Electronic Colloquium on Computational Complexity, Report No. 71 (2013)

# Inapproximability of Minimum Vertex Cover on $k$-Uniform $k$-Partite Hypergraphs*

Venkatesan Guruswami† Sushant Sachdeva ‡ Rishi Saket §

**Abstract**

We study the problem of computing the minimum vertex cover on $k$-uniform $k$-partite hypergraphs when the $k$-partition is given. On bipartite graphs ($k = 2$), the minimum vertex cover can be computed in polynomial time. For $k \geq 3$, this problem is known to be NP-hard. For general $k$, the problem was studied by Lovász [23], who gave a $\frac{k}{2}$-approximation based on the standard LP relaxation. Subsequent work by Aharoni, Holzman, and Krivelevich [1] showed a tight integrality gap of $\left(\frac{k}{2} - o(1)\right)$ for the LP relaxation.

We further investigate the inapproximability of minimum vertex cover on $k$-uniform $k$-partite hypergraphs and present the following results (here $\varepsilon > 0$ is an arbitrarily small constant):

- NP-hardness of obtaining an approximation factor of $\left(\frac{k}{4} - \varepsilon\right)$ for even $k$, and $\left(\frac{k}{4} - \frac{1}{4k} - \varepsilon\right)$ for odd $k$,
- NP-hardness of obtaining a nearly-optimal approximation factor of $\left(\frac{k}{2} - 1 + \frac{1}{2k} - \varepsilon\right)$, and,
- An optimal Unique Games-hardness for approximation within factor $\left(\frac{k}{2} - \varepsilon\right)$, showing the optimality of Lovász's algorithm if one assumes the Unique Games conjecture.

The first hardness result is based on a reduction from minimum vertex cover in $r$-uniform hypergraphs, for which NP-hardness of approximating within $r - 1 - \varepsilon$ was shown by Dinur *et al.* [8]. We include it for its simplicity, despite it being subsumed by the second hardness result. The Unique Games-hardness result is obtained by applying the results of Kumar *et al.* [22], with a slight modification, to the LP integrality gap due to Aharoni *et al.* [1]. The modification ensures that the reduction preserves the desired structural properties of the hypergraph.

The reduction for the nearly optimal NP-hardness result relies on the *Multi-Layered PCP* of [8], and uses a gadget based on biased Long Codes, which is adapted from the LP integrality gap of [1]. The nature of our reduction requires the analysis of several Long Codes with different biases, for which we prove structural properties of the so called *cross-intersecting* collections of set families, variants of which have been studied in extremal set theory.

*A preliminary version of these results appear in [13, 27].

†Computer Science Department, Carnegie Mellon University, PA, USA. guruswami@cmu.edu. Research supported in part by NSF CCF-1115525.

‡Department of Computer Science, Princeton University, NJ, USA. sachdeva@cs.princeton.edu. Research supported by NSF grants CCF-0832797 and CCF-1117309, and Sanjeev Arora's Simons Investigator Grant.

§IBM Thomas J. Watson Research Center, NY, USA. ris.saket@gmail.com

ISSN 1433-8092

# 1 Introduction

A $k$-uniform hypergraph $G = (V, E)$ consists of a set of vertices $V$ and a collection of hyperedges $E \subseteq 2^V$ such that each hyperedge $e \in E$ contains exactly $k$ vertices. A vertex cover for $G$ is a subset of vertices $\mathcal{V} \subseteq V$ such that every hyperedge $e \in E$ contains at least one vertex from $\mathcal{V}$, *i.e.*, $e \cap \mathcal{V} \neq \emptyset$. Equivalently, a vertex cover is a hitting set for the collection of hyperedges $E$. Thus, the complement of a vertex cover is a subset of vertices $\mathcal{I}$ such that no hyperedge $e \in E$ is contained inside $\mathcal{I}$, *i.e.*, $e \nsubseteq \mathcal{I}$. Such a set is called an *Independent Set*.

The $k$-HypVC problem requires us to compute the minimum size of a vertex cover in a $k$-uniform hypergraph $G$. It is an extremely well studied combinatorial optimization problem, especially on graphs ($k = 2$), and is known to be NP-hard. Indeed, the minimum vertex cover problem on graphs was one of Karp's original 21 NP-complete problems [19]. On the other hand, the simple greedy algorithm that picks a maximal collection of disjoint hyperedges and includes all vertices in the edges in the vertex cover gives a $k$-approximation. A $k$-approximation can also be obtained using the standard Linear Programming (LP) relaxation for the problem. The best algorithms known today achieve only a marginally better approximation factor of $(1 - o(1))k$ [18, 15].

On the intractability side, there have been several results. For the case $k = 2$, Dinur and Safra [9] obtained an NP-hardness of approximation factor of 1.36, improving on a $\frac{7}{6} - \varepsilon$ hardness by Håstad [14]. For general $k$, a sequence of successive works yielded improved NP-hardness factors: $\Omega(k^{1/19})$ by Trevisan [28]; $\Omega(k^{1-\varepsilon})$ by Holmerin [16]; $k - 3 - \varepsilon$ by Dinur, Guruswami, and Khot [7]; and the current best of $k - 1 - \epsilon$ due to Dinur, Guruswami, Khot, and Regev [8]. In [8], the authors build upon [7] and the work of Dinur and Safra [9]. Moreover, assuming Khot's Unique Games Conjecture (UGC) [20], Khot and Regev [21] showed an essentially optimal $k - \varepsilon$ inapproximability. This result was further strengthened in different directions by Austrin, Khot, and Safra [5], and by Bansal and Khot [6].

## 1.1 Vertex Cover on $k$-uniform $k$-partite Hypergraphs

In this paper, we study the minimum vertex cover problem on $k$-partite $k$-uniform hypergraphs, when the underlying partition is given. We denote this problem as $k$-HypVC-Partite. This is an interesting problem by itself and its variants have been studied for applications related to databases such as distributed data mining [10], schema mapping discovery [11], and optimization of finite automata [17]. On bipartite graphs ($k = 2$), by König's Theorem, computing the minimum vertex cover is equivalent to computing the maximum matching which can be done efficiently. For general $k$, the problem was studied by Lovász who, in his doctoral thesis [23], proved the following upper bound.

**Theorem 1.1** (Lovász [23])**.** *For every $k$-partite $k$-uniform hypergraph $G$: $\text{vc}(G)/\text{lp}(G) \leq k/2$, where $\text{vc}(G)$ denotes the size of the minimum vertex cover and $\text{lp}(G)$ denotes the value of the standard LP relaxation. This yields an efficient $k/2$ approximation for $k$-HypVC-Partite.*

Note that the standard LP relaxation does not utilize the knowledge of the $k$-partition and therefore, by Lovász's result, $\text{lp}(G)$ is a $k/2$ approximation to $\text{vc}(G)$ even when the $k$-partition is not known. The above upper bound was shown to be tight by Aharoni, Holzman, and Krivelevich [1], who proved the following theorem.

**Theorem 1.2** (Aharoni et al.[1]). *For every $k \geq 3$, there exists a family of $k$-partite $k$-uniform hypergraphs $G$ such that* $\textsc{vc}(G)/\textsc{lp}(G) \geq k/2 - o(1)$*. Thus, the integrality gap of the standard LP relaxation is* $k/2 - o(1)$.

A proof of the above theorem describing the integrality gap construction is included in the appendix.

On the hardness side, the problem was shown to be APX-hard in [17] and [11] for $k = 3$, which can be extended easily to $k \geq 3$.

## 1.2 Our Results

We present the following inapproximability results for $k$-HypVC-Partite. The results of Theorems 1.3 and 1.5 were presented in the conference paper by Guruswami and Saket [13], and the result of Theorem 1.4 appeared in the conference paper by Sachdeva and Saket [27].

**Theorem 1.3.** *For any $\epsilon > 0$ and $k \geq 5$, $k$-HypVC-Partite is NP-hard to approximate within a factor of $\frac{k}{4} - \epsilon$ for even $k$, and $\frac{k}{4} - \frac{1}{4k} - \varepsilon$ for odd $k$.*[1]

The above theorem is obtained via a reduction from the $k$-HypVC problem and using the $k - 1 - \varepsilon$ factor NP-hardness proved for it by Dinur *et al.* [8]. While the result itself is subsumed by the theorem below, the reduction is simple and instructive, and we present it along with the proof of Theorem 1.3 in Section 2.

**Theorem 1.4.** *For any $\epsilon > 0$ and integer $k \geq 4$, it is NP-hard to approximate $k$-HypVC-Partite to within a factor of $\frac{k}{2} - 1 + \frac{1}{2k} - \epsilon$.*

The NP-hardness factor obtained in the above theorem is off by at most an additive constant of 1 from the optimal for any $k \geq 4$. The hardness reduction is based on combining a new combinatorial gadget with the Multi-Layered PCP of Dinur *et al.* [8], and is analyzed using tools from extremal set theory. The hardness factor we obtain is within an additive constant of 1 from the algorithmic upper bound. The sub-optimality is due to fundamental limitations of currently known techniques and is analogous to that of the hardness factor for $k$-HypVC obtained in [8]. The extremal set-theoretic results, the Multi-Layered PCP, and the hardness reduction are presented in Sections 3, 4, and 5 respectively.

**Theorem 1.5.** *Assuming the Unique Games Conjecture of Khot [20], it is NP-hard to approximate $k$-HypVC-Partite to within a factor of $\frac{k}{2} - \varepsilon$ for any $\varepsilon > 0$.*

The above theorem follows by combining a *dictatorship gadget* with an instance of Unique Games via the approach followed in the work of Kumar, Manokaran, Tulsiani, and Vishnoi [22]. The gadget is derived from the LP integrality gap of Aharoni *et al.* [1]. As in most UGC based hardness results, the key element is the construction of an appropriate dictatorship gadget. The crucial observation is that the construction of the dictatorship gadget can be done in a way such that it preserves the structural properties (i.e. the $k$-partiteness) of the integrality gap instance. In Section 7 we describe the dictatorship gadget and its crucial properties. The hardness reduction from Unique Games, using this gadget, follows the standard approach for which we refer the reader to Kumar *et al.* [22].

[1]The conference version of [13] erroneously stated a factor $\frac{k}{4} - \varepsilon$ hardness also for odd values of $k$

### 1.3 Our Techniques

As noted above, the proof of Theorem 1.3 follows from a relatively simple gadget reduction from the $k-1-\varepsilon$ hardness of approximation result for $k$-HypVC in Dinur *et al.* [8]. In this informal description we shall focus on the techniques used to prove Theorems 1.4 and 1.5.

**Nearly Optimal NP-Hardness of $k$-HypVC-Partite**

It is helpful to first briefly review the hardness reduction of [8] for $k$-HypVC.

The main idea of their construction can be illustrated by the following *gadget*. Consider a finite domain $R$ and the set of all its subsets $\mathcal{H} = 2^R$. Sample subsets from $\mathcal{H}$ by choosing each element of $R$ with probability $1 - 1/k - \varepsilon$ (for some small $\varepsilon > 0$), and let the weight of each subset in $\mathcal{H}$ be its corresponding sampling probability, thus making the sum of all weights to be 1. The set $\mathcal{H}$ along with the associated weights is an example of a *biased Long Code* over $R$. Construct a $k$-uniform hypergraph over the vertex set $\mathcal{H}$ by adding an edge between any $k$ subsets whose intersection is empty. In this hypergraph it is easy to see that every element $r \in R$ yields a corresponding independent set (in the hypergraph) of weight $(1 - 1/k - \varepsilon)$, by choosing all subsets which contain that element. On the other hand, Dinur *et al.* [8] show via an analysis based on extremal set theory, that any independent set of weight $\varepsilon$ must contain $k$ subsets in $\mathcal{H}$ which have a small intersection, thus yielding a special *small* subset of $R$. This gap of $1 - 1/k - \varepsilon$ vs $\varepsilon$ for independent set corresponds to a gap of $1/k + \varepsilon$ vs $1 - \varepsilon$ for the complementary minimum vertex cover objective.

