Explicit Optimal hardness via Gaussian stability results

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Abstract

The results of Raghavendra (2008) show that assuming Khot's Unique Games Conjecture (2002), for every constraint satisfaction problem there exists a generic semi-definite program that achieves the optimal approximation factor. This result is existential as it does not provide an explicit optimal rounding procedure nor does it allow to calculate exactly the Unique Games hardness of the problem.

Obtaining an explicit optimal approximation scheme and the corresponding approximation factor is a difficult challenge for each specific approximation problem. An approach for determining the exact approximation factor and the corresponding optimal rounding was established in the analysis of MAX-CUT (KKMO 2004) and the use of the Invariance Principle (MOO 2005). However, this approach crucially relies on results explicitly proving optimal partitions in Gaussian space. Until recently, Borell's result (Borell 1985) was the only non-trivial Gaussian partition result known.

In this paper we derive the first explicit optimal approximation algorithm and the corresponding approximation factor using a new result on Gaussian partitions due to Isaksson and Mossel (2012). This Gaussian result allows us to determine exactly the Unique Games Hardness of MAX-3-EQUAL. In particular, our results show that Zwick algorithm for this problem achieves the optimal approximation factor and prove that the approximation achieved by the algorithm is ≈ 0.796 as conjectured by Zwick.

We further use the previously known optimal Gaussian partitions results to obtain a new Unique Games Hardness factor for MAX-k-CSP : Using the well known fact that pairwise independent Gaussians are independent, we show that the the UGC hardness of Max-k-CSP is $\frac{\lceil (k+1)/2 \rceil}{2^{k-1}}$, improving on results of Austrin and Mossel (2009).

1 Introduction

An important area of research in complexity theory in the past two decades has been the study of inapproximability of Constraint Satisfaction Problems (CSPs). A CSP is specified by a alphabet $[q]$ and a set of predicates $\mathcal{P}$ such that all $P \in \mathcal{P} : [q]^k \to \{0,1\}$[1]. Here k is called the arity of the predicate. An instance of the problem (say G) is given by n variables $x_1, \ldots, x_n$ and a set of constraints $\mathcal{E}$ such that every $e \in \mathcal{E}$ is of the form $e = (S, P)$ where $S \in [n]^k$ and $P \in \mathcal{P}$.

Now, consider any mapping $\mathcal{L} : [n] \to [q]$. A constraint $e = (S, P)$ is said to be ''satisfied'' if $P(\mathcal{L}(S_1), \ldots, \mathcal{L}(S_k)) = 1$ where S_i is the ith element of S. We also define $val_{\mathcal{L}}(G)$ as $val_{\mathcal{L}}(G) = \mathbb{E}_{v \in \mathcal{E}} P(\mathcal{L}(S_1), \ldots, \mathcal{L}(S_k))$. The algorithmic task is to come up with the mapping $\mathcal{L}$ such that $val_{\mathcal{L}}(G)$ is maximized. Towards this, we define $val(G) = \max_{\mathcal{L}} val_{\mathcal{L}}(G)$.

The reason for studying the very general framework of CSPs is because many specific problems of interest say MAX-CUT, MAX-3-SAT etc. fall in this framework. In the past two decades, there have been important important results in the study of inapproximability of CSPs including the monumental work of Håstad [Hås01] who obtained optimal inapproximability results for CSPs like MAX-3-SAT and MAX-3-LIN. Still, there remained a gap between the known algorithms and hardness results for many important CSPs like MAX-CUT and MAX-2-SAT. Towards closing this gap, Khot [Kho02] introduced the Unique Games Conjecture (UGC) which stated the following (equivalent form from [KKMO07]):

Conjecture 1. *Given any $\delta > 0$, there is a prime p such that given a set of linear equations $x_i - x_j = c_{ij} \pmod{p}$, it is NP-hard to decide which one of the following is true :*

- *There is an assignment to the x_i's which satisfies at least $1 - \delta$ fraction of the constraints.*

- *All assignments to the x_i's can satisfy at most δ fraction of the constraints.*

A series of (often optimal) inapproximability results were proven using the Unique Games Conjecture starting with [KR08, KKMO07] which culminated in the beautiful result of Raghavendra [Rag08] who showed that for every CSP of constant arity and alphabet size, there is a simple and generic SDP which is optimal assuming the Unique Games Conjecture. More specifically, he showed the following.

Theorem 2. *Suppose that for the generic SDP, there is an instance G such that $val(G) = s$ while the SDP objective value is c. Then, assuming the UGC, given an instance G' of the CSP such that $val(G') = c - \eta$, it is NP-hard to find a $\mathcal{L}$ such that $val_{\mathcal{L}}(G) \geq s + \eta$ for any $\eta > 0$. Further, there is an efficient rounding algorithm such that given an instance G with value c on the instance G, it finds an assignment $\mathcal{L}$ with value $s - \eta$ (for $\eta > 0$).*

While this result essentially settles the question of approximability of CSPs from an an abstract perspective, perhaps not too surprisingly , it says nothing about the exact hardness factors for specific CSPs. This is in contrast to the situation in the case of MAX-CUT [KKMO07] or MAX-2-SAT [Aus07] where exact inapproximability factors are known. Likewise, even though the rounding algorithm in [Rag08] is efficient, it is a brute force search over a small space that results only in a close to optimal rounding scheme. Thus, in a sense, the result provides implicitly a sequence of rounding algorithm whose approximation factors is guaranteed to converge to the hardness factor. This again is different from the rounding algorithms in [GW95, Zwi98, LLZ02] where the rounding algorithm is far more explicit (in the first two cases, it is simply random hyperplane rounding).

A major reason for difficulty in establishing exact hardness factors is the following : The exact hardness factor in the case of MAX-CUT [KKMO07] and MAX-2-SAT [Aus07] crucially rely on Gaussian Analysis. More specifically, it use the invariance principle [MOO10] together with a result in Gaussian space specifying

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[1]We are assuming a somewhat restricted form of a CSP where all the predicates have the same arity

explicitly an *Optimal Gaussian Partition* for the particular predicate. However, only few optimal Gaussian partitions are known (or even conjectured). In fact, to the best of our knowledge, before this paper, Borell's result [Bor85] was the only non-trivial Gaussian partition result used in hardness of approximation (for e.g., in [KKMO07, Aus07]).

The above issue also explains the "brute-force" search aspect of the rounding scheme in [Rag08]. The optimal rounding scheme and the optimal gaussian partitioning (for a given predicate) are known to be intimately linked to each other (see [Rag08] for a detailed explanation). In absence of knowledge of the optimal partitioning, [Rag08] uses the invariance principle and then resorts to a brute force search over a small space.

1.1 Our contributions In this paper, we consider two maximization CSPs, namely, MAX-3-EQUAL and MAX-k-CSP. Since we are dealing with maximization problems, we set the (usual) convention that an algorithm is said to give an α-approximation (for $\alpha \leq 1$) if it always returns a solution which is at least α times the optimal value.

We first start by describing our result for MAX-3-EQUAL. In MAX-3-EQUAL, the variables are boolean-valued and every constraint consists of three literals and it is satisfied if and only if all the three literals are either all zeros or all ones. We show that assuming the Unique Games Conjecture, the MAX-3-EQUAL problem is $\alpha_{EQU} \approx 0.796$ hard to approximate in polynomial time. On the complementary side, we also provide a polynomial time algorithm for this problem with the approximation ratio α_{EQU}. More formally, we prove

Theorem 3. *There is a polynomial time approximation algorithm for the MAX-3-EQUAL problem which achieves the following approximation ratio :*

$$\alpha := \min_{\delta \in (0,1]} \frac{1 - \frac{3\cos^{-1}(1-\delta)}{2\pi}}{1 - \frac{3\delta}{4}} \approx 0.796$$

Assuming the Unique Games Conjecture, for every $\delta > 0$ there is no polynomial time that provides a better approximation ratio than $\alpha + \delta$.

The hardness proof uses a recent Gaussian noise stability result of Isaksson and Mossel [IM12] which does not seem to have been previously used in the literature for proving hardness of approximation results.

We also give an analytic proof of the performance of the random hyperplane rounding algorithm on the generic SDP for MAX-3-EQUAL (from [Rag08]) showing that the approximation ratio achieved by this rounding algorithm is exactly α_{EQU}.[2] Our proof is computer assisted but completely rigorous. To the best of our knowledge, this is the first complete analytic proof of correctness for the SDP based algorithm for MAX-3-EQUAL problem. We note that while Zwick [Zwi98] also considers this problem and analyzes the performance of this algorithm, the analysis is a computer based search and he notes that there is a possibility of the search having missed the worst instance for the rounding algorithm. Nevertheless, the claimed optimum in [Zwi98] is same as the optimum of our SDP.

We also note that our SDP differs slightly from [Zwi98] in the sense that ours is slightly stronger. This is for convenience of analysis. We have not investigated whether this possible strengthening of the program is indeed a requirement. The next section lists in detail the technical preliminaries required in this paper.

While revisiting the study of the relationship between Gaussian partitions and UGC hardness, we additionally prove hardness results for MAX-k-CSPs. In particular, we investigate the hardness of the MAX-k-AND predicate i.e. every constraint consists of k literals $\ell_1, \ldots, \ell_k$ and the constraint is satisfied if

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[2]We actually do a variant of the random hyperplane rounding algorithm where we sample normal random variables with the covariance matrix given by the SDP vectors. Then each variable is assigned 0 or 1 depending on the sign of the corresponding normal random variable. Our analysis goes through even if the actual random hyperplane algorithm is used.

and only if $\ell_1 = \ldots = \ell_k = 1$. Following [Mos10] and [AM09] by using the fact that in Gaussian space, pair-wise independence implies independence, we prove the following theorem :

Theorem 4. *Assuming the Unique Games Conjecture, for every $\eta > 0$, there is no polynomial time approximation algorithm that provides an approximation ratio better than $\frac{\lceil (k+1)/2 \rceil}{2^{k-1}}$ for the MAX-k-AND problem.*

This improves upon [AM09] where it was shown that MAX-k-CSP is $(k + O(k^{0.525}))/2^k$ hard to approximate. Assuming the Hadamard Conjecture, they could improve it to $(k+4)/2^k$.

