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Clique Problem, Cutting Plane Proofs, and Communication Complexity

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Abstract

Motivated by its relation to the length of cutting plane proofs for the Maximum Biclique problem, here we consider the following communication game on a given graph G, known to both players. Let K be the maximal number of vertices in a complete bipartite subgraph of G, which is not necessarily an induced subgraph if G is not bipartite. Alice gets a set a of vertices, and Bob gets a disjoint set b of vertices such that $|a|+|b|>K$. The goal is to find a nonedge of G between a and b. We show that $O(\log n)$ bits of communication are enough for every n-vertex graph.

1. Introduction

Let $G=(V,E)$ be a graph with vertex set V and edge set E. A *clique* in G is a set $a \subseteq V$ of vertices such that $\{u,v\} \in E$ for all $u \neq v \in a$. A *biclique* in G is a pair $\{a,b\}$ of disjoint subsets of vertices such that $\{u,v\} \in E$ for all $u \in a$ and $v \in b$. Thus, the edges $\{u,v\}$ form a complete bipartite subgraph of G (which is not necessarily an *induced* subgraph if G is not bipartite). The *size* of a clique (or biclique) is the number of its vertices. The maximum size of a clique in G is denoted by $\omega(G)$, and the maximum size of a biclique in G is denoted by $\omega_{\mathrm{b}}(G)$. Note that $\omega(G) \leq \omega_{\mathrm{b}}(G)$ holds for every graph G: every clique of size k contains a biclique (in fact, many bicliques) of size k. A *nonedge* in a graph is a pair of its nonadjacent vertices.

Given an arbitrary (not necessarily bipartite) graph $G=(V,E)$, we are interested in the communication complexity of the following game between two players, Alice and Bob.

> **Biclique Game on** $G=(V,E)$:
> Alice gets $a \subseteq V$, Bob gets $b \subseteq V$ such that $a \cap b = \emptyset$ and $|a|+|b| > \omega_{\mathrm{b}}(G)$. The goal is to find a nonedge of G lying between a and b. This nonedge must be known to both players.

If the underlying graph G is bipartite with a bipartition $V=V_1 \cup V_2$, then we additionally require that $a \subseteq V_1$ and $b \subseteq V_2$. Note that the promise $|a|+|b| > \omega_{\mathrm{b}}(G)$ ensures that there must be at least one nonedge between a and b. The communication complexity, $\mathrm{c}_{\mathrm{b}}(G)$, of this game is the minimum, over all (deterministic) communication protocols for G, of the number of bits communicated on a worst-case input (a,b). We stress that the graph G in this game is *fixed* and is known to both players. The players are not adversaries—they help and trust each other. The difficulty, however, is that Alice cannot see Bob's set b, and Bob cannot see Alice's set a.

To avoid trivialities, we will assume (without mentioning this) that our graphs have no complete stars, that is, vertices adjacent to all remaining vertices—such vertices can be ignored.

> **Clique Game on** $G=(V,E)$:
> Alice gets a set $a \subseteq V$, Bob gets a set $b \subseteq V$ such that $a \cap b = \emptyset$ and $|a|+|b| > \omega(G)$. The goal is to find a nonedge of G lying within $a \cup b$. Again, this nonedge must be known to both players.

Let $\mathrm{c}(G)$ denote the communication complexity of the clique game on G.

Remark 1. The main difference from the biclique game is that now we have a weaker promise $|a|+|b| > \omega(G)$. Note also that the only nontrivial inputs are pairs (a,b), where both a and b are cliques: the found nonedge must then lie *between* a and b (as in the biclique game). Indeed, if one of the sets, say, a is not a clique, then it contains a nonedge. Alice can then send both endpoints of this nonedge to Bob using at most $2\lceil \log_2 n \rceil$ bits, and the game is over.