The construction of [8] combines the above Long Code based gadget with a new *Multi-Layered PCP*. This is a two variable CSP consisting of several *layers* of variables, and constraints between variables from each pair of layers. The work of [8] shows that it is NP-hard to find a labeling of the variables which satisfies a small fraction of the constraints between *any* two layers, even if there is a labeling that satisfies all the constraints of the instance. The reduction to a $k$-uniform hypergraph (as an instance of $k$-HypVC) involves replacing each variable of the PCP with a biased Long Code and adding the edges of the above gadget across different Long Codes.

The starting point for our hardness reduction for $k$-HypVC-Partite is – as in [8] – the Multi-Layered PCP. While we do not explicitly construct a standalone Long Code based gadget, our reduction can be thought of as adapting the integrality gap construction of Aharoni *et al.* [1] into a Long Code based gadget in a manner that preserves the $k$-uniformity and $k$-partiteness of the integrality gap.

Such transformations of integrality gaps into Long Code based gadgets have recently been studied in the works of Raghavendra [25] and Kumar *et al.* [22] who show this for a wide class of CSPs and their appropriate SDP and LP integrality gaps respectively. These Long Code based gadgets can be combined with a Unique Games instance to yield tight UGC based hardness results, where the reduction is analyzed via the Mossel's *Invariance Principle* [24]. Indeed, Theorem 1.5 combines the integrality gap of [1] with (a slight modification) of the approach of Kumar *et al.* [22] to obtain an optimal UGC based hardness result.

Our reduction, on the other hand, combines Long Codes with the Multi-Layered PCP instead of Unique Games and so we cannot adopt a Invariance Principle based analysis. Thus, in a flavor similar to that of [8], our analysis is via extremal combinatorics. However, our *gadget* involves several biased Long Codes with different biases and each hyperedge includes vertices from differently biased Long Codes, unlike the construction in [8]. The different biases are derived from the LP solution to the integrality gap of [1], in such a way that the gap obtained in the gadget corresponds to the value of the integrality gap.

For our analysis, we use structural properties of a *cross-intersecting* collection of set families. A collection of set families is *cross-intersecting* if any intersection of subsets – each chosen from a different family – is large. Variants of this notion have previously been studied in extremal set theory, see for example [2]. We prove an upper bound on the measure of the smallest family in such a collection. This enables a small vertex cover (in the hypergraph of our reduction) to be *decoded* into a good labeling to the Multi-Layered PCP.

**Optimal UGC Based Hardness**

As mentioned above, the techniques for the UGC based hardness result are based on the work of Kumar *et al.* [22], which shows that for a large class of monotone constraint problems – including hypergraph vertex cover – integrality gaps for a natural LP relaxation can be transformed into corresponding hardness of approximation results based on UGC.

The reduction given in (an earlier version of) [22] is analyzed using the general bounds on noise correlation of functions proved by Mossel [24]. For this purpose, the reduction perturbs a "good" solution, say $x^*$, to the LP relaxation for the integrality gap $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}})$, so that $x^*$ satisfies the property that all variables are integer multiples of some $\varepsilon > 0$. Therefore, the number of distinct values in $x^*$ is $m \approx 1/\varepsilon$. The reduction is based on a dictatorship test over the set $[m] \times \{0,1\}^r$ (for some parameter $r$) and the hardness of approximation obtained is related to the performance of a certain (efficient) rounding algorithm on $x^*$, which returns a solution no smaller than the optimum on $G_{\mathcal{I}}$. As described in (the earlier version of) [22] the reduction is not guaranteed to preserve structural properties of the integrality gap instance $G_{\mathcal{I}}$, such as $k$-partiteness.

We make the simple observation that the dictatorship test in the above reduction can analogously be defined over $V_{\mathcal{I}} \times \{0,1\}^r$ which then preserves the partiteness properties of $G_{\mathcal{I}}$ into the final instance. The gap obtained depends directly on the optimum in $G_{\mathcal{I}}$. This observation, combined with the results of [22] and the integrality gap for $k$-HypVC-Partite stated in Theorem 1.2 yields Theorem 1.5. We do not give a full proof of the above theorem; instead we describe the dictatorship test over $V_{\mathcal{I}} \times \{0,1\}^r$ in Section 7 and refer the reader to [22] for the proof. This builds upon two equivalent linear programming relaxations for $k$-HypVC-Partite given in Section 6, and the corresponding integrality gap example [1]. The latter of these relaxations, LP, given in Figure 2, is used to construct the dictatorship test. The integrality gap given in [1] satisfies the property that every value is an integral multiple of a certain $\varepsilon > 0$, which enables us to skip the perturbation step in the construction of the dictatorship test.

**Organization of the Paper.** The next section proves Theorem 1.3. Section 5 gives the hardness reduction and proof of Theorem 1.4, using the extremal set-theoretic results proved in Section 3 and the Multi-Layered PCP defined in Section 4. Building upon a simple equivalence of certain LP relaxations for minimum vertex cover proved in Section 6 and Theorem 1.2, Section 7 gives the dictatorship test used to prove Theorem 1.5. The hardness reduction from Unique Games and its analysis directly follow from the work of Kumar *et al.* [22] to which we refer the reader.

## 2 Reduction from $k$-HypVC to $k$-HypVC-Partite

Let $k$ and $m$ be positive integers such that $m \geq k \geq 2$. In this section, we first give a reduction from an instance of $k$-HypVC to a $m$-partite $k$-uniform hypergraph, and then a reduction from the latter to $k$-HypVC-Partite in Section 2.3 proving Theorem 1.3. Note that the parameter $k$ has different connotations in each of these steps.

### 2.1 Reduction to $m$-partite $k$-uniform hypergraph

Let $H = (U, F)$ be an instance of $k$-HypVC, i.e. $H$ is a $k$-uniform hypergraph with vertex set $U$, and $F$ as the set of hyperedges. The reduction constructs an instance $G = (V, E)$, where $G$ is a $k$-uniform, $m$-partite hypergraph, i.e. $V = \cup_{i=1}^{m} V_i$, where $V_i$ are $m$ disjoint subsets (color classes) such that every hyperedge in $E$ has at most one vertex from each subset. The main idea of the reduction is to let the new vertex set $V$ be the disjoint union of $m$ copies of $U$, and for every hyperedge $e' \in F$, add all hyperedges which contain exactly one copy (in $V$) of every vertex in $e'$, and at most one vertex from any of the $m$ copies of $U$ (in $V$). Clearly every hyperedge 'hits' any of the $m$ copies of $U$ in $V$ at most once, naturally giving an $m$-partition of $V$. It also ensures that if there is a vertex cover in $G$ which is the union of a subset of the copies of $U$, then it must contain at least $m - k + 1$ of the copies. Our analysis will essentially build upon this idea.

To formalize the reduction, we define some notation.

**Definition 2.1.** *Given a hyperedge $e' = \{u_1, \ldots, u_k\}$ in $F$, and a subset $I \subseteq [m]$ where $|I| = k$, a mapping $\sigma : I \mapsto \{u_1, \ldots, u_k\}$ is said to be a "$(I, e')$-matching" if $\sigma$ is a one-to-one map. Let $\Gamma_{I,e'}$ be the set of all $(I, e')$-matchings. Clearly, $|\Gamma_{I,e'}| = k!$ , for all $I \subseteq [m], |I| = k$, and $e' \in F$.*

The steps of the reduction are as follows:

1. For $i = 1, \ldots, m$, let $V_i = U \times \{i\}$.
2. For every hyperedge $e'$ in $F$, for every subset $I \subseteq [m]$ such that $|I| = k$ , for every $(I, e')$-matching $\sigma \in \Gamma_{I,e'}$, we add the hyperedge $e = e(e', I, \sigma)$ defined as follows.

$$\forall i \in [m], \qquad V_i \cap e = \begin{cases} (\sigma(i), i) & \text{if } i \in I, \\ \emptyset & \text{otherwise.} \end{cases} \tag{1}$$

The above reduction outputs the instance $G = (V, E)$, which is $m$-partite and $k$-uniform. Note that the vertex set $V$ is of size $m|U|$ and for every hyperedge $e' \in F$ the number of hyperedges added in $E$ is $\binom{m}{k} \cdot k!$. Therefore the reduction is polynomial time. In the next section, we analyze this reduction.

### 2.2 Analyzing the reduction

We prove the following theorem.

**Theorem 2.2.** *Let $C$ be the size of the optimal vertex cover in $H = (U, F)$, and let $C'$ be the size of the optimal vertex cover in $G = (V, E)$. Then,*

$$(m - (k - 1))C \leq C' \leq mC$$

The above theorem combined with the $k - 1 - \varepsilon$ inapproximability for $k$-HypVC given by [8] implies the following hardness factor for minimum vertex cover in $G$:

$$\frac{(m - (k - 1))C_1}{mC_2},$$

for some integers $C_1, C_2$ (depending on $H$) such that $C_1 \geq C_2(k - 1 - \varepsilon)$ for some $\varepsilon > 0$. It is easy to see that the above expression can be now be simplified to yield the following inapproximability.

**Theorem 2.3.** *For all $m \geq k \geq 2$ and $\varepsilon' > 0$, it is NP-hard to approximate the minimum size of a vertex cover on $m$-partite $k$-uniform hypergraphs to within a factor of*

$$\frac{(m - (k - 1))(k - 1)}{m} - \varepsilon'.$$

*Proof.* (of Theorem 2.2) We first show that there is a vertex cover of size at most $mC$ in $G$, where $C$ is the size of an optimal vertex cover $U^*$ in $H$. To see this consider the set $V^* \subseteq V$, where $V^* = U^* \times [m]$. For every hyperedge $e' \in F$, $e' \cap U^* \neq \emptyset$, and therefore $e \cap U^* \times \{i\} \neq \emptyset$, for some $i \in [m]$, for all $e = e(e', I, \sigma)$. Therefore, $V^* \cap e \neq \emptyset$ for all $e \in E$. The size of $V^*$ is $mC$, which proves the upper bound in Theorem 2.2. In the rest of the proof, we shall prove the lower bound in Theorem 2.2.

Let $S$ be the optimal vertex cover in $G$. Our analysis shall prove a lower bound on the size of $S$ in terms of the size of the optimal vertex cover in $H$. Let $S_i := V_i \cap S$ for $i \in [m]$. Before proceeding, we introduce the following definition. For every $Y \subseteq [m]$, we let $A_Y \subseteq U$ be the set of all vertices which have a copy in $S_i$ for some $i \in Y$. Formally,

$$A_Y := \{u \in U \mid \exists i \in Y \text{ s.t. } (u, i) \in S_i\}.$$

The following simple lemma follows from the construction of the edges $E$ in $G$.

**Lemma 2.4.** *Let $I \subseteq [m]$ be any subset such that $|I| = k$. Then $A_I$, is a vertex cover of the hypergraph $H$.*

*Proof.* Fix any subset $I$ as in the statement of the lemma. Let $e' \in F$ be any hyperedge in $H$. For a contradiction, assume that $A_I \cap e' = \emptyset$. This implies that the sets $S_i$ $(i \in I)$ do not have a copy of any vertex in $e'$. Now choose any $\sigma \in \Gamma_{I,e'}$ and consider the edge $e(e', I, \sigma) \in E$. This edge can be covered only by vertices in $V_i$ for $i \in I$. However, since $S_i$ does not contain a copy of any vertex in $e'$ for $i \in I$, the edge $e(e', I, \sigma)$ is not covered by $S$ which is a contradiction. This completes the proof. ■

The next lemma combines the previous lemma with the minimality of $S$ to show a strong structural statement for $S$, that any $S_i$ is "contained" in the union of any other $k$ sets $S_j$. It will enable us to prove that most of the sets $S_i$ are large.

