



# Attribute-Efficient Learning and Weight-Degree Tradeoffs for Polynomial Threshold Functions

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May 29, 2012

## Abstract

We study the challenging problem of learning decision lists attribute-efficiently, giving both positive and negative results.

Our main positive result is a new tradeoff between the running time and mistake bound for learning length- $k$  decision lists over  $n$  Boolean variables. When the allowed running time is relatively high, our new mistake bound improves significantly on the mistake bound of the best previous algorithm of Klivans and Servedio [8].

Our main negative result is a new lower bound on the *weight* of any degree- $d$  polynomial threshold function (PTF) that computes a particular decision list over  $k$  variables (the “ODD-MAX-BIT” function). The main result of Beigel [1] is a weight lower bound of  $2^{\Omega(k/d^2)}$ , which was shown to be essentially optimal for  $d \leq k^{1/3}$  by Klivans and Servedio. Here we prove a  $2^{\Omega(\sqrt{k/d})}$  lower bound, which improves on Beigel’s lower bound for  $d > k^{1/3}$ . This lower bound establishes strong limitations on the effectiveness of the Klivans and Servedio approach and suggests that it may be difficult to improve on our positive result. The main tool used in our lower bound is a new variant of Markov’s classical inequality which may be of independent interest; it provides a bound on the derivative of a univariate polynomial in terms of both its degree *and* the size of its coefficients.

## 1 Introduction

Learning in the presence of irrelevant information is a central problem both in machine learning theory and in many practical applications. A clean theoretical formulation of this problem is provided by the framework of *attribute-efficient learning* that was introduced by Blum more than twenty years ago [2]. A class  $\mathcal{C}$  of Boolean functions over  $\{0, 1\}^n$  is said to be *attribute-efficiently learnable* if there is an online algorithm that learns any  $k$ -variable function in  $\mathcal{C}$  with a mistake bound that is only  $\text{poly}(k, \log n)$ .<sup>1</sup> While attribute-efficient algorithms are known for simple classes of functions

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<sup>1</sup>Throughout this paper we describe all learning results in terms of mistake bounds for online learning algorithms; standard conversions [12] can be used to translate such statements into sample complexity bounds for PAC-model learning algorithms.

such as  $k$ -variable conjunctions, disjunctions, and linear threshold functions with coefficients that are integers of magnitude at most  $\text{poly}(k)$  [13], researchers have been less successful in developing attribute-efficient algorithms for richer classes of Boolean functions.

A particularly well-studied – and challenging – problem in this area is that of attribute-efficiently learning *decision lists*. This question was first posed by Blum [2] and considered subsequently by many authors [3, 4, 17, 15, 8, 14]. We revisit this problem, presenting both positive and negative results that significantly improve upon prior work. Both our upper and lower bounds rely on a careful study of the achievable tradeoffs between the degree and weight of *polynomial threshold functions* (PTFs) for decision lists. (See Section 2 for the standard definitions of decision lists, polynomial threshold functions, and the ODD-MAX-BIT function that we use throughout the paper.)

**Prior Work on Attribute-Efficiently Learning Decision Lists.** Given the apparent difficulty of obtaining a  $\text{poly}(k, \log n)$ -mistake-bound algorithm that runs in time  $\text{poly}(n)$  for learning length- $k$  decision lists, it is natural to try to obtain some nontrivial tradeoff between running time and mistake bound for this problem. It has long been known that the Winnow algorithm [13] can learn any length- $k$  decision list running in  $\text{poly}(n)$  time per trial but with a mistake bound of  $2^{\Theta(k)} \log n$ . On the other hand, it is also well known that by running Winnow over an “expanded feature space” of all monomials (i.e. monotone conjunctions) of length at most  $k$ , the algorithm can learn any length- $k$  decision list with a mistake bound of  $\text{poly}(k, \log n)$  but running in time  $n^k$  per trial. Is it possible to trade off between these two simple results?

Klivans and Servedio [8] observed that by running the Winnow algorithm over an expanded feature space of all monomials of length at most  $d$ , weight upper bounds for degree- $d$  PTFs that compute decision lists directly yield tradeoffs between running time and mistake bound for learning decision lists.

More precisely, they showed the following:

**Fact 1** *Let  $\mathcal{C}$  be a class of Boolean functions with the property that each  $f \in \mathcal{C}$  has a polynomial threshold function (over  $\{0, 1\}^n$  or  $\{-1, 1\}^n$ ) of degree at most  $d$  and weight at most  $W$ . Then the Expanded-Winnow algorithm runs in  $n^{O(d)}$  time per trial and has mistake bound  $O(W^2 \cdot d \cdot \log n)$  for  $\mathcal{C}$ .*

As their main result, Klivans and Servedio showed that for every  $1 \leq d \leq \sqrt{k}$ , any length- $k$  decision list has a PTF of degree  $d$  and weight  $2^{\tilde{O}(k/d^2+d)}$ . Together with Fact 1, for  $d \leq k^{1/3}$  this yields a nontrivial tradeoff between running time ( $n^{O(d)}$ ) and mistake bound ( $2^{\tilde{O}(k/d^2)} \log n$ ) for learning length- $k$  decision lists over  $\{0, 1\}^n$ . The tradeoff breaks down for larger settings of  $d$ , though, because for  $d > k^{1/3}$  the weight bound of Klivans and Servedio is worse than the bound obtained at  $d = k^{1/3}$ .

In earlier work Beigel [1] had already proved that any degree- $d$  PTF for the ODD-MAX-BIT decision list over  $k$  variables must have weight  $2^{\Omega(k/d^2)}$ . The Klivans and Servedio upper bound shows that this lower bound is essentially optimal for  $d \leq k^{1/3}$ , but again it was not clear what is the right bound for larger values of  $d$ .

Several natural questions present themselves given this state of prior work: for  $d > k^{1/3}$ , is there an  $n^d$ -time algorithm that can learn length- $k$  decision lists with a better mistake bound than  $2^{\tilde{O}(k^{1/3})} \log n$ ? Is there a stronger lower bound than  $2^{\Omega(k/d^2)}$  for the weight of degree- $d$  PTFs that compute length- $k$  decision lists? As the main results of this paper we give affirmative answers to both these questions.

**Our Positive Result for Learning Decision Lists.** For any  $d \geq k^{1/3}$ , we give an algorithm that learns an unknown length- $k$  decision list in time  $n^{O(d)}$  per trial with a mistake bound of  $\tilde{O}(kd^2) \cdot 2^{\tilde{O}(\sqrt{k/d})} \log n$ . For all  $d \geq k^{1/3}$  this improves on the  $2^{\tilde{O}(k/d^2+d)} \log n$  mistake bound of Klivans and Servedio [8], and it also improves on the  $O(d \log(n)) \cdot 2^{O(k/d)}$  mistake bound, which holds for all  $d < k$ , achieved by an intermediate result of that paper [8, Page 593].

Our approach uses a simple extension of Fact 1 to “generalized PTFs” (GPTFs), which are linear combinations of *arbitrary* conjunctions of length at most  $d$  (note that in contrast a “standard” degree- $d$  PTF over  $\{0, 1\}^n$  is a linear combination of *monotone* conjunctions of length at most  $d$ ). In order to use this extension of Fact 1, we prove a new weight upper bound on degree- $d$  GPTFs that compute decision lists; see Section 3.

**Our Lower Bounds for Decision Lists.** In Section 4, we show that for  $d > k^{1/3}$ , any degree- $d$  PTF for the length- $k$  decision list ODD-MAX-BIT $_k$  must have weight  $2^{\Omega(\sqrt{k/d})}$ . (This weight lower bound holds over both domains  $\{0, 1\}^k$  and  $\{-1, 1\}^k$ .) This strictly improves Beigel’s lower bound for  $k^{1/3} < d \leq k$ , and suggests that our positive result may be essentially optimal for  $d$  in this range.

Our lower bound proof augments the approach of Beigel with a new technical ingredient which may be of independent interest, namely a variant of the classical Markov’s inequality for real polynomials. While the original inequality gives a uniform upper bound on the derivative of a real univariate polynomial in terms of its degree, our new variant also takes into account the *size of the coefficients* of the polynomial to obtain a sharper bound when the coefficients are small; see Lemma 1. As described in Sections 3 and 4, a common intuition underlies both the key lower bound ingredient, Lemma 1, and the GPTF construction that gives our positive result.

We also improve Beigel’s  $2^{\Omega(k/d^2)}$  lower bound in another direction, by extending it to a wider space of allowable features. Beigel’s result may be interpreted as saying that any sign-representation of ODD-MAX-BIT $_k$  as an integer combination of *length- $d$  monotone conjunctions* must have total weight  $2^{\Omega(k/d^2)}$ . We show that if  $\mathcal{F}$  is any family of  $2^d$  Boolean functions over  $k$ -bit inputs that is closed under restrictions, then there is some decision list over at most  $k$  variables such that any sign-representation as an integer combination of functions from  $\mathcal{F}$  must have total weight  $2^{\Omega(k/d^2)}$ . This result can be viewed as a refinement of a closely related bound due to Buhrman, Vereshchagin, and de Wolf [6]. See Section 5.

## 2 Preliminaries: Decision lists, PTF weight and degree, and Markov’s inequality

A *decision list*  $L : \{0, 1\}^n \rightarrow \{-1, 1\}$  of length  $k$  over the Boolean variables  $x_1, \dots, x_n$  computes a Boolean function in the following way: given an input assignment, the output bit is determined according to the rule

“if  $\ell_1$  then output  $b_1$  else ... else if  $\ell_k$  then output  $b_k$  else output  $b_{k+1}$ ,”

where each  $\ell_i$  is a literal and each  $b_i$  is either  $-1$  or  $1$ .

