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On the sum of $L1$ influences

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Abstract

For a multilinear polynomial $p(x_1, ...x_n)$, over the reals, the $L1$-influence is defined to be $\sum_{i=1}^{n} \mathrm{E}_x \left[\frac{|p(x)-p(x^i)|}{2}\right]$, where x^i is x with i-th bit swapped. If p maps $\{-1,1\}^n$ to $[-1,1]$, we prove that the $L1$-influence of p is upper bounded by a function of its degree (and independent of n). This resolves affirmatively a question of Aaronson and Ambainis (Proc. Innovations in Comp. Sc., 2011).

We give an application for this theorem for maximal deviation of cut-value of graphs. We also present the connection between the sum of $L1$ influences and quantum query complexity which was the original context where Aaronson and Ambainis encountered this question.

1 Introduction

Let $p : \{-1,1\}^n \to \mathbb{R}$. The $L1$ influence of i-th variable of a function $p : \{-1,1\}^n \to \mathbb{R}$ is defined as

$$\mathrm{Inf}_i(p) = \mathrm{E}_x[|p(x) - p(x^i)|/2],$$

where x^i is x with i-th variable swapped. The total $L1$ influence is $\mathrm{Inf}(p) = \sum_{i=1}^{n} \mathrm{Inf}_i(p)$. For any function p we define $A_p = \max_x p(x) - \min_x p(x)$.

Every function $p : \{-1,1\}^n \to \mathbb{R}$ has a unique representation as a multilinear polynomial. Let

$$p(x) = \sum_{S \subseteq [n]} \beta_S \, \chi_S(x),$$

where $\chi_S(x) = \prod_{i \in S} x_i$, is the monomial corresponding to the set S, be such a representation of p. Let $\deg(p)$ be the degree of the polynomial of p or, in other words, the maximum of $|S|$ such that $\beta_S \neq 0$.

In [1] Aaronson and Ambainis asked whether the total $L1$ influence of p can be bounded by a polynomial in $\deg(p)$ and A_p. One of the motivations for this question is the analogy to well-known $L2$ version of this inequality. $L2$ version of this inequality says that if $p : \{-1,1\}^n \to \mathbb{R}$ is a degree-d polynomial such that $-1 \le p(x) \le 1$ for all $x \in \{-1,1\}^n$, then $\mathrm{Inf}^{sq}(p) \le d\,\mathrm{E}[p(x)^2] \le d$, where $\mathrm{Inf}^{sq}(p) = \sum_{i=1}^{n} \mathrm{Inf}_i^{sq}(p)$ and $\mathrm{Inf}_i^{sq}(p) = \mathrm{E}_x\left[\left(\frac{p(x)-p(x^i)}{2}\right)^2\right]$. For a proof of this simple fact and some of its applications, see the surveys by O'Donnel and de Wolf [5, 9] and also the paper by Shi [8] for an application in quantum complexity.

In this work we resolve the question Aaronson and Ambainis affirmatively.

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Theorem 1.1 *Let $p : \{-1,1\}^n \to \mathbb{R}$ and $\deg(p) = d$. Then we have*

$$\mathrm{Inf}(p) = O(A_p d^3 \log d).$$

In section 5, we show how we can use our inequality to give a proof of the following result of Erdös, Goldberg and Pach [4] on cut-deviation of the graphs.

Theorem 1.2 *Given a graph $G = (V, E)$ with density $\rho_G = |E|/\binom{n}{2}$ there always exist a cut (S, S^c) such that*

$$|\mathrm{E}(S, S^c) - \rho_G |S||S^c|| = \Omega(\min(\rho_G, 1 - \rho_G) n^{\frac{3}{2}}).$$

We prove this result by applying our inequality to the following polynomial,

$$g_G(x) := \frac{|E|}{2} - \rho_G \frac{|V|(|V|-1)}{4} + \frac{\rho_G}{2} \sum_{i<j} x_i x_j - (1/2 - \rho_G/2) \sum_{i \sim j} x_i x_j.$$

One nice feature of this example is that it shows that Theorem 1.1 can be more powerful than its $L2$ counterpart. More precisely, applying the $L2$ inequality to polynomial g_G would only show the existence of cuts with deviation $\Omega(n)$ rather than $\Omega(n^{3/2})$.

