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On Two-Level Poset Games

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Abstract

We consider the complexity of determining the winner of a finite, two-level poset game. This is a natural question, as it has been shown recently that determining the winner of a finite, three-level poset game is PSPACE-complete. We give a simple formula allowing one to compute the status of a type of two-level poset game that we call *parity-uniform*. This class includes significantly more easily solvable two-level games than was known previously. We also establish general equivalences between various two-level games. These equivalences imply that for any n, only finitely many two-level posets with n minimal elements need be considered, and a similar result holds for two-level posets with n maximal elements.

1 Introduction

Partially ordered set (poset) games are a class of two-player impartial games that have been widely studied. The game rules are pretty simple to describe. Given a partially ordered set, in a move, a player can remove any element from the set. Removal of any element e will also remove any element in the set which is greater than e. If a player has no move left, i.e., if the set is empty, then the player loses. Several subclasses of poset games have been considered over the years, the most familiar being Nim, but also Chomp, Divisors, and Hackendot, to name a few.

Given a finite poset game, the problem we consider is to decide whether it is a winning game, i.e., whether the first player has a strategy to win, whatever the second player plays. For some subclasses of poset games, this problem is known to be easy. The strongest result of this sort is by Deuber & Thomassé [DT96], who showed that N-free poset games (where the poset does not have an "N" as an induced subposet) are solvable in polynomial time.

Recently, Daniel Grier has shown this problem to be PSPACE-hard for general poset games [Gri12]. (The problem is easily seen to be in PSPACE.) In fact, Grier's reduction shows that the problem remains PSPACE-complete even restricted to games played on posets with three levels, i.e., games where the poset is the union of (at most) three antichains.[1] It

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[1]In an earlier version of this paper, we used the term "height" to denote the number of levels, that is, the minimum size of any partition of the poset into antichains (eqivalently, the largest number of points in any chain). This alternate terminology is a bit ambiguous and can be misinterpreted, however, since the size of a chain is sometimes taken to mean the number of "edges" in the chain, i.e., one less than the number of points of the chain.

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is a natural question to ask if two-level poset games are easy, as no hardness result is known for them. We address this question here.

Fraenkel and Scheinerman [FS91] have solved some cases of a certain class of poset games known as hypergraph games. Some of these—P-regular hypergraphs—correspond to a special class of two-level poset games. Until now, their paper was the only one giving results applying specifically to two-level games. In Section 4, we generalize their result to a significantly larger class of two-level poset games. Also, we prove in Section 3 that if a set of elements of a poset have exactly the same connections in the poset, then only the parity of the set matters for the status of the game. This implies that we only need consider a finite number of poset games with any given set of bottom elements, and similarly for a given set of top elements. Also in Section 3, as a warm-up, we compute the Grundy numbers (see Definition 2.2) of some simple two-level poset games.

The formulas for the g-numbers that we present in this paper are all easily computed in polynomial time, in fact in TC^0. Hence, these games are solvable in polynomial time. Whether two-level poset games are easy in general is still an open question.

2 Preliminaries

Let us first define a two-player impartial game. This is a game where allowed moves only depend on the current state of the game and not on which player is playing or the number of turns already taken. Given a game, the question is to find the status of the game: "winning" or "losing," i.e., whether the first player has or does not have a winning strategy.

In a poset game, a partially ordered set P is given initially. Two players then alternate turns, each choosing an element x from the remaining poset and removing all $y \geq x$ from the poset. We call this move "playing x." For any $x \in P$, we define

$$P_x = \{y \in P \mid x \not\leq y\} .$$

This is the portion of P that remains after a player plays x starting with P. The player who makes the last move, leaving an empty poset, wins.

In this paper, we use directed acyclic graphs (Hasse diagrams) to represent partially ordered sets. The vertices of the graph are the elements of the poset, and $y \geq x$ if and only if vertex y is reachable from vertex x in the graph.

We say that a winning game is an $\exists$-*game* as there exists a winning strategy for the first player, and a losing game is a $\forall$-*game* as any strategy of the first player will lose.[2] Here is the formal, recursive definition.

Definition 2.1. *Let P be any finite poset.*

- *P is an* $\exists$-game *iff there exists $x \in P$ such that P_x is a $\forall$-game.*
- *P is a* $\forall$-game *iff it is not an $\exists$-game.*

So for example, $\emptyset$ is a $\forall$-game, and the one-element poset is an $\exists$-game.