A decision list of particular interest to us is the function ODD-MAX-BIT $_n$ :  $\{0, 1\}^n \rightarrow \{-1, 1\}$ . This function outputs 1 on an  $n$ -bit input string  $z \in \{0, 1\}^n$  if and only if the rightmost bit of  $z$  that is set to 1 is in an odd bit position, i.e.  $z$  is of the form  $x10^j$  where the length of  $x$  is even,

In the above definition, decision lists are defined over the domain  $\{0, 1\}^n$ , but we may just as well view these functions as being defined over domain  $\{-1, 1\}^n$ .

**PTFs.** Let  $p(x_1, \dots, x_n)$  be an  $n$ -variable polynomial of total degree  $d$  with integer coefficients whose absolute values sum to  $W$ . Let  $f$  be any function from  $\{0, 1\}^n$  to  $\{-1, 1\}$ . If  $f(x) = \text{sign}(p(x))$  for all  $x \in \{0, 1\}^n$ , then we say that  $p$  is a *polynomial threshold function for  $f$*  (PTF) over  $\{0, 1\}^n$  of degree  $d$  and weight  $W$ . Similarly, for  $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$  we say that  $p$  is a *PTF for  $f$  over  $\{-1, 1\}^n$  of degree  $d$  and weight  $W$*  if  $f(x) = \text{sign}(p(x))$  for all  $x \in \{-1, 1\}^n$ . Finally, we say that a PTF  $p$  has *length  $s$*  if the polynomial  $p$  has at most  $s$  non-zero coefficients.

It is easy to see that a PTF of degree  $d$  and weight  $W$  over  $\{0, 1\}^n$  corresponds to a depth-2 circuit with a majority gate as the output gate and  $W$  AND gates each of fan-in at most  $d$  at the bottom, where we allow negations between each AND gate and the majority gate. A PTF over  $\{-1, 1\}^n$  corresponds to a similar circuit but with  $W$  parity gates of fan-in at most  $d$ , rather than AND gates of fan-in at most  $d$ , at the bottom layer.

**Markov's Inequality.** We will also need the following well-known inequality due to Markov.

**Fact 2 (Markov)** *Let  $P(t)$  be a univariate real polynomial of degree  $k$ . Then for all  $t \in [-1, 1]$  we have*

$$|P'(t)| \leq k^2 \cdot \max_{t \in [-1, 1]} |P(t)|.$$

### 3 An improved tradeoff between runtime and mistake bound for large $d$ .

In this section we present our main positive result, giving an improved mistake bound for learning length- $k$  decision lists in time  $n^d$  when  $k^{1/3} \leq d \leq k$ :

**Corollary 1** *For any  $k^{1/3} \leq d \leq k$ , there is an algorithm that learns length- $k$  decision lists over  $n$  Boolean variables in time  $n^{O(d)}$  with a mistake bound of  $\tilde{O}(kd^2) \cdot 2^{\tilde{O}(\sqrt{k/d})} \cdot \log n$ .*

We first introduce our new approach of learning via GPTFs and state a GPTF analogue of Fact 1. Then, after briefly reviewing the Klivans and Servedio construction, we modify their construction to fit the new GPTF framework and thereby prove the main positive result. We close this subsection by noting that our GPTF approach can be combined with ingredients from Krause and Pudlák [10] to obtain a weight upper bound for honest-to-goodness PTFs over the domain  $\{-1, 1\}^n$  (which is somewhat weaker, though, than the weight upper bound that we establish for GPTFs.)

**Learning using GPTFs.** Let  $\mathcal{C}_d$  denote the class of all conjunctions (not necessarily monotone) of at most  $d$  literals over Boolean variables  $x_1, \dots, x_n$ , where we view conjunctions as outputting either 0 (false) or 1 (true). A *degree- $d$  generalized PTF (GPTF)* for  $f : \{0, 1\}^n \rightarrow \{-1, 1\}$  is an expression of the form

$$p(x_1, \dots, x_n) = \sum_{c \in \mathcal{C}_d} w_c \cdot c(x_1, \dots, x_n),$$

where each coefficient  $w_c$  is an integer, which is such that  $f(x) = \text{sign}(p(x))$  for all  $x \in \{0, 1\}^n$ . The *weight* of  $p$  is  $\sum_{c \in \mathcal{C}_d} |w_c|$ .

What is the point of using GPTFs instead of PTFs over  $\{0, 1\}^n$ ? It is clear that if  $f$  has a GPTF of degree  $d$  and weight  $W$ , then  $f$  also has a PTF of degree  $d$  and weight  $W2^d$  over  $\{0, 1\}^n$ : simply replace each conjunction  $c(x)$  in the GPTF with its interpolating 0/1-valued polynomial (which has integer coefficients whose magnitudes sum to at most  $2^d$ ). Thus for weight bounds  $W \geq 2^d$ , GPTFs cannot have much smaller weight than PTFs over  $\{0, 1\}^n$ ; but if  $2^d \gg W$ , then it is possible that GPTFs may offer a significant savings in weight over ordinary PTFs. We exploit precisely this phenomenon to obtain our main positive result. For  $d \geq n^{1/3}$  we do not know how to prove a stronger weight bound than  $2^{\tilde{O}(n^{1/3})}$  for degree- $d$  PTFs over  $\{0, 1\}^n$  that compute length- $n$  decision lists, but for GPTFs we will show the following (which, together with Fact 3, immediately gives Corollary 1):

**Theorem 1** *Let  $f$  be any length- $n$  decision list. For all  $n^{1/3} \leq d \leq n$ , there is a degree- $d$  GPTF for  $f$  that has weight  $\tilde{O}(\sqrt{nd}) \cdot 2^{\tilde{O}(\sqrt{n/d})}$ .*

The following fact is a straightforward analogue of Fact 1:

**Fact 3** *Let  $\mathcal{F}$  be a class of Boolean functions over  $\{0, 1\}^n$  with the property that each  $f \in \mathcal{F}$  has a degree- $d$  GPTF of weight at most  $W$ . Then there is an online learning algorithm for  $\mathcal{F}$  which runs in  $n^{O(d)}$  time per trial and has mistake bound  $O(W^2 \cdot d \cdot \log n)$ .*

The algorithm simply runs Winnow over the feature space of all (not just monotone) conjunctions of length at most  $d$  over  $x_1, \dots, x_n$ . Fact 3 follows directly from [8, Theorem 1] since there are  $\sum_{i=0}^d \binom{n}{i} 2^i \leq (2n)^d$  such conjunctions.

**Proof outline for Theorem 1.** We first recall the high-level structure of the Klivans and Servedio [8] construction of low-weight PTFs for decision lists. Given a decision list  $L$  over  $n$  variables and a degree bound  $d$ , Klivans and Servedio first break the decision list into  $n/\tilde{O}(d^2)$  “inner lists” of length  $\tilde{O}(d^2)$ , and view  $L$  as consisting of an “outer” decision list over the  $n/\tilde{O}(d^2)$  inner lists. (This is referred to as the “Outer Construction.”) Next, using Chebyshev polynomials, they construct a polynomial of degree  $d$  that gives a very accurate uniform approximation for each  $\tilde{O}(d^2)$ -length inner list; they call this the “Inner Construction.” The final PTF is obtained by composing the inner and outer constructions; it has weight  $2^{\tilde{O}(n/d^2+d)}$ .

As mentioned in the Introduction, the lower bound of Beigel [1] shows that the Klivans and Servedio weight upper bound is essentially optimal for  $d \leq n^{1/3}$ . However, for  $d \geq n^{1/3}$  the weight bound of Klivans and Servedio is essentially  $2^d$ ; roughly speaking this is because the degree- $d$  Chebyshev polynomial used in the inner construction has weight essentially  $2^d$ , which is quite high. In our construction we avoid this  $2^d$  weight factor by exploiting the fact that, with GPTFs at our disposal, we can instead use a *lower*-degree Chebyshev polynomial within the inner construction. Intuitively, this is because with GPTFs we can “use non-monotone ANDs as variables in our polynomial” without paying for it in the weight bound, as each such AND has GPTF weight 1. (This is reminiscent of the intuition that underlies the main technical lemma for our lower bounds, see Section 4.)

**Outer construction.** Fix any decision list  $f$ , and recall that  $f$  can be written as “if  $\ell_1$  then output  $b_1$  else  $\dots$  else if  $\ell_n$  then output  $b_n$  else output  $b_{n+1}$ ,” where each  $\ell_i$  is a Boolean literal and each  $b_i$  is an output bit in  $\{-1, 1\}$ .