In section 6, we present the connection between sum of $L1$ influences and quantum query complexity which was the original context where the question of the relation between the degree and the sum of $L1$ influences originally arose. [1] This connection is as follows: Given a quantum algorithm that queries X and accepts X with probability $p(X)$, the number of queries will be at least $\Omega\left(\left(\frac{\mathrm{Inf}(p)}{\log(\mathrm{Inf}(p))}\right)^{1/3}\right)$. Improving our inequality will immediately improve this bound which in turn improves some of the theorems in Aaronson and Ambainis paper [1].

In the last section, we give some future directions and open problem related to sum of $L1$ influences.

2 Preliminaries

In this work, we use concepts from the analysis of function over the hypercubes $\{-1,1\}^n$. For an introduction to analysis of Boolean functions and its application to complexity theory we refer to the surveys [5, 9]. It is well-known that any function $f : \{-1,1\}^n \to \mathbb{R}$ can be represented as a polynomial with real coefficients over the monomials $\chi_S(x) = \prod_{i \in S} x_i$. The notion of influence of a variable is well-known in the context of the analysis of Boolean functions. For a Boolean valued function $g : \{-1,1\}^n \to \{-1,1\}$ the influence of ith coordinate is defined to be $\mathrm{inf}_i(g) = \mathrm{Pr}_x[g(x) \neq g(x^i)]$, where $x^i \in \{-1,1\}^n$ is the point x with ith coordinate flipped.

For more more general non-Boolean functions $p : \{-1,1\}^n \to \mathbb{R}$ the notion of influence is usually extended in $L2$ form by following definition,

$$\mathrm{Inf}_i^{sq}(p) = \mathrm{E}_x \left[\left(\frac{p(x) - p(x^i)}{2} \right)^2 \right]$$

In this work, we work with a different generalization of notion of influences to non-Boolean functions, that of $L1$ influences.

$$\mathrm{Inf}_i(p) = \mathrm{E}_x\left[\frac{|p(x) - p(x^i)|}{2}\right]$$

Although for Boolean functions $p : \{-1,1\}^n \to \{-1,1\}$ the $L2$ and $L1$ influences will always coincide, for a general bounded function $p : \{-1,1\}^n \to [-1,1]$ the sum of $L1$ influences can be much larger than sum of $L2$ influences.

This separation is one reason why upper bounding the sum of $L1$ influences in terms of degree is a harder problem that in the $L2$ case.

We will use the noise operator.

Definition 2.1 *Noise operator with rate $\rho \in \mathbb{R}$ applied to polynomial p is the following polynomial:*

$$T_\rho p(x) = \sum_{S \subseteq [n]} \beta_S \rho^{|S|} \chi_S(x).$$

We will later use the fact that for $|\rho| \leq 1$ we have $T_\rho\, p(x) = \mathrm{E}_{y \sim_\rho x}[p(y)]$, where $y \sim_\rho x$ means that for every i: $y_i = x_i$ with probability $(1+\rho)/2$ and $y_i = -x_i$ with the remaining probability.

3 The case of homogeneous polynomials

Recall that $p(x) = \sum_{S \subseteq [n]} \beta_s \chi_S(x)$ is a homogeneous polynomial if $\beta_S = 0$ for all S such that $|S| \leq \deg(p)$. In this section we prove the following theorem.