It is worth mentioning that [AM09] proves the aforementioned hardness for a very general class of predicates (ones whose satisfying assignments support pairwise independent distributions) but MAX-k-AND is not included in that class of CSPs. Another important point of difference is that [AM09] shows that given a MAX-k-CSP with optimal value $1 - \eta$, it is (Unique Games) hard to find an assignment which satisfies $\frac{k+O(k^{0.525})}{2^k} + \eta$ fraction of the constraints (for any $\eta > 0$). In terms of PCPs, the PCP in [AM09] has near perfect completeness. This in fact is true even for an earlier paper on hardness of MAX-k-CSPs by Samorodnitsky and Trevisan [ST06]. In contrast, our result shows that given an instance of MAX-k-CSP with optimal value $\frac{1}{2^{\lceil (k+1)/2 \rceil}} - \eta$, it is hard to find an assignment satisfying more than $\frac{1}{2^k} + \eta$ fraction of the constraints.

We do remark that while our improvement over [AM09] might seem very minor, Charikar *et al.* [CMM09] give a $0.44k/2^k$ approximation algorithm for MAX-k-CSP over boolean alphabet. This shows that in some sense, the scope of improvement in the existing hardness results for MAX-k-CSPs is rather limited. Of course, the question of closing the gap between our hardness result and the performance of the algorithm of Charikar *et al.* remains open.

Overview of proofs of hardness: The two main novelties in our paper are :

- Use of the new Gaussian stability result of Isaksson and Mossel [IM12] to construct a ''dictatorship'' test for MAX-3-EQUAL.

- Use of the ''obvious'' Gaussian stability result (i.e.stable partitions for independent gaussians) in a new context to construct a ''dictatorship'' test for MAX-k-AND.

Given the dictatorship test, getting the corresponding Unique Games hardness result is rather standard (see [KKMO07, Rag08]). For the sake of completeness, we give a complete proof of for hardness of MAX-3-EQUAL using the corresponding dictatorship test. For MAX-k-AND, we do not show the conversion of the dictatorship test to a Unique Games hardness result as the proof is completely analogous to that of MAX-3-EQUAL.

For MAX-3-EQUAL, we also devote a major part of the paper to show that the performance of our rounding algorithm on the generic SDP from [Rag08] indeed matches the Unique Games hardness result.

1.2 Organization Section 2 states all the fourier analytic and other technical preliminaries required for this paper. Section 3 describes a dictatorship test where the tester checks for equality of three literals. Section 4 describes a dictatorship test where the tester checks if all the k literals are 1. Section 5 has the two main theorems of this paper, namely a UG-hardness result for the MAX-3-EQUAL problem and a UG-hardness result for MAX-k-AND. Section 6 describes a SDP relaxation and a rounding algorithm for the MAX-3-EQUAL problem showing the tightness of the hardness result.

2 Preliminaries

2.1 Basics of Fourier analysis Our proofs are significantly dependent on fourier analysis. We start by giving several important definitions. For a more extensive reference, see lecture notes by Mossel [Mos05].

We recall that any function $f : \{-1, 1\}^n \to \mathbb{R}$ can be written as a multi-linear polynomial.

$$f(x) = \sum_{S \subset [n]} \hat{f}(S) x_S,$$

where $x_S = \prod_{i \in S} x_i$. Moreover, considering the uniform measure over $\{-1, 1\}^n$, we have:

$$\mathbf{E}[f] = \hat{f}(\emptyset), \quad Var[f] = \sum_{S \neq \emptyset} \hat{f}^2(S).$$

The i'th influence of f is given by

$$I_i(f) := Var[f | x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = \sum_{S : i \in S} \hat{f}^2(S).$$

2.2 Noise operators and their properties We will also require the notion of noise operators. We consider a particularly important instantiation of the Bonami-Beckner operator namely that on functions over the boolean hypercube $\{-1, 1\}^n$ equipped with the uniform measure.

Definition 5. *For $\rho \in [-1, 1]$, we define the Bonami-Beckner operator T_ρ on functions $f : \{-1, 1\}^n \to \mathbb{R}$ as follows.*

$$T_\rho f(x) = \mathbb{E}_{y \sim_\rho x}[f(y)]$$

where each coordinate y_i is set to be x_i independently with probability $(1 + \rho)/2$ and $-x_i$ with probability $(1 - \rho)/2$.

The effect of the Bonami-Beckner operator T_ρ can be conveniently expressed in terms of the fourier spectrum of a function. In particular, if f is as above, then

$$T_\rho f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \rho^{|S|} \chi_S(x)$$

The following standard lemma proves a bound on the number of coordinates with high influence on a function after applying the Bonami-Beckner operator on it, see e.g. [KKMO07].

Lemma 6. *Let $f : \{-1, 1\}^n \to [0, 1]$ and $\tau, \gamma > 0$. If $\mathcal{A}(f) = \{i : \text{Inf}_i(T_{1-\gamma} f) \geq \tau\}$, then $|\mathcal{A}(f)| \leq 1/(\gamma\tau)$.*

The next lemma is a specialization of Lemma 6.2 from [Mos10]. It says that expected value of product of polynomials does not change by a lot when noise is added provided individual coordinates come from correlated probability spaces such that no coordinate is absolutely fixed given rest of the coordinates.

Lemma 7. *For $1 \leq i \leq n$, let $(\Omega_i, \mu_i) = (\{-1, 1\}^k, \mu_i)$ where*

$$\min_{x \in \{-1, 1\}^k} \mu_i(x) \geq \alpha > 0.$$

Let $(\Omega, \mu) = \prod_{i=1}^n \Omega_i, \mu_i)$. For $1 \leq a \leq k$, let μ_i^a be the a'th marginal of μ_i, in other words

$$\mu_i^a(x) = \mu_i(\{(x_1, \ldots, x_k) : x_a = x\}).$$

Let $\mu^a = \prod_{i=1}^n \mu_i^a$. An element $x \in \Omega$ is a $k \times n$ matrix. We write x^a for the a'th column of x which is distributed according to μ^a. For $1 \leq a \leq k$, let Q_a be a multilinear polynomials $Q_a : \{-1, 1\}^n \to [-1, 1]$. Then, for all $\epsilon > 0$, $\exists \gamma = \gamma(\epsilon, \alpha) > 0$ such that

$$\left| \mathbf{E} \prod_{a=1}^k Q_a(x^a) - \mathbf{E} \prod_{i=1}^k T^{1-\gamma} Q_a(x^a) \right| \leq \epsilon k$$

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2.3 Gaussian Stability results The following theorem from Isaksson and Mossel [IM12] is the main technical result that we use here.

Theorem 8. *Let $\Omega = \{-1, 1\}^k$, $\rho \in [0, 1]$ and let μ be a probability distribtion over Ω such that*

- *$\mu(x) \geq \alpha > 0$ for all x.*

- *For $s, t \in \{-1, 1\}$ and all $1 \leq a \neq b \leq k$:*

$$\mu(x_a = s, x_b = t) = \frac{1}{2}\rho\delta(s, t) + \frac{1}{4}(1 - \rho).$$

Consider the space (Ω^n, μ^n). An element $x \in \Omega^n$ may be viewed as a $k \times n$ matrix. Write x^a for the a'th column of this matrix for $1 \leq a \leq k$. Note that x^a is uniformly distributed in $\{-1, 1\}^n$.

Then for every $\epsilon > 0$, $\exists \tau = \tau(\epsilon, k, \alpha) > 0$ such that for any $f_1, \ldots, f_k : \{-1, 1\}^n \to [0, 1]$ satisfying $\max_{i,j} \mathrm{Inf}_i(f_j) \leq \tau$,

$$\mathbf{E} \prod_{a=1}^k f_a(x^a) \leq \mathbf{Pr}[\forall a \in [k] : \mathcal{Z}_a \leq t_j] + \epsilon$$

where $\mathcal{Z}_1, \ldots, \mathcal{Z}_k \sim \mathcal{N}(0, 1)$ are jointly normal and $\mathrm{Cov}(\mathcal{Z}_a, \mathcal{Z}_{a'}) = \rho$ for all $a \neq a'$ such that each t_j is chosen so that $\mathbf{Pr}[\mathcal{Z}_a \leq t_a] = \mathbf{E}[f_a]$

We also consider the corollary of the above theorem when $\rho = 0$. We do remark that the following corollary can actually be obtained using the Invariance principle from Mossel [Mos10] and does not require the full strength of [IM12].

Corollary 9. *Let $\Omega = \{-1, 1\}^k$ and let μ be a probability distribtion over Ω such that*

- *$\mu(x) \geq \alpha > 0$ for all x.*

- *For $s, t \in \{-1, 1\}$ and all $1 \leq a \neq b \leq k$:*

$$\mu(x_a = s, x_b = t) = \frac{1}{4}.$$

Consider the space (Ω^n, μ^n). An element $x \in \Omega^n$ may be viewed as a $k \times n$ matrix. Write x^a for the a'th column of this matrix for $1 \leq a \leq k$. Note that x^a is uniformly distributed in $\{-1, 1\}^n$.

Then for every $\epsilon > 0$, $\exists \tau = \tau(\epsilon, k, \alpha) > 0$ such that for any $f_1, \ldots, f_k : \{-1, 1\}^n \to [0, 1]$ satisfying $\max_{i,j} \mathrm{Inf}_i(f_j) \leq \tau$,

$$\mathbf{E} \prod_{a=1}^k f_a(x^a) \leq \prod_{a=1}^k \mathbf{E}[f_a] + \epsilon$$

Proof. The corollary follows by putting $\rho = 0$ in Theorem 8 and then observing that $\mathcal{Z}_1, \ldots, \mathcal{Z}_k \sim \mathcal{N}(0, 1)$ in the conclusion of Theorem 8 are simply i.i.d. $\mathcal{N}(0, 1)$ random variables. $\qquad\square$

2.4 Useful facts We will require the following very useful fact about Gaussians. For a reference, see [Bac63].

Fact 10. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \sim \mathcal{N}(0, 1)$ such that $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ are jointly normal and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = \rho_1$, $\mathrm{Cov}(\mathcal{Z}, \mathcal{Y}) = \rho_2$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Z}) = \rho_3$. Then,*

$$\mathbf{Pr}[X, Y, Z \leq 0] = \mathbf{Pr}[X, Y, Z \geq 0] = \frac{1}{2} - \frac{\cos^{-1} \rho_1 + \cos^{-1} \rho_2 + \cos^{-1} \rho_3}{4\pi}$$

We will also use the following very useful construction of distributions from [BGGP12] though the basic construction dates back to [BP89].