**Claim 1** *Let  $f$  be a length- $n$  decision list. For every  $r \leq n$  there is a degree- $r$  GPTF for  $f$  that has weight  $r \cdot 3^{O(n/r)}$ .*

**Proof:** We assume below that  $r$  divides  $n$  evenly and prove an  $O(r \cdot 3^{n/r})$  weight bound; this implies the claim as stated above. We break  $f$  into length- $r$  “modified decision lists”  $f_1, \dots, f_{n/r}$  which output values in  $\{-1, 0, 1\}$  as described below. The first function  $f_1$  may be expressed as a  $\pm 1$  sum of  $r$  conjunctions as follows:

$$f_1(x) = b_1 \ell_1 + b_2 (\bar{\ell}_1 \wedge \ell_2) + \dots + b_r (\bar{\ell}_1 \wedge \dots \wedge \bar{\ell}_{r-1} \wedge \ell_r)$$

(recall that each  $b_i \in \{-1, 1\}$  and each conjunction gives a value in  $\{0, 1\}$ ). It is clear that  $f_1$  outputs the correct value  $b_i$  of the decision list  $f$  if any literal  $\ell_i \in \{\ell_1, \dots, \ell_r\}$  is satisfied by input  $x$ , and otherwise  $f_1$  outputs the value 0 (since no conjunction is satisfied). The other functions  $f_2, \dots, f_{n/r}$  are defined similarly. Notice that each modified decision list  $f_i$  has GPTF weight  $r$  and GPTF degree  $r$ . Combining these expressions for the modified decision lists as

$$P(x) = 3^{n/r} f_1 + 3^{n/r-1} f_2 + \dots + 3^1 f_{n/r} + b_{n+1}, \quad (1)$$

we get that  $P(x)$  is a GPTF for  $f$  that has weight  $O(r \cdot 3^{n/r})$  and degree  $r$ . ■

**Inner Construction.** In the outer construction we broke  $f$  into  $\ell := n/r$  modified decision lists  $f_1, \dots, f_\ell$  each of length  $r$ , as in Eq. (1). We set parameters  $\ell = \sqrt{n/d}$  and  $r = \sqrt{nd}$ .

**Claim 2** *Let  $n^{1/3} \leq d \leq n$  and let  $f_i$  be a modified decision list of length  $r = \sqrt{nd}$ . There is an integer linear combination  $p_i$  of conjunctions of width  $2d \log(n/d)$ , such that*

- *The weight of  $p_i$  is  $2^{\tilde{O}(\sqrt{n/d})}$ ;*
- *There is an integer  $C = 2^{\tilde{O}(\sqrt{n/d})}$  such that for every input  $x \in \{0, 1\}^r$  that satisfies at least one literal in  $f_i$ , we have  $|p_i(x) - C f_i(x)| \leq C/r$ ; and*
- *$f_i(x) = 0$  implies  $p_i(x) = 0$ .*

We note that in [8, Corollary 9] the Chebyshev polynomial is used to construct a polynomial of degree  $2\sqrt{r} \log r$  and weight  $2^{O(\sqrt{r} \log^2 r)}$  that satisfies the last two properties of the lemma. Here, exploiting the fact that our  $p_i$  may be a linear combination of conjunctions, we show how to achieve the same properties using higher weight but lower degree; this will be useful for us later when we combine with the outer construction.

**Proof of Claim 2:** Recall that  $f_i$  may be expressed as  $f_i(x) = b_1 T_1(x) + \dots + b_r T_r(x)$ , where each  $T_j$  is a conjunction of length  $j \leq r$ . We view  $T_j$  as an AND-of-ANDs, where

- The top AND has fanin  $t := \lceil jr/d^2 \rceil \leq \lceil r^2/d^2 \rceil = \lceil n/d \rceil$ , and
- Each bottom AND has fanin at most  $b := \lceil j/t \rceil \leq \lceil d^2/r \rceil = \lceil \sqrt{d^3/n} \rceil$ . (Note that  $d^3/n \geq 1$  since  $d \geq n^{1/3}$  by assumption.)

Let  $\text{AND}_t(y_1, \dots, y_t)$  denote the top AND gate, so the inputs  $y_1, \dots, y_t$  are the bottom-level ANDs. To construct  $p$  we first apply the construction of [8, Corollary 9] to  $\text{AND}_t$  (which is itself a decision list on  $t$  variables). This yields a polynomial  $q$  (in 0/1 variables  $y_1, \dots, y_t$ ) of degree at most  $2\sqrt{n/d}\log(n/d)$  and weight  $2^{O(\sqrt{n/d}\log^2(n/d))}$ , and an integer  $C = 2^{O(\sqrt{n/d}\log^2(n/d))}$  such that  $|q(y_1, \dots, y_t) - C \cdot \text{AND}_t(y_1, \dots, y_t)| \leq \frac{C}{t}$  for all  $y \in \{0, 1\}^t \setminus \{0^t\}$ . Moreover, if  $y = 0^t$  then  $q(y) = 0$ .

The inputs to the top AND gate are themselves conjunctions of width  $b \leq \sqrt{d^3/n}$ , so we obtain our final representation  $p_i$  by simply composing  $q$  with the exact representation of each bottom AND gate as a conjunction of width  $b$ . (For example, if the polynomial  $q$  contains a monomial  $y_1 y_3 y_4$  we simply AND together the first, third and fourth bottom-level AND gates.) Since  $q$  has degree at most  $2\sqrt{n/d}\log(n/d)$ ,  $p_i$  is a linear combination of conjunctions of width at most  $b \cdot 2\sqrt{n/d}\log(n/d) = 2d\log(n/d)$  as required. The weight of the linear combination equals the weight of  $q$ , which is  $2^{O(\sqrt{n/d}\log^2(n/d))}$ . ■

**Proof of Theorem 1:** The remainder of the analysis necessary to prove Theorem 1 follows exactly as in [8, Theorem 10]; we present the details for completeness. We assume that  $f$  is the decision list  $(x_1, b_1), \dots, (x_n, b_n), b_{n+1}$  (the case when  $f$  contains negated literals is entirely similar). We begin with the outer construction and note that

$$f(x) = \text{sign} \left( C \cdot \left( \sum_{i=1}^{\ell} 3^{\ell-i+1} f_i(x) + b_{n+1} \right) \right),$$

where  $C$  is the value from Claim 2 and each  $f_i$  is a modified decision list of length  $r := \sqrt{nd}$  computing the restriction of  $f$  to its  $i$ -th block (so the total number of modified decision lists is  $\ell := n/r = \sqrt{n/d}$ ). Next we replace each  $Cf_i$  by  $p_i$ , the approximating polynomial given by the inner construction in Claim 2, and consider

$$H(x) = \sum_{i=1}^{\ell} (3^{\ell-i+1} p_i(x)) + Cb_{n+1}.$$

We will argue that  $\text{sign}(H(x))$  is a GPTF for  $f$  with the claimed degree and weight. Fix any  $x \in \{0, 1\}^n$ . If  $x = 0^n$  then by Claim 2 each  $p_i(x) = 0$  and so  $H(x) = Cb_{n+1}$  agrees in sign with  $b_{n+1}$ . Now suppose that  $t = (i-1)r + c$  is the first index such that  $x_t = 1$  (i.e. the input  $x$  “exits the overall decision list” at the modified decision list  $f_t$ , and  $f(x) = f_t(x) = b_t$ ). By Claim 2, we have

- $3^{\ell-j+1} p_j(x) = 0$  for all  $j < i$ ,
- $3^{\ell-i+1} p_i(x)$  differs from  $3^{\ell-i+1} Cb_t$  by at most  $C3^{\ell-i+1} \cdot \frac{1}{r}$ , and
- the magnitude of  $3^{\ell-j+1} p_j(x)$  is at most  $C3^{\ell-j+1} (1 + \frac{1}{r})$  for all  $j > i$ .

Combining these bounds, the value of  $H(x)$  differs from  $3^{\ell-i+1} Cb_t$  by at most

$$C \left( \frac{3^{\ell-i+1}}{r} + \left( 1 + \frac{1}{r} \right) (3^{\ell-i} + 3^{\ell-i-1} + \dots + 3) + 1 \right),$$

which can be checked to be less than  $C3^{\ell-i+1}$  in magnitude for  $r > 1$ . Consequently,  $\text{sign}(H(x)) = b_t = f(x)$  as claimed. Finally, applying the degree and weight bounds for each  $p_i$  given by Claim 2 completes the proof. ■

### 3.1 A New Upper Bound for PTFs over $\{-1, 1\}^n$

We do not know whether  $n$ -variable decision lists have degree- $d$  PTFs (as opposed to GPTFs) of weight  $\tilde{O}(\sqrt{nd}) \cdot 2^{\tilde{O}(\sqrt{n/d})}$ , though in the next section we will prove that they do *not* have PTFs of weight  $2^{o(\sqrt{n/d})}$ . For PTFs over  $\{-1, 1\}^n$ , though, we can combine the approach of Theorem 1 with ideas from Krause and Pudlák [10] to give a new upper bound on weight which improves on the Klivans and Servedio weight upper bound for  $d \geq n^{1/2}$ . In Appendix A we prove:

**Theorem 2** *For any  $n^{1/4} \leq d \leq n$ , any decision list  $f$  on  $n$  variables has a degree- $d$  PTF over the domain  $\{-1, 1\}^n$  of weight  $2^{(n/d)^{2/3} \cdot \text{polylog}(n)}$ .*

## 4 Degree- $d$ PTFs for ODD-MAX-BIT $_n$ Require Weight $2^{\Omega(\sqrt{n/d})}$

In this section we prove our main lower bound:

**Theorem 3** *For any  $d = o\left(\frac{n}{\log^2 n}\right)$ , any degree- $d$  PTF for ODD-MAX-BIT $_n$  requires weight  $2^{\Omega(\sqrt{n/d})}$ . This holds for PTFs over both domains  $\{0, 1\}^n$  and  $\{-1, 1\}^n$ .*

**Proof Outline.** To explain our proof it is helpful to first recall the  $2^{\Omega(n/d^2)}$  lower bound proof of Beigel [1]. Beigel first breaks the list of  $n$  variables into  $k = n/\ell$  blocks of length  $\ell := \Theta(d^2)$  each. His argument proceeds for  $k$  stages; in the  $i$ -th stage he argues that there exists an  $n$ -bit input  $x_i$  with  $|p(x_i)| \geq 2|p(x_{i-1})|$  which is such that  $x_i$  and  $x_{i-1}$  differ only in the  $i$ -th block. After the  $k$ -th stage this yields an input  $x_k$  that has  $|p(x_k)| \geq 2^k$ ; such an input clearly implies that the weight of  $p$  is at least  $2^k$ .