Theorem 3.1 *Let $p : \{-1,1\}^n \to \mathbb{R}$ of $\deg(p) = d$ be a homogeneous polynomial. Then we have,*

$$\mathrm{Inf}(p) = O(A_p d^2 \log d).$$

Let $p(x) = \sum_{R \subseteq [n]} \beta_R \chi_R(x)$ be homogeneous polynomial of degree d. Let $S \subseteq [n]$. Critical to our analysis is the following polynomial

$$q_S(x) = \sum_{R \subseteq [n] : |R \cap S| = 1} \beta_R \chi_R(x).$$

Lemma 3.2 *For all $S \subseteq [n]$,*

$$q_S(x) = O(A_p d \log d).$$

Proof

We define $v_\alpha \in \mathbb{R}^d$ such that for any $1 \leq k \leq d$: $(v_\alpha)_k = \mathrm{Pr}_P[|P \cap [k]| \equiv 1 (\mathrm{mod}\ 2)]$, where we choose set P by putting each $i \in [n]$ in it independently with probability α.

$$(v_\alpha)_k = \sum_{i : i \equiv 1 (\mathrm{mod}\ 2)} \alpha^i (1-\alpha)^{k-i} \binom{k}{i}$$

$$= \frac{1}{2}\left(((1-\alpha)+\alpha)^k - ((1-\alpha)-\alpha)^k)\right) = (1-(1-2\alpha)^k)/2.$$

Let $S \subseteq [n]$ and $S' \subseteq S$ be chosen by including every $i \in S$ independently with probability α. Then

$$A_p \geq \mathrm{E}_{S'}[p(x) - p(x^{S'})] = 2 \sum_{R \subseteq [n] : |R \cap S| \geq 1} \beta_R \cdot (v_\alpha)_{|R \cap S|} \chi_R(x).$$

We will choose $\alpha_1, \alpha_2, ..., \alpha_d$ and $x_1, x_2, ..., x_d$ such that $\sum_{i=1}^n v_{\alpha_i} x_i = \overrightarrow{e}$ and $\sum_{i=1}^d |x_i| = O(d \log d)$, where $\overrightarrow{e}$ is d-dimensional vector with the first entry 1 and with the remaining entries 0. This gives

$$q_S(x) = \sum_{R \subseteq [n] : |R \cap S| = 1} \beta_R \chi_R(x) = \sum_{i=1}^d x_i \sum_{R \subseteq [n] : |R \cap S| = 1} \beta_R \chi_R(x) (v_{\alpha_i})_{|R \cap S|}$$

$$\leq \frac{A_p}{2} \sum_{i=1}^d |x_i| = O(A d \log d).$$

For any $-1 \leq \gamma \leq 1$ we consider vectors v'_γ with $(v'_\gamma)_k = \gamma^k$ for $-1 \leq \gamma \leq 1$. Notice that since $v_{1/2}$ is the vector of all $1/2$ entries, $v'_\gamma = 2\left(v_{1/2} - v_{(1-\gamma)/2}\right)$. So instead of working with v_α's directly we instead choose to work with v'_γ.

If d is even we choose $\gamma_i = -1/2 + (i-1)/d$ for $i \leq d/2$ and $\gamma_i = (i - d/2)/d$ for $i > d/2$. If d is odd we choose $\gamma_i = -1/2 + (i - 1/2)/d$ for $i \leq (d-1)/2$ and $\gamma_i = (i + 1/2 - d/2)/d$ for $i \geq (d+1)/2$. Consider matrix M with v'_γ as columns. We have to solve $Mx = \overrightarrow{e}$. Notice that M is similar to Vandermonde matrix. Using Cramer's rule we obtain

$$|x_k| = \left| \frac{\gamma_1 ... \gamma_{k-1} \gamma_{k+1} ... \gamma_d}{\gamma_k} \frac{1}{\prod_{j \neq k} (\gamma_j - \gamma_k)} \right|.$$

Now because of the choice of γs we get $|\gamma_1 ... \gamma_{k-1} \gamma_{k+1} ... \gamma_d| \leq |\prod_{j \neq k} (\gamma_j - \gamma_k)|$. Thus $\sum_i |x_i| \leq \sum_i \frac{1}{|\gamma_i|} = O(d \log d)$. [1]

■

Now we shall prove Theorem 3.1.