A natural way to construct a new game is to take two disjoint games P_1 and P_2 and put them together side-by-side, so that no element of P_1 is comparable with any element of P_2. A valid move in this game is thus either a move in P_1 or a move in P_2. We will denote this

[2] In the literature, these are often called N-games and P-games, respectively.

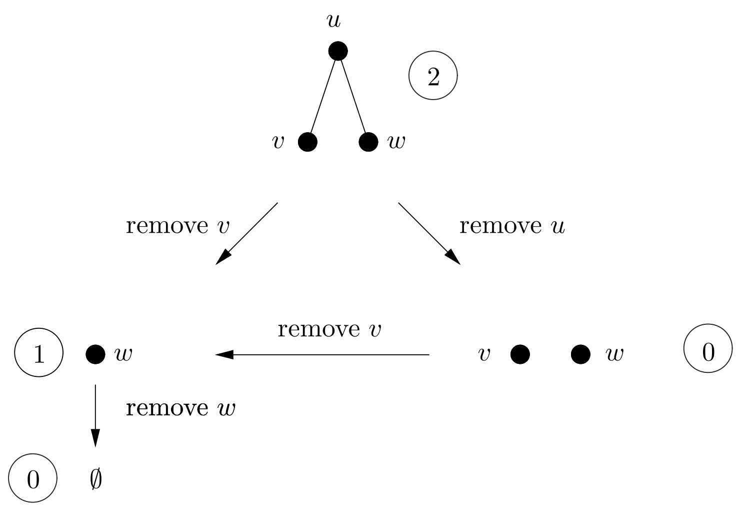


Figure 1: Computation of the g-number of a poset with its game tree. The numbers in the circles show the g-numbers for the corresponding games.

game by $(P_1 \parallel P_2)$ and call it the *parallel union* of P_1 and P_2.[3] For example, for any game P, the parallel union $(P \parallel P)$ is a $\forall$-game: for any x the first player plays in one copy of P, the second player can imitate the same move in the other copy, leaving the game $(P_x \parallel P_x)$ for the first player. Thus the second player can always win.

It is natural to ask if we can find the status of $(P_1 \parallel P_2)$, given the status of P_1 and P_2. A classic result of Sprague and Grundy [Spr36, Gru39] says that this can be done if we are given more information about P_1 and P_2, namely, their Grundy numbers, or g-numbers. The g-number is defined as follows, letting $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers:

Definition 2.2. *The* Grundy number *(or g-*number*) $g(P)$ of a poset game P is given by the following recursive formula:*

$$g(P) = \min(\mathbb{N} - \{g(P_x) \mid x \in P\}) \ .$$

We call the set $\{g(P_x) \mid x \in P\}$ the *g-set* of the poset P. In other words, the Grundy number $g(P)$ is the least natural number not present in g-set(P). Notice that according to the definition, $g(\emptyset) = 0$. More generally, if C_n is a chain, i.e., a linearly ordered set of n points, then $g(C_n) = n$ and g-set$(C_n) = \{0, \ldots, n-1\}$. Clearly, the number n of nodes of a poset game is an upper bound on its g-number.

Figure 1 shows the computation of the g-number of a particular poset from its game tree. By the definition we get that given a poset game with Grundy number $g \neq 0$, one can always choose a move to make it a game with Grundy number i, for any $0 \leq i \leq g - 1$. Also, if the Grundy number of the given game is 0, then whatever move one chooses, it becomes a game with nonzero Grundy number. Thus we get the following criterion, which was proved for more general games by Grundy [Gru39].

Observation 2.3 (Grundy [Gru39])**.** *A poset game P is an $\exists$-game if and only if $g(P) \neq 0$.*

[3] This is sometimes denoted $P_1 + P_2$ in the literature.

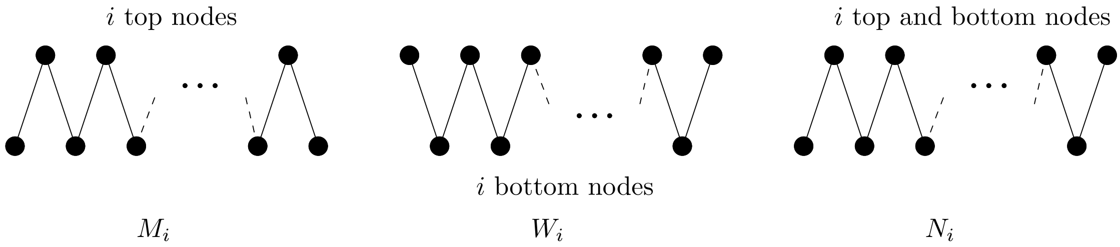


Figure 2: The zig-zag posets M_i, W_i, and N_i.