The existence of the desired  $x_i$  is established in each stage using Markov’s inequality for univariate real polynomials (see Fact 2 of Section 2). Beigel shows that if no such  $x_i$  existed, then by using  $x_{i-1}$  and symmetrizing  $p$  with respect to an appropriate distribution, one would obtain a univariate polynomial  $P$  of degree  $d$  such that (roughly speaking)  $P$  “stays bounded” on the interval  $[0, 1]$ , and  $P$  has large derivative at some point in this interval (i.e.  $|P'(x)| = \Omega(\ell)$  for some  $x \in [0, 1/\ell]$ ). But Markov’s inequality implies that any such polynomial must have  $\Omega(\sqrt{\ell}) > d$ ; consequently, the desired  $x_i$  must indeed exist.

While Beigel’s lower bound is essentially optimal for  $d \leq n^{1/3}$ , it becomes vacuous for  $d \geq n^{1/2}$  (and we show below that it is not tight even for  $d = \omega(n^{1/3})$ .) Intuitively, the reason that the bound is loose in the high-degree regime is that Markov’s inequality only takes the *degree* of the polynomial  $P$  into account and does not use information about the *weight* of  $P$ . With this intuition in mind, we prove a generalization of Markov’s inequality that takes into account  $P$ ’s weight. More specifically, we show that if  $P$  has degree  $d$  and weight  $W$ , “stays bounded” in the interval  $[0, 1]$ , and satisfies  $\max_{x \in [0, 1]} |P(x)| \geq 1/2$ , then  $|P'(x)| = O(d \log W)$  for all  $x \in [0, 1]$ . Notice that for  $W = 2^d$  this bound matches Markov’s inequality, but – crucially – it is significantly stronger if  $W$  is subexponential in  $d$ .

With this Markov-type inequality in hand, our proof follows the same outline as Beigel. We break the list into blocks of length  $\ell$ ; with our stronger inequality we are able to take  $\ell$  to be  $o(d^2)$ , which is why we improve on Beigel’s bound. We iteratively construct inputs  $x_i$  with  $|p(x_i)| \geq 2|p(x_{i-1})|$ , arguing at each stage  $i$  that if no such  $x_i$  existed, then one could obtain a univariate polynomial  $P$  of weight  $2^{O(\ell/d)}$  which satisfies the conditions of our Markov-type inequality. By



invoking our inequality, we conclude that  $P$  must have degree greater than  $d$ , and so this implies that the desired  $x_i$  must exist at each stage.

**The Markov-Type Inequality.** Below we state our Markov-type inequality as Lemma 1. This lemma generalizes a result of Borwein and Erdélyi [5, Corollary 3.2] and may be of independent interest.

**Lemma 1** *Let  $P : \mathbb{R} \rightarrow \mathbb{R}$  be a degree- $d$  polynomial satisfying the following:*

1. *The coefficients of  $P$  (which need not be integers) each have absolute value at most  $W$ ; and*
2.  *$1/2 \leq \max_{x \in [0,1]} |P(x)| \leq R$ .*

*Then  $\max_{x \in [0,1]} |P'(x)| = O(d \cdot R \cdot \max\{\log W, \log d\})$ .*

At a high level, the proof first shows that  $|P(y)|$  is not just “small” for all  $y \in [0, 1]$ , but in fact is small everywhere within a sufficiently large ellipse  $B_\rho$  surrounding  $[0, 1]$  in the complex plane. Cauchy’s Integral Formula then completes the result, as this formula expresses  $P'(y)$  as an average of  $P$ ’s values on  $B_\rho$ , scaled by (the squared reciprocal of) the diameter of  $B_\rho$ .

**Proof:** Borwein and Erdélyi proved the following lemma, which is a consequence of Hadamard’s Three Circle Theorem, a classical result in complex analysis.

**Lemma 2** [5, Corollary 3.2] *Let  $M \in \mathbb{R}$  and  $d \in \mathbb{N}$  satisfy  $1 \leq M \leq 2d$ . Suppose  $f$  is analytical inside and on the complex ellipse  $A_{d,M}$  with foci at 0 and 1 and major axis  $[-\frac{M}{d}, 1 + \frac{M}{d}]$ . Let  $B_{d,M}$  be the complex ellipse with foci at 0 and 1 and with major axis  $[-\frac{1}{dM}, 1 + \frac{1}{dM}]$ . Then there is an absolute constant  $c_1 > 0$  such that*

$$\max_{z \in B_{d,M}} \ln |f(z)| \leq \max_{z \in [0,1]} \ln |f(z)| + \frac{c_1}{M} \left( \max_{z \in A_{d,M}} \ln |f(z)| - \max_{z \in [0,1]} \ln |f(z)| \right).$$

Note that for all  $z \in A_{d,M}$ , we have

$$\begin{aligned} \ln |P(z)| &\leq \ln \left( (d+1)W \left( 1 + \frac{M}{d} \right)^d \right) \\ &\leq \ln(d+1) + \ln(W) + d \cdot \frac{M}{d} = \ln(d+1) + \ln(W) + M, \end{aligned}$$

where the first inequality holds because  $P$  has at most  $d+1$  coefficients, each of which has absolute value at most  $W$ , and  $|z| \leq 1 + \frac{M}{d}$  for all  $z \in A_{d,M}$  and each monomial in  $P$  has degree at most  $d$ . If we take  $M \geq \ln(W) + \ln(d+1)$ , the upper bound on  $\ln |P(z)|$  is at most  $2M$ .

Now applying Lemma 2 to the function  $P$  and  $M = \ln(W) + \ln(d+1)$ , we get that there is an absolute constant  $c_1$  such that

$$\max_{z \in B_{d,M}} \ln |P(z)| \leq \ln R + \frac{c_1}{M} (3M + \ln 2) \leq \ln R + c_2$$

for some universal constant  $c_2$ , where we have used the assumption that  $1/2 \leq \max_{z \in [0,1]} |P(z)| \leq R$ . As a result, we have that

$$\max_{z \in B_{d,M}} |P(z)| \leq c_3 R$$

for some universal constant  $c_3 > 0$ .

Now for any  $y \in [0, 1]$ , there is a positive number  $\rho = \Theta(\frac{1}{dM})$  such that  $B_\rho(y) = \{z \in \mathbb{C} : |z - y| = \rho\}$ , the closed disk of radius  $\rho$  around  $y$ , is contained in  $B_{d,M}$ . Consequently Cauchy's integral formula implies that for all  $y \in [0, 1]$ , we have

$$\begin{aligned} |P'(y)| &= \left| \frac{1}{2\pi i} \int_{B_{d,M}} \frac{P(z)}{(z-y)^2} dz \right| = \left| \frac{1}{2\pi i} \int_{B_\rho} \frac{P(z)}{(z-y)^2} dz \right| \\ &\leq \frac{1}{2\pi} c_3 R \left| \int_{B_\rho} \frac{1}{(z-y)^2} dz \right| \leq c_3 R \frac{1}{\rho} = \Theta(R \cdot d \cdot M). \end{aligned}$$

Here, the first equality holds by Cauchy's Theorem, because  $\frac{P(z)}{(z-y)^2}$  is analytical except at  $y$ , and the penultimate inequality combines parametric evaluation of the complex integral  $\int_{B_\rho} \frac{1}{(z-y)^2} dz$  with the fact that  $|z - y| = \rho$  for all  $z \in B_\rho$ . Since we took  $M = \log(W) + \log(d + 1)$ , the lemma follows.  $\blacksquare$

**Discussion.** For  $R = 1$ , Lemma 1 is tight up to constant factors for a wide range of values of  $W$ . To see this, let  $W \in [d, 3^d]$  be a power of 3, and let  $d' = \lceil \frac{d}{\log_3 W} \rceil$ . Consider the function  $P(x) = T_{\log_3 W}(x^{d'})$ , where  $T_k$  denotes the degree- $k$  Chebyshev polynomial of the first kind. Basic properties of  $T_k$  give that all of  $P$ 's coefficients have absolute value at most  $3^{\log_3 W} = W$ ; that  $P$  has degree  $d' \cdot \log_3 W = O(d)$ ; and that  $\max_{x \in [0,1]} |P(x)| = 1$ . Since  $T'_k(1) = k^2$  for any  $k$ , the chain rule implies that  $P'(1) = (\log_3 W)^2 \cdot d' = \Omega(d \log W)$ .

This tight example formalizes the following intuition: Recalling that the Chebyshev polynomials are a tight example for Markov's inequality, Markov's inequality can be understood to say that in order to maximize the derivative of a polynomial  $P$  of degree  $d$  which stays bounded on the interval  $[0, 1]$ , the best approach is to "spend all the allowed degree on a Chebyshev polynomial." Lemma 1 says that in order to maximize the derivative of a polynomial of degree  $d$  and weight  $W$  which stays bounded on the interval  $[0, 1]$ , the best approach is to "spend all of the allowed weight on a Chebyshev polynomial  $T$ , and then spend any remaining allowed degree by composing  $T$  with a suitable monomial  $x^{d'}$ ." This is the intuition that underlies our construction in the positive result Claim 2, where we use conjunctions (the bottom-level AND gates) as inputs for the top-level polynomial  $q$  (which is essentially a Chebychev polynomial).