Our motivation to consider clique and biclique games cames from their connection to the length of so-called "tree like" cutting plane proofs for the Maximum Clique problem on a fixed graph $G=(V,E)$. Cliques in G are exactly the 0-1 solutions of the system $\mathrm{Cl}(G)$ consisting of linear inequalities $x_u + x_v \leq 1$ for all nonedges $\{u,v\} \notin E$, and $x_v \geq 0$ for all vertices $v \in V$. If the graph is bipartite with bipartition $V=V_1 \cup V_2$, then we only have inequalities $x_u + x_v \leq 1$ for all nonedges $\{u,v\}$ with $u \in V_1$ and $v \in V_2$. In the "find a hurt axiom" game, given a 0-1 assignment α to the variables such that $\sum_{v \in V} \alpha_v \geq \omega(G)+1$, we (the adversary) first split the bits of α between Alice and Bob, and their goal is to find a nonedge $\{u,v\}$ such that $\alpha_u = \alpha_v = 1$. In the bipartite case, the promise is $\sum_{v \in V} \alpha_v \geq \omega_{\mathrm{b}}(G)+1$.

Results of [8] imply that, if a clique (or biclique) game requires K bits of communication, then every tree-like cutting planes proof of the 0-1 unsatisfiability of the system $\mathrm{Cl}(G)$ augmented by the inequality $\sum_{v \in V} x_v \geq \omega(G)+1$ (or $\sum_{v \in V} x_v \geq \omega_{\mathrm{b}}(G)+1$) must either use super-polynomially large coefficients, or must produce at least $2^{\Omega(K/\log n)}$ inequalities; see [9,

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Section 19.3 and Research Problem 19.12] for details. It was therefore a hope that n-vertex graphs G with $\mathrm{c_b}(G) \gg \log^2 n$ or at least $\mathrm{c}(G) \gg \log^2 n$ exist.

Our main result (Theorem 1 below) destroys the first hope: for every (not necessarily bipartite) n-vertex graph, $\mathrm{c_b}(G) = O(\log n)$ bits of communication are enough.

Since the found nonedge must be known to both players, at least $\log n$ bits of communication are necessary for any nontrivial graph on n-vertices. However, if the graph is complicated enough, then (intuitively) this trivial number of bits should be not sufficient. If, say, there are many nonedges leaving the sets a and b, but only one of them lies between a and b, how should the players quickly localize this unique nonedge?

It turns out that, somewhat surprisingly, a logarithmic number of bits is sufficient for *any* graph! That is, up to constant factors, the communication complexity of the biclique game does not depend on the structure of the underlying graph.

Theorem 1. *For every n-vertex graph G, we have* $\mathrm{c_b}(G) \leq 7.3\log n + O(1)$.

The situation with the clique game is more complicated. Here we are only able to show that $O(\log n)$ bits are enough for many graphs. Interestingly, the clique game is related to the monotone complexity of the following decision problem.

The *induced k-clique function* of an n-vertex graph G is a monotone boolean function of n variables which, given a subset of vertices, outputs 1 if and only if some k of these vertices form a clique in G. Thus, this function is just a version of the well-known **NP**-complete Clique function restricted to only *spanning* subgraphs of one *fixed* graph G. Let $\mathrm{Depth}(G)$ denote the maximum, over all integers $1 \leq k \leq n$, of the minimum depth of a monotone circuit with fanin-2 AND and OR gates computing the induced k-clique function of G.

Theorem 2. *For every n-vertex graph G, we have* $\mathrm{c}(G) \leq \mathrm{Depth}(G) + 2\log n + O(1)$.

The measure $\mathrm{Depth}(G)$ is related to the number $\kappa(G)$ of *maximal* cliques in G; a clique is *maximal*, if it cannot be extended by adding a new vertex. It can be shown (see Lemma 3 below) that

$$\mathrm{Depth}(G) \leq \log \kappa(G) + 5.3\log n + O(1)\,. \qquad (1)$$

There are many n-vertex graphs $G = (V,E)$ for which $\kappa(G)$ is polynomial in n. In particular, $\kappa(G) \leq n(d/2)^{p-2}$ holds for every K_p-free graph of maximal degree $d \geq 2$ [13]; $\kappa(G) \leq n^p$, where p is the chromatic number of G [11]; $\kappa(G) \leq (|E|/p + 1)^p + |E|$, where p is the maximum number of edges in an induced matching in the complement of G [4, 2]. If $p = O(\log n)$ then Theorem 2 gives $\mathrm{c}(G) = O(\log^2 n)$ for all such graphs, implying that communication complexity arguments will fail for such graphs, even for the Maximum Clique problem (not just for the Maximum Biclique problem).

