_i in the MAX-CUT linear program (for eg. 6.2 or 6.3) for the vertex pairs $(a, b), (c, d)$ and (e, f). Consider an inequality obtained after r lifts of SA_*:

$$\Pi_{p \in \mathcal{I}} R_p \geq 0. \tag{6.5}$$

Let $Q^{\mathcal{I}} = \bigcup_{i \in R_p, p \in \mathcal{I}} i$ denote the set of vertices present as indices in the above lifted inequality. Let $Q_S \subseteq Q^{\mathcal{I}}$ such that $Q_I \subseteq Q_S$ for all x_I in the lifted constraint 6.5. The valuation of the LHS of inequation 6.5 is a sum of expectations of the form

$$\sum_{I \subseteq V_G \times V_G, Q_I \subseteq Q_S} \alpha_I E_{\mu_{Q_I}}(\Pi_{(i,j)\in I} 1^I_{ij})$$

where $\alpha_I \in \mathbb{R}$. Using linearity of expectation and k-local consistency one can simplify (recombine) the above expression as follows

$$V(\Pi_{p \in \mathcal{I}} R_j(x_{ab}, x_{cd}, x_{ef})) = E_{\mu_{Q_S}}(\Pi_{p \in \mathcal{I}} R_j(\{1^S_{ab}, 1^S_{cd}, 1^S_{ef})\}). \tag{6.6}$$

Since $\Pi_{p \in \mathcal{I}} R_p(X) \geq 0$ for any given cut X on Q_S the expectation on RHS above is non-negative. Since we only have k-locally consistent distributions we need $|Q_S| \leq k$ in 6.6, which is implied by $|Q^{\mathcal{I}}| \leq k$ i.e.

$$3(r + 1) \leq k. \tag{6.7}$$

Therefore the image of V lies in the SA_* polytope for MAX-CUT obtained after $\frac{k}{3} - 1$ rounds. □

Therefore following Theorem 5.3 in [5] we get the following lower bounds.

Theorem 6.5. *The SA_* rank of MAX-CUT is $\Omega(n^{\gamma(\varepsilon)})$ and the LS_* rank of MAX-CUT is $\Omega(\log_2 n)$ and the integrality gap is $2 - \varepsilon$ in each case for a given $\varepsilon > 0$.*

The two step approach of Charikar et al [5] (esp. Lemma 6.2) is quite modular and can be used to prove integrality gaps for problems other than MAX-CUT (for example Vertex Cover, Sparsest Cut and other SA rank bounds which proceed by constructing locally consistent distributions as above). Unlike SA, an exact generalization of Lemma 6.2 to SA_* i.e., existence of k-locally consistent distributions implies $\Omega(k)$ rank lower bound in SA_* hierarchy, is not likely for different problems (see the last example in Section 4). However, given k-locally consistent distributions a proof of $\frac{k}{ct}$ rank lower bound in the SA_* hierarchy follows from the proof of the $\frac{k}{c}$ SA rank lower bound by suitably changing the coefficient on the LHS of equation 6.7 in the SA proof. The constant c is usually the ratio between the size of indices of the set over which we take the linear combination in the objective function and the size of indices of the set underlying the locally consistent distribution and it depends on the canonical linear programming formulation of the problem. The parameter t can be bounded by the maximum number of variables in any inequality of the initial linear program. In the case of MAX-CUT, $c = 2$ and $t = 3$ so $ct = 6$. However, the total number of vertices involved in any inequality of the cut polytope is 3 instead of 6 so we are able to reduce the denominator above to 3 for the case of MAX-CUT.

7 Open Problems

Some interesting problems, besides the ones posed in [14], remain. For example, is it possible to optimize over LS_* relaxations of even restricted classes of polytopes more efficiently? Do small rounds of SA_* or LS_* relaxation produce a smaller integrality gap for any version of the travelling salesman problem then the corresponding linear programming relaxation (or its LS, SA relaxations)? Finally, for the LS_* rank of blossom inequaities we conjecture the right answer to be $\omega(\log n)$ due to the similarities of the problem with symmetric knapsack [14].

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References

[1] Yu-Hin Au and Levent Tunçel. Complexity Analyses of Bienstock-Zuckerberg and Lasserre Relaxations on the Matching and Stable Set Polytopes. In *IPCO*, pages 14–26, 2011.