Proof. We prove it by induction. The base case is when $P = \emptyset$, where the claim holds trivially.

If $g(P) \neq 0$, then we know that $\exists x \in P$ such that $g(P_x) = 0$. Now, P_x must be a $\forall$-game by our inductive hypothesis. Hence, P is a $\exists$-game.

If $g(P) = 0$, then we know that $\forall x \in P$ $g(P_x) \neq 0$. So, for all $x \in P$, P_x is a $\exists$-game. Hence, P is a $\forall$-game. $\square$

Now, we will look at the Sprague-Grundy theorem applied to poset games, which relates $g(P_1 \parallel P_2)$ with $g(P_1)$ and $g(P_2)$ for any finite posets P_1 and P_2. For natural numbers x and y, let $x \oplus y$ denote the natural number whose binary representation is the bitwise XOR of the binary representations of x and y.

Theorem 2.4 (Sprague, Grundy [Spr36, Gru39])**.** $g(P_1 \parallel P_2) = g(P_1) \oplus g(P_2)$.

Corollary 2.5. $g(P_1) = g(P_2) \iff g(P_1 \parallel P_2) = 0 \iff P_1 \parallel P_2$ *is a* $\forall$*-game.*

Computing the g-number of a poset game is polynomial-time equivalent to finding its status: Finding the status from the g-number is trivial by the Observation above. For the other direction, by the Sprague-Grundy theorem, given a finite poset P with n points, $g(P)$ is the unique $g \in \{0, \ldots, n\}$ such that $(P \parallel C_g)$ is a $\forall$-game. Hence, our focus will be to compute the g-number of a given poset.

A poset P has k levels iff it can be partitioned into k antichains. As mentioned earlier, for three-level posets, finding the status is PSPACE-hard. So, a natural question is whether two-level posets are easy. In the next section we will give explicit formulas for the g-number for some specific classes of two-level posets.

3 Two-level posets

A two-level poset game can be represented by a two-layer graph, say top and bottom layers (both antichains), which partition the nodes into *top nodes* and *bottom nodes*, respectively. Any edge in the Hasse diagram joins a bottom node to a top node. Two nodes are comparable if and only if there is an edge between them, and moreover, if they are comparable then the top node is greater than the bottom node. Hence we can write $G = (B, T, E)$, where B is the set of bottom nodes, T is the set of top nodes, and E are the edges.

We first look at a particular class of two-level posets which just have a zig-zag pattern as shown in Figure 2. We can classify them into three categories, M_i, W_i, and N_i, depending on their starting and ending edge orientation. Here $i \geq 0$ represents the number of bottom nodes, or top nodes in case of M_i. We define M_0 and W_0 as a single node, and N_0 as the empty set. Their g-numbers follow a nice pattern which we give in the next theorem. This pattern was found independently by Rogers [Rog12].

Theorem 3.1. *The Grundy numbers of the zig-zag posets are given by the following formulas:*

(1) $g(M_i) = (i \bmod 2) + 1$

(2) $g(N_i) = \begin{cases} 4\lfloor \frac{i}{3} \rfloor + 2 & \text{if } i \equiv 1 \pmod 3, \\ 4\lfloor \frac{i}{3} \rfloor & \text{otherwise.} \end{cases}$

(3) $g(W_i) = 1$

Proof. We prove it by induction. The base cases are easy to verify.

For (1), let us construct the g-set for M_i for $i > 1$. Removing the first or the last bottom node leads to M_{i-1}. So, $((i-1) \bmod 2) + 1 \in g\text{-set}(M_i)$. Removing any other node leads to two M-posets. When the two posets have a different parity of top nodes, the g-number of the combined poset is $1 \oplus 2 = 3$ by Theorem 2.4. When they have same the parity of top nodes, the g-number is 0. Note that removing a top node from M_i leads to two M-posets of different top parity if and only if removing an inner bottom node leads to two M-posets of the same top parity. Hence, both cases are covered, and thus $0, 3 \in g\text{-set}(M_i)$. Hence, $g(M_i) = (i \bmod 2) + 1$.