Still, it remains unknown whether $\mathrm{c}(G) = O(\log^2 n)$ holds for *all* graphs. We can only show that $O(\log n)$ bits are always enough in the following relaxed version of the clique game. This version is no more related to cutting plane proofs, but may be of independent interest.

A *common neighbor* of a subset $b \subseteq V$ of vertices is a vertex $v \notin b$ which is adjacent to all vertices in b.

> **Relaxed Clique Game on** $G = (V,E)$:
> Alice gets a set $a \subseteq V$ on vertices, Bob gets a set $b \subseteq V$ of vertices such that $a \cap b = \emptyset$ and $|a| + |b| > \omega(G)$. The goal is to find a nonedge of G which lies either within $a \cup b$ or between a and some common neighbor of b.

Theorem 3. *In the relaxed clique game,* $7.3\log n + O(1)$ *bits of communication are enough for every n-vertex graph.*

The rest is devoted to the proofs of these results.

2. The biclique game: proof of Theorem 1

Let $G = (V,F)$ be a graph on $|V| = n$ vertices with edge set F. Inputs to the biclique game on G are pairs (a,b) of disjoint subsets of vertices such that $|a| + |b| > \omega_{\mathrm{b}}(G)$. Hence, there must be at least one nonedge lying between a and b. The goal is to find such a "crossing" nonedge.

To solve this task, let $E := \binom{V}{2} \setminus F$ be the set of all nonedges of G, and take a set $X = \{x_e \colon e \in E\}$ of boolean variables, one for each nonedge. Say that a nonedge e is *incident* with a subset $a \subseteq V$, if $e \cap a \neq \emptyset$. For a subset $a \subseteq V$ of vertices, let $E(a) \subseteq E$ denote the set of all nonedges incident with a. Finally, we associate with every subset $a \subseteq V$ two vectors p_a and q_a in $\{0,1\}^{|E|}$ whose coordinates correspond to nonedges $e \in E$:

- $p_a(e) = 1$ if and only if $e \in E(a)$;
- $q_a(e) = 0$ if and only if $e \in E(a)$.

Thus, p_a is the characteristic vector of $E(a)$, and q_a is the complement of p_a. Given an input (a,b), the goal in the biclique game is to find a position (a nonedge) e such that $p_a(e) = 1$ (e is incident with a) and $q_b(e) = 0$ (e is incident with b). To do this, we will use monotone circuits for threshold functions. Recall that a *threshold-k* function Th_k^n accepts a 0-1 vector of length n if and only if it contains at least k ones. By a *monotone circuit* we will mean a circuit consisting of fanin-2 AND and OR gates; no negated variables are allowed as inputs. The *depth* of a circuit is the length of a longest path from an input to the output gate.

Theorem 4 (Valiant [17]). *Every threshold function* Th_k^n *can be computed by a monotone circuit of depth at most* $5.3\log n + O(1)$.

We will use this result to show that there exist at most n small-depth monotone circuits such that every given pair of vectors (p_a, q_b) is separated by at least one of them. Then we use these circuits to design the desired protocol.

Lemma 1. *For every $1 \leq k \leq n$, there is a monotone circuit $C(X)$ of depth at most $6.3\log n + O(1)$ such that $C(p_a) = 1$ and $C(q_b) = 0$ for all subsets a and b of vertices of size $|a| = k$ and $|b| > \omega_{\mathrm{b}}(G) - k$.*

Proof. Associate with each subset $c \subseteq V$ the monomial

$$M_c(X) := \bigwedge_{e \in E(c)} x_e ,$$

and let $f_k(X)$ be the OR of these monomials over all k-element subsets $c \subseteq V$. Then f_k clearly accepts vector p_a for every k-element subset of vertices a. So, let $b \subseteq V$ be a subset of $|b| > \omega_b(G) - k$ vertices. To show that the function f_k rejects the vector q_b, it is enough to show that every its monomial M_c does this.