[2] Paul Beame, Trinh Huynh, and Toniann Pitassi. Hardness Amplification in Proof Complexity. *ECCC*, 2009.

[3] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[4] Josh Buresh-Oppenheim, Nicola Galesi, Shlomo Hoory, Avner Magen, and Toniann Pitassi. Rank Bounds and Integrality Gaps for Cutting Planes Procedures. In *FOCS*, 2003.

[5] Moses Charikar, Konstantin Makarychev, and Yury Makarychev. Integrality Gaps for Sherali-Adams Relaxations. In *STOC*, pages 283–292, 2009.

[6] Kevin K. H. Cheung. Computation of the Lasserre Ranks of Some Polytopes. *Math. Oper. Res.*, 32:88–94, February 2007.

[7] Eden Chlamatac and Madhur Tulsiani. Convex Relaxations and Integrality Gaps. In *Handbook on Semidefinite, Cone and Polynomial Optimization*, 2010.

[8] William Cook and Sanjeeb Dash. On the Matrix-Cut Rank of Polyhedra. *Math. Oper. Res.*, 26:19–31, February 2001.

[9] Stefan Dantchev, Barnaby Martin, and Mark Rhodes. Tight Rank Lower Bounds for the Sherali-Adams Proof System. *Theoretical Computer Science*, 410(21-23):2054 – 2063, 2009.

[10] Sanjeeb Dash. *On the Matrix-Cut of Polyhedra and their use in Integer Programming*. Department of Computer Science, Rice University, March 2001.

[11] Wenceslas Fernandez de la Vega and Claire Kenyon-Mathieu. Linear programming relaxations of MAX-CUT. In *SODA*, pages 53–61, 2007.

[12] Friedrich Eisenbrand. On the Membership Problem for the Elementary Closure of a Polyhedron. *Combinatorica*, 19(2):297–300, 1999.

[13] Michel X. Goemans and Levent Tunçel. When Does the Positive Semidefiniteness Constraint Help in Lifting Procedures? *Math. Oper. Res.*, 26(4):796–815, 2001.

[14] Dima Grigoriev, Edward Hirsch, and Dmitrii Pasechnik. Complexity of Semialgebraic Proofs. In *STACS 2002*, volume 2285 of *Lecture Notes in Computer Science*, pages 734–734, 2002.

[15] Jan Krajíček. *Bounded Arithmetic, Propositional Logic, and Complexity Theory*. Cambridge University Press, New York, NY, USA, 1995.

[16] Monique Laurent. A Comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre Relaxations for 0-1 Programming. *Math. Oper. Res.*, 28(3):470–496, 2003.

[17] László Lovász and Alexander Schrijver. Cones of Matrices and Set-Functions and 0-1 Optimization. *SIAM Journal on Optimization*, 1:166–190, 1991.

[18] Claire Mathieu and Alistair Sinclair. Sherali-adams relaxations of the matching polytope. In *STOC*, pages 293–302, 2009.

[19] Toniann Pitassi and Nathan Segerlind. Exponential Lower Bounds and Integrality Gaps for Tree-like Lovász-Schrijver Procedures. In *SODA*, pages 355–364, 2009.

[20] Pavel Pudlák. On the Complexity of Propositional Calculus, Sets and Proofs. *London Mathematical Society, Lecture Note Series*, 258:197–218, 1999.

[21] Alexander A. Razborov. Proof Complexity of Pigeonhole Principles. In *Developments in Language Theory*, pages 100–116, 2001.

[22] Steven Rudich and Avi Wigderson. *Computational Complexity Theory.* AMS, IAS/Park City Mathematics Series, 2004.

[23] Nathan Segerlind. The Complexity of Propositional Proofs. *Bulletin of Symbolic Logic*, 13(4):417–481, 2007.

[24] Hanif D. Sherali and Warren P. Adams. A Hierarchy of Relaxations between the Continuous and Convex Hull Representations for Zero-One Programming Problems. *SIAM J. Disc. Math.*, 3:411–430, 1990.

[25] Vijay V. Vazirani. *Approximation Algorithms.* Springer, 2004.

A Matching polytope

A proof of Lemma 4.5, repeated from [18] for convenience.