For (2), the pattern given by the formula looks as follows

i	0	1	2	3	4	5	6	7	8	$\cdots$
$g(N_i)$	0	2	0	4	6	4	8	10	8	$\cdots$

We determine the g-set of N_i:

- Removing the leftmost bottom node from N_i leads to an N_{i-1}-poset, and removing the leftmost top node leads to an N_{i-1}-poset with one node in parallel. Hence, $g(N_{i-1}),\ g(N_{i-1}) \oplus 1 \in g\text{-set}(N_i)$.
- Let $2 \leq j \leq i$.
  - Removing the jth bottom node from left from N_i leads to an N_{i-j}-poset to the right of this node, and a M_{j-2}-poset to its left.
  - Removing the jth top node from left from N_i leads again to an N_{i-j}-poset to the right of this node, whereas we get a M_{j-1}-poset to its left.

  From formula (1) of the theorem, setting $k = i - j$, we obtain $g(N_k) \oplus 1,\ g(N_k) \oplus 2 \in g\text{-set}(N_i)$, for $k = 0, 1, \ldots, i-2$.

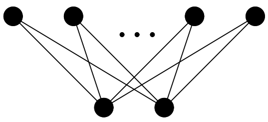
Hence we get that $g\text{-set}(N_i) = \{0, 1, \ldots, 4\lfloor \frac{i}{3} \rfloor - 1\} \cup S_i$, where

$$S_i = \begin{cases} \emptyset & \text{for } i \equiv 0 \pmod 3, \\ \{4\lfloor \frac{i}{3} \rfloor, 4\lfloor \frac{i}{3} \rfloor + 1\} & \text{for } i \equiv 1 \pmod 3, \\ \{4\lfloor \frac{i}{3} \rfloor + 1, 4\lfloor \frac{i}{3} \rfloor + 2, 4\lfloor \frac{i}{3} \rfloor + 3\} & \text{for } i \equiv 2 \pmod 3. \end{cases}$$

Hence, the g-number is as claimed in the theorem.

For (3), let us now construct the g-set for W_i. Removing any node from it would give us one or two N-posets. There always exists a middle bottom or top node whose removal will lead to the parallel union of two identical N-posets, which has g-number 0. Since by (2), the g-number of N-posets is always even, $1 \notin g\text{-set}(W_i)$. Hence, $g(W_i) = 1$. $\square$

t top nodes



$g = 2\,(t \bmod 2)$

Figure 3: A 2-level poset with 2 nodes at the bottom layer connected with all the t nodes in the top layer. The g-number of the poset is $2\,(t \bmod 2)$.

Now, let us look at another poset game. Figure 3 shows a particular class of two-level poset games where there are 2 bottom nodes and t top nodes, connected as a complete bipartite graph. One can show by induction that the g-number of such a game is $2\,(t \bmod 2)$. So, the g-number or the status only depends on the parity of the top nodes. Hence if we remove all but $t \bmod 2$ of the top nodes in the game shown in Figure 3, we get a game with the same g-number. Our next lemma shows how we can generalize this pattern for any two-level poset.

Let $P = (B, T, E)$ be a two-level poset game. For nodes $x \in B$ and $y \in T$ we define the set of neighbor nodes as

$$\begin{aligned} T_x &= \{\, v \in T \mid (x, v) \in E \,\} \\ B_y &= \{\, v \in B \mid (v, y) \in E \,\} \end{aligned}$$

Theorem 3.2. *Let $P = (B, T, E)$ be a two-level poset game and $S \subseteq T$ be a set of top nodes that all have the same set B_0 of neighbors, i.e., $B_y = B_0$ for all $y \in S$. Construct P' from P by removing all but $|S| \bmod 2$ of the nodes in S. Then $g(P) = g(P')$.*

Proof. By Corollary 2.5 it suffices to show that $g(P \parallel P') = 0$, i.e., $(P \parallel P')$ is a $\forall$-game. We consider the case that we have removed only two nodes from S. Then the claim follows by induction. Let us denote the two nodes removed from P by α and β. We show that the second player has a winning strategy in $(P \parallel P')$.