Case 1: $c \cap b = \emptyset$. Since $|c| = k$ and $c \cap b = \emptyset$, our assumption $|c| + |b| > |c| + (\omega_b(G) - k) = \omega_b(G)$ implies that there must be a nonedge between c and b, that is, a nonedge e in $E(c) \cap E(b)$. But vector q_b sets all variables x_e with $e \in E(b)$ to 0, implying that $M_c(q_b) = 0$.

Case 2: $c \cap b \neq \emptyset$. Since we assumed that G contains no complete stars, there must be a nonedge e incident to some vertex in $a \cap b$. So, $e \in E(c) \cap E(b)$, and we again obtain that $M_c(q_b) = 0$.

Thus, $f_k(p_a) = 1$ and $f_k(q_b) = 0$ for all disjoint subsets a and b of vertices of size $|a| = k$ and $|b| > \omega_b(G) - k$. It therefore remains to show that the function f_k can be computed by a monotone circuit C of depth at most $6.3 \log n + O(1)$.

The function f_k accepts a set $E' \subseteq E$ of nonedges if and only if $E(c) \subseteq E'$ holds for some subset $c \subseteq V$ of $|c| = k$ nodes, which happens if and only if E' contains at least k of the sets $E(v) = \{e \in E : v \in e\}$ of nonedges incident to vertices v. We can therefore construct a monotone circuit $C(X)$ computing $f_k(X)$ as follows.

The circuit, testing whether $E(v) \subseteq E'$, is just the AND $M_v(X) = \bigwedge_{e \in E(v)} x_e$ of at most n variables. Thus, by taking the threshold-k of the outputs of these ANDs, we obtain an unbounded fanin circuit of depth-2 computing f_k. Each M_v has a monotone fanin-2 circuit of depth at most $\log n + 1$. By Theorem 4, the function Th_k^n has such a circuit of depth at most $5.3 \log n + O(1)$. Thus the depth of the entire circuit is at most $6.3 \log n + O(1)$. □

We can now describe our protocol for the biclique game on the graph $G = (V, F)$. Recall that inputs to this game are pairs (a, b) of disjoint subsets of vertices such that $|a| + |b| > \omega_b(G)$.

Alice first uses at most $\log n + 1$ bits to communicate Bob the size $k = |a| \leq \omega_b(G)$ of her set a; hence $|b| > \omega_b(G) - k$. The players then take a minimal monotone circuit C guaranteed by Lemma 1. Hence, $C(p_a) = 1$ and $C(q_b) = 0$. After that they traverse (as in [10]) the circuit C backwards starting at the output gate by keeping the invariant: $C'(p_a) = 1$ and $C'(q_b) = 0$ for every reached subcircuit C'.

Namely, suppose the output gate of C is an AND gate, that is, we can write $C = C_0 \wedge C_1$. Then Bob sends a bit i corresponding to a function C_i such that $C_i(q_b) = 0$; if both $C_0(q_b)$ and $C_1(q_b)$ output 0, then Bob sends 0. Since $C(p_a) = 1$, we know that $C_i(p_a) = 1$. If $C = C_0 \vee C_1$, then it is Alice who sends a bit i corresponding to a function C_i such that $C_i(p_a) = 1$; again, if both $C_0(p_a)$ and $C_1(p_a)$ output 1, then Alice sends 0. Since $C(q_b) = 0$, we know that $C_i(q_b) = 0$.

Alice and Bob repeat this process until they reach an input of the circuit. Since the circuit is monotone (there are no negated inputs), this input is some variable x_e. Hence, $x_e(p_a) = 1$ and $x_e(q_b) = 0$. By the definition of vectors p_a and q_b (and since $a \cap b = \emptyset$), this means that the nonedge e lies between a and b, as desired.