**Proof of Theorem 3.** Here we prove the result for PTFs over the domain  $\{0, 1\}^n$ ; the proof over the domain  $\{-1, 1\}^n$  is similar and is given in Appendix B.

Let  $p$  be a degree  $d$ , weight  $W$  PTF for ODD-MAX-BIT $_n$  over  $\{0, 1\}^n$ . Suppose further that  $W < 2^{\sqrt{n/d}-1}$ . We partition the  $n$  variables  $x_1, \dots, x_n$  into  $n/\ell$  blocks of length  $\ell := 2^{\sqrt{n/d}}$ . We refer to a string  $y_i \in \{0, 1\}^{\ell \cdot i}$  as a *partial input* to  $p$ . Let  $V(y_i)$  denote  $p(y_i \circ 0^{n-\ell \cdot i})$ , where  $\circ$  denotes concatenation.

We say that partial input  $y_i \in \{0, 1\}^{\ell \cdot i}$  is *good* if  $|V(y_i)| \geq 2^{i-1}$ . We will attempt to iteratively construct good partial inputs  $y_1, \dots, y_{n/\ell}$ ; if we succeed at all iterations, then  $y = y_{n/\ell}$  is an input such that  $|p(y)| \geq 2^{\sqrt{n/d}-1}$ , which clearly contradicts the assumption that  $W < 2^{\sqrt{n/d}-1}$ . Thus, there must be some  $i^* < n/\ell$  such that there is a good partial input  $y_{i^*} \in \{0, 1\}^{\ell \cdot i^*}$ , but no good partial input  $y_{i^*+1} \in \{0, 1\}^{\ell \cdot (i^*+1)}$ . Moreover we have  $i^* \geq 1$ , because it is easy to construct  $y_1$ : since  $p$  has integer coefficients we may assume without loss of generality that  $|p(x)| \geq 1$  for all

$x \in \{-1, 1\}^n$  (if  $p(x) = 0$  for some  $x$  we can simply add  $1/2$  to  $p$  and multiply the resulting polynomial by two). So we may take  $y_1$  to be any string in  $\{0, 1\}^\ell$ .

Below we show how to use the  $y_{i^*}$  whose existence was established above to construct a certain univariate polynomial  $P$ . By applying Lemma 1 to  $P$ , we will conclude that  $W = 2^{\Omega(\sqrt{n/d})}$  as desired.

We may assume without loss of generality that  $V(y_{i^*}) < 0$ . Given a string  $w \in \{0, 1\}^{\ell/2}$ , let  $z_w \in \{0, 1\}^n$  be the string defined as follows:

- the first  $i^*$  blocks of  $z_w$  agree with  $y_{i^*}$ ;
- the even bits in the  $(i^* + 1)$ -th block are all 0;
- the odd bits in the  $(i^* + 1)$ -th block are successively  $w_1, w_2, \dots, w_{\ell/2}$ ; and
- the remaining blocks  $i^* + 2, \dots, n/\ell$  have all bits set to 0.

Define  $q$  to be the function which, on input  $w \in \{0, 1\}^{\ell/2}$ , outputs  $p(z_w)$ , i.e.  $q(w) := p(z_w)$ . Note that  $\deg(q) \leq \deg(p)$ .

Define the univariate polynomial

$$Q(t) := \mathbb{E}_{x \leftarrow \mu_t}[q(x)].$$

Here  $\mu_t$  is the product distribution over the discrete cube  $\{0, 1\}^{\ell/2}$  where each coordinate  $j$  satisfies  $\mathbb{E}_{x \leftarrow \mu_t}[x_j] = t$  (that is, each  $x_j$  is independently set to 1 with probability  $t$ ). The following properties of  $Q$  are easily verified:

1.  $Q(0) = q(0^{\ell/2})$  and  $Q(1) = q(1^{\ell/2})$ .
2.  $|Q(t)| \leq 2^{i^*}$  for all  $t \in [0, 1]$ . This holds by the assumption that there is no good partial input  $y_{i^*+1}$ , i.e.  $|V(y_{i^*+1})| \leq 2^{i^*}$  for all  $y_{i^*+1} \in \{0, 1\}^{(i^*+1) \cdot \ell}$ .
3.  $\deg(Q) \leq \deg(q) \leq \deg(p)$ . Indeed, given the multivariate polynomial expansion of  $q(w_1, \dots, w_{\ell/2})$ , we can obtain  $Q$  easily just by “erasing all the subscripts in each variable”. For example, if  $q(w) = 5w_1w_2w_3 - 8w_1w_2 - 4w_1w_3w_4 - 3w_3$  then  $Q(t) = 5t^3 - 8t^2 - 4t^3 - 3t = t^3 - 8t^2 - 3t$ . This follows from linearity of expectation along with the fact that  $\mu_t$  is defined to be the product distribution satisfying  $\mathbb{E}_{x \leftarrow \mu_t}[x_i] = t$  for all coordinates  $i$ . So indeed we have  $\deg(Q) \leq \deg(q) \leq \deg(p)$ .
4. The sum of the absolute value of the coefficients of  $Q$  is at most  $W$ . (Since this holds for  $p$  it holds for  $q$ , and since it holds for  $q$  it holds for  $Q$  by the previous remark.)

Note that we have  $q(0^{\ell/2}) = V(y_{i^*}) < -2^{i^*-1}$ . Crucially, we also have  $q(w) \geq 1/2$  for all  $w \in \{0, 1\}^{\ell/2} \setminus \{0^{\ell/2}\}$  (this is because for any such  $w$ , every input  $z \in Z_w$  has  $p(z) \geq 1$ , because  $p$  is a PTF for ODD-MAX-BIT $_n$ ).

The final polynomial we seek is the univariate polynomial

$$P(t) := Q(t) - Q(0).$$

We trivially have  $P(0) = 0$ . On the other hand, for any  $t \in [2/\ell, 1]$  we have  $P(t) = \mathbb{E}_{x \leftarrow \mu_t}[q(x) - q(0^{\ell/2})]$ . Recalling the bounds on  $q(0^{\ell/2})$  and on  $q(w)$  from the previous paragraph, and observing

that for  $t \in [2/\ell, 1]$  we have  $\Pr_{x \leftarrow \mu_t}[x = 0^{\ell/2}] \leq (1 - \frac{2}{\ell})^{\ell/2} < 1/e$ , we get that  $P(t) \geq 2^{i^*-1} \cdot (1 - \frac{1}{e}) \geq 2^{i^*-2}$  for  $t \in [2/\ell, 1]$ .

Thus,  $P(t)$  is a degree- $d$  polynomial that has the following properties:

1. The coefficients of  $P$  each have absolute value at most  $W + |V(y_{i^*})| \leq 2W$ ;
2.  $1/2 \leq \max_{x \in [0,1]} |P(x)| \leq 2^{i^*+1}$ ;
3. There exists some  $t \in [0, 2/\ell]$  such that  $P'(t) \geq \ell \cdot 2^{i^*-3}$ . This follows from the Mean Value Theorem, as  $P(0) = 0$  and  $P(2/\ell) \geq 2^{i^*-2}$ .

By Lemma 1, the first two properties of  $P$  imply that  $\max_{x \in [0,1]} |P'(x)| = O(d \cdot 2^{i^*} \cdot \max\{\log W, \log d\})$ . Combining this with the third property of  $P$ , we get that  $d \cdot \max\{\log W, \log d\} = \Omega(\ell) = \Omega(\sqrt{nd})$ . Recalling that  $d = o\left(\frac{n}{\log^2 n}\right)$ , we conclude that  $W = 2^{\Omega(\sqrt{n/d})}$  as desired. This concludes the proof of Theorem 3.  $\blacksquare$

## 5 Extending Beigel's Lower Bound to More General Features

Our next result extends Beigel's lower bound to significantly more general feature spaces. We say that a class  $\mathcal{F}$  of functions over  $\{0, 1\}^n$  is *closed under restrictions* if for all  $i \in [n]$  and all  $b \in \{0, 1\}$ , we have that if  $f(x_1, \dots, x_n) \in \mathcal{F}$  then  $f(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n) \in \mathcal{F}$ . Note that many natural classes of functions are closed under restrictions, including the set of parities of degree at most  $d$ , as well as the set of conjunctions of degree at most  $d$ . The following theorem can be understood as a refinement of a closely related result of Buhrman, Vereshchagin, and de Wolf [6] (see the discussion following the theorem). The proof is in Appendix C.

**Theorem 4** *Fix  $d > 0$ , and let  $\mathcal{F}$  be any set of at most  $2^d$  boolean-valued functions from  $\{0, 1\}^n$  to  $\{0, 1\}$ . Assume  $\mathcal{F}$  is closed under restrictions. There exists a decision list  $L$  on  $n$  variables such that, for any integer-weight representation  $L(x) = \text{sign}(\sum_{f_i \in \mathcal{F}} w_i f_i + \theta)$  of  $L$  as a linear threshold function over the feature space  $\mathcal{F}$ , the absolute value of the  $w_i$ 's and  $\theta$  sum to  $2^{\Omega(n/d^2)}$ .*

Several remarks are in order. First, Buhrman, Vereshchagin, and de Wolf [6, Section 3] show a related lower bound that applies to *arbitrary* classes of Boolean functions  $\mathcal{F}$ , not just to classes closed under restrictions. They couch their argument in terms of communication complexity, studying a natural formulation of the ODD-MAX-BIT $_n$  function as a communication problem. They show that any (quantum) communication protocol for this problem that requires the exchange of at most  $d$  bits and achieves *bias*  $\beta$  satisfies  $d + \log(1/\beta) = \Omega(n^{1/3})$ . Examination of their proof in fact reveals the slightly stronger result that  $m^2 \log(1/\beta) = \Omega(n)$ , where  $m = d + \log(1/\beta)$ . Translating this bound into our terminology yields the following theorem.