**Lemma 2.5.** *Let $I \subseteq [m]$ be any set of indices such that $|I| = k$. Then, for any $j' \in [m]$, $S_{j'} \subseteq A_I \times \{j'\}$.*

*Proof.* Let $I$ be any choice of a set of $k$ indices in $[m]$ as in the statement of the lemma. From Lemma 2.4 we know that $A_I$ is a vertex cover in $H$ and is therefore non-empty. Let $j' \in [m]$ be an arbitrary index for which we shall verify the lemma for the above choice of $I$. If $j' \in I$, then the lemma is trivially true. Therefore, we may assume that $j' \notin I$. For a contradiction, assume that

$$(u, j') \in S_{j'} \setminus (A_I \times \{j'\}) \tag{2}$$

From the minimality of $S$, we deduce that there must be a hyperedge, say $e \in E$ such that $e$ is covered by $(u, j')$ and by no other vertex in $S$; otherwise $S \setminus \{(u, j')\}$ would be a smaller vertex cover in $G$. Now, $e = e(e', I', \sigma)$ for some $e' \in F$, $I' \subseteq [m]$ ($|I'| = k$), and $\sigma \in \Gamma_{I',e'}$. Since $(u, j')$ covers $e$, we obtain that $j' \in I'$ and $\sigma(j') = u \in e'$. Combining this with the fact that $j' \notin I$, and that $|I| = |I'| = k$, we obtain that $I \setminus I' \neq \emptyset$.

Let $j \in I \setminus I'$. We claim that $(u, j) \notin S_j$. To see this, observe that if $(u, j) \in S_j$ then $u \in A_I$, which would contradict our assumption in Equation (2).

We now consider the following hyperedge $\tilde{e} = \tilde{e}(e', \tilde{I}, \tilde{\sigma}) \in E$ where the quantities are defined as follows. The set $\tilde{I}$ simply replaces the index $j'$ in $I'$ with the index $j$, i.e.

$$\tilde{I} = (I' \setminus \{j'\}) \cup \{j\}. \tag{3}$$

Analogously, $\tilde{\sigma} \in \Gamma_{\tilde{I},e'}$ is identical to $\sigma$ except that it is defined on $j$ instead of $j'$ where $\tilde{\sigma}(j) = \sigma(j') = u$. Formally,

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } i \in \tilde{I} \setminus \{j\}, \\ u & \text{if } i = j. \end{cases} \tag{4}$$

Equations (3) and (4) imply the following,

$$V_i \cap \tilde{e} = V_i \cap e \qquad \forall i \in [m] \setminus \{j, j'\}, \tag{5}$$
$$V_j \cap \tilde{e} = (u, j), \tag{6}$$
$$V_{j'} \cap \tilde{e} = \emptyset. \tag{7}$$

Since $(u, j') \in S$ uniquely covers $e$, Equation (5) implies that $\tilde{e}$ is not covered by any vertex in $S_i$ for all $i \in [m] \setminus \{j, j'\}$. Moreover, since $j' \notin \tilde{I}$, no vertex in $S_{j'}$ covers $\tilde{e}$. On the other hand, by our assumption in Equation (2), $(u, j) \notin S_j$, which, along with Equation (6), implies that no vertex in $S_j$ covers $\tilde{e}$. Therefore, $\tilde{e}$ is not covered by $S$. This is a contradiction to the fact that $S$ is a vertex cover in $G$, and therefore our assumption in Equation (2) is incorrect. This implies that $S_{j'} \subseteq A_I \times \{j'\}$. This holds for every $j'$, thus proving the lemma. ∎

Note that the above lemma immediately implies the following corollary.

**Corollary 2.6.** *For every $I \subseteq [m], |I| = k$, we have $A_{[m]} = A_I$.*

It is easy to see the following simple lemma.

**Lemma 2.7.** *For any vertex $u \in A_{[m]}$, let $I_u \subseteq [m]$ be the largest set such that $u \notin A_{I_u}$. Then, $|I_u| < k$.*

*Proof.* Suppose the above does not hold. Then $I_u$ (or any subset of $I_u$ of size $k$) would violate Corollary 2.6, which is a contradiction. This completes the proof. ■

The above lemma immediately implies the desired lower bound on the size of $S$.

**Lemma 2.8.** *Let $C$ be the size of the optimal vertex cover in $H$. Then,*

$$|S| \geq (m - (k-1))C.$$

*Proof.* For convenience, let $q = |A_{[m]}|$. Note that, by Lemma 2.4 $A_{[m]}$ is a vertex cover in $H$. Therefore, $q \geq C$. From Lemma 2.7 we deduce that every vertex $u \in A_{[m]}$ has a copy $(u, i)$ in at least $m - (k-1)$ of the sets $S_i$. Therefore, $S$ contains $m - (k-1)$ copies of every vertex in $A_{[m]}$ which yields,

$$|S| \geq (m - (k-1))q \geq (m - (k-1))C,$$

thus completing the proof. ■

The above also completes the proof of the lower bound of Theorem 2.2. ■

### 2.3 Proof of Theorem 1.3

We prove Theorem 1.3 by giving a simple reduction from an instance $G = (V, E)$ of minimum vertex cover on $k$-partite $k'$-uniform hypergraphs to an instance $G' = (V', E')$ of $k$-HypVC-Partite where the parameter $k'$ will be chosen later.

Given $G = (V, E)$, construct $V'$ by adding $k$ "dummy" vertices $b_1, \ldots, b_k$ to $V$, i.e. $V' = V \cup \{b_1, \ldots, b_k\}$. Let $V_1, \ldots, V_k$ be the $k$ color classes of $V$. For any hyperedge $e \in E$, construct a corresponding hyperedge $e' \in E'$ which contains all the vertices in $e$ in addition to $b_i$ if $e \cap V_i = \emptyset$, for all $i \in [k]$. It is easy to see that $G'$ is a $k$-partite $k$-uniform hypergraph, with the $k$-partition given by the subsets $V_i \cup \{b_i\}$. As a final step, set the weight of the dummy vertices $b_1, \ldots, b_k$ to be much larger than $|V'|$ so that no dummy vertex is chosen in any optimal vertex cover in $G'$. This is because $V$ is always a vertex cover in $G$. Note that the hypergraph can be made unweighted by the (standard) technique of replicating each dummy vertex many times and multiplying the hyperedges appropriately.

Since no optimal vertex cover in $G'$ contains a dummy vertex, we deduce that an optimal vertex cover in $G'$ is an optimal vertex cover in $G$ and vice versa. From Theorem 2.3, for any $\varepsilon > 0$, we obtain a hardness factor of,

$$\frac{(k - (k'-1))(k'-1)}{k} - \varepsilon,$$

for approximating $k$-HypVC-Partite. Let $\alpha := \frac{(k'-1)}{k}$. The above expression is maximized in terms of $k$ when $\frac{(1-\alpha)\alpha}{2}$ attains a maximum where $\alpha \in [0, 1]$. Clearly, the maximum is obtained when $\alpha = \frac{(k'-1)}{k} = \frac{1}{2}$. Choosing $k' - 1 = \lfloor \frac{k}{2} \rfloor$ yields the hardness of approximation factor:

$$\frac{1}{k}\left(\left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{k}{2} \right\rfloor\right) - \varepsilon,$$

which proves Theorem 1.3.

## 3 Cross-Intersecting Set Families

We use the notation $[n] = \{1, \ldots, n\}$ and $2^{[n]} = \{F \mid F \subseteq [n]\}$. A subset of $2^{[n]}$ is called a *set family.* We begin by defining cross-intersecting set families:

**Definition 3.1.** *A collection of $k$ families $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$, is called $k$-wise $t$-cross-intersecting if for every choice of sets $F_i \in \mathcal{F}_i$ for $i = 1, \ldots, k$, we have $|F_1 \cap \ldots \cap F_k| \geq t$.*

In this section, we prove that if a collection of $k$ families $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$ is $k$-wise $t$-cross-intersecting, then at least one of the families is small in size, under an appropriately picked measure. Before we formally state the claim in Lemma 3.3, we define a family of $p$-biased measures on subsets of $[n]$ that we shall work with.

**Definition 3.2.** *Given a bias parameter $0 < p < 1$, we define the measure $\mu_p$ on subsets of $[n]$ as: $\mu_p(F) := p^{|F|} \cdot (1-p)^{n-|F|}$ . The measure of a family $\mathcal{F}$ is defined as $\mu_p(\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu_p(F)$.*

We can now state the main result of this section.

**Lemma 3.3.** *For arbitrary $\epsilon, \delta > 0$, there exists some $t = O\left(\frac{1}{\delta^2}\left(\log\frac{1}{\epsilon} + \log\left(1 + \frac{1}{2\delta^2}\right)\right)\right)$ such that the following holds: Given $k$ numbers $0 < q_i < 1$ such that $\sum_i q_i \geq 1$ and $k$ families, $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$, that are $k$-wise $t$-cross-intersecting, there exists a $j$ such that $\mu_{1-q_j-\delta}(\mathcal{F}_j) < \epsilon$.*

*Proof.* Let $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$ be a collection of $k$-wise $t$-cross-intersecting families. We will specify our choice of $t(\epsilon, \delta)$ at the end of the proof.

For the proof, we introduce an important technique for analyzing cross-intersecting families – the shift operation (see Def 4.1, pg. 1298 [12]). Given a family $\mathcal{F}$, define the $(i, j)$-shift as follows:

$$S_{ij}^{\mathcal{F}}(F) = \begin{cases} (F \cup \{i\} \backslash \{j\}) & \text{if } j \in F,\ i \notin F \text{ and } (F \cup \{i\} \backslash \{j\}) \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$$

Let the $(i, j)$-shift of a family $\mathcal{F}$ be defined as $S_{ij}(\mathcal{F}) = \{S_{ij}^{\mathcal{F}}(F) \mid F \in \mathcal{F}\}$. Given a family $\mathcal{F} \subseteq 2^{[n]}$, we repeatedly apply $(i, j)$-shift for $1 \leq i < j \leq n$ to $\mathcal{F}$ until we obtain a family that is invariant under these shifts. Such a family is called a *left-shifted family* and we will denote it by $S(\mathcal{F})$.

The following lemma, whose proof is included later in the section, shows that the cross-intersecting property is preserved under left-shifting.

**Lemma 3.4.** *Consider families $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$ that are $k$-wise $t$-cross-intersecting. Then, the families $S(\mathcal{F}_1), \ldots, S(\mathcal{F}_k)$ are also $k$-wise $t$-cross-intersecting.*

Moreover, by definition, there is a bijection between the sets in $\mathcal{F}$ and $S(\mathcal{F})$ that preserves the size of the set. Thus, for any fixed $p$, the measures of $\mathcal{F}$ and $S(\mathcal{F})$ are the same under $\mu_p$, i.e., $\mu_p(\mathcal{F}) = \mu_p(S(\mathcal{F}))$. Combining this observation with Lemma 3.4, it suffices to prove that there exists a $j$ such that $\mu_{1-q_j-\delta}(S(\mathcal{F}_j)) < \varepsilon$. Thus, we can assume that the families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are left-shifted (if not, we can replace then with $S(\mathcal{F}_1), \ldots, S(\mathcal{F}_k)$).

Next, we prove a key structural property about left-shifted cross-intersecting families which states that for at least one of the families, all of its subsets have a dense prefix. A similar fact for a single left-shifted family was shown in [12] (pg. 1311, Lemma 8.3), which was reproved and used in [8]. However, our case of multiple families with varying biases is different for which we need to prove the following combinatorial lemma.