The number of communicated bits in this last step is at most the depth $6.3 \log n + O(1)$ of the circuit C. Thus, the total number of communicated bits is at most $7.3 \log n + O(1)$. This completes the proof of Theorem 1. □

Remark 2. One could presume that the main reason, why the biclique game has small communication complexity, is just the fact that the biclique problem *is* solvable in polynomial time via, say, the maximum matching algorithm. In the biclique problem, we are given a graph G and a positive integer K; the goal is to decide whether G contains a biclique $a \times b$ of size $|a| + |b| \geq K$. However, it is known [12] that a similar *maximum edge biclique* problem is already **NP**-complete, even for bipartite graphs. In this problem, the goal is to decide whether G contains a biclique $a \times b$ with $|a \times b| \geq K$ edges. If G is a graph, in which every biclique has at most K edges, then the corresponding to this latter problem game is, given two disjoint sets a, b of vertices such that $|a \times b| > K$, to find a nonedge between a and b. It is easy to see that $O(\log n)$ bits of communication are enough also in this game. For this, it is enough just to replace the condition $|b| > \omega_b(G) - k$ in Lemma 1 by the condition $|b| > K/k$. The rest of the proof is the same.

3. The clique game: proof of Theorem 2

Consider the clique game for a given n-vertex graph $G = (V, F)$. Inputs to this game are pairs (a, b) of disjoint subsets of vertices such that $|a| + |b| > \omega(G)$, and the goal is to find a nonedge lying within $a \cup b$. Hence, now the promise is weaker, but also the task is (apparently) easier: it is allowed that the found nonedge lies within a or within b.

Let us first see why we cannot use the same function f_k as in the biclique game. Recall that f_k is the OR of monomials $M_c(X) = \bigwedge_{e \in E(c)} x_e$ over *all* k-element subsets $c \subseteq V$. Now, even if $b \subseteq V \setminus c$ is a clique, the condition $|c| + |b| > \omega(G)$ does not imply that $M_c(q_b) = 0$. If, for example, there are no nonedges lying between c and b, that is, when all nonedges in $c \cup b$ lie within the set c, then $q_b(e) = 1$ for all nonedges $e \in E(c)$, implying that $M_c(q_b) = 1$, that is, the function f_k wrongly accepts the vector q_b. To get rid of this problem, we use more complicated circuits.

Lemma 2. *For every $1 \leq k \leq n$, there is a monotone circuit $C(X)$ of depth at most* $\mathrm{Depth}(G) + \log n$ *such that $C(p_a) = 1$ and $C(q_b) = 0$ for all cliques a and b of size $|a| = k$ and $|b| > \omega(G) - k$.*

Proof. As before, associate with each subset $c \subseteq V$ the monomial $M_c(X) := \bigwedge_{e \in E(c)} x_e$, and let $g_k(X)$ be the OR of such monomials over all k-cliques $c \subseteq V$. That is, we now take the OR only over sets c containing no nonedges. Let $b \subseteq V$ be a clique of size $|b| > \omega(G) - k$. If $c \cap b \neq \emptyset$, then the star-freeness of G implies $E(c) \cap E(b) \neq \emptyset$, and hence, also $M_c(q_b) = 0$. If $c \cap b = \emptyset$, then $|c| + |b| > \omega(G)$ implies that there must be a nonedge in $c \cup b$. But since both c and b are cliques, this nonedge must lie between c and b, that is belong to $E(c) \cap E(b)$, and we again obtain that $M_c(q_b) = 0$. Thus, $g_k(p_a) = 1$ and $g_k(q_b) = 0$ for all cliques a and b of size $|a| = k$ and $|b| > \omega(G) - k$.

To design a monotone circuit of desired depth for the function g_k, recall that g_k accepts a set $E' \subseteq E$ of nonedges if and only if there is a k-clique $c \subseteq V$ such that $M_v(E') = 1$ for all $v \in c$. Thus, applying the induced k-clique function of G to the outputs of the monomials M_v, we obtain a monotone circuit for g_k of depth at most $\mathrm{Depth}(G)$. ☐

We can now describe our protocol for the clique game on a given graph $G = (V, F)$. By Remark 1, we can assume that the inputs are pairs (a, b) of disjoint cliques such that $|a| + |b| > \omega(G)$. The goal is to find a nonedge lying between a and b.