If the first player plays anything other than α or β, the second player just imitates the same move in the other game. If the first player removes α, the second player plays β, and vice versa. Note that by our assumption, at the first player's turn, either both nodes α and β are still there or none of them: since $B_\alpha = B_\beta = B_0$, any bottom node move will remove either both or none of them. Hence, the second player is always the last to play. $\square$

Theorem 3.2 tells us that for any two-level poset game we can assume that it has the following "extensional" property: any two top nodes have a different set of neighbors in B. All other cases can be reduced to this one. Such a poset is essentially a set system, that is, a collection $\{S_1, \ldots, S_t\}$ of subsets of some finite domain B. In the poset, the bottom points form the domain, and each top point corresponds to one of the sets S_i, containing those bottom points less than it. There are 2^{2^n} many set systems on a domain of size n, and so there are at most 2^{2^n} many two-level posets to consider with n bottom elements. Many of these are isomorphic, so the number of distinct posets is much fewer. For example, of the 256 extensional posets with 3 bottom elements, there are 40 isomorphism types.

A similar theorem can be obtained for the bottom nodes too, but a slightly weaker version. For any set $S \subseteq B$ such that all nodes in S have the same set of neighbors in T, we can remove all but one node from S, if $|S|$ is odd, and obtain an equivalent game. This is analogous to Theorem 3.2. However, if $|S| \geq 2$ is even, then we can remove all but two nodes from S. It does *not* follow that the cases $|S| = 2$ and $|S| = 0$ are equivalent. Therefore, $|S|$ is either 0, 1 or 2.

Theorem 3.3. *Let $P = (B, T, E)$ be a two-level poset game and $S \subseteq B$, $|S| \geq 3$, be a set of bottom nodes that all have the same set T_0 of neighbors, i.e., $T_y = T_0$ for all $y \in S$. Construct P' from P by removing two nodes from S. Then $g(P) = g(P')$.*

Proof. We show again that $g(P \parallel P') = 0$. We will describe the winning strategy for the second player.

The second player again just imitates the first player's moves in the other game while he can. If the first player removes a node, say from S in P, then the second player will remove a node from the corresponding set S' in P', or vice versa. At some point after the second player moves we will have $|S| = 2$ in P, and $|S'| = 0$ in the corresponding set in P'. By our assumption that $|S| \geq 3$ originally, we know that some node from S has already been removed, and so the set T_0 is also removed from both P and P' in some previous moves. Now if the first player then removes a node from S, the second player just removes the other node from S. As the set T_0 was already removed, now the games P and P' are exactly same, and the second player can continue his imitating strategy. So the second player is always the last player. □

4 Parity-uniform games

In this section, we present our other main result, which computes the g-number for a certain class of two-level games which we define below.

Fraenkel & Scheinerman [FS91] have studied a related class of games called hypergraph games. There, the game is a hypergraph, and a valid move is to either remove a vertex and all hyper-edges containing it or remove a hyper-edge and all hyper-edges containing it. A two-level poset can also be seen as a hypergraph, with bottom nodes being the vertices and the top nodes being the hyper-edges. A valid move here, however, would be a little different from that in a hypergraph game. A valid move is to either remove a vertex and hyper-edges incident on it or else remove a hyper-edge. Fraenkel & Scheinerman gave an explicit formula for the g-number for p-uniform, p-partite hypergraphs. A p-uniform p-partite hypergraph has a p-partition of its vertices such that every hyperedge is incident on exactly one vertex from each partition. This class of hypergraphs correspond to two-level posets where the bottom layer has p partitions and every top node is connected to exactly one node in each partition of the bottom layer. For this class, the hypergraph game and the poset game have the same rule for a valid move.

We generalize their result to a larger class of two-level posets defined as follows:

Definition 4.1. *We say that a two-level poset is* parity-uniform *if*

(i) all top nodes have the same degree parity, and

(ii) there is a bipartition of the bottom nodes such that every top node has an odd number of connections to at least one of the partitions.

Note that parity-uniformity is preserved under valid moves on the poset. This class of posets is clearly a generalization of the uniform p-partite hypergraphs. Note that when every top node has odd degree, then the poset trivially has property (ii) of Definition 4.1: we just take $(B, \emptyset)$ as the bipartition. So property (ii) is significant only in the other case, when every top node has an even degree, and in this case every top node has an odd number of connections to *both* partitions. The next theorem gives the g-number for this class of posets.