Fact 11. *For any $k \in \mathbb{N}$, there is a distribution D_k on $\{-1, 1\}^k$ such that the following holds :*

- *For any $i \in [k]$, $\mathbf{E}[x_i] = 0$*

- *For any $i, j \in [k]$ and $i \neq j$, $\mathbf{E}[x_i x_j] = 0$ i.e. any two coordinates are pairwise independent.*

- $\mathbf{Pr}_{x \in D_k}[x_1 = \ldots = x_k = 1] = \frac{1}{2^{\lceil (k+1)/2 \rceil}}$

3 Dictatorship test for MAX-3-EQUAL

In this section, we will construct a dictatorship test where the tester checks for equality of 3 literals. More precisely, we will prove the following theorem :

Theorem 12. *For any $0 < \delta < 1$ and $\epsilon > 0$, there is a distribution D_δ^n over $(\{-1, 1\}^n)^3$ such that if $(X, Y, Z) \sim D_\delta^n$ such that for every $f : \{-1, 1\}^n \to [0, 1]$ with $\mathbf{E}[f] = 1/2$*

- *If $f(x) = (1 + x_i)/2$ for some $i \in [n]$, then*

$$\mathbf{Pr}_{(X,Y,Z) \sim D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))] = 1 - \frac{3\delta}{4}$$

- $\exists \tau = \tau(\delta, \epsilon) > 0$ *and* $\eta = \eta(\delta, \epsilon) > 0$ *such that if* $\max_i \mathrm{Inf}_i(T_{1-\eta} f) \leq \tau$

$$\mathbf{Pr}_{(X,Y,Z) \sim D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))] \leq 1 - (3 \cos^{-1}(1 - \delta))/2\pi + \epsilon$$

Before starting the proof, we note that if f were a boolean function with range $\{0, 1\}$, then $f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))$ is 1 if and only if $f(X) = f(Y) = f(Z)$. Thus, we have a dictatorship test which checks for equality of 3 bits.

Proof. Let us define D_δ as a distribution over $\{-1, 1\}^3$ which outputs $(1, 1, 1)$ with probability $(1 - \delta)/2$, $(-1, -1, -1)$ with probability $(1 - \delta)/2$ and uniform 3 bit string with probability δ. Let $D_1, D_2, \ldots, D_n$ be n i.i.d. samples of D_δ. Let $D_i(j)$ denote the jth bit of D_i. With this, let us define $X, Y, Z \in \{-1, 1\}^n$ as

$$X = (D_1(1), \ldots, D_n(1)) \quad Y = (D_1(2), \ldots, D_n(2)) \quad Z = (D_1(3), \ldots, D_n(3))$$

We let the joint distribution (X, Y, Z) as defined here be D_δ^n. We start with the proof of the first item. Completeness : Note for any particular $i \in [n]$, the ith coordinate of D has the same string with probability $1 - 3\delta/4$. Now, if $f(x) = (1 + x_i)/2$, then it means that $f(x) = 1$ if $x_i = 1$ and 0 otherwise. Hence, we have

$$\mathbf{E}_{(X,Y,Z) \in D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))$$
$$= \mathbf{E}_{(X,Y,Z) \in D}[\mathbf{I}[X_i = Y_i = Z_i]] = 1 - 3\delta/4$$

This finishes the proof of the first item. We next do the proof of the second item.
Soundness : Let $\mathcal{Q}$ be the multilinear polynomial representation of f. Note that for any $x \in \{-1, 1\}^n$, $|\mathcal{Q}(x)| \leq 1$. Let Ω be the probability space with domain $\{-1, 1\}^3$ and probability measure D_δ on it. Note that $\forall x \in \{-1, 1\}^3$, $D_\delta(x) \geq \delta/8$. Hence, by Lemma 7, we get that $\exists \eta = \eta(\delta, \epsilon) > 0$,

$$|\mathbf{E}_{(X,Y,Z) \in D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) - T_{1-\eta}f(X) \cdot T_{1-\eta}f(Y) \cdot T_{1-\eta}f(Z)]| \leq \epsilon/4 \qquad (1)$$

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Likewise, we get that

$$|\mathbf{E}_{(X,Y,Z)\in D_\delta^n}[(1-f(X))\cdot(1-f(Y))\cdot(1-f(Z))-(1-T_{1-\eta}f(X))\cdot(1-T_{1-\eta}f(Y))\cdot(1-T_{1-\eta}f(Z))]| \le \epsilon/4 \tag{2}$$

We can now apply Theorem 8. In particular, note that if $(X, Y, Z) \sim D$, then the variables (X_i, Y_i, Z_i) are identical and independently distributed. Also, for any $i \in [n]$, X_i, Y_i and Z_i are pairwise $\rho = (1 - \delta)$ correlated. Also, for any $i \in [n]$ and $(x, y, z) \in \{-1, 1\}^3$, $\mathbf{Pr}[(X_i, Y_i, Z_i) = (x, y, z)] \ge \delta/8 > 0$. Finally, note that X, Y and Z are distributed as U_n. Hence

$$\mathbf{E}_X[f(X)] = \mathbf{E}_Y[f(Y)] = \mathbf{E}_Z[f(Z)] = 1/2$$

As the Bonami Beckner operator preserves expectation of the function under the uniform distribution, we get

$$\mathbf{E}_X[T_{1-\eta}f(X)] = \mathbf{E}_Y[T_{1-\eta}f(Y)] = \mathbf{E}_Z[T_{1-\eta}f(Z)] = 1/2$$

Thus, by Theorem 8, $\exists \tau = \tau(\delta, \epsilon)$ such that if $\max_i \mathrm{Inf}_i(T_{1-\eta}f) \le \tau$, then we have

$$|\mathbf{E}_{(X,Y,Z)\in D_\delta^n}[T_{1-\eta}f(X) \cdot T_{1-\eta}f(Y) \cdot T_{1-\eta}f(Z)]| \le \mathbf{Pr}[\mathcal{X}, \mathcal{Y}, \mathcal{Z} \le 0] + \epsilon/4$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \sim \mathcal{N}(0,1)$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = \mathrm{Cov}(\mathcal{Z}, \mathcal{Y}) = \mathrm{Cov}(\mathcal{X}, \mathcal{Z}) = 1 - \delta$. Here, we again assume that τ in the hypothesis of the theorem is sufficiently small so that the hypothesis of Theorem 8 is valid. Likewise, we get that

$$|\mathbf{E}_{(X,Y,Z)\in D_\delta^n}[(1 - T_{1-\eta}f(X)) \cdot (1 - T_{1-\eta}f(Y)) \cdot (1 - T_{1-\eta}f(Z))]| \le \mathbf{Pr}[\mathcal{X}, \mathcal{Y}, \mathcal{Z} \le 0] + \epsilon/4$$

Combining the above with (2) and (1), we get that

$$\mathbf{E}_{(X,Y,Z)\in D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))] \le 2\,\mathbf{Pr}[\mathcal{X}, \mathcal{Y}, \mathcal{Z} \le 0] + \epsilon$$

Using Fact 10, we conclude that

$$\mathbf{E}_{(X,Y,Z)\in D_\delta^n}[f(X) \cdot f(Y) \cdot f(Z) + (1 - f(X)) \cdot (1 - f(Y)) \cdot (1 - f(Z))] \le 1 - \frac{3\cos^{-1}(1 - \delta)}{2\pi} + \epsilon$$

completing the proof. $\qquad\square$

4 Dictatorship test for MAX-k-AND

In this section, we construct a dictatorship test for MAX-k-AND i. e. the tester checks if a particular set of k literals are all set to 1. For the purposes of this section, let us assume $\rho(k) = \frac{1}{2\lceil(k+1)/2\rceil}$.

Theorem 13. *For any $k \ge 3$ and $\delta > 0$, there is a distribution D over $(\{-1, 1\}^n)^k$ such that if $(X_1, \ldots, X_k) \sim D$ such that for every $f : \{-1, 1\}^n \to [0, 1]$ with $\mathbf{E}[f] = 1/2$*

- *If $f(x) = (1 + x_i)/2$ for some $i \in [n]$, then*

$$\mathbf{Pr}_{(X_1,\ldots,X_k)\sim D}[f(X_1) \cdot \ldots \cdot f(X_k)] \ge \rho(k) - \delta$$

- *$\exists \tau = \tau(\delta, k) > 0$ and $\eta = \eta(\delta, k) > 0$ such that if $\max_i \mathrm{Inf}_i(T_{1-\eta}f) \le \tau$*

$$\mathbf{Pr}_{(X_1,\ldots,X_k)\sim D}[f(X_1) \cdot \ldots \cdot f(X_k)] \le \frac{1}{2^k} + \delta$$

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We remark that if f were to take values in $\{0,1\}$, then we note that $f(X_1) \cdot \ldots \cdot f(X_k) = 1$ if and only if $f(X_1) \wedge \ldots \wedge f(X_k) = 1$.

Proof. Let D_k be the distribution from Fact 11. We let $\xi = \delta/4$. Now, we let $D_\xi = (1 - \xi)D_k + \xi U_k$. Let $D_1, \ldots, D_n$ be n i.i.d. samples from D_ξ. Let $D_i(j)$ be the jth bit of D_i. Having done this, we define X_j for $1 \leq j \leq k$ as $X_j = (D_1(j), \ldots, D_n(j))$. Let D be defined as the joint distribution of $(X_1, \ldots, X_k)$.

As before, we start with the proof of the first item.