**Lemma 3.5.** *Let $q_1, \ldots, q_k \in (0,1)$ be $k$ numbers such that $\sum_i q_i \geq 1$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^{[n]}$ be left-shifted families that are $k$-wise $t$-cross-intersecting for some $t \geq 1$. Then, there exists a $j \in [k]$ such that for all sets $F \in \mathcal{F}_j$, there exists a positive integer $r_F \leq n - t$ such that $|F \cap [t + r_F]| > (1 - q_j)(t + r_F)$.*

We defer the proof of this lemma to the end of this section. Assuming this lemma for now, we conclude that there must exist a $j$ such that for all sets $F \in \mathcal{F}_j$, there exists an $r_F$ such that $|F \cap [t + r_F]| > (1 - q_j)(t + r_F)$. We now apply a Chernoff bound argument to deduce that $\mu_{1-q_j-\delta}(\mathcal{F}_j)$ must be small.

Note that $\mu_{1-q_j-\delta}(\mathcal{F}_j)$ is equal to the probability that a random set $F$ chosen according to $\mu_{1-q_j-\delta}$ lies in $\mathcal{F}_j$. Thus, $\mu_{1-q_j-\delta}(\mathcal{F}_j)$ is bounded by the probability that for a random set $F$ chosen according to $\mu_{1-q_j-\delta}$, there exists an $r_F$ that satisfies $|F \cap [t + r_F]| \geq (1 - q_j)(t + r_F)$.

Chernoff bound states that for a set of $m$ independent Bernoulli random variables $X_i$, with $\Pr[X_i = 1] = 1 - q_j - \tau$,

$$\Pr\left[\sum_{i=1}^{m} X_i \geq (1 - q_j)m\right] \leq e^{-2m\tau^2}$$

Thus, we get that for any $r \geq 0$, $\Pr[|F \cap [t + r]| \geq (1 - q_j)(t + r)] \leq e^{-2(t+r)\delta^2}$. Summing over all $r$, we get that,

$$\mu_{1-q_j-\delta}(\mathcal{F}_j) \leq \sum_{r \geq 0} e^{-2(t+r)\delta^2} \leq \frac{e^{-2t\delta^2}}{1 - e^{-2\delta^2}} \leq e^{-2t\delta^2}\left(1 + \frac{1}{2\delta^2}\right).$$

Thus, if $t$ is large enough ($t = \Omega\left(\frac{1}{\delta^2}\left(\log\frac{1}{\epsilon} + \log\left(1 + \frac{1}{2\delta^2}\right)\right)\right)$ suffices), then $\mu_{1-q_j-\delta}(\mathcal{F})$ must be smaller than $\epsilon$. This completes the proof of Lemma 3.3. ■

We now give a proof of Lemma 3.4 that shows that left-shifting preserves the cross-intersecting property.

*Proof. (of Lemma 3.4)* Given the assumption, we will prove that $S_{ij}(\mathcal{F}_1), \ldots, S_{ij}(\mathcal{F}_k)$ are $k$-wise $t$-cross-intersecting. A simple induction then implies the statement of the lemma.

Consider arbitrary sets $F_i \in \mathcal{F}_i$. By our assumption, $|F_1 \cap \ldots \cap F_k| \geq t$. It suffices to prove that $|S_{ij}^{\mathcal{F}_1}(F_1) \cap \ldots \cap S_{ij}^{\mathcal{F}_k}(F_k)| \geq t$. If $j \notin F_1 \cap \ldots \cap F_k$, the claim is true since the only element being removed is $j$. Thus, for all $l \in [k]$, $j \in F_k$. If for all $l \in [k]$, $S_{ij}^{\mathcal{F}_l}(F_l) = F_l$, the claim is trivial. Thus, let us assume WLOG that $S_{ij}^{\mathcal{F}_1}(F_1) \neq F_1$. Thus, $i \notin F_1$ and hence $i \notin F_1 \cap \ldots \cap F_k$. Now, if $i \in S_{ij}^{\mathcal{F}_1}(F_1) \cap \ldots \cap S_{ij}^{\mathcal{F}_k}(F_k)$, we get that $j$ is replaced by $i$ in the intersection and we are done. Thus, we can assume WLOG that $i \notin S_{ij}^{\mathcal{F}_2}(F_2)$. This implies that $i \notin F_2$ and $F_2 \cup \{i\} \backslash \{j\} \in \mathcal{F}_2$. Now consider $F_1 \cap (F_2 \cup \{i\} \backslash \{j\}) \cap F_3 \cap \ldots \cap F_k$. Since we are picking one set from each $\mathcal{F}_i$, it must have at least $t$ elements, but this intersection does not contain $j$ and hence it is a subset of $S_{ij}^{\mathcal{F}_1}(F_1) \cap \ldots \cap S_{ij}^{\mathcal{F}_k}(F_k)$, implying that $|S_{ij}^{\mathcal{F}_1}(F_1) \cap \ldots \cap S_{ij}^{\mathcal{F}_k}(F_k)| \geq t$. ■

Finally, we give a proof of Lemma 3.5 that gives a structural property of cross-intersecting families.

*Proof. (of Lemma 3.5)* Let us assume to the contrary that for every $i \in [k]$, there exists a set $F_i \in \mathcal{F}_i$ such that for all $r \geq 0$, $|F_i \cap [t+r]| \leq (1-q_i)(t+r)$. The following combinatorial argument shows that the families $\mathcal{F}_i$ cannot be $k$-wise $t$-cross-intersecting.

Let us construct an arrangement of balls and bins where each ball is colored with one of $k$ colors. Create $n$ bins labeled $1, \ldots, n$. For each $i$ and for every $x \in [n] \backslash F_i$, we place a ball with color $i$ in the bin labeled $x$. Note that a bin can have several balls, but they must have distinct colors. Given such an arrangement, we can recover the sets it represents by defining $F_i^c$ to be the set of bins that contain a ball with color $i$.

For all $r$, our initial assumption implies that $|F_i^c \cap [t+r]| \geq q_i(t+r)$. Thus, there are at least $\lceil q_i(t+r) \rceil$ balls with color $i$ in bins labeled $1, \ldots, t+r$. The total number of balls in bins labeled $1, \ldots, t+r$ is,

$$\sum_{i=1}^{k} |F_i^c \cap [t+r]| \ \geq \ \sum_{i=1}^{k} \lceil q_i(t+r) \rceil \ \geq \ \sum_{i=1}^{k} q_i(t+r) \ \geq \ t+r \ \geq \ r+1,$$

where the last two inequalities follow using $\sum_i q_i \geq 1$ and $t \geq 1$.

Next, we describe a procedure to manipulate the above arrangement of balls.

**for** $r := 0$ to $n-t$
  **if** bin $t+r$ is empty
  **then if** a bin labeled from 1 to $t-1$ contains a ball **then** move it to bin $t+r$
    **else if** a bin labeled from $t$ to $t+r-1$ contains two balls **then** move one of them to bin $t+r$
      **else** output "error"

We need the following lemma.

**Lemma 3.6.** *The above procedure satisfies the following properties:*
*1. The procedure never outputs* error*.*
*2. At every step, any two balls in the same bin have different colors.*
*3. At step $r$, define $G_i^{(r)}$ to be the set of labels of the bins that do not contain a ball of color $i$. Then, for all $i \in [k]$, $G_i^{(r)} \in \mathcal{F}_i$.*
*4. After step $r$, the bins $t$ to $t+r$ have at least one ball each.*

*Proof.* 1. If it outputs error at step $r$, there must be at most $r$ balls in bins 1 to $t+r$. At the start of the procedure, there are at least $r+1$ balls in these bins and during the first $r$ steps, the number of balls in these bins remain unchanged. This is a contradiction.
2. Note that this is true at $r = 0$ and a ball is only moved to an empty bin, which proves the claim.
3. We first note that for all $i \in [k]$, $G_i^{(0)} \in \mathcal{F}_i$. Next, observe that for any left-shifted family $\mathcal{F} \subseteq 2^{[n]}$, and $F \in \mathcal{F}$ such that $i \notin F$ and $j \in F$ where $i < j$, the set $(F \cup \{i\} \backslash \{j\})$ must be in $\mathcal{F}$. Whenever we move a ball from bin $i$ to $j$, we have $i < j$. Since $\mathcal{F}_i$ are left-shifted, by repeated application of the above observation, we get that at step $r$, $G_i^{(r)} \in \mathcal{F}_i$.
4. Since the procedure never outputs error, at step $r$, if the bin $t+r$ is empty, the procedure places a ball in it while not emptying any bin labeled between $[t, t+r-1]$. This proves the claim. ∎

The above lemma implies that at the end of the procedure (after $r = n - t$), there is a ball in each of the bins labeled from $[t, n]$. Thus, the sets $G_i = G_i^{(n-t)}$ satisfy $\cap_i G_i \subseteq [t-1]$ and hence $|\cap_i G_i| \leq t - 1$. Also, we know that $G_i \in \mathcal{F}_i$. Thus, the families $\mathcal{F}_i$ cannot be $k$-wise $t$-cross-intersecting. This completes the proof of Lemma 3.5. ■

# 4 Multi-Layered PCP

In this section, we describe the Multi-Layered PCP constructed in [8] and its useful properties. An instance $\Phi$ of the Multi-Layered PCP is parametrized by integers $L, R > 1$. The PCP consists of $L$ sets of variables $X_1, \ldots, X_L$. The label set (or range) of the variables in the $l^{\text{th}}$ set $X_l$ is a set $R_{X_l}$, where $|R_{X_l}| = R^{O(L)}$. For any two integers $1 \leq l < l' \leq L$, the PCP has a set of constraints $\Phi_{l,l'}$ in which each constraint depends on one variable $x \in X_l$, and one variable $x' \in X_{l'}$. The constraint (if it exists) between $x \in X_l$ and $x' \in X_{l'}$ $(l < l')$ is denoted and characterized by a projection $\pi_{x \to x'} : R_{X_l} \to R_{X_{l'}}$. A labeling to $x$ and $x'$ satisfies the constraint $\pi_{x \to x'}$ if the projection (via $\pi_{x \to x'}$) of the label assigned to $x$ coincides with the label assigned to $x'$.

The following useful 'weak-density' property of the Multi-Layered PCP was defined in [8].

**Definition 4.1.** *An instance $\Phi$ of the Multi-Layered PCP with $L$ layers is* weakly-dense *if for any $\delta > 0$, given $m \geq \lceil \frac{2}{\delta} \rceil$ layers $l_1 < l_2 < \cdots < l_m$ and given any sets $S_i \subseteq X_{l_i}$, for $i \in [m]$ such that $|S_i| \geq \delta |X_{l_i}|$; there always exist two layers $l_{i'}$ and $l_{i''}$ such that the constraints between the variables in the sets $S_{i'}$ and $S_{i''}$ is at least $\frac{\delta^2}{4}$ fraction of the constraints between the sets $X_{l_{i'}}$ and $X_{l_{i''}}$.*

The following inapproximability of the Multi-Layered PCP was proven by Dinur et al. [8] based on the PCP Theorem ([4], [3]) and Raz's Parallel Repetition Theorem [26].

**Theorem 4.2.** *There exists a universal constant $\gamma > 0$ such that, for any integer parameters $L, R > 1$, there is a weakly-dense $L$-layered PCP $\Phi = \bigcup \Phi_{l,l'}$ such that it is NP-hard to distinguish between the following two cases:*

- **YES** Case: *There exists an assignment of labels to the variables of $\Phi$ that satisfies all the constraints.*
- **NO** Case: *For every $1 \leq l < l' \leq L$, not more that $1/R^{\gamma}$ fraction of the constraints in $\Phi_{l,l'}$ can be satisfied by any assignment.*

# 5 Hardness Reduction for HypVC-Partite

## 5.1 Construction of the Hypergraph

Fix an integer $k \geq 3$, an arbitrarily small parameter $\varepsilon > 0$, and let $r = \lceil 10\varepsilon^{-2} \rceil$. We shall construct a $(k+1)$-uniform $(k+1)$-partite hypergraph as an instance of $(k+1)$-HypVC-Partite. Our construction will be a reduction from an instance $\Phi$ of the Multi-Layered PCP with number of layers $L = 32\varepsilon^{-2}$, and parameter $R$ which shall be chosen later to be large enough. It involves creating, for each variable of the PCP, several copies of the Long Code endowed with different biased measures as explained below.