Proof of Theorem 3.1

Consider

$$B = \mathrm{E}_S \left[\mathrm{E}_{x^{S^c}} \left[\sum_{i \in S} \left| \sum_{R : R \cap S = \{i\}} \beta_R \chi_{R \setminus \{i\}}(x) \right| \right] \right],$$

where in the first expectation we choose S by putting each $i \in [n]$ in it with probability $1/d$ and in the second expectation we choose values of variables in complement of S uniformly and independently at random. Now if for every $i \in S$ we choose

$$x_i = \mathrm{sgn}\left(\sum_{R : R \cap S = \{i\}} \beta_R \chi_{R \setminus \{i\}}(x) \right),$$

[1] Numerical experiments shows that if we choose $\gamma_k = \cos\left(\frac{k\pi}{d}\right)$ then $\sum_i |x_i| = d$ for odd d. Thus it is probably possible to remove log factor from our upper bound.

we can use the previous upper bound and conclude that $B = O(A_p d \log d)$ as well.

Now we lower bound B:

$$B = \frac{1}{d-1}\sum_{i=1}^{n} \mathrm{E}_{x,z}\left|\sum_{R:i\in R}\beta_R\chi_R(x)\chi_R(z)\right|$$
$$= \Omega\left(\frac{1}{d}\sum_{i=1}^{n}\mathrm{E}_x\left|\sum_{R:i\in R}\beta_R\chi_R(x)\right|\right),$$

where we choose each $z_i = 0$ with probability $1/d$ and 1 with the remaining probability. In the last equality we moved expectation over z inside the absolute value and then used $\mathrm{E}_z[\chi_R(z)] = (1-1/d)^d = \Omega(1)$. (We use the fact that there is no $\alpha_R \neq 0$ with $|R| < d$.)

Now it remains to notice that

$$\mathrm{Inf}(p) = \sum_{i=1}^{n}\mathrm{E}_x\left[|p(x)-p(x^i)|/2\right] = \sum_{i=1}^{n}\mathrm{E}_x\left|\sum_{R:i\in R}\beta_R\chi_R(x)\right|.$$

■

4 Proof of Theorem 1.1

Now we will modify the proof of Theorem 3.1 to solve the case of non-homogeneous polynomial (Theorem 1.1).

To prove the theorem we also need another lemma:

Lemma 4.1 *Let q be a degree-d polynomial in ρ such that $|q(\rho)| \leq 1$ for $-1 \leq \rho \leq 1$. Then the following equality holds: $q\left(\frac{1}{1-\frac{1}{d^2}}\right) = O(1)$.*

Lemma 4.1 follows from properties of Chebyshev polynomials and lemma of Paturi [6].

Proposition 4.2 ([7]) *Define $\mathcal{P}_d$ as follows*

$$\mathcal{P}_d = \{p \in \mathbb{R}[x] \mid \deg(p) \leq d\,,\ \max_{x\in[-1,1]}|p(x)| \leq 1\}$$

Then we have

$$\forall p \in \mathcal{P}_d, x \notin [-1,1] \qquad |p(x)| \leq |T_d(x)|$$

Where T_d is the d-th Chebychev polynomial of the first kind.

$$T_d(\rho) = \frac{1}{2}\left(\left(\rho+\sqrt{\rho^2-1}\right)^d + \left(\rho-\sqrt{\rho^2-1}\right)^d\right).$$

Lemma 4.3 (Paturi) *$T_d(1+\gamma) \leq e^{2d\sqrt{2\gamma+\gamma^2}}$ for all $\gamma \geq 0$.*

Proof [3] $T_d(1+\gamma) \le (1+\gamma+\sqrt{2\gamma+\gamma^2})^d \le (1+2\sqrt{2\gamma+\gamma^2})^d \le e^{2d\sqrt{2\gamma+\gamma^2}}$. ■

Proof of Theorem 1.1

Now Combining Lemma 3.2 and Lemma 4.1 with the fact that for all $-1 \le \rho \le 1$ we have $\max_x T_\rho p(x) \le \max_x p(x)$ and $\min_x T_\rho p(x) \ge \min_x p(x)$ (Those inequalities follows from $T_\rho p(x) = \mathrm{E}_{y\sim_\rho x}[p(y)]$.) we get that

$$\sum_{R\subseteq[n]:|R\cap S|=1} \beta_R\, \rho'^{|R|}\, \chi_R(x) = O(A_p d \log d),$$

where $\rho' = 1/(1-1/d^2)$. (We fix x and consider $T_\rho q_S(x)$ as a polynomial in ρ and then apply Lemma 4.1.)