Completeness : Since $f(x) = (1 + x_i)/2$ (for some $i \in [n]$), it means that $f(x) = 1$ if $x_i = 1$ and 0 otherwise. Hence, we have

$$
\begin{aligned}
\mathbf{E}_{(X_1,\ldots,X_k) \in D}[f(X_1) \cdot \ldots \cdot f(X_k)] &= \mathbf{E}_{(X_1,\ldots,X_k) \in D}[\mathbf{I}[X_1(i) = \ldots = X_k(i) = 1]] \\
&= \rho(k)(1 - \xi) + \xi 2^{-k} \geq \rho(k) - \delta
\end{aligned}
$$

Soundness : Let $\mathcal{Q}$ be the multilinear polynomial representation of f. Note that for any $x \in \{-1,1\}^n$, $|\mathcal{Q}(x)| \leq 1$. Let Ω be the probability space with domain $\{-1,1\}^k$ and probability measure D_ξ on it. Observe that $D_\xi(x) \geq \xi \cdot 2^{-k}$ for all $x \in \{-1,1\}^k$. Hence, by Lemma 7, we get that $\exists \eta = \eta(\xi, k) > 0$ (note because $\xi = \delta/4$, we can also express η as a function of δ and k as required by the theorem),

$$
|\mathbf{E}_{(X_1,\ldots,X_k) \in D}[f(X_1) \cdot \ldots \cdot f(X_k) - T_{1-\eta}f(X_1) \cdot \ldots \cdot T_{1-\eta}f(X_k)]| \leq \frac{\xi}{4} \tag{3}
$$

We can now apply Corollary 9 to the function $T_{1-\eta}f$ and the random variables $(X_1, \ldots, X_k) \sim D$. Much like in the proof of Theorem 12, it is easy to check that all the conditions are satisfied (In particular, note that for any $i \in [n]$, $X_1(i), X_2(i), \ldots, X_k(i)$ are pairwise independent). By Corollary 9, $\exists \tau = \tau(\xi, k)$ such that if $\max_i Inf_i(f) \leq \tau$, we have

$$
|\mathbf{E}_{(X_1,\ldots,X_k) \in D}[T_{1-\eta}f(X_1) \cdot \ldots \cdot T_{1-\eta}f(X_k)]| \leq 2^{-k} + \frac{\xi}{4} \tag{4}
$$

As before, we note that $\tau(\xi, k)$ can be expressed as $\tau(\delta, k)$. Here, we are assuming that the $\eta(\xi, k)$ and $\tau(\xi, k)$ chosen to be sufficiently small so that the hypothesis of Corollary 9 is valid. Combining (3) and (4), we get that

$$
|\mathbf{E}_{(X_1,\ldots,X_k) \in D}[f(X_1) \cdot \ldots \cdot f(X_k)]| \leq 2^{-k} + \frac{\xi}{2}
$$

$\square$

5 Unique games hardness from Dictatorship test

In this section, we use the dictatorship tests constructed in Section 3 and Section 4 to show the following theorems.

Theorem 14. *Assuming the Unique Games Conjecture, for every $0 < \delta < 1$ and $\epsilon > 0$, it is NP-hard to distinguish an instance of MAX-3-EQUAL with value $1 - 3\delta/4 - \epsilon$ from an instance of value $1 - \frac{3\cos^{-1}(1-\delta)}{2\pi} + \epsilon$. In other words, for every $\epsilon > 0$, MAX-3-EQUAL is $\alpha_{EQU} + \epsilon$ hard to approximate where*

$$
\alpha_{EQU} = \min_{\delta \in (0,1)} \frac{1 - \frac{3\cos^{-1}(1-\delta)}{2\pi}}{1 - \frac{3\delta}{4}} \approx 0.796
$$

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Theorem 15. *Assuming the Unique Games Conjecture, for every $\epsilon > 0$, it is NP-hard to distinguish an instance of MAX-k-AND with value $\frac{1}{2^{\lceil (k+1)/2 \rceil}} - \epsilon$ from an instance of value $2^{-k} + \epsilon$. In other words, for every $\epsilon > 0$, MAX-k-AND is $\frac{\lceil (k+1)/2 \rceil}{2^{k-1}} + \epsilon$ hard to approximate.*

Theorem 14 uses the dictatorship test in Theorem 12 to reduce Unique Games to MAX-3-EQUAL. Similarly, Theorem 15 uses the dictatorship test in Theorem 13 to reduce Unique Games to MAX-k-AND. As we said in the introduction, these reductions are by now very standard and can be found at several places. For the sake of convenience of the reader, we include the full proof of Theorem 14. The proof of Theorem 15 is exactly analogous and hence, we do not do it here.

We begin by defining the Unique Label cover problem followed by stating Khot's Unique Games Conjecture (slightly differently stated than Conjecture 1).

Definition 16. *A unique Label cover problem (G, Σ) on alphabet size k is defined by a graph $G = (V, E)$ and a set of permutations $\Sigma = \{\sigma_{(u,v)} : [k] \to [k]\}_{(u,v) \in E}$. For any map $\mathcal{L} : V \to [k]$ and $(u, v) \in E$, $\mathcal{A}_{\mathcal{L}}(u, v) = 1$ if and only if $\mathcal{L}(v) = \sigma_{(u,v)}(\mathcal{L}(u))$, otherwise it is zero. For a map $\mathcal{L} : V \to [k]$, $val_{\mathcal{L}}(G) = \mathbf{E}_{(u,v) \sim E}[\mathcal{A}_{\mathcal{L}}(u, v)]$. Value of the label cover problem (denoted by) $val(G) = \max_{\mathcal{L}:V \to [k]} val_{\mathcal{L}}(G)$.*

Conjecture 17. *[Kho02] Unique Games Conjecture : For every $\epsilon > 0$, there is a $k = k(\epsilon)$ such given a unique Label cover problem (G, Σ) on alphabet size k, distinguishing whether $val(G) \leq \epsilon$ or $val(G) \geq 1 - \epsilon$ is NP-hard. We can also assume that the graph G is regular.*

Having stated the unique games conjecture, we describe a PCP verifier for the unique Label cover problem which checks for equality of 3 bits. By the standard reduction between PCP verifiers and hardness of approximation, we will get a hardness result for the MAX-3-EQUAL problem.

Description of the PCP verifier: Given the unique games instance (G, Σ) (on alphabet size k), we assume that $V = [n]$ and build a PCP tester over $n \cdot 2^k$ boolean variables as follows : For every $i \in [n]$, we have a function $f_i : \{-1, 1\}^k \to \{0, 1\}$. Note that each such truth table can be described by 2^k boolean variables and hence the family of functions $\{f_i\}$ can be described in all by $n \cdot 2^k$ variables.

For a given $\delta \in (0, 1)$, let D_δ^n be the distribution in the hypothesis of Theorem 12. With this, the tester is as follows :

- Pick $v \in V$ uniformly at random and choose three random neighbors of v, say, w_1, w_2, w_3 uniformly at random.

- Choose $(X, Y, Z) \sim D_\delta^n$ (described above) and accept if and only if

$$f_{w_1} \circ \sigma_{(w_1,v)}(X) = f_{w_2} \circ \sigma_{(w_2,v)}(Y) = f_{w_3} \circ \sigma_{(w_3,v)}(Z)$$

Remark 18. *We will also assume the functions are folded i.e. for any x, $f(x) \neq f(-x)$. Note that this can be done without loss of generality, because whenever a tester needs to query $f(x)$, if $x_1 = 1$, it queries $f(x)$. Else it queries $f(-x)$ and flips the output. Also, we observe that dictators (and as such, any linear function) satisfy this requirement.*

We next show the correctness of this tester. In other words, we prove the following two lemmas.

Lemma 19. *If $val(G) \geq 1 - \epsilon$, then there is a set of functions $\{f_i : \{-1, 1\}^k \to \{0, 1\}\}_{i \in [n]}$ such that the above tester accepts with probability at least $(1 - 3\epsilon)(1 - 3\delta/4)$.*

Lemma 20. *For any $\epsilon > 0$, if the above tester passes with probability more than $1 - \frac{3\cos^{-1}(1-\delta)}{2\pi} + \epsilon$, then $\exists \mathcal{L} : V \to [k]$ such that $val_{\mathcal{L}}(G) = \kappa(\epsilon, \delta) > 0$.*

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Since $val_{\mathcal{L}}(G)$ in conclusion of $\kappa(\epsilon, \delta)$ does not depend on k, hence by combining Lemma 19 and Lemma 20 and the standard reduction between PCPs and hardness of CSPs, we prove Theorem 14. We next prove Lemma 19 and Lemma 20.

Proof of Lemma 19. Since $val(G) \geq (1 - \epsilon)$, $\exists \mathcal{L} : V \to [k]$ such that $val_{\mathcal{L}}(G) \geq 1 - \epsilon$. Let $\mathcal{L}$ be such a labeling of the vertices. We let $f_i : \{-1, 1\}^k \to \{0, 1\}$ be the dictator function corresponding to $\mathcal{L}(i)$. In other words, $f_i(x) = (1 + x_{\mathcal{L}(i)})/2$. Now, since $val_{\mathcal{L}}(G) \geq 1 - \epsilon$ and the constraint graph G is regular, hence if we choose v uniformly at random and then a uniform random neighbor w_i, then $\mathcal{L}(v, w_i) = 1$ with probability $1 - \epsilon$. By a union bound, with probability at least $1 - 3\epsilon$, $\mathcal{A}_{\mathcal{L}}(v, w_1) = \mathcal{A}_{\mathcal{L}}(v, w_2) = \mathcal{A}_{\mathcal{L}}(v, w_3) = 1$. If it indeed the case i.e. $\mathcal{A}_{\mathcal{L}}(v, w_1) = \mathcal{A}_{\mathcal{L}}(v, w_2) = \mathcal{A}_{\mathcal{L}}(v, w_3) = 1$, then

$$f_{w_1} \circ \sigma_{(w_1, v)} = f_{w_2} \circ \sigma_{(w_2, v)} = f_{w_3} \circ \sigma_{(w_3, v)}$$

Now, applying the first part of Theorem 12, we get that in this case, the test accepts with probability $1 - 3\delta/4$. Thus, the total probability that the test accepts is at least $(1 - 3\epsilon)(1 - 3\delta/4)$. $\qquad \square$

We next move to the more difficult case of soundness. In particular, we prove the following lemma.

Proof. The proof of this is exactly the same as that in [KKMO07]. We first describe the labeling $\mathcal{L}$ and then describe its correctness. Our labeling is a randomized scheme. Let $\eta, \tau > 0$ be two parameters which are chosen according to the second part of the hypothesis of Theorem 12 for parameters $\epsilon/2$ and δ. First, for every $v \in V$, we define $g_v : \{-1, 1\}^n \to [0, 1]$ as

$$g_v(x) = \mathbb{E}_{(w, v) \in E}[f_w \circ \sigma_{(w, v)}(x)]$$

Again for every $v \in V$ we define $\mathcal{A}(v) \subseteq V$ as

$$\mathcal{A}(v) = \{i : \mathrm{Inf}_i(T_{1-\eta} f_v) \geq \tau/2\} \cup \{i : \mathrm{Inf}_i(T_{1-\eta} g_v) \geq \tau\}$$

The randomized labeling scheme is the following : If the set $\mathcal{A}(v)$ is empty, $\mathcal{L}(v)$ is chosen arbitrarily. Else, it is chosen to be a uniformly random element from the set $\mathcal{A}(v)$. The following claim gives us the desired result.