Over any domain $T$, a Long Code $\mathcal{H}$ is a collection of all subsets of $T$, i.e., $\mathcal{H} = 2^T$. A bias $p \in [0,1]$ defines a measure $\mu_p$ on $\mathcal{H}$ such that $\mu_p(v) = p^{|v|}(1-p)^{|T \setminus v|}$ for any $v \in \mathcal{H}$. In our construction, we need several different biased measures defined as follows. For all $j = 1, \ldots, r$, define $q_j := \frac{2j}{rk}$, and biases $p_j := 1 - q_j - \varepsilon$. Each $p_j$ defines a biased measure $\mu_{p_j}$ over a Long Code over any domain. Next, we define the vertices of the hypergraph.

**Vertices.** We shall denote the set of vertices by $V$. Consider a variable $x$ in the layer $X_l$ of the PCP. For $i \in [k+1]$ and $j \in [r]$, let $\mathcal{H}_{ij}^x$ be a Long Code on the domain $R_{X_l}$ endowed with the bias $\mu_{p_j}$, i.e., $\mu_{p_j}(v) = {p_j}^{|v|}(1-p_j)^{|R_{X_l} \setminus v|}$ for all $v \in \mathcal{H}_{ij}^x = 2^{R_{X_l}}$. The set of vertices corresponding to $x$ is $V[x] := \bigcup_{i=1}^{k+1} \bigcup_{j=1}^{r} \mathcal{H}_{ij}^x$. We define the weights on vertices to be proportional to its biased measure in the corresponding Long Code. Formally, for any $v \in \mathcal{H}_{ij}^x$,

$$\mathrm{wt}(v) := \frac{\mu_{p_j}(v)}{L|X_l|r(k+1)}. \tag{8}$$

The above conveniently ensures that for any $l \in [L]$,

$$\sum_{x \in X_l} \mathrm{wt}(V[x]) = 1/L \ , \quad \text{and} \quad \sum_{l \in [L]} \sum_{x \in X_l} \mathrm{wt}(V[x]) = 1 \ .$$

In addition to the vertices for each variable of the PCP, the instance also contains $k+1$ *dummy* vertices $d_1, \ldots, d_{k+1}$, each with a very large weight given by $\mathrm{wt}(d_i) := 2$ for $i \in [k+1]$. Clearly, this ensures that the total weight of all the vertices in the hypergraph is $2(k+1)+1$. As we shall see later, the edges shall be defined in such a way that, together with vertex weights, we would be guaranteed that the maximum weight independent set will contain all the dummy vertices. Before defining the edges we define the $(k+1)$ partition $(V_1, \ldots, V_{k+1})$ of $V$ to be:

$$V_i = \left( \bigcup_{l=1}^{L} \bigcup_{x \in X_l} \bigcup_{j=1}^{r} \mathcal{H}_{ij}^x \right) \bigcup \{d_i\}, \tag{9}$$

for all $i = 1, \ldots, k+1$. We now define the hyperedges of the instance. In the rest of the section, we shall think of the vertices of the long codes as subsets of their respective domains.

**Hyperedges.** For every pair of variables $x$ and $y$ of the PCP such that there is a constraint $\pi_{x \to y}$, we construct edges as follows.

(1.) Consider all permutations $\sigma : [k+1] \to [k+1]$ and sequences $(j_1, \ldots, j_k, j_{k+1})$ such that, $j_1, \ldots, j_k \in [r] \cup \{0\}$ and $j_{k+1} \in [r]$ such that: $\sum_{i=1}^{k} \mathbb{1}_{\{j_i \neq 0\}} \, q_{j_i} \geq 1$.

(2.) Add all possible hyperedges $e$ such that for all $i \in [k]$:

(2.a) If $j_i \neq 0$ then $e \cap V_{\sigma(i)} =: v_{\sigma(i)} \in \mathcal{H}_{\sigma(i), j_i}^x$, and,

(2.b) If $j_i = 0$ then $e \cap V_{\sigma(i)} = d_{\sigma(i)}$ and,

(2.c) $e \cap V_{\sigma(k+1)} =: u_{\sigma(k+1)} \in \mathcal{H}_{\sigma(k+1), j_{k+1}}^y$,

which satisfy,

$$\pi_{x \to y}\left( \bigcap_{\substack{i:\ i \in [k] \\ j_i \neq 0}} v_{\sigma(i)} \right) \bigcap u_{\sigma(k+1)} = \emptyset. \tag{10}$$

Let us denote the hypergraph constructed above by $G(\Phi)$. From the construction it is clear that $G(\Phi)$ is $(k+1)$-partite with partition $V = \cup_{i\in[k+1]} V_i$.

The role of the dummy vertices $\{d_1, \ldots, d_{k+1}\}$ is to ensure that each hyperedge contains exactly $k+1$ vertices – without them we would have hyperedges with fewer than $k+1$ vertices. Also note that the hyperedges are defined in such a way that the set $\{d_1, \ldots, d_{k+1}\}$ is an independent set in the hypergraph. Moreover, since the weight of each dummy vertex $d_i$ is 2, while the total weight of all except the dummy vertices is 1, this implies that any maximum independent set $\mathcal{I}$ contains all the dummy vertices. Thus, $V \setminus \mathcal{I}$ is a minimum vertex cover that does not contain any dummy vertices. For convenience, the analysis of our reduction, presented in the rest of this section, shall focus on the weight of $(\mathcal{I} \cap V) \setminus \{d_1, \ldots, d_{k+1}\}$.

The rest of this section is devoted to proving the following theorem which, together with Theorem 4.2, implies Theorem 1.4.

**Theorem 5.1.** *Let $\Phi$ be the instance of Multi-Layered PCP from which the hypergraph $G(\Phi)$ is derived as an instance of $(k+1)$-HypVC-Partite. Then,*

- *Completeness: If $\Phi$ is a YES instance, then there is an independent set $\mathcal{I}^*$ in $G(\Phi)$ such that,*

$$\mathrm{wt}\left(\mathcal{I}^* \cap (V \setminus \{d_1, \ldots, d_{k+1}\})\right) \geq 1 - \frac{1}{k} - 2\varepsilon.$$

- *Soundness: If $\Phi$ is a NO instance, then for all independent sets $\mathcal{I}$ in $G(\Phi)$,*

$$\mathrm{wt}\left(\mathcal{I} \cap (V \setminus \{d_1, \ldots, d_{k+1}\})\right) \leq 1 - \frac{k}{2(k+1)} + \varepsilon.$$

The completeness case of the above theorem is proved in Section 5.2, and the soundness case in Section 5.3.

## 5.2 Completeness

In the completeness case, the instance $\Phi$ is a YES instance, i.e., there is a labeling $A$ which maps each variable $x$ in layer $X_l$ to an assignment in $R_{X_l}$ for all $l = 1, \ldots, L$, such that all the constraints of $\Phi$ are satisfied.

Consider the set of vertices $\mathcal{I}^*$ which satisfies the following properties:
(1) $d_i \in \mathcal{I}^*$ for all $i = 1, \ldots, k+1$.
(2) For all $l \in [L]$, $x \in X_l$, $i \in [k+1], j \in [r]$,

$$\mathcal{I}^* \cap \mathcal{H}^x_{ij} = \{v \in \mathcal{H}^x_{ij} : A(x) \in v\}. \tag{11}$$

Suppose $x$ and $y$ are two variables in $\Phi$ with a constraint $\pi_{x\to y}$ between them. Consider any $v \in \mathcal{I}^* \cap V[x]$ and $u \in \mathcal{I}^* \cap V[y]$. The above construction of $\mathcal{I}^*$ along with the fact that the labeling $A$ satisfies the constraint $\pi_{x\to y}$ implies that $A(x) \in v$ and $A(y) \in u$ and $A(y) \in \pi_{x\to y}(v) \cap u$. Therefore, Equation (10) of the construction is not satisfied by the vertices in $\mathcal{I}^*$, and so $\mathcal{I}^*$ is an independent set in the hypergraph. By Equation (11), the fraction of the weight of the Long Code $\mathcal{H}_{ij}^x$ which lies in $\mathcal{I}^*$ is $p_j$, for any variable $x$, $i \in [k+1]$ and $j \in [r]$. Therefore,

$$\frac{\mathrm{wt}(\mathcal{I}^* \cap V[x])}{\mathrm{wt}(V[x])} = \frac{1}{r}\sum_{j=1}^{r} p_j = 1 - \frac{1}{k}\left(1 + \frac{1}{r}\right) - \varepsilon, \tag{12}$$

by our setting of $p_j$ in Section 5.1. The above yields that

$$\mathrm{wt}\left(\mathcal{I}^* \cap (V \setminus \{d_1, \ldots, d_{k+1}\})\right) = 1 - \frac{1}{k}\left(1 + \frac{1}{r}\right) - \varepsilon \geq 1 - \frac{1}{k} - 2\varepsilon, \tag{13}$$

for a small enough value of $\varepsilon > 0$, and our setting of the parameter $r$.

### 5.3 Soundness

For the soundness analysis we have that $\Phi$ is a NO instance as given in Theorem 4.2, and we wish to prove that the size of the maximum weight independent set in $G(\Phi)$ is appropriately small. For a contradiction, we assume that there is a maximum independent set $\mathcal{I}$ in $G(\Phi)$ such that,

$$\mathrm{wt}(\mathcal{I} \cap (V \setminus \{d_1, \ldots, d_{k+1}\})) \geq 1 - \frac{k}{2(k+1)} + \varepsilon. \tag{14}$$

Define the set of variables $X'$ to be as follows:

$$X' := \left\{ x \text{ a variable in } \Phi : \frac{\mathrm{wt}(\mathcal{I} \cap V[x])}{\mathrm{wt}(V[x])} \geq 1 - \frac{k}{2(k+1)} + \frac{\varepsilon}{2} \right\}. \tag{15}$$

An averaging argument shows that $\mathrm{wt}(\bigcup_{x \in X'} V[x]) \geq \varepsilon/2$. A further averaging implies that there are $\frac{\varepsilon}{4}L = \frac{8}{\varepsilon}$ layers of $\Phi$ such that $\frac{\varepsilon}{4}$ fraction of the variables in each of these layers belong to $X'$. Applying the Weak Density property of $\Phi$ given by Definition 4.1 and Theorem 4.2 yields two layers $X_{l'}$ and $X_{l''}$ ($l' < l''$) such that $\frac{\varepsilon^2}{64}$ fraction of the constraints between them are between variables in $X'$. The rest of the analysis shall focus on these two layers and for convenience we shall denote $X' \cap X_{l'}$ by $X$ and $X' \cap X_{l''}$ by $Y$, and denote the respective label sets by $R_X$ and $R_Y$.

Consider any variable $x \in X$. For any $i \in [k+1], j \in [r]$, call a Long Code $\mathcal{H}_{ij}^x$ *significant* if $\mu_{p_j}(\mathcal{I} \cap \mathcal{H}_{ij}^x) \geq \frac{\varepsilon}{2}$. From Equation (15) and an averaging argument we obtain that,

$$\left|\{(i,j) \in [k+1] \times [r] : \mathcal{H}_{ij}^x \text{ is } \textit{significant.}\}\right| \geq \left(1 - \frac{k}{2(k+1)}\right)(r(k+1)) = \frac{rk}{2} + r. \tag{16}$$

Using an analogous argument we obtain a similar statement for every variable $y \in Y$ and corresponding Long Codes $\mathcal{H}_{ij}^y$. The following structural lemma follows from the above bound.