Using at most $\log n + 1$ bits, Alice first communicates Bob the size $k = |a| \leq \omega(G)$ of her clique a; hence $|b| > \omega(G) - k$. The players then take a minimal monotone circuit C guaranteed by Lemma 2. Hence, $C(p_a) = 1$ and $C(q_b) = 0$. By traversing this circuit, the players will find a variable x_e (an input of C) such that $x_e(p_a) = 1$ and $x_e(q_b) = 0$. By the definition of vectors p_a and q_b (and since $a \cap b = \emptyset$), this means that the nonedge e lies between a and b, as desired. ☐

We now prove the inequality (1). Note that Theorem 4 states that $\mathrm{Depth}(K_n) \leq 5.3 \log n + O(1)$. The graph K_n has only one maximal clique—the graph itself. But Valiant's theorem can be easily extended to graphs with a larger number of maximal cliques. Recall that $\kappa(G)$ denotes the number of maximal cliques in G.

Lemma 3. *For every n-vertex graph G,* $\mathrm{Depth}(G) \leq \log \kappa(G) + 5.3 \log n + O(1)$.

Proof. Let $G = ([n], E)$ be a graph, and $\mathrm{Cliq}(x)$ be its induced k-clique function. That is, $\mathrm{Cliq}(x) = 1$ if and only if the set $S_x = \{i \colon x_i = 1\}$ contains a k-clique of G. Since every clique is contained in some maximal clique, we have that $\mathrm{Cliq}(x) = 1$ if and only if $|S_x \cap C| \geq k$ for at least one maximal clique C. Thus, if $\delta_C \in \{0,1\}^n$ is the characteristic vector of C, and if $\delta_C \wedge x$ is a component-wise AND, then $\mathrm{Cliq}(x) = 1$ if and only if $\mathrm{Th}^n_k(\delta_C \wedge x) = 1$ holds for at least one maximal clique C. By taking the OR, over all $\kappa(G)$ maximal cliques C, of monotone circuits computing the threshold functions $\mathrm{Th}^n_k(\delta_C \wedge x)$, and using Theorem 4, we obtain a monotone circuit of depth at most $\log \kappa(G) + 5.3 \log n + O(1)$ computing $\mathrm{Cliq}(x)$. ☐

4. Relaxed clique game: proof of Theorem 3

Let $G = (V, F)$ be a graph on $|V| = n$ vertices. Inputs to the relaxed clique game on G are pairs (a, b) of disjoint subsets of vertices with the same promise $|a| + |b| > \omega(G)$ as in the clique game. The task, however, is easier: the found nonedge must either lie within $a \cup b$ (as in the clique game) or between a and some common neighbor of b. We will argue as before, but will use a modified definition of Bob's vectors q_b.

Namely, say that a nonedge is a *common neighbor* of set $b \subseteq V$, if both its endpoints are common neighbors of b, that is, are connected (by edges of G) to all vertices in b. Now define the vector q'_b by: $q'_b(e) = 0$ if and only if $e \in E(b)$ or e is a common neighbor of b.

Lemma 4. *For every $1 \leq k \leq n$, there is a monotone circuit $C(X)$ of depth at most $6.3 \log n + O(1)$ such that $C(p_a) = 1$ and $C(q'_b) = 0$ for all cliques a and b of size $|a| = k$ and $|b| > \omega(G) - k$.*

Proof. Let $f_k(X)$ be the monotone boolean function defined in the proof of Lemma 1. That is, f_k is the OR of monomials $M_c(X) = \bigwedge_{e \in E(c)} x_e$ over all k-element sets $c \subseteq V$. Let $b \subseteq V$ be a clique of size $|b| > \omega(G) - k$. It is enough to show that every monomial M_c rejects the vector q'_b. This clearly holds if $E(c) \cap E(b) \neq \emptyset$, because q'_b sets to 0 all variables x_e with $e \in E(b)$.

So, assume that $E(c) \cap E(b) = \emptyset$, that is, $c \cap b = \emptyset$ and there are no nonedges between c and b. Since b is a clique, the condition $|c| + |b| > |c| + (\omega(G) - k) = \omega(G)$ implies that both endpoints of some nonedge e must belong to c. But the absence of nonedges between c and b implies that e is common neighbor of b. Hence, again, the vector q'_b sets the variable x_e to 0, and $M_c(q'_b) = 0$ holds.