Claim 21. *Over the choice of randomness for choosing $\mathcal{L}$, $\mathbf{E}[val_{\mathcal{L}}(G)] \geq (\eta^2 \tau^3)/32$.*

By fixing the randomness in the above claim desirably, we get Lemma 20. So, the proof boils down to proving Claim 21.

Proof. Note that the probability of acceptance of the tester is given by

$$\mathop{\mathbf{E}}_{v \in V} \mathop{\mathbb{E}}_{w_1, w_2, w_3:(w_i, v) \in E} \mathop{\mathbf{E}}_{(X, Y, Z) \in D} [\mathbf{I}(f_{w_1} \circ \sigma_{(w_1, v)}(X) = f_{w_2} \circ \sigma_{(w_2, v)}(Y) = f_{w_3} \circ \sigma_{(w_3, v)}(Z))]$$

$$= \mathop{\mathbf{E}}_{v \in V} \mathop{\mathbb{E}}_{w_1, w_2, w_3:(w_i, v) \in E} \mathop{\mathbf{E}}_{(X, Y, Z) \in D} [f_{w_1} \circ \sigma_{(w_1, v)}(X) \cdot f_{w_2} \circ \sigma_{(w_2, v)}(Y) \cdot f_{w_3} \circ \sigma_{(w_3, v)}(Z)]$$

$$+ \mathop{\mathbf{E}}_{v \in V} \mathop{\mathbb{E}}_{w_1, w_2, w_3:(w_i, v) \in E} \mathop{\mathbf{E}}_{(X, Y, Z) \in D} [(1 - f_{w_1} \circ \sigma_{(w_1, v)}(X)) \cdot (1 - f_{w_2} \circ \sigma_{(w_2, v)}(Y)) \cdot (1 - f_{w_3} \circ \sigma_{(w_3, v)}(Z))]$$

$$= \mathop{\mathbf{E}}_{v \in V} \mathop{\mathbf{E}}_{(X, Y, Z) \in D} [g_v(X) \cdot g_v(Y) \cdot g_v(Z) + (1 - g_v(X)) \cdot (1 - g_v(Y)) \cdot (1 - g_v(Z))] \geq 1 - \frac{3 \cos^{-1}(1 - \delta)}{2\pi} + \epsilon$$

A Markov argument gives that for at least an $\epsilon/2$ fraction of vertices $v \in V$

$$\mathop{\mathbf{E}}_{(X, Y, Z) \in D} [g_v(X) \cdot g_v(Y) \cdot g_v(Z) + (1 - g_v(X)) \cdot (1 - g_v(Y)) \cdot (1 - g_v(Z))] \geq 1 - \frac{3 \cos^{-1}(1 - \delta)}{2\pi} + \epsilon/2$$

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Call this subset of V as A. Note that by the second part of Theorem 12, for every $v \in A$, $\exists i \in [k]$, such that $\mathrm{Inf}_i(T_{1-\eta}g_v) \geq \tau$. For $v \in A$, choose any such i which satisfies $\mathrm{Inf}_i(T_{1-\eta}g_v) \geq \tau$.

$$\tau \leq \mathrm{Inf}_i(T_{1-\eta}g_v) \;=\; \sum_{S:i\in S}(1-\eta)^{|S|}\widehat{g_v}(S)^2 = \sum_{S:i\in S}(1-\eta)^{|S|}\left(\mathbf{E}_{(w,v)\in E}[\widehat{f_w\circ\sigma_{(w,v)}}(S)]\right)^2$$

$$= \sum_{S:i\in S}(1-\eta)^{|S|}\left(\mathbf{E}_{(w,v)\in E}[\widehat{f_w}(\sigma^{-1}_{(w,v)}(S))]\right)^2$$

Here $\sigma^{-1}_{(w,v)}(S)$ denotes the set which under the map $\sigma_{(w,v)}$ gets mapped to S. Now, by Jensen's inequality we get that

$$\sum_{S:i\in S}(1-\eta)^{|S|}\left(\mathbf{E}_{(w,v)\in E}[\widehat{f_w}(\sigma^{-1}_{(w,v)}(S))]\right)^2 \;\leq\; \mathbf{E}_{(w,v)\in E}\sum_{S:i\in S}(1-\eta)^{|S|}\widehat{f_w}^2\left(\sigma^{-1}_{(w,v)}(S)\right)$$

$$= \mathbf{E}_{(w,v)\in E}\mathrm{Inf}_{\sigma^{-1}_{(w,v)}(i)}(T_{1-\eta}f_w)$$

This implies that for such a $v \in A$ and i such that $\mathrm{Inf}_i(T_{1-\eta}g_v) \geq \tau$, at least for $\tau/2$ fraction of neighbors w of v,

$$\mathrm{Inf}_{\sigma^{-1}_{(w,v)}(i)}(T_{1-\eta}f_w) \geq \tau/2$$

Call such a pair (v,w) of vertices as "good". Using Lemma 6, it can be easily shown that for every $v \in V$, $|\mathcal{A}(v)| \leq 4/(\tau\eta)$. This means that for every $v \in A$, the randomized scheme $\mathcal{L}$ assigns $\mathcal{L}(v) = i$ such that $\mathrm{Inf}_i(T_{1-\eta}g_v) \geq \tau$ with probability at least $(\eta\tau)/4$. Observe that for any such $v \in A$, at least $\tau/2$ fraction of its neighbors w are such that (v,w) is "good". Note that for any good pair (v,w), if $\mathrm{Inf}_i(T_{1-\eta}g_v) \geq \tau$, then $\mathrm{Inf}_{\sigma^{-1}_{(w,v)}(i)}(T_{1-\eta}f_w) \geq \tau/2$. This implies that $\sigma^{-1}_{(w,v)}(i) \in \mathcal{A}(w)$. Thus, with probability at least $(\tau\eta)/4$, $\mathcal{L}(w) = \sigma^{-1}_{(w,v)}(i)$. Thus, overall the probability that $\mathcal{L}(v,w) = 1$ is at least $\tau^3\eta^2/32$. This proves the Claim 21 and concludes the proof. $\qquad\square$

$\square$

6   Approximation algorithm for the MAX-3-EQUAL problem

In this section, we give a SDP based approximation algorithm for MAX-3-EQUAL whose performance matches the hardness result from the last section. In particular, we prove the following theorem.

Theorem 22. *There is a polynomial time approximation algorithm for the MAX-3-EQUAL problem which achieves the following approximation ration :*

$$\min_{\delta\in(0,1)}\frac{1-\frac{3\cos^{-1}(1-\delta)}{2\pi}}{1-\frac{3\delta}{4}} \approx 0.796$$

Thus, this theorem shows that we have an approximation algorithm whose performance ratio matches the Unique Games hardness for this problem. Towards proving Theorem 22, we state a SDP relaxation for the MAX-3-EQUAL problem followed by a rounding procedure and then analyze the performance of this algorithm. The SDP formulation is the generic SDP by Raghavendra [Rag08] specialized to the MAX-3-EQUAL problem. We assume that the variables are $x_1,\ldots,x_n \in \{-1,1\}$. The constraint set $E \subseteq [n]^3 \times \{-1,1\}^3$ such that for every $(i,j,k) \times (\eta_i, \eta_j, \eta_k) \in E$, we have a constraint that $\eta_i x_i = \eta_j x_j = \eta_k x_k$. The SDP relaxation is given in Figure 1.

___

SDP formulation

1. $\forall\, i \in [n]$, $v_i \in \mathbb{R}^n$ and $\|v_i\|_2 = 1$.

2. $\forall\, i,j,k \in [n]^3$, $i < j < k$, $\alpha_{(i,j,k)}, \beta_{(i,j,k)}, \gamma_{(i,j,k)}, \delta_{(i,j,k)} \in \mathbb{R}^+ \cup \{0\}$ such that

$$\alpha_{(i,j,k)} + \beta_{(i,j,k)} + \gamma_{(i,j,k)} + \delta_{(i,j,k)} = 1$$

3. $\forall\, i,j,k \in [n]^3$, $i < j < k$,

$$\alpha_{(i,j,k)} + \beta_{(i,j,k)} - \gamma_{(i,j,k)} - \delta_{(i,j,k)} = \langle v_i, v_j \rangle$$
$$\alpha_{(i,j,k)} - \beta_{(i,j,k)} + \gamma_{(i,j,k)} - \delta_{(i,j,k)} = \langle v_j, v_k \rangle$$
$$\alpha_{(i,j,k)} - \beta_{(i,j,k)} - \gamma_{(i,j,k)} + \delta_{(i,j,k)} = \langle v_i, v_k \rangle$$

4. For $e \in E$, where $e = (i,j,k) \times (\eta_i, \eta_j, \eta_k)$, define

$$\lambda(e) = \begin{cases} \alpha_{(i,j,k)} & \text{if } \eta_i = \eta_j = \eta_k, \\ \beta_{(i,j,k)} & \text{if } \eta_i = \eta_j = -\eta_k, \\ \gamma_{(i,j,k)} & \text{if } -\eta_i = \eta_j = \eta_k, \\ \delta_{(i,j,k)} & \text{if } \eta_i = -\eta_j = \eta_k, \end{cases}$$

5. Subjected to the above, Maximize $\mathbf{E}_{(i,j,k) \times (\alpha, \beta, \gamma) \in E}\, \lambda(e)$

___

Figure 1: SDP relaxation for MAX-3-EQUAL problem

Remark 23. *We note that Zwick [Zwi98] describes a SDP relaxation and a similar rounding procedure for the MAX-3-EQUAL problem. The paper also gives numerical evidence towards showing that the performance ratio of their algorithm is approximately 0.796. However, the paper notes that they do not have an analytical proof of this and to the best of our knowledge, no analytical proof has appeared ever since. We analyze a slightly different SDP and analytically show that the performance of it is indeed what we claim. The reason we do not analyze Zwick's SDP is because it appears to be more difficult to analyze though we are not aware of any counterexample showing that the performance of Zwick's algorithm is not what it is claimed in [Zwi98].*

To see why the SDP in Figure 1 is a relaxation, consider a particular assignment to the variables $x_1, \ldots, x_n$. Let us define $\mathbf{v_0} \in \mathbb{R}^n$ as 1 in the top coordinate and 0 everywhere else. If $x_i = 1$, set $v_i = \mathbf{v_0}$. Else, if $x_i = -1$, set $v_i = -\mathbf{v_0}$. The rest of the variables are set as follows. For every triple (i,j,k), $i < j < k$,

- If $x_i = x_j = x_k$, then $\alpha_{(i,j,k)} = 1$, $\beta_{(i,j,k)} = \gamma_{(i,j,k)} = \delta_{(i,j,k)} = 0$.