**Lemma 5.2.** *Consider any variable $x \in X$. Then there exists a sequence $(j_1, \ldots, j_{k+1})$ with $j_i \in [r] \cup \{0\}$ for $i \in [k+1]$; such that the Long Codes $\{\mathcal{H}_{i,j_i}^x \mid i \in [k+1] \text{ where } j_i \neq 0\}$, are all* significant. *Moreover,*

$$\sum_{i=1}^{k+1} j_i \geq \frac{rk}{2} + r\ . \tag{17}$$

*Proof.* For all $i \in [k+1]$ choose $j_i$ as follows: if none of the Long Codes $\mathcal{H}^x_{ij}$ for $j \in [r]$ are *significant* then let $j_i := 0$, otherwise let $j_i := \max\{j \in [r] : \mathcal{H}^x_{ij} \text{ is } \textit{significant}\}$. It is easy to see that $j_i$ is an upper bound on the number of significant Long Codes in $\{\mathcal{H}^x_{ij}\}_j$. Therefore,

$$\sum_{i=1}^{k+1} j_i \;\geq\; \left|\{(i,j) \in [k+1] \times [r] : \mathcal{H}^x_{ij} \text{ is } \textit{significant.}\}\right| \;\geq\; \frac{rk}{2} + r \qquad \text{(From Equation (16))} \tag{18}$$

which proves the lemma. ∎

Next, we define the decoding procedure to define a label for any given variable $x \in X$.

### 5.3.1 Labeling for variable $x \in X$

The label $A(x)$ for each variable $x \in X$ is chosen independently via the following three step (randomized) procedure.

Step 1. Choose a sequence $(j_1, \ldots, j_{k+1})$ yielded by Lemma 5.2 applied to $x$.

Step 2. Choose an element $i_0$ uniformly at random from $[k+1]$.

Before describing the third step of the procedure, we require the following lemma.

**Lemma 5.3.** *There exist vertices $v_i \in \mathcal{I} \cap \mathcal{H}^x_{ij_i}$ for every $i : i \in [k+1] \setminus \{i_0\}, j_i \neq 0$, and an integer $t := t(\varepsilon)$ satisfying:*

$$\left| \bigcap_{\substack{i: i \in [k+1]\setminus\{i_0\},\\ j_i \neq 0}} v_i \right| < t \ . \tag{19}$$

*Proof.* Since $j_{i_0} \leq r$ it is easy to see,

$$\sum_{i \in [k+1]\setminus\{i_0\}} j_i \geq \frac{rk}{2} \;\Rightarrow\; \sum_{\substack{i: i \in [k+1]\setminus\{i_0\},\\ j_i \neq 0}} q_{j_i} \geq 1. \tag{20}$$

Moreover, since the sequence $(j_1, \ldots, j_{k+1})$ was obtained by Lemma 5.2 applied to $x$, we know that $\mu_{p_{j_i}}(\mathcal{I} \cap \mathcal{H}^x_{ij_i}) \geq \frac{\varepsilon}{2}$, $\forall i : i \in [k+1] \setminus \{i_0\}, j_i \neq 0$. Combining this with Equation (20) and Lemma 3.3 we obtain that for some integer $t := t(\varepsilon)$ the collection of set families $\{\mathcal{H}^x_{ij_i} : i \in [k+1] \setminus \{i_0\}, j_i \neq 0\}$ is not $k'$-wise $t$-cross-intersecting, where $k' = |\{i \in [k+1] \setminus \{i_0\} : j_i \neq 0\}|$. This proves the lemma. ∎

The third step of the labeling procedure is as follows:

Step 3. Apply Lemma 5.3 to obtain the vertices $v_i \in \mathcal{I} \cap \mathcal{H}^x_{ij_i}$ for every $i : i \in [k+1] \setminus \{i_0\}, j_i \neq 0$ satisfying Equation (19). Define $B(x)$ as,

$$B(x) := \bigcap_{\substack{i: i \in [k+1]\setminus\{i_0\},\\ j_i \neq 0}} v_i, \tag{21}$$

noting that $|B(x)| < t$. Assign a random label from $B(x)$ to the variable $x$ and call the assigned label $A(x)$.

### 5.3.2 Labeling for variable $y \in Y$

After labeling the variables $x \in X$ via the procedure above, we construct a labeling $A(y)$ for any variable $y \in Y$ by defining,

$$A(y) := \text{argmax}_{a \in R_Y} \left| \{ x \in X \cap N(y) \;\mid\; a \in \pi_{x \to y}(B(x)) \} \right|, \tag{22}$$

where $N(y)$ is the set of all variables that have a constraint with $y$. The above process selects a label for $y$ which lies in maximum number of projections of $B(x)$ for variables $x \in X$ which have a constraint with $y$.

The rest of this section is devoted to lower bounding the number of constraints satisfied by the labeling process, and thus obtaining a contradiction to the fact that $\Phi$ is a NO instance.

### 5.3.3 Lower bounding the number of satisfied constraints

Fix a variable $y \in Y$. Let $U(y) := X \cap N(y)$, i.e., the variables in $X$ which have a constraint with $y$. Further, define the set $P(y) \subseteq [k+1]$ as follows,

$$P(y) = \{ i \in [k+1] \;\mid\; \exists j \in [r] \text{ such that } \mu_{p_j}(\mathcal{I} \cap \mathcal{H}^y_{ij}) \geq \varepsilon/2 \}. \tag{23}$$

In other words, $P(y)$ is the set of all those indices in $[k+1]$ such that there is a *significant* Long Code corresponding to each of them. Applying Equation (16) to $y$ we obtain that there at least $\frac{r(k+2)}{2}$ *significant* Long Codes corresponding to $y$, and therefore $|P(y)| \geq \frac{k+2}{2} \geq 1$. Next we define subsets of $U(y)$ depending on the outcome of Step 2 in the labeling procedure for variables $x \in U(y)$. For $i \in [k+1]$ define,

$$U(i, y) := \{ x \in U(y) \;\mid\; i \text{ was chosen in Step 2 of the labeling procedure for } x \}, \tag{24}$$

and,

$$U^*(y) := \bigcup_{i \in P(y)} U(i, y). \tag{25}$$

Note that $\{U(i,y)\}_{i \in [k+1]}$ is a partition of $U(y)$. Also, since $|P(y)| \geq \frac{k+1}{2}$, and the labeling procedure for each variable $x$ chooses the index in Step 2 uniformly and independently at random, we have,

$$\mathbb{E}[|U^*(y)|] \geq \frac{|U(y)|}{2}, \tag{26}$$

where the expectation is over the random choice of the indices in Step 2 of the labeling procedure for all $x \in U(y)$. Before continuing, we need the following simple lemma (proved as Claim 5.4 in [8]).

**Lemma 5.4.** *Let $A_1, \ldots, A_N$ be a collection of $N$ sets, each of size at most $T \geq 1$. If there are not more than $D$ pairwise disjoint sets in the collection, then there is an element that is contained in at least $\frac{N}{TD}$ sets.*

Now consider any $i' \in P(y)$ such that $U(i', y) \neq \emptyset$, and a variable $x \in U(i', y)$. Since $i' \in P(y)$, there is a *significant* Long Code $\mathcal{H}^y_{i'j'}$ for some $j' \in [r]$. Furthermore, since $\mathcal{I}$ is an independent set there cannot be a $u \in \mathcal{I} \cap \mathcal{H}^y_{i',j'}$ such that $\pi_{x\to y}(B(x)) \cap u = \emptyset$, otherwise the following set of $k+1$ vertices,

$$\{v_i \ \mid \ i \in [k+1] \setminus \{i'\}, j_i \neq 0\} \cup \{d_i \ \mid \ i \in [k+1] \setminus \{i'\}, j_i = 0\} \cup \{u\}$$

form an edge in $\mathcal{I}$, where $v_i, j_i$ ($i \in [k+1]$) are as constructed in the labeling procedure for $x$.

Consider the collection of sets $\pi_{x\to y}(B(x))$ for all $x \in U(i', y)$. Clearly, each set is of size less than $t$. Let $D$ be the maximum number of pairwise disjoint sets in this collection. Each disjoint set independently reduces the measure of $\mathcal{I} \cap \mathcal{H}^y_{i',j'}$ by a factor of $(1-(1-p_{j'})^t)$. However, since $\mu_{p_{j'}}(\mathcal{I} \cap \mathcal{H}^y_{i',j'})$ is at least $\frac{\varepsilon}{2}$, this implies that $D$ is at most $\log(\frac{\varepsilon}{2})/\log(1-(2/rk)^t)$, since $p_{j'} \leq 1 - \frac{2}{rk}$. Moreover, since $t$ and $r$ depends only on $\varepsilon$, the upper bound on $D$ also depends only on $\varepsilon$.

Therefore by Lemma 5.4, there is an element $a \in R_Y$ such that $a \in \pi_{x\to y}(B(x))$ for at least $\frac{1}{Dt}$ fraction of $x \in U(i', y)$. Noting that this bound is independent of $j'$ and that $\{U(i', y)\}_{i' \in P(y)}$ is a partition of $U^*(y)$, we obtain that there is an element $a \in R_Y$ such that $a \in \pi_{x\to y}(B(x))$ for at least $\frac{1}{(k+1)Dt}$ fraction of $x \in U^*(y)$. Therefore, in Step 3 of the labeling procedure when a label $A(x)$ is chosen uniformly at random from $B(x)$, in expectation, $a = \pi_{x\to y}(A(x))$ for at least $\frac{1}{(k+1)Dt^2}$ fraction of $x \in U^*(y)$. Combining this with Equation (26) gives us that there is a labeling to the variables in $X$ and $Y$, which satisfies at least $\frac{1}{2(k+1)Dt^2}$ fraction of the constraints between variables in $X$ and $Y$ which is in turn at least $\frac{\varepsilon^2}{64}$ fraction of the constraints between the layers $X_{l'}$ and $X_{l''}$. Since $D$ and $t$ depend only on $\varepsilon$, choosing the parameter $R$ of $\Phi$ to be large enough we obtain a contradiction to our assumed lower bound on the size of the independent set. Therefore in the Soundness case, for any independent set $\mathcal{I}$,

$$\mathrm{wt}(\mathcal{I} \cap (V \setminus \{d_1, \ldots, d_{k+1}\})) \leq 1 - \frac{k}{2(k+1)} + \varepsilon.$$

Combining the above with Equation (13) of the analysis in the Completeness case yields a factor $\frac{k^2}{2(k+1)} - \delta$ (for any $\delta > 0$) hardness for approximating $(k+1)$-HypVC-Partite .

Thus, we obtain a factor $\frac{k}{2} - 1 + \frac{1}{2k} - \delta$ hardness for approximating $k$-HypVC-Partite.

## 6 LP Relaxations for Hypergraph Vertex Cover

In this section, we give two natural linear programming (LP) relaxations for Hypergraph Vertex Cover and show that they are equivalent. The first one, $\mathtt{LP}_0$, is the natural linear programming relaxation for vertex cover, while the second relaxation, $\mathtt{LP}$, was used in [22] to convert integrality gaps for it into corresponding UGC based hardness of approximation results. On the other hand, Aharoni *et al.* [1] give integrality gaps for $\mathtt{LP}_0$ applied to instances of different variants of the hypergraph vertex cover problem. This motivates us to present a fairly simple argument that the two relaxations are indeed equivalent, and the integrality gaps for $\mathtt{LP}_0$ hold for $\mathtt{LP}$ as well. Note that the relaxations are oblivious to the structure of the hypergraph.