Thus, $f_k(p_a) = 1$ and $f_k(q'_b) = 0$ holds for all disjoint cliques a and b of size $|a| = k$ and $|b| > \omega(G) - k$. Since, as shown in the proof of Lemma 1, the function f_k can be computed by a monotone circuits of depth at most $6.3 \log n + O(1)$, we are done. ☐

The protocol for the relaxed clique game on a graph G is now the same as for the clique game. As in that game, interesting are only inputs (a, b), where both a and b are cliques. In this case, the players take the circuit guaranteed by Lemma 4, and traverse it until they find a nonedge $e = \{u, v\}$ such that $x_e(p_a) = 1$ and $x_e(q'_b) = 0$. By the definition of vectors p_a and q'_b, this means that one endpoint of e, say, vertex u belongs to the clique a, and the second endpoint v either belongs to b or is a common neighbor of b (because in this latter case the nonedge e must be a common neighbor of b). In both cases, the nonedge e is a legal answer in the relaxed clique game. ☐

Remark 3. Note that if a and b are disjoint cliques such that $|a| + |b| > \omega(G)$, then there *must* be a "crossing" nonedge (between a and b), which would be a legal answer in the clique game. However, the protocol for the relaxed game may output a "wrong" nonedge—a common neighbor of b.

5. Conclusion and open problems

Note that our communication protocol is not explicit because the construction of a small-depth monotone circuits for the majority function in [17] is probabilistic. To get an explicit protocol, one can use the construction of a circuit of depth $K \log n$ for the majority function given in [1]. But the constant K resulting from this construction is huge, it is about 5000.

The main message of Theorem 1 is that communication complexity arguments cannot yield any non-trivial lower bounds on the length of cutting plane proofs for systems corresponding to the Maximum Biclique problem, because $c_b(G) = O(\log n)$ holds for all n-vertex graphs G. However, the case of the Maximum Clique problem remains unclear. Do n-vertex graphs G requiring $c(G) \gg \log^2 n$ bits of communication in the clique game exist? We have only shown that $c(G) = O(\log n)$ holds for a lot of graphs, and that this number of communicated bits is enough for all graphs in the relaxed clique game (which is no more related to cutting plane proofs).

Let us mention that a different type of (adversarial) games, introduced in [14], was recently used in [5] to derive strong lower bounds for tree like *resolution* proofs for the Maximum Clique problem. Is there some analogue of these games in the case of cutting plane proofs?

The clique and biclique games on a given graph G are special cases of a *monotone* Karchmer–Wigderson game [10]: given a pair (A,B) of two intersecting subsets of a fixed n-element set, find an element in their intersection $A \cap B$. (In our case we have $A = E(a)$ and $B = E(b)$.) In the *non-monotone* game, inputs are pairs of distinct sets, and the goal is to find an element in the symmetric difference $A \oplus B := (A \setminus B) \cup (B \setminus A)$. It is usually much easier to find an element in the symmetric difference than in the intersection. Say, if the players know that $|A| \neq |B|$, $O(\log n)$ bits are also enough to find an element in $A \oplus B$ [6]. However, monotone games (with the goal to find an element in the intersection) usually require much more bits of communication. For example, the *monotone* game corresponding to the matching problem requires $\Omega(n)$ bits of communication [15], whereas [6] implies that $O(\log n)$ bits are enough in the *non-monotone* game for this problem. It is therefore interesting that, in the biclique game, a logarithmic number of communicated bits is enough even to find an element in the intersection $A \cap B$, not just in $A \oplus B$.

Finally, it would be interesting to understand the (monotone) complexity of the induced k-clique functions $\mathrm{CLIQUE}[G,k]$, that is, to prove nontrivial lower bounds on $\mathrm{Depth}_k(G)$, the smallest depth of a monotone circuit computing this function for individual graphs G. Recall that $\mathrm{CLIQUE}[G,k]$ accepts a set of vertices if and only if the induced subgraph of G on these vertices contains a k-clique.

The minterms of $\mathrm{CLIQUE}[G,k]$ are k-cliques of G, and maxterms are *k-clique transversals*, that is, minimal sets of vertices intersecting all k-cliques of G. Thus, the result of Karchmer and Wigderson [10] implies that $\mathrm{Depth}_k(G)$ is exactly the communication complexity of the following game for $\mathrm{CLIQUE}[G,k]$: Alice gets a k-clique, Bob a k-clique transversal, and the goal is to find a common vertex. Theorem 2 shows that the communication complexity $c(G)$ of the clique game is at most $\mathrm{Depth}_k(G)$ plus an additive logarithmic factor. Does some reasonable converse (up to an additive $\log^2 n$ factor) of this inequality hold? What is $\mathrm{Depth}_k(G)$ for random graphs G?