- If $x_i = x_j = -x_k$, then $\beta_{(i,j,k)} = 1$, $\alpha_{(i,j,k)} = \gamma_{(i,j,k)} = \delta_{(i,j,k)} = 0$.

- If $-x_i = x_j = x_k$, then $\gamma_{(i,j,k)} = 1$, $\alpha_{(i,j,k)} = \beta_{(i,j,k)} = \delta_{(i,j,k)} = 0$.

- If $x_i = -x_j = x_k$, then $\delta_{(i,j,k)} = 1$, $\alpha_{(i,j,k)} = \beta_{(i,j,k)} = \gamma_{(i,j,k)} = 0$.

It is easy to verify that with these assignments of $\alpha_{(i,j,k)}$, $\beta_{(i,j,k)}$, $\gamma_{(i,j,k)}$, $\delta_{(i,j,k)}$ and v_i, constraints 1, 2 and 3 are indeed satisfied. Further, for this assignment, if a constraint $e \in E$ is satisfied, then it is easy to see that

$\lambda(e) = 1$. Also, if a constraint e is not satisfied, then $\lambda(e) = 0$. Thus, the objective value of the program for this assignment is exactly the fraction of constraints $e \in E$ which are satisfied and hence its a relaxation.

6.1   Rounding algorithm Our rounding algorithm is as follows : Let $\Sigma \in \mathbb{R}^{n \times n}$ be the matrix such that $\Sigma_{i,j} = \langle v_i, v_j \rangle$. Note that Σ is positive semidefinite. So, we let $\mathcal{X} \sim \mathcal{N}(\mathbf{0}, \Sigma)$ i.e. $\mathcal{X}$ be a jointly normal distribution in $\mathbb{R}^n$ with mean at the origin and the covariance matrix Σ. The rounding algorithm gets a sample $\mathcal{X}$ and assigns $x_i = 1$ if $\mathcal{X}_i \geq 0$ and -1 otherwise. Here $\mathcal{X}_i$ denotes the ith coordinate of $\mathcal{X}$. We will call this rounding as the "random gaussian" rounding. We now prove Theorem 22 by analyzing the performance of the aforedescribed algorithm.

  We would also like to remark that (perhaps not too surprisingly), if instead of the "random gaussian" rounding, we would have used "random hyperplane" rounding, the performance of the algorithm would have been the same and our analysis would have also gone through without any changes.

Proof of Theorem 22. We start by considering a particular constraint $e \in E$. Without loss of generality, assume that $e = (i, j, k) \times (1, 1, 1)$. We note that if the latter part of the argument were not $(1, 1, 1)$ any other $(\eta_i, \eta_j, \eta_k) \in \{-1, 1\}^3$ the analysis will remain unchanged.

  Now, for the particular edge e, its contribution to the SDP objective is $\lambda(e) = \alpha_{(i,j,k)}$. On the other hand, let the expected contribution to the true objective from this edge be $\kappa(e)$. Note that

$$\kappa(e) = \mathbf{Pr}[(\mathcal{X}_i, \mathcal{X}_j, \mathcal{X}_k \geq 0) \cup (\mathcal{X}_i, \mathcal{X}_j, \mathcal{X}_k < 0)] \tag{5}$$

As is standard, the performance ratio of the algorithm is lower bounded by $\inf \kappa(e)/\lambda(e)$. Hence, we will simply aim to prove a lower bound on $\inf \kappa(e)/\lambda(e)$. Now, plugging Fact 10 into (5) and using that for any (i, j, k), $\alpha_{(i,j,k)} + \beta_{(i,j,k)} + \gamma_{(i,j,k)} + \delta_{(i,j,k)} = 1$, we get that

$$
\begin{aligned}
\kappa(e) &= 1 - \frac{\cos^{-1}(\langle v_i, v_j \rangle) + \cos^{-1}(\langle v_j, v_k \rangle) + \cos^{-1}(\langle v_i, v_k \rangle)}{2\pi} \\
&= 1 - \frac{\cos^{-1}(2(\alpha_{(i,j,k)} + \beta_{(i,j,k)}) - 1) + \cos^{-1}(2(\alpha_{(i,j,k)} + \gamma_{(i,j,k)}) - 1) + \cos^{-1}(2(\alpha_{(i,j,k)} + \delta_{(i,j,k)}) - 1)}{2\pi}
\end{aligned}
$$

Thus, for $a, b, c, d \in \mathbb{R}^+ \cup \{0\}$, if we define

$$g(a, b, c, d) \equiv \frac{1 - \frac{\cos^{-1}(2(a+b)-1) + \cos^{-1}(2(a+c)-1) + \cos^{-1}(2(a+d)-1)}{2\pi}}{a}$$

$$\frac{\kappa(e)}{\lambda(e)} \geq \min_{a,b,c,d} g(a, b, c, d) \text{ subjected to } a + b + c + d = 1 \text{ and } a, b, c, d \geq 0$$

For the purposes of the analysis, it is helpful to fix the value of a, and then find the optimum choice of b, c, d for that value of a to minimize $g(a, b, c, d)$. Subsequently, one optimizes over the choice of a. In other words, let us define $h_a(b, c, d)$ as

$$h_a(b, c, d) = \cos^{-1}(2(a+b) - 1) + \cos^{-1}(2(a+c) - 1) + \cos^{-1}(2(a+d) - 1)$$

$$\Psi(a) = \max_{b,c,d} h_a(b, c, d) \text{ subjected to } b + c + d = 1 - a \text{ and } b, c, d \geq 0 \text{ where } a > 0 \tag{6}$$

Hence, we now get that

$$\frac{\kappa(e)}{\lambda(e)} \geq \inf_{0 < a \leq 1} \frac{1 - \frac{\Psi(a)}{2\pi}}{a} \tag{7}$$

Thus, we now focus on finding $\Psi(a)$ for every $a \in (0, 1]$. In order to find out $\Psi(a)$, we find out the local minima by evaluating the partial derivatives of the function $h_a(b, c, d)$ and also investigate the value of $h_a(b, c, d)$ at the boundaries of the domain.

6.2 Minimum value of $h_a(b, c, d)$ at the boundary of the domain : The next claim gets the maximum of $h_a(b, c, d)$ when b, c, d lie on the boundary of the domain defined in Equation 6.

Claim 24. *The maximum of $h_a(b, c, d)$ when b, c and d lie on the boundary of the domain defined in (6) is* $\cos^{-1}(2a - 1) + 2\cos^{-1}(a)$.

Proof. Note that because $b, c, d \geq 0$ and $b + c + d = 1 - a$, the boundary of the domain is defined by at least one of the variables being 0. Without loss of generality, we assume $b = 0$. In that case,

$$h_a(0, c, d) = \cos^{-1}(2a - 1) + \cos^{-1}(2(a + c) - 1) + \cos^{-1}(2(a + d) - 1)$$

with $c + d = 1 - a$ and $c, d \geq 0$. Doing the substitution $d = 1 - a - c$, we get

$$h_a(0, c, d) = \cos^{-1}(2a - 1) + \cos^{-1}(2(a + c) - 1) + \cos^{-1}(1 - 2c) \tag{8}$$

where $0 \leq c \leq 1 - a$. Now, note that since a is fixed, hence $h_a(0, c, d)$ is solely a function of c. Hence, to find out the maximum of $h_a(0, c, d)$, we evaluate it at the end points of the domain i.e. at $c = 0$, $c = 1 - a$ and at its critical points.

- If $c = 0$, then $d = 1 - a$. Hence, at this point, $h_a(b, c, d) = h_a(0, 0, 1 - a) = \cos^{-1}(2a - 1) + \cos^{-1}(2a - 1) + \cos^{-1}(1) = 2\cos^{-1}(2a - 1)$.

- If $c = 1 - a$, then $d = 0$. Hence, at this point, $h_a(b, c, d) = h_a(0, 1 - a, 0) = \cos^{-1}(2a - 1) + \cos^{-1}(1) + \cos^{-1}(2a - 1) = 2\cos^{-1}(2a - 1)$.

Having evaluated $h_a(0, c, d)$ at the boundary points, we now find out the critical points of this function. Differentiating the expression in (8), we get

$$\frac{\partial h_a(0, c, d)}{\partial c} = \frac{-2}{\sqrt{1 - (2(a + c) - 1)^2}} + \frac{2}{\sqrt{1 - (1 - 2c)^2}} = 0$$

This implies that

$$1 - (2(a + c) - 1)^2 = 1 - (1 - 2c)^2$$
$$\Rightarrow (2(a + c) - 1) = \pm(1 - 2c)$$

This means that either $a = 0$ or $a + 2c = 1$. Since $a > 0$, we can neglect the first condition. Thus, the only condition we need to consider is $a + 2c = 1$. Because $a + c + d = 1$, this means that $c = d = (1 - a)/2$. Thus, $h_a(0, c, d) = \cos^{-1}(2a - 1) + 2\cos^{-1}(a)$. Thus, we get that

$$\max_{c,d} h_a(0, c, d) = \max\{\cos^{-1}(2a - 1) + 2\cos^{-1}(a), 2\cos^{-1}(2a - 1)\} = \cos^{-1}(2a - 1) + 2\cos^{-1}(a) \tag{9}$$

The last equality uses Fact 26. □

6.3 Evaluation of $h_a(b, c, d)$ at the critical points : The next claim evaluates the maximum of $h_a(b, c, d)$ at the critical points of the domain.