Let $G = (V, E)$ be a $k$-uniform hypergraph. Let $x_v$ be a real variable for every vertex $v \in V$. The first relaxation, $\mathtt{LP}_0$, the natural relaxation for vertex cover in hypergraphs and is given in Figure 1. Before proceeding, let us define the set $Q(k) := \{\sigma \in \{0,1\}^k \ \mid \ \sigma \neq \mathbf{0}\}$. Informally speaking, $Q(k)$ is the set of all valid assignments to the $k$ vertices of any hyperedge in $E$ in any

$$\min \sum_{v \in V} x_v \tag{27}$$

subject to,

$$\forall e = \{v_1, \dots, v_k\} \in E, \quad \textstyle\sum_{v_i \in e} x_{v_i} \geq 1 \tag{28}$$

$$\forall v \in V, \quad 1 \geq x_v \geq 0. \tag{29}$$

Figure 1: Relaxation $\mathtt{LP}_0$.

integral solution to the LP for $G = (V, E)$. For the relaxation, we can constrain the $k$-tuple of variables for every hyperedge to lie inside the convex hull of $Q(k)$. Keeping this in mind we write the relaxation LP which is given in Figure 2. We now prove the equivalence of the above relaxations.

$$\min \sum_{v \in V} x_v \tag{30}$$

subject to,

$$\forall e = \{v_1, \dots, v_k\} \in E, \quad (x_{v_1}, \dots, x_{v_k}) = \textstyle\sum_{\sigma \in Q(k)} \lambda^e_\sigma \sigma \tag{31}$$

$$\forall e \in E, \quad \textstyle\sum_{\sigma \in Q(k)} \lambda^e_\sigma = 1 \tag{32}$$

$$\forall e \in, \sigma \in Q(k), \quad 1 \geq \lambda^e_\sigma \geq 0 \tag{33}$$

$$\forall v \in V, \quad 1 \geq x_v \geq 0. \tag{34}$$

Figure 2: Relaxation LP.

**Lemma 6.1.** *The relaxations* $\mathtt{LP}_0$ *and* LP *are equivalent.*

*Proof.* We observe the constraints (31)-(34) are equivalent to $(x_{v_1}, \dots, x_{v_k}) \in \mathrm{conv}(Q(k))$, where $\mathrm{conv}(A)$ is the convex hull of a set $A$ of vectors.

Furthermore, the set $\{0,1\}^k \cap \{(y_1, \dots, y_k) \in \mathbb{R}^k \mid \sum_{i \in [k]} y_i = 1\}$ consists of exactly the $k$ unit coordinate vectors. Therefore, the corner points of the (bounded) polyhedron $\mathrm{conv}(\{0,1\}^k) \cap \{(y_1, \dots, y_k) \in \mathbb{R}^k \mid \sum_{i \in [k]} y_i \geq 1\}$ are exactly the elements of $Q(k)$. This implies,

$$\mathrm{conv}(Q(k)) = \mathrm{conv}(\{0,1\}^k) \cap \{(y_1, \dots, y_k) \in \mathbb{R}^k \mid \textstyle\sum_{i \in [k]} y_i \geq 1\}.$$

Observing that the constraints (28) and (29) are equivalent to

$$(x_{v_1}, \ldots, x_{v_k}) \in \mathrm{conv}(\{0,1\}^k) \cap \{(y_1, \ldots, y_k) \in \mathbb{R}^k \mid \sum_{i \in [r]} y_i \geq 1\},$$

we obtain that the two relaxations $\mathtt{LP}_0$ and $\mathtt{LP}$ are equivalent, thus proving Lemma 6.1. ■

We now restate the integrality gap of [1], augmenting Theorem 1.2 with the fact that the lower bound holds also for the relaxation $\mathtt{LP}$.

**Theorem 6.2** (Aharoni *et al.* [1])**.** *The integrality gap of* $\mathtt{LP}_0$ *and of* $\mathtt{LP}$ *for an instance of* $k$-HypVC-Partite *is at least* $k/2 - o(1)$.

## 7 Construction of the Dictatorship Test for $k$-HypVC-Partite

Before we begin, let us restate a more detailed version of Theorem 6.2 which abstracts out some useful properties of the integrality gap instance. The construction of the integrality gap is included in Appendix A.

**Theorem 7.1.** *For any positive integers $k, m$ ($k \geq 2$), there is an instance $G_k = (V_k, E_k)$ of $k$-HypVC-Partite on $O(mk^2)$ vertices such that there exists a solution $\{x_v^*\}_{v \in V_k}$ to* $\mathtt{LP}$ *applied to $G_k$ such that,*

$$\frac{\mathrm{OPT}_{\mathrm{VC}}(G_k)}{\mathrm{VAL}_{\mathtt{LP}}(\{x_v^*\}_{v \in V_k}, G_k)} \geq \frac{k}{2} - O\left(\frac{1}{m}\right), \tag{35}$$

*where $\mathrm{OPT}_{\mathrm{VC}}(G_k)$ is the optimal size of the vertex cover in $G_k$, and $\mathrm{VAL}_{\mathtt{LP}}(\{x_v^*\}_{v \in V_k}, G_k)$ is the value of the solution $\{x_v^*\}_{v \in V_k}$ for the relaxation* $\mathtt{LP}$ *on the instance $G_k$. Moreover, the value of any $x_v^*$ ($v \in V_k$) is an integral multiple of $\varepsilon := \frac{1}{mk}$.*

For convenience, we drop the subscript and denote as $G = (V, E)$ the integrality gap instance of $k$-HypVC-Partite, which we shall utilize for the rest of the reduction along with the solution $\{x_v^*\}_{v \in V}$ given in Theorem 7.1 to the relaxation $\mathtt{LP}$ for the minimum vertex cover in $G$. Before proceeding to the reduction we need the following definitions.

**Definition 7.2.** *For every hyperedge $e \in E$, let $P_e^{x^*}$ be the distribution induced on $Q(k)$ by choosing $\sigma \in Q(k)$ with probability $\lambda_\sigma^e$, where the values $\{\lambda_\sigma^e\}_{e \in E, \sigma \in Q(k)}$ are obtained along with $x^*$ as the solution to the relaxation* $\mathtt{LP}$. *Let $M_\delta(P_e^{x^*})$ be a distribution over $\{0,1\}^k$ obtained by sampling $z \in \{0,1\}^k$ from $P_e^{x^*}$ and then independently letting every coordinate $z_i$ remain unchanged with probability $1 - \delta$ and setting it to be 1 with probability $\delta$.*

Note that the support of $P_e^{x^*}$ is a subset of $Q(k)$. Since $Q(k)$ is a monotone set, the support of $M_\delta(P_e^{x^*})$ is also a subset of $Q(k)$.

The following lemma were proved in [22].

**Lemma 7.3** ([22])**.** *Since $x^*$ satisfies the property that every $x_v^*$ is an integral multiple of $\varepsilon = \frac{1}{mk}$, there exists a set of values $\{\lambda_k^e\}_{e \in E}$ which along with $x^*$ forms a solution to the relaxation* $\mathtt{LP}$ *for $G$, with the property that for every $e \in E$, the minimum probability of any atom in $P_e^{x^*}$ is at least $\frac{\varepsilon}{(2^k)!}$.*

For the rest of this section, we assume that the property given by the above lemma is satisfied by the values $\{\lambda^e_\sigma\}_{e\in E,\sigma\in Q(k)}$ associated with the solution $x^*$. The following lemma is a simple consequence which we state without proof.

**Lemma 7.4** ([22]). *For any hyperedge $e \in E$, the minimum probability of any atom in the distribution $M^{x^*}_e$ is at least $\frac{\varepsilon\delta^k}{(2^k)!}$.*

For a given parameter $r$, the construction of a dictatorship test shall use the integrality gap $G = (V, E)$ of $k$-HypVC-Partite along with the solution $x^*$ for the relaxation LP given by Theorem 7.1, to output an instance $\mathcal{D}$ of $k$-HypVC-Partite consisting of a weighted $k$-uniform $k$-partite hypergraph $G_{\mathcal{D}} = (V_{\mathcal{D}}, E_{\mathcal{D}})$ where every vertex in $V_{\mathcal{D}}$ is indexed by $V \times \{0,1\}^r$.

Informally, the instance $\mathcal{D}$ serves as a dictatorship test in the following manner:

- (Completeness) Every *dictator* boolean function on $\{0,1\}^r$ gives a vertex cover in $G_{\mathcal{D}}$ of weight $\approx (\mathrm{VAL}_{\mathtt{LP}}(x^*, G))/(|V|)$.
- (Soundness) Every vertex cover of weight substantially smaller than $(\mathrm{OPT}_{\mathrm{VC}}(G_k))/(|V|)$ is "close to a dictator".

## 7.1 Construction of $G_{\mathcal{D}} = (V_{\mathcal{D}}, E_{\mathcal{D}})$ as an instance of $k$-HypVC-Partite

The integrality gap instance $G = (V, E)$ and the solution $x^*$ are fixed along with the parameter $m$ to depend only on $k$. Additionally, $\delta > 0$ is a small enough constant and $r$ is a parameter to the procedure and is the size of the domain of the dictatorship test. It corresponds to the size of the label set in the eventual reduction from Unique Games. We use the distributions $P^{x^*}_e$ and $M_\delta(P^{x^*}_e)$ over $\{0,1\}^k$ as given in Definition 7.2. The following steps describe the construction of the instance $\mathcal{D}$.

1. The set of vertices $V_{\mathcal{D}} = V \times \{0,1\}^r$. Each vertex in $V_{\mathcal{D}}$ is of the form $(v, y)$ where $v \in V$ and $y \in \{0,1\}^r$.
2. Let $a_v := x^*_v(1-\delta) + \delta$, for $v \in V$. Let $\mu_a$ be the $a$-biased probability measure on $\{0,1\}^r$, where every coordinate is chosen independently to be 1 with probability $a$. The weight $\mathrm{wt}_{\mathcal{D}}$ of any vertex $(v, y) \in V_{\mathcal{D}}$ is given by,
$$\mathrm{wt}_{\mathcal{D}}(v, y) := \frac{\mu_{a_v}(y)}{|V|}.$$
Note that $\sum_{(v,y)\in V_{\mathcal{D}}} \mathrm{wt}_{\mathcal{D}}((v, y)) = 1$.
3. The set of hyperedges $E_{\mathcal{D}}$ is the union of all the hyperedges output with positive probability in the following randomized procedure.
   a. Pick a random hyperedge $e = (v_1, \ldots, v_k)$ from $E$.
   b. Sample $r$ independent copies $(z^j_{v_1}, z^j_{v_2}, \ldots, z^j_{v_k})$ for $j = 1, \ldots, r$ from the distribution $M_\delta(P^{x^*}_e)$. Let $z_{v_i} \in \{0,1\}^r$ be defined as $z_{v_i} := (z^1_{v_i}, z^2_{v_i}, \ldots, z^r_{v_i})$ for $i = 1, \ldots, k$.

d. Add the following hyperedge in $E_{\mathcal{D}}$,

$$((v_1, z_{v_1}), (v_2, z_{v_2}), \ldots, (v_k, z_{v_k})).$$

4. Output the hypergraph $G_{\mathcal{D}} = (V_{\mathcal{D}}, E_{\mathcal{D}})$.

We first observe that the hypergraph $G_{\mathcal{D}}$ is indeed $k$-partite. To see this, recall that $G$ is $k$-partite. Since $V_{\mathcal{D}} = V \times \{0,1\}^r$, any partition of $V$ extends naturally to a partition of $V_{\mathcal{D}}$. Let $\{V_i\}_{i=1}^k$ be the $k$ disjoint subsets of $V$ comprising the $k$-partition. Then, $\{V_i \times \{0,1\}^r\}_{i=1}^k$ gives the $k$-partition of $V_{\mathcal{D}}$. Moreover, since any hyperedge $e' \in E_{\mathcal{D}}$ is of the form $((v_1, z_{v_1}), (v_2, z_{v_2}), \ldots, (v_k, z_{v_k}))$, where $(v_1, \ldots, v_k) \in E$, it is easy to see that $e'$ hits each set $V_i \times \{0,1\}^r$ $(1 \leq i \leq k)$ exactly once. Therefore, $G_{\mathcal{D}}$ is $k$-partite.

Next, we shall formally state the completeness and the soundness of the dictatorship test given by $\mathcal{D}$. Their proofs are completely analogous to the corresponding ones in [22]. While we shall present the proof of the completeness, we refer the reader to [22] for a proof of the soundness of the test. We need the notions of a function being a "dictator" and "far from a dictator".