In the communication game for the **NP**-complete problem $\mathrm{CLIQUE}(n,k)$, inputs are pairs (A,B) of subsets of edges (not vertices) of K_n such that edges in A form a k-clique, and edges in B form a *k-coclique*, that is, B consists of $k-1$ vertex-disjoint cliques covering all vertices of K_n. The goal is to find an edge in $A \cap B$. It is known that, for particular choices of $k = k(n)$, this game requires $\Omega(\sqrt{k}\log n)$ bits [16, 3], and even $\Omega(n^{1/3})$ bits [7] of communication. Can the arguments of [16, 3, 7] be adopted to the game for $\mathrm{CLIQUE}[G,k]$? The problem in this latter game is with Bob's inputs: how to find a large family of k-clique transversals in G such that only a small fraction of them will contain a fixed set of, say, $\sqrt{n}$ vertices? Actually, it is even not clear whether there exist a sequence $(G_n : n = 1,2,\ldots)$ of n-vertex graphs G_n for which $\mathrm{CLIQUE}[G_n,k]$ is an **NP**-complete problem.

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References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, Sorting in $c\log n$ parallel steps, Combinatorica 3(1) (1983) 1–19.
[2] V. E. Alekseev, An upper bound for the number of maximal independent sets in a graph, Discrete Math. and Appl. 17(4) (2007) 355–359.
[3] N. Alon and R. Boppana, The monotone circuit complexity of boolean functions, Combinatorica 7(1) (1987) 1–22.
[4] E. Balas and C. S. Yu, On graphs with polynomially solvable maximal-weight clique problem, Networks 19(2) (1989) 247–253.
[5] O. Beyersdorff, N. Galesi, and M. Lauria, Parameterized complexity of DPLL search procedures, Springer Lect. Notes in Comput. Sci. 6695 (2011) 19–22.
[6] G. S. Brodal and T. Husfeldt, A communication complexity proof that symmetric functions have logarithmic depth. Technical Report RS-96-1, BRICS, 1996.
[7] M. Goldmann and J. Håstad, A simple lower bound for monotone clique using a communication game, Inf. Process. Lett. 41(4) (1992) 221–226.
[8] R. Impagliazzo, T. Pitassi, and A. Urquhart, Upper and lower bounds for tree-like cutting planes proofs, in: Proc. of 9th Ann. IEEE Symp. on Logics in Computer Science (1994) 220–228.
[9] S. Jukna, Boolean Function Complexity: Advances and Frontiers, Springer 2012.
[10] M. Karchmer and A. Wigderson, Monotone circuits for connectivity require super-logarithmic depth, SIAM J. Discrete Math. 3 (1990) 255–265.
[11] J. W. Moon and L. Moser, On cliques in graphs, Israel J. Math. (1965) 23–28.
[12] R. Peeters, The maximum edge biclique problem is NP-complete, Discrete Appl. Math 131 (2003) 651–654.
[13] E. Prisner, Graphs with few cliques, in: Graph Theory, Combinatorics, and Applications, Proc. of 7th Quadrennial Int. Conf. on the Theory and Applications of Graphs (Y. Alavi, A. Schwenk ed.) John Wiley and Sons, (1995) 945–956.
[14] P. Pudlák and R. Impagliazzo, A lower bound for DLL algorithms for SAT, in: Proc. of 11th Symp. on Discrete Algorithms (2000) 128–136.
[15] R. Raz and A. Wigderson, Monotone circuits for matching require linear depth, J. ACM 39(3) (1992), 736–744.
[16] A. A. Razborov, Lower bounds for the monotone complexity of some boolean functions, Soviet Math. Dokl. 31 (1985) 354–357.
[17] L. G. Valiant, Short monotone formulae for the majority function, J. Algorithms 5 (1984) 363–366.

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