Consider

$$B = \mathrm{E}_S\left[\mathrm{E}_{x^{S^c}}\left[\sum_{i\in S}\left|\sum_{R:R\cap S=\{i\}} \beta_R \rho'^{|R|} \chi_{R\setminus\{i\}}(x)\right|\right]\right],$$

where in the first expectation we choose S by putting each $i \in [n]$ in it with probability $1/d^2$ and in the second expectation we choose values of variables in complement of S uniformly and independently at random. Now if for every $i \in S$ we choose

$$x_i = \operatorname{sgn}\left(\sum_{R:R\cap S=\{i\}} \beta_R \rho'^{|R|} \chi_{R\setminus\{i\}}(x)\right),$$

we can use the previous upper bound and conclude that $B = O(A_p d \log d)$ as well.

Now we lower bound B:

$$\begin{aligned} B &= \frac{1}{d^2-1}\sum_{i=1}^{n} \mathrm{E}_{x,z}\left|\sum_{R:i\in R} \beta_R \rho'^{|R|} \chi_R(x)\chi_R(z)\right| \\ &= \Omega\left(\frac{1}{d^2}\sum_{i=1}^{n} \mathrm{E}_x\left|\sum_{R:i\in R} \beta_R \chi_R(x)\right|\right), \end{aligned}$$

where we choose each $z_i = 0$ with probability $1/d^2$ and 1 with the remaining probability. In the last equality we moved expectation over z inside the absolute value and then used $\mathrm{E}_z[\rho'^{|R|}\chi_R(z)] = \rho'^{|R|}(1-1/d^2)^{|R|} = 1$. ■

5 A Corollary on Maximal Deviation of Cut-value of Graphs

In this section we use a very special case of our original theorem to reprove a theorem of Erdös et al [4] on the maximum discrepancy of cut-values in graphs. In a graph $G = (V, E)$, by S^c we denote $V \setminus S$ and we write $u \sim v$ if and only if $(u, v) \in E$.

Definition 5.1 *For any graph $G = (V, E)$ and $0 \le p \le 1$ the cut-deviation $D_p(G)$ is the maximum over all cuts $(S, V \backslash S)$ of the discrepancy between the cut-value $|E(S, S^c)|$ and the expected cut-value $p|S|(|V| - |S|)$ (where we choose each edge independently with probability p), i.e.,*

$$D_p(G) = \max_{S \subseteq V} ||E(S, S^c)| - p|S||S^c|| .$$

We are interested in lower bounding the quantity $D_p(G)$. Given a $G = (V, E)$, let $\rho_G := |E|/\binom{|V|}{2}$ be the edge density. Notice that for any $p \ne \rho_G$ a random cut will already give a deviation of $\Omega(n^2)$ for $D_p(G)$. So the interesting case is when $p = \rho_G$. For this choice of the parameter we prove the following Theorem,

Theorem 5.2 *For every graph $G = (V, E)$,*

$$D_{\rho_G}(G) = \Omega(\min(\rho_G, 1 - \rho_G)n^{\frac{3}{2}}).$$

We note that the above inequality is tight as it follows by applying standard tail inequalities to Erodös-Renyi graphs $G(n, p)$. Moreover, the one-sides variants of this inequality

$$\max_{S \subseteq V} E(S, S^c) - \rho_G|S||S^c| = \Omega(\min(\rho_G, 1 - \rho_G)n^{\frac{3}{2}})$$

which holds for random graphs, does not hold in general as can be seen from the example of the complement of complete bipartite graph $K_{n/2,n/2}$.

To prove this result we will use the following lemma.