**Theorem 5 (Buhrman, Vereshchagin, and de Wolf [6])** *Fix  $d > 0$ , and let  $\mathcal{F}$  be any set of at most  $2^d$  boolean-valued functions from  $\{0, 1\}^n$  to  $\{0, 1\}$ . Suppose that every decision list  $L$  can be written as the threshold of an integer linear combination of functions from  $\mathcal{F}$ ,  $L(x) = \text{sign}(\sum_{f_i \in \mathcal{F}} w_i f_i + \theta)$ , with  $|\theta| + \sum_i |w_i| \leq W$ . Then*

$$m^2 \log W = \Omega(n),$$

where  $m = d + \log W$ . Elementary manipulations then yield the following bound on  $\log W$ :

$$\log(W) = \begin{cases} \Omega(n/d^2) & \text{if } d \geq n^{1/3}, \\ \Omega(n^{1/3}) & \text{otherwise.} \end{cases}$$

We explain formally how to derive Theorem 5 from the communication lower bound of [6, Section 3] in Appendix D. The proof is similar to Lemma 5.7 of Feldman [7], and is relatively standard.

In contrast to Theorem 5, our bound in Theorem 4 shows that if  $\mathcal{F}$  is closed under restrictions, then  $\log W = \Omega(n/d^2)$  for all  $d$ . Thus, for classes closed under restrictions, our lower bound matches that of Buhrman *et al.* when  $d$  is relatively large, but is a strict improvement when  $d = o(n^{1/3})$ . At a high level, our improvement can be traced to the fact that in Buhrman *et al.*'s argument, they break each decision list up into “blocks” of length  $\ell = \Theta(m^2) = \Theta((d + \log W)^2)$ , while we are able to use blocks of length  $\ell = \Theta(d^2)$ . For  $d = o(n^{1/3})$ , our block length is strictly smaller than that of Buhrman *et al.*

Second, we clarify that given any set  $\mathcal{F}$  of boolean-valued functions, the decision list  $L$  whose existence is guaranteed by Theorem 4 is actually a restriction of ODD-MAX-BIT $_n$  – this is easily seen by examining the proof of the theorem (and holds for Theorems 5 as well). However, the specific decision list  $L$  necessarily depends on  $\mathcal{F}$ . Indeed for any fixed decision list  $L$ , there exists a trivial set  $\mathcal{F}$  which computes  $L$  exactly (specifically, the unique function in  $\mathcal{F}$  is  $L$  itself).

Third, we note that Theorem 4 implies that any degree- $d$  GPTF that computes ODD-MAX-BIT $_n$  must have weight  $2^{\Omega(n/(d^2 \log^2 n))}$ . Indeed, the set  $\mathcal{F}$  of all conjunctions of width at most  $d$  has size  $n^{O(d)} = 2^{O(d \log n)}$  and is closed under restrictions; applying Theorem 4 to  $\mathcal{F}$  guarantees the existence of some decision list  $L$  such that any degree  $d$  GPTF for  $L$  requires weight  $2^{\Omega(n/(d^2 \log^2 n))}$ . Since  $L$  is in fact a restriction of ODD-MAX-BIT $_n$ , this lower bound applies to ODD-MAX-BIT $_n$  itself. This lower bound could actually have been derived from prior work: it follows immediately from Beigel’s weight lower bound for degree- $d$  PTFs over  $\{0, 1\}^n$  when  $d \leq n^{1/3}$  (see the discussion preceding Theorem 1), while for  $d > n^{1/3}$ , the bound follows from Theorem 5. Theorem 4 has the appealing property of providing a “unified” proof of an almost matching bound.

## 6 Conclusion

We have given new constructions of low-weight GPTFs, and new weight lower bounds for ordinary PTFs, for decision lists. Our new constructions yield improved tradeoffs between runtime and mistake bounds for learning decision lists, while our lower bounds imply that new algorithms or analyses are needed to improve over current results.

A number of interesting questions remain open. For example, it would be interesting to generalize our  $2^{\Omega(\sqrt{n/d})}$  weight lower bound for decision lists to more general feature spaces. In particular, generalizing the lower bound to arbitrary length- $d$  conjunctions would show that the weight upper bound of Theorem 1 is the best possible for GPTFs. It would also be interesting to prove stronger lower bounds for PTF weight-degree tradeoffs for richer classes of functions such as DNFs or decision trees.

**Acknowledgements.** We are grateful to Vitaly Feldman, Ronald de Wolf, and Alexander Sherstov for helping us to clarify the relationship of Theorem 4 to prior work.

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## A Proof of Theorem 2: Weight Upper Bounds for PTFs over $\{-1, 1\}^n$ .

Since Theorem 1 employs GPTFs rather than PTFs, it does not show that the weight lower bound of Theorem 3 is tight. It is thus a natural question to determine whether the lower bound of Theorem 3 is tight for PTFs. Towards this goal, Theorem 2 gives an upper bound showing that Theorem 3 is close to tight for PTFs over  $\{-1, 1\}^n$ . This result improves on the PTF weight upper bound in Klivans and Servedio [8] for any  $d \geq n^{1/2}$ . We restate Theorem 2 for the reader's convenience:

**Theorem 2** *For any  $n^{1/4} \leq d \leq n$ , any decision list  $f$  on  $n$  variables has a degree- $d$  PTF over the domain  $\{-1, 1\}^n$  of weight  $2^{(n/d)^{2/3} \cdot \text{polylog}(n)}$ .*

Intuitively, the difficulty in extending Theorem 1 to PTFs over  $\{0, 1\}^n$  is that in the inner construction, the bottom AND gates in the proof of Claim 2 may not have low-weight exact representations over  $\{0, 1\}^n$  even if we allow very high degree. Instead, we will use the fact that over  $\{-1, 1\}^n$  any AND gate has a low-weight *uniform approximation*. More specifically, the following fact is an easy extension of [11, Proposition 2.1] (we note explicitly that the function  $\text{AND}^{(j)}$  in the following fact does not have any dependence on  $j$ ; we use this notation because it will be useful in the context of the proof of Theorem 2):

**Fact 4** *Let  $\text{AND}^{(j)} : \{-1, 1\}^n \rightarrow \{0, 1\}$  denote some conjunction of exactly  $n$  literals, so there is a distinguished input  $z \in \{-1, 1\}^n$  such that  $\text{AND}^{(j)}(z) = 1$  and  $\text{AND}^{(j)}(x) = 0$  for all  $x \in \{-1, 1\}^n \setminus \{z\}$ . Then for any  $\epsilon > 0$ , there is a polynomial  $g(x_1, \dots, x_n)$  of degree at most  $n$  such that*

1. *All coefficients of  $g$  are integers, and the sum of their absolute values is  $O(n/\epsilon^2)$ ;*
2. *There is an integer  $C_1 = O(n/\epsilon^2)$  such that  $|g_j(y) - C_1 \text{AND}^{(j)}(y)| \leq C_1 \epsilon$  for all  $y \in \{-1, 1\}^n$ ; and*
3.  *$g_j(-z) = 0$ . That is,  $g$  evaluates to precisely 0 on the input in  $\{-1, 1\}^n$  that satisfies none of the literals in  $\text{AND}^{(j)}$ .*

With this fact in hand we now prove Theorem 2.

**Proof:** We break  $f$  into  $\ell := (n/d)^{2/3}$  modified decision lists as in Equation 1, each of length  $r = n/\ell = n^{1/3}d^{2/3}$ . The proof uses the following variant of Claim 2.

**Claim 3** *Fix  $d \geq n^{1/3}$  and let  $f_i$  be a modified decision list of length  $r = n^{1/3}d^{2/3}$ . There is an degree- $\tilde{O}(d)$  polynomial  $p_i(x_1, \dots, x_r)$  with integer coefficients such that*

- *The weight of  $p_i$  is  $2^{\tilde{O}((n/d)^{2/3})}$ ;*
- *There is an integer  $C = 2^{\tilde{O}((n/d)^{2/3})}$ , such that for every input  $x \in \{-1, 1\}^r$  that satisfies at least one literal in  $f_i$ , we have  $|p(x) - C f_i(x)| \leq C/r$ ; and*
- *$f_i(x) = 0$  implies  $p_i(x) = 0$ .*

**Proof:** Recall that  $f_i$  may be expressed as  $f_i(x) = b_1 T_1(x) + \dots + b_r T_r(x)$ , where  $T_j$  is a conjunction of length  $j \leq r$ . We view  $T_j$  as an AND-of-ANDs where

- The top AND has fanin  $t := \lceil jr/d^3 \rceil \leq \lceil r^2/d^2 \rceil = \lceil (n/d)^{2/3} \rceil$ ,
- Each bottom AND has fanin at most  $b := \lceil j/t \rceil \leq \lceil d^2/r \rceil = \lceil \frac{d^{4/3}}{n^{1/3}} \rceil$ . Note that  $\frac{d^{4/3}}{n^{1/3}} \geq 1$  since  $d \geq n^{1/4}$  by assumption.

Compared to the proof of Claim 2, we have reduced the top fan-in to minimize the propagation of error due to the approximations we will use for the bottom AND gates.