Claim 25. *The maximum value of $h_a(b, c, d)$ at the critical points inside the domain defined in (6) is given by*

$$\max h_a(b, c, d) = \begin{cases} \pi + \cos^{-1}(4a - 1) & \text{if } 0 \leq a \leq 1/4, \\ 3\cos^{-1}((4a - 1)/3) & \text{if } 1/4 < a \leq 1. \end{cases}$$

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Proof. Note that $b + c + d = 1 - a$. Thus, we get

$$h_a(b, c, d) = \cos^{-1}(1 - 2c - 2d) + \cos^{-1}(2(a + c) - 1) + \cos^{-1}(2(a + d) - 1)$$

Now, that since a is fixed, $h_a(b, c, d)$ is a function of c and d alone. At the critical point,

$$\frac{\partial h_a(b, c, d)}{\partial c} = \frac{2}{\sqrt{1 - (1 - 2c - 2d)^2}} - \frac{2}{\sqrt{1 - (1 - 2a - 2c)^2}} = 0$$

$$\frac{\partial h_a(b, c, d)}{\partial d} = \frac{2}{\sqrt{1 - (1 - 2c - 2d)^2}} - \frac{2}{\sqrt{1 - (1 - 2a - 2d)^2}} = 0$$

Thus, at the critical point,

$$(1 - 2c - 2d)^2 = (1 - 2a - 2c)^2 = (1 - 2a - 2d)^2$$

$$\Rightarrow \pm(1 - 2c - 2d) = \pm(1 - 2a - 2c) = \pm(1 - 2a - 2d)$$

We now solve for c, d for the various possibilities listed above.

- $1 - 2c - 2d = 1 - 2a - 2c = 1 - 2a - 2d$. In this case, we get $a = c = d$ and hence $b = 1 - 3a$. Since $b \geq 0$, this possibility occurs only when $0 \leq a \leq (1/3)$. If this indeed holds,

$$h_a(b, c, d) = \cos^{-1}(1 - 4a) + \cos^{-1}(4a - 1) + \cos^{-1}(4a - 1) = \pi + \cos^{-1}(4a - 1)$$

- $1 - 2c - 2d = -(1 - 2a - 2c) = 1 - 2a - 2d$. In this case, we get $a = c = b$ and $d = 1 - 3a$. Again as $d \geq 0$, this possibility occurs only when $0 \leq a \leq (1/3)$. As before,

$$h_a(b, c, d) = \cos^{-1}(1 - 4a) + \cos^{-1}(4a - 1) + \cos^{-1}(4a - 1) = \pi + \cos^{-1}(4a - 1)$$

- $1 - 2c - 2d = 1 - 2a - 2c = -(1 - 2a - 2d)$. This goes exactly the same way as the previous case. Here again, we have

$$h_a(b, c, d) = \cos^{-1}(1 - 4a) + \cos^{-1}(4a - 1) + \cos^{-1}(4a - 1) = \pi + \cos^{-1}(4a - 1)$$

- $-(1 - 2c - 2d) = 1 - 2a - 2c = 1 - 2a - 2d$. In this case, $b = c = d = (1 - a)/3$. Now, we get

$$h_a(b, c, d) = \cos^{-1}((4a - 1)/3) + \cos^{-1}((4a - 1)/3) + \cos^{-1}((4a - 1)/3) = 3\cos^{-1}((4a - 1)/3)$$

That means that for the critical points, $\max h_a(b, c, d) = \max\{\pi + \cos^{-1}(4a - 1), 3\cos^{-1}((4a - 1)/3)\}$ if $0 < a \leq 1/3$. On the other hand, if $a > 1/3$, then $\max h_a(b, c, d) = 3\cos^{-1}((4a - 1)/3)$.

However, using Fact 27, the above simplifies to saying that at the critical points,

$$\max h_a(b, c, d) = \begin{cases} \pi + \cos^{-1}(4a - 1) & \text{if } 0 \leq a \leq 1/4, \\ 3\cos^{-1}((4a - 1)/3) & \text{if } 1/4 < a \leq 1. \end{cases}$$

$\square$

Combining Claims 24 and 25 along with (7), we get that

$$\frac{\kappa(e)}{\lambda(e)} \geq \min\left\{ \inf_{a\in(0,1/4]} \frac{1 - \frac{\pi+\cos^{-1}(4a-1)}{2\pi}}{a}, \inf_{a\in(1/4,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a}, \inf_{a\in(0,1]} \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a} \right\}$$

By Fact 28, the first quantity inside the minimum simplifies to 1 as follows :

$$\inf_{a\in(0,1/4]} \frac{1 - \frac{\pi+\cos^{-1}(4a-1)}{2\pi}}{a} = \frac{1 - \frac{\pi+\cos^{-1}(4*(1/4)-1)}{2\pi}}{(1/4)} = 1 \tag{10}$$

At this point, we are left with the task of finding the following quantities :

$$\inf_{a\in(0,1]} \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a} \qquad \inf_{a\in(1/4,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a}$$

Thus, we are now left with the task of finding the infimum of two single-variable functions and then taking the minima of these two quantities. We do this computation by evaluating these two functions at sufficiently many points and then taking the infimum of these. For a mathematical justification, see Appendix B.

$$\inf_{a\in(0,1]} \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a} = [0.802225, 0.804225] \tag{11}$$

$$\inf_{a\in(1/4,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a} = [0.795070, 0.796070] \tag{12}$$

Further, the value of a achieving the infimum in (12) is $a = 0.700296 \pm 0.000001$. Hence, we have that

$$\frac{\kappa(e)}{\lambda(e)} \geq \inf_{a\in(1/4,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a} = \inf_{a\in(0,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a}$$

The second equality (i.e. making the domain $(0,1]$ instead of $(1/4,1]$) follows because Fact 27 and (10) can be combined as:

$$\forall 0 < a \leq 1/4 \quad \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a} \geq \frac{1 - \frac{\pi+\cos^{-1}(4a-1)}{2\pi}}{a} \geq 1$$

Put $\delta = 4(1-a)/3$. Then, we get that

$$\inf_{a\in(0,1]} \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a} = \inf_{0\leq\delta<4/3} \frac{1 - \frac{3\cos^{-1}(1-\delta)}{2\pi}}{1 - \frac{3\delta}{4}} = \inf_{0\leq\delta\leq1} \frac{1 - \frac{3\cos^{-1}(1-\delta)}{2\pi}}{1 - \frac{3\delta}{4}}$$

Here the last equality is true because we have earlier observed that the infimum of the expression in (12) is obtained when $a \approx 0.700$. This means the corresponding value of $\delta \approx 0.400 < 1$. Thus, making the domain of δ to be $(0,1]$ instead of $(0,4/3]$ does not affect the value of the infimum. This also conclude the proof of the theorem.

$\square$

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7 Conclusion

Our results illustrate the importance of Gaussian partition results in establishing exact optimal UGC hardness and rounding schemes. Not only did we show that a new Gaussian partition result allows to obtain exact UGC hardness of MAX-3-EQUAL, we also showed how the trivial Gaussian partition gives near optimal hardness for MAX-k-CSPs.

There are many interesting open problem that emerge from our work and previous work. Perhaps the most natural open problem is regarding the hardness of MAX-k-EQUAL. In particular, is it true that the generic SDP from [Rag08] followed by the random gaussian / hyperplane rounding is optimal for MAX-k-EQUAL (assuming the Unique Games Conjecture)?

A more general challenge it to obtain further optimal Gaussian partition results. In particular we recall the Standard Simplex Conjecture from [IM12] which says that if (X, Y) are jointly normal random variables in $\mathbb{R}^n$ such that $X, Y \sim \mathcal{N}(0, 1)$ and $\mathrm{Cov}(X, Y) = \rho I_n$ where $\rho > 0$, then a partitioning of the gaussian space into k parts of equal measure such that (X, Y) fall in the same partition is maximized when the partition corresponds to a k-simplex centered at the origin. Proving this, will have consequences for hardness of MAX-k-CUT.

Acknowledgements

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A   Useful Trigonometric facts

Fact 26. *For every* $0 \le a \le 1$, $2\cos^{-1}(a) - \cos^{-1}(2a - 1) \ge 0$.

Proof. Note that

$$\cos(2\cos^{-1}(a)) = 2a^2 - 1 \le 2a - 1 = \cos(\cos^{-1}(2a - 1))$$

Now recall that if $0 \le \theta, \phi \le \pi$, then $\cos\theta \le \cos\phi$ if and only if $\theta \ge \phi$. Clearly, as $a \ge 0$, $0 \le 2\cos^{-1}(a) \le \pi$. And also, $0 \le \cos^{-1}(2a - 1) \le \pi$. This concludes the proof.   □

Fact 27. *Let* $-1 \le x \le 1$. *Then, if* $x \ge 0$, *then* $\pi + \cos^{-1}(x) \le 3\cos^{-1}(x/3)$. *Else if*, $x \le 0$, *then* $\pi + \cos^{-1}(x) \ge 3\cos^{-1}(x/3)$.

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Proof. Consider $f(x) = 3\cos^{-1}(x/3) - \pi - \cos^{-1}(x)$. Then, note that within the domain $(-1, 1)$, the function is differentiable and hence

$$\frac{df(x)}{dx} = \frac{-1}{\sqrt{1 - x^2/9}} + \frac{1}{\sqrt{1 - x^2}}$$

This means that $df(x)/dx = 0$ only at $x = 0$. Also, note that $f(0) = 0$. Also $f(1) > 0$ and $f(-1) < 0$. This proves the conclusion of the claim. $\qquad\square$

Fact 28. *Let $f : (0, 1/4] \to \mathbb{R}$ be defined as*

$$f(x) = \frac{1 - \frac{\pi + \cos^{-1}(4x - 1)}{2\pi}}{x}$$

Then, $f(x)$ is decreasing in the interval $(0, 1/4]$.