Lemma 5.3 *Let $G = (V, E)$. For any $S \subseteq V$ let $x_S \in \{-1, 1\}^{|V|}$ be such that $(x_S)_i = 1$ iff $i \in S$ (assume that $V = [n]$). Then*

$$g_p(x) := \frac{|E|}{2} - p\frac{|V|(|V| - 1)}{4} + \frac{p}{2}\sum_{i<j} x_i x_j - (1/2 - p/2)\sum_{i\sim j} x_i x_j$$

satisfies $g_p(x_S) = E(S, S^c) - p|S||S^c|$.

Proof

We check that $|E(S, S^c)| = 1/2|E| - 1/2\sum_{i\sim j} x_i x_j$ and $|S||S^c| = \frac{|V|(|V|-1)}{4} - \frac{1}{2}\sum_{i<j} x_i x_j$. ■

Proof of Theorem 5.2 Set $p := \rho_G$. First we notice that,

$$A_{g_p} = \max_x g_p(x) - \min_x g_p(x) \le 2\max_x |g_p(x)| = 2\max_{S\subseteq[n]} |E(S, S^c) - p|S||S^c|| = 2D_p,$$

where in the third equality we use the previous lemma.

Theorem 1.1 implies

$$\mathrm{Inf}(g_p) = \sum_{i=1}^{n} \mathrm{E}_x\left[\left|p/2\sum_{j\nsim i} x_j - (1-p)/2\sum_{j\sim i} x_j\right|\right] = O\Big(\max_{S\subseteq[n]} \big|E(S, S^c) - p|S||S^c|\big|\Big),$$

where we use the fact that $\deg(g_p) = 2$.

Now we just need to lower bound the left hand side. For a particular i we have,

$$\mathrm{E}_x \left[\left| p/2 \sum_{j \nsim i} x_j - (1-p)/2 \sum_{j \sim i} x_j \right| \right] = \Omega(\min(p, 1-p)\sqrt{n}),$$

where the last equality follows when we consider random walk of length $n-1$ with steps of length $p/2$ or $(1-p)/2$. ■

6 Relation to quantum query complexity

We use the following well-known lemma,

Lemma 6.1 *Let Q be a quantum algorithm that makes T queries to black-box $X : [n] \to \{-1, 1\}$. Then there exists a real-valued multilinear polynomial $p(X)$ of degree at most $2T$, which equals the acceptance probability of Q when it is run on a black box containing X.*

The proof of this Lemma 6.1 can be found for example in [3]. The idea of the proof is that if you look at the expansion of the state of the computation $|\psi\rangle$ in computational basis right after the kth query to X, the amplitude of each basis state is a polynomial of degree k in the coordinates of X. This fact can be easily shown as each query will only multiply the amplitudes by a linear factors and each unitary operation does not affect the degree. Since the final acceptance probability is proportional to the square of the amplitudes of accepting states the lemma follows. Now combining this lemma together with our result gives the following corollary,

Corollary 6.2 *Any quantum algorithm with acceptance probability $p(X)$ has to make $\Omega\left(\left(\frac{\mathrm{Inf}(p)}{\log(\mathrm{Inf}(p))}\right)^{1/3}\right)$ queries.*

7 Future Directions

The main open problem is to improve the bound in theorem 1.1. We believe that the bound is far from optimal. The best lower bound we know is achieved by a single Fourier character: $p(x) = \chi_S(x)$. For this polynomial equality $\mathrm{Inf}(p) = \deg(p)$ holds. We conjecture that $\mathrm{Inf}(p) \leq deg(p)$ for any p. We believe that by optimizing our techniques it might be possible to improve the bound 1.1 by at least a logarithmic factor. It would also be very interesting to flesh out the connection to quantum query complexity more. One possible candidate here is applications to quantum property testing.

We also believe that this inequality can have further application for discrepancy theory. The important fact here is that our result can be applied in black-box fashion in the sense that one just needs to find the appropriate polynomial that captures the combinatorics of the problem and estimate its $L1$ influence. We leave the exploration of applications of this inequality for discrepancy theory for future work.

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