Let  $\text{AND}_t(y_1, \dots, y_t)$  denote the top AND gate, so the inputs  $y_1, \dots, y_t$  are the bottom-level ANDs (which take values in  $\{0, 1\}$ ). To construct  $p$  we apply [8, Corollary 9] to  $\text{AND}_t$  to obtain a polynomial  $q$  of degree  $d' = \tilde{O}(\sqrt{t})$  and weight  $2^{\tilde{O}(\sqrt{t})}$ , and an integer  $C_2 = 2^{\tilde{O}(\sqrt{t})}$  such that  $|q(y_1, \dots, y_t) - C_2 \cdot \text{AND}_t(y_1, \dots, y_t)| \leq \frac{C_2}{t}$  for all  $y \in \{0, 1\}^t$ . Moreover,  $q(0^t) = 0$ .

The inputs to the top AND gate are themselves conjunctions of width  $b$  over  $\{-1, 1\}$  inputs. Denote the  $j$ -th bottom-level AND gate by  $\text{AND}^{(j)}$ . Set  $\epsilon = 2^{-\tilde{O}(\sqrt{t})}$ , and let  $g_j$  denote the polynomials for the functions  $\text{AND}^{(j)}$  given by Fact 4. We obtain our final representation  $p$  by composing  $q$  with the functions  $g_j/C_1$ , and then scaling the resulting coefficients to be integers. More precisely, viewing the polynomials  $g_j$  as functions over  $r$  variables (though only  $t$  variables are relevant for each  $g_j$ ), the polynomial  $p$  is  $p(x_1, \dots, x_r) = (C_1)^{d'} q(\frac{g_1(x)}{C_1}, \dots, \frac{g_t(x)}{C_1})$ . This polynomial  $p$  has degree  $\tilde{O}(\sqrt{t}) \cdot b = \tilde{O}((n/d)^{1/3}) \cdot \frac{d^{4/3}}{n^{1/3}} = \tilde{O}(d)$  as desired. It has integer coefficients, and the sum of the absolute values of its coefficients is at most  $(n/\epsilon^2)^{\tilde{O}(\sqrt{t})} = 2^{\tilde{O}(t) \cdot \log n} = 2^{(n/d)^{2/3} \cdot \text{polylog}(n)}$ .

All that remains to show is that there is an integer  $C = 2^{(n/d)^{2/3} \cdot \text{polylog}(n)}$  such that for every input  $x \in \{-1, 1\}^r$ , it holds that  $|p(x) - C f_i(x)| \leq C/r$ , and that  $f_i(x) = 0$  implies  $p(x) = 0$ . The latter property follows from the fact that each  $g_j(0^t) = 0$  for all  $j$ , and  $q(0^t) = 0$ . To see the first property, notice that  $q$  has at most  $2^{\tilde{O}(\sqrt{t})} \leq \frac{1}{r\epsilon}$  monomials, since it has degree  $d' = \tilde{O}(\sqrt{t})$ . Since each  $\frac{g_j}{C_1}$  is a pointwise  $\epsilon$ -approximation to the  $j$ 'th bottom gate  $\text{AND}^{(j)}$ , for each monomial the difference between its value evaluated at  $(\text{AND}^{(1)}(x), \dots, \text{AND}^{(t)}(x))$  and at  $(\frac{g_1(x)}{C_1}, \dots, \frac{g_t(x)}{C_1})$  is most  $\epsilon$ . Thus, letting  $C = C_2 \cdot C_1^{d'}$ , the total error  $|p(x) - C \cdot f_i(x)|$  at any point is at most  $C/r$ . This concludes the proof of the claim.  $\blacksquare$

Theorem 2 now follows by replacing each modified decision list  $f_i$  with the corresponding polynomial  $p_i$  from Claim 3. The result is a PTF  $P$  over  $\{-1, 1\}^n$  for  $f$  which has weight  $2^{(n/d)^{2/3} \cdot \text{polylog}(n)}$  and has degree  $\tilde{O}(d)$ . The analysis showing that  $\text{sign}(P(x)) = f(x)$  for all  $x \in \{-1, 1\}^n$  follows exactly as in Theorem 1 and is omitted.  $\blacksquare$

## B Proof of Theorem 3 over domain $\{-1, 1\}^n$ .

We describe the changes to the proof for domain  $\{0, 1\}^n$  that are required to obtain the lower bound for a PTF  $p$  over domain  $\{-1, 1\}^n$ . As in the  $\{0, 1\}^n$  proof we may assume that  $V(y_{i^*}) < 0$ . Given a string  $w \in \{-1, 1\}^{\ell/2}$  we now let  $z_w \in \{-1, 1\}^n$  be the input defined as follows:

- the first  $i^*$  blocks of  $z_w$  agree with  $y_{i^*}$ ;



- the even bits in the  $(i^* + 1)$ -th block are all 1;
- the odd bits in the  $(i^* + 1)$ -th block are successively  $w_1, w_2, \dots, w_{\ell/2}$ ; and
- the remaining blocks  $i^* + 2, \dots, n/\ell$  have all bits set to 1.

As before, define  $q$  to be the function which, on input  $w \in \{-1, 1\}^{\ell/2}$ , outputs  $p(z_w)$ . Also as before, define

$$Q(t) := \mathbb{E}_{x \leftarrow \mu_t}[q(x)],$$

where now  $\mu_t$  is the product distribution over the Hamming cube  $\{-1, 1\}^{\ell/2}$  where each coordinate  $j$  satisfies  $\mathbb{E}_{x \leftarrow \mu_t}[x_j] = t$  (so each  $x_j$  is independently 1 with probability  $\frac{1+t}{2}$  rather than with probability  $t$  as before). The earlier arguments show that  $Q$  satisfies the following properties:

1.  $Q(-1) = q(-1, \dots, -1)$  and  $Q(1) = q(1, \dots, 1)$ .
2.  $|Q(t)| \leq 2^{i^*}$  for all  $t \in [-1, 1]$ .
3.  $\deg(Q) \leq \deg(q) \leq \deg(p)$ .
4. The sum of the absolute value of the coefficients of  $Q$  is at most  $W$ .

Finally, we now define  $P$  as

$$P(t) := Q(t) - Q(-1).$$

The earlier arguments imply that  $P(t)$  is a degree- $d$  polynomial with the following properties:

1. The coefficients of  $P$  have absolute value at most  $W + |V(y_{i^*})| \leq 2W$ .
2.  $1/2 \leq \max_{x \in [-1, 1]} |P(x)| \leq 2^{i^*+1}$ .
3.  $P(-1) = 0$  and  $P(-1 + (4/\ell)) \geq 2^{i^*-2}$ . In particular, there exists some  $t \in [-1, -1 + (4/\ell)]$  such that  $P'(t) \geq \ell \cdot 2^{i^*-4}$ .

Note that the main difference compared to the argument for the  $\{0, 1\}^n$  domain is that in item (3) above, the “jump” now occurs near  $-1$ , not in the domain  $[0, 1]$ , so we cannot directly invoke Lemma 1. On the other hand, the polynomial  $P$  stays bounded on the larger domain  $[-1, 1]$  rather than  $[0, 1]$ , and so we can easily transform  $P$  into a polynomial which satisfies all of the conditions of Lemma 1. Indeed, defining  $P_1(x) = P(-x)$ , it is easy to check that we may apply Lemma 1 to  $P_1$  to conclude that  $W = 2^{\Omega(\sqrt{n/d})}$  as desired. ■

## C Proof of Theorem 4

**Proof Outline.** Fix a polynomial  $p$  which sign-represents ODD-MAX-BIT. In proving his lower bound, Beigel breaks the  $n$  variable domain into blocks of length  $\ell$ , and iteratively constructs inputs  $x_i$  such that  $|p(x_i)|$  grows exponentially. We explained in Section 4 that at each stage the existence of an input  $x_i$  with  $|p(x_i)| \geq 2|p(x_{i-1})|$  follows from Markov’s inequality. However, another way to understand the existence proof is as follows. If no such  $x_i$  existed, then we could in fact obtain a

degree  $d$  polynomial which uniformly approximates the OR function on  $\ell$  variables. By a result of Nisan and Szegedy [16], such a polynomial requires degree  $\Omega(\sqrt{\ell})$ .

In [9, Theorem 4.2], the authors prove a generalization of Nisan and Szegedy’s result, showing that if one wants to uniformly approximate *all* disjunctions on  $\ell$  variables using *any* feature space, then the size of the feature space must be  $2^{\Omega(\sqrt{\ell})}$ . It is therefore plausible that Beigel’s argument can be extended to more general feature spaces, using the feature-independent result of Klivans and Sherstov in place of Nisan and Szegedy, and indeed this is how our proof proceeds. Details follow.

**Proof of Theorem 4.** Fix  $\ell := \Theta(d^2)$ . We will iteratively consider decision lists defined on  $i \cdot \ell$  variables, for each  $i \in \{1, \dots, n/\ell\}$ . Refer to the set of all such decision lists as  $\mathcal{DL}_i$ , and refer to the function ODD-MAX-BIT defined over  $i \cdot \ell$  variables as ODD-MAX-BIT $_i$ . Furthermore, let  $\mathcal{F}_i$  denote the set of features over  $i \cdot \ell$  variables obtained from  $\mathcal{F}$  by “ignoring” the last  $n - i$  variables in each feature. That is,

$$\mathcal{F}_i = \{\psi : \{0, 1\}^{i \cdot \ell} \rightarrow \{0, 1\} \mid \psi_i(x) = \phi_i(x \circ \mathbf{0}) \text{ for some } \phi \in \mathcal{F}\}.$$

Notice that since  $\mathcal{F}$  is closed under restrictions,  $\mathcal{F}_i$  is as well.