Theorem 4.2. *Let $P = (B, T, E)$ be a parity-uniform, two-level poset with every top node having degree parity $p \in \{0, 1\}$. Let $b = |B| \bmod 2$ and $t = |T| \bmod 2$. Then*

$$g(P) = b \ \oplus \ t\,(p \oplus 2).$$

Proof. We prove the formula by induction on $|P|$. In the base case when $|B| = 0$ and $|T| = 0$, the formula is trivially true. As parity-uniformity is preserved under a valid move in the poset, our inductive hypothesis is that the theorem holds for the game P' that we get after a single move. We consider four inductive cases, below. In each, we use b' and t' to denote the bottom and top parities in P', respectively.

- $b = 0$ and $t = 0$. We need to show that $g(P) = 0$. If we remove a bottom node from P to get P', then by the inductive hypothesis, we have $g(P') > 0$, since $b' = 1$. If we remove a top node, then $t' = 1$ and $b' = 0$, so $g(P') \geq 2$. Hence $g(P) = 0$.

- $b = 1$ and $t = 0$. We need to show that $g(P) = 1$. If we remove a top node, then $b' = 1$ and $t' = 1$. By induction, $g(P') \geq 2$. If we remove a bottom node with odd degree, we get a game P' with $b' = 0$ and $t' = 1$. So, again $g(P') \geq 2$. If we remove a bottom node with even degree (assuming for the moment that one exists), then $b' = 0$ and $t' = 0$, and by induction, $g(P') = 0$. Hence $g(P) = 1$.

  We claim that there always exists a bottom node with even degree. We note that $|E| \equiv pt \equiv 0 \pmod 2$. As there are an odd number of bottom nodes, and sum of their degrees should be even, there must be at least one bottom node with even degree.

In the next two cases, $t = 1$. Also, there is a subset $S \subseteq B$ of the bottom nodes such that every top node has an odd number of connections to S. Therefore, the number of edges incident on S, $|E(S)| \equiv t \cdot 1 \equiv 1 \pmod 2$. Thus there must be a node in S with odd degree.

- $b = 0$ and $t = 1$. If we remove a top node, then $b' = 0$ and $t' = 0$. Hence $g(P') = 0$. If we remove a bottom node with odd degree, which always exists, then $b' = 1$ and $t' = 0$. So, by induction, $g(P') = 1$. Now we consider two subcases:

  1. $p = 0$. If there is a bottom node with even degree, removing it would lead to $b' = 1$ and $t' = 1$, and thus $g(P') = 3$ by induction. Hence, $g(P) = 2$.
  2. $p = 1$. The total number of edges $|E| \equiv tp \equiv 1 \pmod 2$. So there must be odd number of bottom nodes with odd degree. And as $b = 0$, there is a bottom node with even degree. Removing it would lead to $b' = 1$ and $t' = 1$, and thus $g(P') = 2$ by induction. Hence $g(P) = 3$.

- $b = 1$ and $t = 1$. If we remove a top node then $b' = 1$ and $t' = 0$. So, $g(P') = 1$. If we remove a bottom node with odd degree, which always exists, we get $b' = 0$ and $t' = 0$, and thus $g(P') = 0$. Again we consider two subcases:

  1. $p = 1$. If there is a bottom node of even degree, removing it would lead to $b' = 0$, $t' = 1$ and $g(P') = 3$. Hence, $g(P) = 2$.

2. $p = 0$. The total number of edges $|E| \equiv tp \equiv 0 \pmod 2$. As $b = 1$, there must be a bottom node of even degree. Removing it would lead to $b' = 0$, $t' = 1$, and $g(P') = 2$. Hence $g(P) = 3$.

□

As already mentioned above, when every top node has odd degree, then the game is parity-uniform. Hence, with the same notation as in Theorem 4.2, we get the following special case:

Corollary 4.3. *Let $P = (B, T, E)$ be a two-level poset with every top node having odd degree. Then*

$$g(P) = b \oplus 3t.$$

5 Discussion

We have given an explicit formula for the g-numbers of some classes of two-level posets. As a consequence, we can determine in polynomial time whether these games are winning games. For general two-level posets, the question is still open. A natural extension of our work would be to weaken the hypotheses of Theorem 4.2 and still get an easy formula for the g-number.

We conjecture that two-level poset games can be solved in polynomial time, and consequently, that three levels are necessary for a reduction from PSPACE. But as long as there is no proof for this conjecture, one should also look at the other side of the fence, as it were, and look for a reduction from a NP- or PSPACE-complete language to two-level posets.

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