Proof. Given an optimal solution y of the lifted maximum matching linear program permute the vertices of K_{2n+1} to obtain, by symmetry, another optimal solution $\sigma(y)$. Define y^s to be the solution obtained after componentwise averaging over all $(2n+1)!$ solutions over permuted instances and we observe that y^s is still optimal and feasible. $\square$

B LS_+ rank lower bound for K_α^n

The following defintions are required for the LS_+ rank lower bound from Section 4. Let e_i denote the ith standard unit vector (where the dimension will be clear from the context) and let e denote the all 1s vector. For $a \in \mathbb{R}^{n+1}$ define $\bar{a} \in \mathbb{R}^n$ as $a = (1, \bar{a})$. Let F_i^0 denote the face of Q with ith coordinate set to 0.

Definition B.1 ([8])**.** *Define embedding $emb_I : \mathbb{R}^n \to \mathbb{R}^{n+k}$ such that if $y = emb_I(x)$ then $y_{i_j} = x_j$ for $I := \{i_j \in [n+k] | j \in [n]\}$, and $y_i \in \{0, 1\}$ for $i \notin I$.*

For a face F of Q_n let emb_F denote the embedding where $Q_{dim(F)} \mapsto F$ (i.e. emb_F is short for emb_I, $I = [n] \setminus \{i\}$ where i is the coordinate fixed to $\{0, 1\}$ in F). Lemma 2.1 in Cook and Dash [8] is restated below

Lemma B.2 ([8])**.** *Given polytope $P \subseteq Q$ and embedding $emb : \mathbb{R}^n \to \mathbb{R}^m$, $N_+(emb(P)) = emb(N_+(P))$.*

Theorem B.3. *The LS_+ rank of K_α^n is n.*

Proof. It suffices to show $y^{k,n} e_0 \in N_+^k(K_\alpha^n)$ for $k < n$ and some $y^{k,n} \in \mathbb{R}^{(n+1)\times(n+1)}$ defined below. Let

$$y_{0,0}^{k,n} = 1, y_{i,i}^{k,n} = y_{0,i}^{k,n} = y_{i,0}^{k,n} = \frac{\alpha}{n - (1-\alpha)k}$$

and $y^{k,n}$ is 0 elsewhere. Hence $y^{k,n}$ is a symmetric, positive semidefinite (diagonally dominant) matrix for all $k < n$. Note that $\sum_{i \in [n]} y_{\{i\}}^{k,n} < 1$ as required.

The proof proceeds by induction on k, n. In the base case $n = 1$ and $k = 0$ the hypothesis holds. Suppose the induction hypothesis (i.e. $y^{k,n}e_0 \in N_+^k(K_\alpha^n)$ for $k < n$) holds for K_α^{n-1} with $n \geq 2$ and $k < n$.

For brevity let $y^{k,n}$ defined above be denoted by y. Observe that $\overline{ye_i}$ is a positive multiple of $e_i \in (K_\alpha^n)_I$ therefore $\overline{ye_i} \in N_+^{k-1}(K_\alpha^n)$ for $k < n$.

Let $z_i = y(e_0 - e_i)$. Then

$$z_i = \frac{n-(1-\alpha)k-\alpha}{n-(1-\alpha)k} e_0 + \sum_{j=1, j\neq i}^n \frac{\alpha}{n-(1-\alpha)k} e_j \Rightarrow \bar{z}_i = \sum_{j=1, j\neq i}^n \frac{\alpha}{n-(1-\alpha)k-\alpha} e_j.$$

So $\bar{z}_i \in F_i^0$. Also $\frac{\alpha}{n-(1-\alpha)k-\alpha} e = \frac{\alpha}{n-1-(1-\alpha)(k-1)} e$ hence by induction hypothesis

$$\frac{\alpha}{n-(1-\alpha)k-\alpha} e \in N_+^{k-1}(K_\alpha^{n-1}) \Rightarrow \bar{z}_i \in emb_{F_i^0}(N_+^{k-1}(K_\alpha^{n-1})) = N_+^{k-1}(K_\alpha^n \cap F_i^0) \subseteq N_+^{k-1}(K_\alpha^n).$$

The equality on the the RHS of the implication above follows from Lemma B.2 and the observation $K_\alpha^n \cap F_i^0 = emb_{F_i^0}(K_\alpha^{n-1})$. Hence $y \in M_+(N_+^{k-1}(K_\alpha^n))$ for $k < n$ and so $(y_{\{i\}}) \in N_+^k(K_\alpha^n)$. Hence the proof follows. $\square$

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