Proof. We do a change of variables. Put $\cos\theta = 4x - 1$. In other words, $\pi/2 \leq \theta \leq \pi$ and we need to show that $g(\theta)$ is increasing in θ in the said interval.

$$g(\theta) = 4 \cdot \frac{\frac{1}{2} - \frac{\theta}{2\pi}}{1 + \cos\theta}$$

We now evaluate $g'(\theta)$ and show that it is an increasing function.

$$g'(\theta) = 4 \cdot \frac{(1 + \cos\theta) \cdot \frac{-1}{2\pi} + \sin\theta \cdot \left(\frac{1}{2} - \frac{\theta}{2\pi}\right)}{(1 + \cos\theta)^2}$$

That means we need to show that

$$(1 + \cos\theta) \cdot \frac{-1}{2\pi} + \sin\theta \cdot \left(\frac{1}{2} - \frac{\theta}{2\pi}\right) \geq 0$$

which is equivalent to showing

$$(\pi - \theta)\sin(\theta/2) \geq \cos(\theta/2) \equiv \pi - \theta - \cot(\theta/2) \geq 0$$

So, we finally need to show that $h(\theta) = \pi - \theta - \cot(\theta/2)$ is non-negative in the interval $\theta \in [\pi/2, \pi)$. But $h'(\theta) = -\cot^2(\theta) < 0$. This means that $h(\theta) \geq h(\pi) = 0$ proving our claim. $\qquad\square$

Fact 29. *For $0 \leq x \leq 1$, $\cos^{-1}(x) \leq \pi/2 - x$*

Proof.

$$\sin x \leq x \quad \Rightarrow \quad \cos(\pi/2 - x) \leq x \quad \Rightarrow \quad \pi/2 - x \geq \cos(x)$$

$\qquad\square$

Fact 30. *For $0 \leq x \leq 1$, $\cos^{-1}(x - 1) \leq \pi - \sqrt{x}$*

Proof. Let $g(x) = \cos(\sqrt{x}) - 1 + x$. Observe that $g(0) = 0$. Also,

$$g'(x) = -\frac{\sin\sqrt{x}}{2\sqrt{x}} + 1 > 0$$

This implies that $g(x) \geq 0$ for all $0 \leq x \leq 1$. This implies

$$\cos(\sqrt{x}) - 1 + x \geq 0 \quad \Rightarrow \quad x - 1 \geq -\cos(\sqrt{x}) = \cos(\pi - \sqrt{x})$$
$$\Rightarrow \quad \cos^{-1}(x - 1) \leq \pi - \sqrt{x}$$

$\qquad\square$

Fact 31. *For $0.9 \leq x \leq 1$, $\cos^{-1}(x) \leq 3\sqrt{1-x}$.*

Proof. Put $x = 1 - \epsilon$ and then, we need to prove that for $0 \leq \epsilon \leq 0.1$, $\cos^{-1}(1 - \epsilon) \leq 3\sqrt{\epsilon}$. Now consider $g(\epsilon) = 3\sqrt{\epsilon} - \cos^{-1}(1 - \epsilon)$. Clearly, $g(0) = 0$. Next, we note that

$$g'(\epsilon) = \frac{3}{2\sqrt{\epsilon}} - \frac{1}{\sqrt{1 - (1 - \epsilon)^2}} = \frac{3}{2\sqrt{\epsilon}} - \frac{1}{\sqrt{2\epsilon - 4\epsilon^2}}$$

As $g'(\epsilon) \geq 0$ if $0 \leq \epsilon \leq 0.1$, so is $g(\epsilon)$ thus proving the proposition. $\qquad\square$

Fact 32. *For $0.9 \leq x \leq 1$, $\cos^{-1}(2x - 1) \leq 5\sqrt{1-x}$.*

Proof. Note that putting $x = 1 - \epsilon$, this is equivalent to proving that for $0 \leq \epsilon \leq 0.1$, $\cos^{-1}(1 - 2\epsilon) \leq 5\sqrt{\epsilon}$. To prove this, consider the function $g(\epsilon) = 5\sqrt{\epsilon} - \cos^{-1}(1 - 2\epsilon)$. Clearly, $g(0) = 0$. Also,

$$g'(\epsilon) = \frac{5}{2\sqrt{\epsilon}} - \frac{2}{\sqrt{1 - (1 - 2\epsilon)^2}} = \frac{5}{2\sqrt{\epsilon}} - \frac{1}{\sqrt{\epsilon - \epsilon^2}}$$

As $g'(\epsilon) > 0$ for $\epsilon \leq 0.1$, so is $g(\epsilon)$, thus proving the proposition. $\qquad\square$

B   Justification for numerically finding the minima

In Section 6, we numerically evaluate the minimum of two single variable functions using the software "Mathematica". We now give a detailed explanation of how we find the minima of these functions to the desired error and the mathematical soundness of this computer-assisted procedure.

B.1   Infimum of $h_1(a)$ Given a function $h_1 : (0, 1] \to \mathbb{R}$ from Section 6 (which is defined as)

$$h_1(a) = \frac{1 - \frac{2\cos^{-1}(a) + \cos^{-1}(2a - 1)}{2\pi}}{a}$$

to find $\inf_{a \in (0, 1]} h_1(a)$, we do the following :

- Show that for the interval $A_1 = (0, x_s]$ and $A_2 = [x_t, 1]$ (where $x_s = 0.179$ and $x_t = 0.99$), $\inf_{x \in A_1} h_1(x) \geq 0.85$ and $\inf_{x \in A_2} h_1(x) \geq 0.83$

- Show that for the interval $A_3 = (x_s, x_t)$, and $x \in A_3$, $|h_1'(x)| \leq \Delta$ where $\Delta = 500$

- Divide the interval A_3 into Δ/η (with $\eta = 10^{-4}$) intervals of equal length and evaluate h_1 at each of these points where $h_1(a)$ is evaluated at each point with an error of $\epsilon = 10^{-6}$. Subsequently, take the minimum of all these numbers.

It is clear that the above procedure returns the infimum of h_1 in the interval $(0, 1]$ to within error $\epsilon + \eta \leq 10^{-3}$. Following this procedure, $\inf_{a \in (0, 1]} h_1(a)$ was obtained to be 0.803225. Since, we note that the error can be at most 10^{-3}, hence $\inf_{a \in (0, 1]} h_1(a) \in [0.802225, 0.804225]$.

We now give proofs for the first and the second item in the above procedure.

Proposition 33. *Let $h_1 : [0, 1] \to \mathbb{R}$ be defined as*

$$h_1(a) = \frac{1 - \frac{2\cos^{-1}(a) + \cos^{-1}(2a - 1)}{2\pi}}{a}$$

Then, for $0 \leq a \leq 0.179$, $h(a) \geq 0.85$.

Proof. Using Fact 29 and Fact 30, we have

$$h_1(a) = \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a} \geq \frac{2a + \sqrt{2a}}{2\pi a} = \frac{1}{\pi} + \frac{1}{\pi\sqrt{2a}}$$

Plugging in the values, this implies that as long as $a \leq 0.179$, $h_1(a) \geq 0.85$. $\qquad\square$

Proposition 34. *Let $h_1 : [0,1] \to \mathbb{R}$ be defined as*

$$h_1(a) = \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a}$$

Then, for $0.99 \leq a \leq 1$, $h(a) \geq 0.83$.

Proof. Using Fact 31 and Fact 32, we have

$$h_1(a) = \frac{1 - \frac{2\cos^{-1}(a)+\cos^{-1}(2a-1)}{2\pi}}{a} \geq \frac{1 - \frac{6\sqrt{1-a}+5\sqrt{1-a}}{2\pi}}{a}$$

Plugging in the values, this implies that as long as $0.99 \leq a \leq 1$, $h_1(a) \geq 0.83$. $\qquad\square$

Proposition 33 and Proposition 34 imply the proof of the first item. The next proposition implies the correctness of the third item.

Proposition 35. *For every $a \in [0.179, 0.99]$, $|h_1'(a)| \leq 500$.*

Proof.

$$h_1'(a) = \frac{\frac{a}{\pi\sqrt{1-a^2}} + \frac{a}{\pi\sqrt{1-(1-2a)^2}} - 1 + \frac{\cos^{-1}(a)}{\pi} + \frac{\cos^{-1}(2a-1)}{2\pi}}{a^2}$$

This implies that

$$|h_1'(a)| \leq \frac{\frac{a}{\pi\sqrt{1-a^2}} + \frac{a}{2\pi\sqrt{a-a^2}} + 3}{a^2} \leq \frac{3}{a^2} + \frac{\frac{1}{\pi\sqrt{1-a^2}} + \frac{1}{2\pi\sqrt{a-a^2}}}{a}$$

To bound the value of $|h_1'(a)|$, we consider the two cases : when $0.179 \leq a \leq 0.5$ and when $0.99 \geq a > 0.5$. Splitting into these two cases, it is easy to show

$$|h_1'(a)| \leq 500$$

$\qquad\square$

B.2   Infimum of $h_2(a)$ Recall that we need to find the following quantity :

$$\inf_{a \in (1/4, 1]} h_2(a) \quad \text{where} \quad h_2(a) = \frac{1 - \frac{3\cos^{-1}((4a-1)/3)}{2\pi}}{a}$$

We do the following change of variables : We put $(4a - 1)/3 = \cos x$. Then, the problem becomes finding the quantity

$$\inf_{x \in [0, \pi/2)} g(x) \quad \text{where} \quad g(x) = 4 \cdot \frac{1 - \frac{3x}{2\pi}}{1 + 3\cos x}$$

To find $\inf_{x \in [0, \pi/2)} g(x)$, we do the following :

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- Show that for $x \in [0, \pi/2)$, $|g'(x)| \leq \Delta$ where $\Delta = 50$

- Divide the interval $[0, \pi/2)$ into Δ/η (with $\eta = 10^{-4}$) intervals of equal length and evaluate $g(x)$ at each of these points where $g(x)$ is evaluated at each point with an error of $\epsilon = 10^{-6}$. Subsequently, take the minimum of all these numbers.

It is clear that the above procedure returns the infimum of h_2 in the interval $(0, 1]$ to within error $\epsilon + \eta \leq 10^{-3}$. Following this procedure, $\inf_{a \in (0,1]} h_2(a)$ was obtained to be 0.796070. Since the error is bounded by 10^{-3}, we know $\inf_{a \in (0,1]} h_2(a) \in [0.795070, 0.796070]$ We now give proof for the first item in the above procedure.

Proposition 36. *Let $g : [0, \pi/2) \to \mathbb{R}$ be defined as above. Then, for $x \in [0, \pi/2)$, $|g'(x)| \leq 50$*

Proof.

$$g'(x) = 12 \cdot \frac{\sin x - \frac{3x \sin x}{2\pi} - \frac{1}{2\pi} - \frac{3\cos x}{2\pi}}{(1 + 3\cos x)^2}$$

It is now trivial to see that the absolute value of $g'(x)$ is bounded by 50 at all points in $[0, \pi/2)$. $\qquad \square$