We iteratively construct decision lists  $L_i \in \mathcal{DL}_i$  such that for *every* representation  $L_i(x) = \text{sign}(h(x))$  of  $L_i$  as the sign of a integer linear combination of the features in  $\mathcal{F}_i$ , there exists a  $y_i(L_i) \in \{0, 1\}^{i \cdot \ell}$  such that  $|h(y_i(L_i))| \geq 2^{i-1}$ . Moreover, each  $L_i$  will be obtained from ODD-MAX-BIT $_i$  by restricting some of the variables in block  $i$ . If we succeed at all iterations, it clearly follows that the decision list in the final iteration satisfies the properties guaranteed by the theorem. Notice we can take  $L_1$  to simply be ODD-MAX-BIT $_1$ , as we can assume all sign-representations of  $L_1$  have margin at least 1.

Suppose this process fails at stage  $i > 1$ . Then

1. There is a decision list  $L_i \in \mathcal{DL}_i$  such that for *every* sign-representation  $h$  for  $L_i$  as an integer linear combination of features in  $\mathcal{F}_i$ , there is an input  $y_i \in \{0, 1\}^{i \cdot \ell}$  such that  $|h(y_i)| \geq 2^{i-1}$ . Moreover,  $L_i$  is obtained from ODD-MAX-BIT $_i$  by restricting some of the variables.
2. For all decision lists  $L \in \mathcal{DL}_{i+1}$ , there exists a sign-representation  $L(x) = \text{sign}(h(x))$  such that  $|h(y_{i+1})| \leq 2^i$  for all  $y_{i+1} \in \{0, 1\}^{(i+1) \cdot \ell}$ .

We show that there is a feature space  $\mathcal{F}'$  with  $|\mathcal{F}'| \leq |\mathcal{F}|^5$  which uniformly approximates the set of *all* disjunctions on  $\ell/2$  variables within additive error  $1/3$ . By [9, Theorem 4.2], this implies that  $|\mathcal{F}'| = 2^{\Omega(d)}$ , and hence  $\mathcal{F} = 2^{\Omega(d)}$  as well.

Let  $y_i \in \{0, 1\}^{i \cdot \ell}$  be any vector, and assume that  $L_i(y_i) < 0$ . Consider the set  $\mathcal{S}$  of decision lists in  $\mathcal{DL}_{i+1}$  which are obtained from ODD-MAX-BIT $_{i+1}$  by

- Restricting variables in blocks  $\{1, \dots, i\}$  in the same manner as  $L_i$ ;
- Restricting all even variables in block  $i + 1$  to 0;
- Restricting some subset of the odd variables in block  $i + 1$  to 0.

There is a natural one-to-one correspondence between decision lists in  $\mathcal{S}$  and disjunctions on  $\ell/2$  variables: if  $L_S \in \mathcal{S}$  denotes the decision list such that the “active” odd variables in block  $i$  are precisely those in  $S$ , then  $L_S$  naturally corresponds to the disjunction  $\text{OR}_S(x_1, \dots, x_{\ell/2}) := \bigvee_{j \in S} x_j$ .

The key observation is that any sign-representation  $h_{L_S}$  for  $L_S$  as an integer linear combination of features in  $\mathcal{F}_{i+1}$  can be transformed into a uniform approximation for  $\text{OR}_S$ , using features which are themselves (products of) restrictions of features in  $\mathcal{F}_{i+1}$ . Since  $\mathcal{F}_i$  is closed under restrictions, this means that we have in fact obtained a uniform approximation for  $\text{OR}_S$  using features which are themselves products of those in  $\mathcal{F}_{i+1}$ .

Indeed, consider the disjunction  $\text{OR}_S(x_1, \dots, x_{\ell/2}) := \bigvee_{j \in S} x_j$ . Let  $h_{L_S}(x) = \sum_{\phi_j \in \mathcal{F}_{i+1}} c_j \phi_j(x)$  be the sign-representation of  $L_S$  guaranteed by Property 2 above. Then we conclude the following.

- There is some  $y_i$  such that  $W := |h_{L_S}(y_i \circ \mathbf{0})| \geq 2^{i-1}$ . This holds by Property 1 above because  $L_i(y_i) = \text{sign}(h_{L_S}(y_i \circ \mathbf{0}))$ , and the function  $g(y) = h_{L_S}(y \circ \mathbf{0})$  is an integer linear combination of features in  $\mathcal{F}_i$ .
- $h_{L_S}(y_{i+1}) \leq 2^i$  for all  $y_{i+1} \in \{0, 1\}^{(i+1) \cdot \ell}$  by definition of  $h_{L_S}$ .
- Assume without loss of generality that  $L_i(y_i) < 0$ . Then  $\text{OR}_S(x_1, \dots, x_{\ell/2}) = L_S(y_i^* \circ \tilde{x})$ , where  $\tilde{x} \in \{0, 1\}^\ell$  denotes the vector obtained by placing  $x_i$  in the  $i$ th odd coordinate, and placing 0 in all even coordinates.

Consider the function  $h_1 : \{0, 1\}^{(i+1) \cdot \ell} \rightarrow \{0, 1\}$  given by  $h_1(x) = h_{L_S}(x) - h_{L_S}(y_i \circ \mathbf{0})$ . Trivially,  $h_1(y_i \circ \mathbf{0}) = 0$ , and the first and second bullets above imply that,  $W \leq h_1(x) \leq 3W$  for all  $x$ . Combining this with the third bullet point above, we conclude that the function

$$h_2(x_1, \dots, x_{\ell/2}) = 2 \cdot \left( \frac{h_{L_S}(y_i \circ \tilde{x})}{3W} - 1 \right)^5 + 1$$

uniformly approximates  $\text{OR}_S$  to within error  $1/3$ . Indeed,  $h_2(\mathbf{0}) = -1$ , and for all  $x \neq \mathbf{0}$ ,  $h_2(x) \in [2/3, 1]$ . Notice  $h_2$  is a linear combination of features which can be written as the product of at most 5 features in  $\mathcal{F}_{i+1}$  (possibly after adding the constant-function to  $\mathcal{F}_{i+1}$ ). As *every* disjunction over  $\ell/2$  variables can be uniformly approximated in this way, we conclude from [9, Theorem 4.2], that  $\mathcal{F}_{i+1} \geq 2^{\Omega(d)}$ . Since  $|\mathcal{F}_{i+1}| \leq |\mathcal{F}|$ , we conclude that  $|\mathcal{F}| = 2^{\Omega(d)}$ , completing the proof.

## D Proof of Theorem 5

We restate Theorem 5 here for the reader's convenience.

**Theorem 5** (Buhrman, Vereshchagin, and de Wolf [6]) Fix  $d > 0$ , and let  $\mathcal{F}$  be any set of at most  $2^d$  boolean-valued functions from  $\{0, 1\}^n$  to  $\{0, 1\}$ . Suppose that every decision list  $L$  can be written as the threshold of an integer linear combination of functions from  $\mathcal{F}$ ,  $L(x) = \text{sign}(\sum_{f_i \in \mathcal{F}} w_i f_i + \theta)$ , with  $|\theta| + \sum_i |w_i| \leq W$ . Then

$$m^2 \log W = \Omega(n),$$

where  $m = d + \log W$ . Elementary manipulations then yield the following bound on  $\log W$ :

$$\log(W) = \begin{cases} \Omega(n/d^2) & \text{if } d \geq n^{1/3}, \\ \Omega(n^{1/3}) & \text{otherwise.} \end{cases}$$

**Proof:** Buhrman, Vereshchagin, and de Wolf prove a communication lower bound for the function  $\text{ODD-MAX-BIT}_n(x \wedge y)$ , which outputs 1 if and only if the smallest  $i$  such that  $x_i = y_i = 1$  is odd.

Specifically, they show that any quantum communication protocol for  $\text{ODD-MAX-BIT}_n(x \wedge y)$  that requires the exchange of at most  $c$  bits and achieves *bias*  $\beta$  satisfies  $m^2 \log(1/\beta) = \Omega(n)$ , where  $m = c + \log(1/\beta)$ . Here, a protocol is said to have *bias*  $\beta$  if it outputs  $\text{ODD-MAX-BIT}_n(x \wedge y)$  with probability at least  $1/2 + \beta$ .

Suppose by way of contradiction that every decision list can be written as a threshold of a weight- $W$  integer linear combination of functions in  $\mathcal{F}$ . We show how to obtain a (classical) one-way communication protocol for  $\text{ODD-MAX-BIT}_n(x \wedge y)$  as follows. This protocol is similar to that described in Lemma 5.7 of [7].

For a fixed  $x$ , the decision list (in the entries of  $y$ ) given by  $\text{ODD-MAX-BIT}_n(x \wedge y)$  has a representation as  $\text{sign}(\sum_{f_i \in \mathcal{F}} w_i f_i(y))$  with  $\sum_i |w_i| = W$ . Alice picks a function  $f_i$  with probability  $w_i/W$ , and with  $d + 1$  bits of communication she tells Bob the number  $i$  as well as  $\text{sign}(w_i)$ . Bob outputs  $\text{sign}(w_i) \cdot \text{sign}(f_i(y))$ . This protocol is easily seen to have communication  $d + 1$  and bias  $\frac{1}{2W}$ . It follows from the communication lower bound of Buhrman *et al.* that  $(d + \log W)^2 \cdot \log W = \Omega(n)$ . This completes the proof. ■