**Definition 7.5** (Dictator)**.** *A subset $S \subseteq V \times \{0,1\}^r = V_{\mathcal{D}}$ is said to be a dictator if there is an index $i \in [r]$ such that $S = \{(v, y) \mid y_i = 1\}$.*

Given a subset $S \subseteq V \times \{0,1\}^r$, let $S_v := S \cap (\{v\} \times \{0,1\}^r)$ the function $f_v^S : \{0,1\}^r \mapsto \{0,1\}$ be the complement of $S_v$, i.e. $f_v^S = \{y \in \{0,1\}^r \mid (v, y) \notin S_v\}$. For a given $i \in [r]$ and $v \in V$, let $\mathrm{Inf}_i^{\leq d}(f_v^S)$ be the degree $d$ influence of the $i$-th coordinate, with respect to the measure $\mu_{a_v}$. The quantity $\mathrm{Inf}_i^{\leq d}(f)$ for any function $f$ measures the likelihood of the value of the function $f$ changing if the $i$th coordinate in the input is changed. We refer the reader to [24] for a formal definition of $\mathrm{Inf}_i^{\leq d}(f)$. Using this we have the following notion of a function being far from a dictator.

**Definition 7.6** ($(\tau, d)$-pseudorandom)**.** *For $\tau, d \geq 0$, a subset $S \subseteq V \times \{0,1\}^r$ is said to be $(\tau, d)$-pseudorandom if for every $v \in V$ and $i \in [r]$, $\mathrm{Inf}_i^{\leq d}(f_v^S) \leq \tau$.*

We first prove the completeness of the dictatorship test.

**Lemma 7.7.** *(Completeness) Suppose $S \subseteq V \times \{0,1\}^r = V_{\mathcal{D}}$ is a dictator. Then $S$ is a vertex cover in $G_{\mathcal{D}}$ and*

$$\mathrm{wt}_{\mathcal{D}}(S) \leq \frac{\mathrm{VAL}_{\mathtt{LP}}(x^*, G)}{|V|} + \delta.$$

*Proof.* Let $i \in [r]$ be such that $S = \{(v, y) \mid y_i = 1\}$. Let $e \in E$ be any hyperedge, where $e = (v_1, \ldots, v_k)$. Since the tuple $(z_{v_1}^i, z_{v_2}^i, \ldots, z_{v_k}^i)$ sampled in Step 3b is in the support of $M_\delta(P_e^{x^*})$ and therefore an element of $Q(k)$, we deduce that there is some $j \in [r]$ such that $z_{v_j}^i = 1$ i.e. $(v_j, z_{v_j}) \in S$. This implies that $S$ covers all hyperedges obtained after choosing $e$ in Step 3a. As this holds for any choice of $e \in E$, we obtain that $S$ is a vertex cover in $G_{\mathcal{D}}$. Also,

$$\mathrm{wt}_{\mathcal{D}}(S) = \frac{\sum_{v \in V} \sum_{y | y_i = 1} \mu_{a_v}(z)}{|V|} = \frac{\sum_{v \in V} (x_v^*(1-\delta) + \delta)}{|V|} \leq \frac{\mathrm{VAL}_{\mathtt{LP}}(x^*, G)}{|V|} + \delta,$$

which completes the proof of the lemma. ■

We state the following theorem regarding the soundness of the dictatorship test and refer the reader to [22] for the proof and the hardness reduction from Unique Games.

**Theorem 7.8** (Soundness)**.** *For every $\delta > 0$, there exists $d, \tau > 0$ such that if $S \subseteq V \times \{0,1\}^r$ is a vertex cover of $G_{\mathcal{D}}$ and is $(\tau, d)$-pseudorandom, then,*

$$\mathrm{wt}_{\mathcal{D}}(S) \geq \frac{\mathrm{OPT}_{VC}(G)}{|V|} - \delta.$$

## References


[1] R. Aharoni, R. Holzman, and M. Krivelevich. On a theorem of Lovász on covers in r-partite hypergraphs. *Combinatorica*, 16(2):149–174, 1996. 1, 2, 3, 4, 5, 19, 21, 25

[2] Noga Alon and Eyal Lubetzky. Uniformly cross intersecting families. *Combinatorica*, 29(4):389–431, 2009. 5

[3] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. *J. ACM*, 45(3):501–555, 1998. 13

[4] S. Arora and S. Safra. Probabilistic checking of proofs: A new characterization of NP. *J. ACM*, 45(1):70–122, 1998. 13

[5] Per Austrin, Subhash Khot, and Muli Safra. Inapproximability of vertex cover and independent set in bounded degree graphs. *Theory of Computing*, 7(1):27–43, 2011. 2

[6] N. Bansal and S. Khot. Inapproximability of hypergraph vertex cover and applications to scheduling problems. In *ICALP*, pages 250–261, 2010. 2

[7] I. Dinur, V. Guruswami, and S. Khot. Vertex cover on k-uniform hypergraphs is hard to approximate within factor (k-3-$\epsilon$). ECCC Technical Report TR02-027, 2002. 2

[8] Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. A new multilayered PCP and the hardness of hypergraph vertex cover. *SIAM J. Comput.*, 34:1129–1146, May 2005. 1, 2, 3, 4, 7, 11, 13, 18

[9] Irit Dinur and Samuel Safra. On the hardness of approximating minimum vertex cover. *Annals of Mathematics*, 162:439–485, 2004. 2

[10] T. Feder, R. Motwani, L. O'Callaghan, R. Panigrahy, and D. Thomas. Online distributed predicate evaluation. Technical Report, Stanford University, 2003. 2

[11] G. Gottlob and P. Senellart. Schema mapping discovery from data instances. *J. ACM*, 57(2), January 2010. 2, 3

[12] R. L. Graham, M. Grötschel, and L. Lovász, editors. *Handbook of combinatorics (vol. 2)*. MIT Press, Cambridge, MA, USA, 1995. 10

[13] V. Guruswami and R. Saket. On the inapproximability of vertex cover on $k$-partite $k$-uniform hypergraphs. In *ICALP*, pages 360–371, 2010. 1, 3

[14] J. Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, 2001. 2

[15] E. Halperin. Improved approximation algorithms for the vertex cover problem in graphs and hypergraphs. *SIAM J. Comput.*, 31(5):1608–1623, 2002. 2

[16] J. Holmerin. Improved inapproximability results for vertex cover on k -uniform hypergraphs. In *Proc. ICALP*, pages 1005–1016, 2002. 2

[17] L. Ilie, R. Solis-Oba, and S. Yu. Reducing the size of NFAs by using equivalences and preorders. In *Proc. CPM*, pages 310–321, 2005. 2, 3

[18] G. Karakostas. A better approximation ratio for the vertex cover problem. In *ICALP*, pages 1043–1050, 2005. 2

[19] R. M. Karp. Reducibility among combinatorial problems. *Complexity of Computer Computations*, pages 85–103, 1972. 2

[20] S. Khot. On the power of unique 2-prover 1-round games. In *Proc.* $34^{th}$ *ACM STOC*, pages 767–775, 2002. 2, 3

[21] S. Khot and O. Regev. Vertex cover might be hard to approximate to within 2-epsilon. *J. Comput. Syst. Sci.*, 74(3):335–349, 2008. 2

[22] A. Kumar, R. Manokaran, M. Tulsiani, and N. K. Vishnoi. On LP-based approximability for strict CSPs. In *Proc. SODA*, pages 1560–1573, 2011. 1, 3, 4, 5, 19, 21, 22, 23, 24

[23] L. Lovász. On minimax theorems of combinatorics. *Doctoral Thesis, Mathematiki Lapok*, 26:209–264, 1975. 1, 2

[24] E. Mossel. Gaussian bounds for noise correlation of functions and tight analysis of long codes. In *Proc.* $49^{th}$ *IEEE FOCS*, pages 156–165, 2008. 4, 5, 23

[25] P. Raghavendra. Optimal algorithms and inapproximability results for every CSP? In *Proc.* $40^{th}$ *ACM STOC*, pages 245–254, 2008. 4

[26] R. Raz. A parallel repetition theorem. *SIAM J. Comput.*, 27(3):763–803, 1998. 13

[27] Sushant Sachdeva and Rishi Saket. Nearly optimal NP-hardness of vertex cover on k-uniform k-partite hypergraphs. In *APPROX-RANDOM*, pages 327–338, 2011. 1, 3

[28] L. Trevisan. Non-approximability results for optimization problems on bounded degree instances. In *Proc.* $33^{rd}$ *ACM STOC*, pages 453–461, 2001. 2

# A LP Integrality Gap for $k$-HypVC-Partite

This section describes the $\frac{k}{2} - o(1)$ integrality gap construction of Aharoni *et al.* [1] for the standard LP relaxation for $k$-HypVC-Partite. The hypergraph that is constructed is unweighted.

Let $r$ be a (large) positive integer. The vertex set $V$ of the hypergraph is partitioned into subsets $V_1, \ldots, V_k$ where, for all $i = 1, \ldots, k$,

$$V_i = \{x_{ij} \mid j = 1, \ldots, r\} \cup \{y_{il} \mid l = 1, \ldots, rk + 1\}. \tag{36}$$

Before we define the hyperedges, for convenience we shall define the LP solution. The LP values of the vertices are as given by the function $h : V \to [0, 1]$ as follows: for all $i = 1, \ldots, k$,

$$\begin{aligned} h(x_{ij}) &= \frac{2j}{rk}, \qquad \forall j = 1, \ldots, r \\ h(y_{il}) &= 0, \qquad \forall l = 1, \ldots, rk + 1. \end{aligned}$$

The set of hyperedges is naturally defined to be the set of all possible hyperedges, choosing exactly one vertex from each $V_i$ such that the sum of the LP values of the corresponding vertices is at least 1. Formally,

$$E = \{e \subseteq V \mid \forall i \in [k], \ |e \cap V_i| = 1 \text{ and } \sum_{v \in e} h(v) \geq 1\}. \tag{37}$$

Clearly the graph is $k$-uniform and $k$-partite with $\{V_i\}_{i \in [k]}$ being the $k$-partition of $V$.

The value of the LP solution is

$$\sum_{v \in V} h(v) = k \sum_{j \in [r]} \frac{2j}{rk} = r + 1. \tag{38}$$

Now let $V'$ be a minimum vertex cover in the hypergraph. To lower bound the size of the minimum vertex cover, we first note that the set $\{v \in V \mid h(v) > 0\}$ is a vertex cover of size $rk$, and therefore $|V'| \leq rk$. Also, for any $i \in [k]$ the vertices $\{y_{il}\}_{l \in [rk+1]}$ have the same neighborhood. Therefore, we can assume that $V'$ has no vertex $y_{il}$, otherwise it will contain at least $rk + 1$ such vertices.

For all $i \in [k]$ let define indices $j_i \in [r] \cup \{0\}$ as follows:

$$j_i = \begin{cases} 0 & \text{if: } \forall j \in [r], \ x_{ij} \in V', \\ \max\{j \in [r] \mid x_{ij} \notin V'\} & \text{otherwise.} \end{cases} \tag{39}$$

It is easy to see that since $V'$ is a vertex cover,

$$\sum_{i \in [k]} h(x_{ij_i}) < 1,$$

which implies,

$$\sum_{i \in [k]} j_i < \frac{rk}{2}.$$

Also, the size of $V'$ is lower bounded by $\sum_{i \in [k]} (r - j_i)$. Therefore,

$$|V'| \geq \sum_{i \in [k]} (r - j_i) \geq rk - \sum_{i \in [k]} j_i \geq rk - \frac{rk}{2} = \frac{rk}{2}. \tag{40}$$

The above combined with the value of the LP solution yields an integrality gap of $\frac{rk}{2(r+1)} \geq \frac{k}{2} - o(1)$ for large enough $r$.



ECCC ISSN 1433-8092
http://eccc.hpi-web.de